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Statistical properties of a kernel-type estimator of the intensity function of a cyclic Poisson process

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Abstract

We consider a kernel-type nonparametric estimator of the intensity function of a cyclic Poisson process when the period is unknown. We assume that only a single realization of the Poisson process is observed in a bounded window which expands in time. We compute the asymptotic bias, variance, and the mean-squared error of the estimator when the window indefinitely expands.

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1. Introduction, main assumptions and definitions

In [4] we constructed a consistent estimator (cf. (1.3) and Theorem 1.1 below) of a cyclic Poisson intensity function λ under the following assumptions:

(a) The period (i.e., cycle) of the intensity function λ is unknown.

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(b) Only a single realization of the Poisson process X is available in a window $W_n \subset \mathbf{R}$.
(c) The window W_n is bounded for any "time" instance n but expands when n increases.

There are many practical situations where estimating cyclic Poisson intensity functions under assumptions (a)–(c) is of importance. In [4] we presented a review of such applications, and a number of them can also be found in the monographs by Cox and Lewis [2], Lewis [8], Daley and Vere-Jones [3], Karr [6], Snyder and Miller [11], Reiss [10], and Kutoyants [7].

We shall now introduce and discuss further notations and assumptions to be used throughout the paper.

Let X be a Poisson point process on the real line $\mathbf{R}$ with (unknown) locally integrable intensity function λ. We assume throughout that λ is periodic with (unknown) period

$$\tau > 0, \tag{1.1}$$

that is, $\lambda(z + k\tau) = \lambda(z)$ for any real $z \in \mathbf{R}$ and any integer $k \in \mathbf{Z}$.

Let the windows $W_1, W_2, \ldots \subset \mathbf{R}$ be intervals of finite length $|W_n|$ such that

$$|W_n| \to \infty$$

when $n \to \infty$. (Unless confusion is likely, we shall suppress $n \to \infty$ throughout the paper.)

Assume that the Poisson process X has been observed in W_n and a consistent estimator $\hat{\tau}_n \geqslant 0$ of τ has been constructed. That is, let

$$\hat{\tau}_n \to_P \tau, \tag{1.2}$$

where $\to_P$ stands for the convergence in probability. For example, one may consider using the estimators constructed by Vere-Jones [12], Mangku [9], Bebbington and Zitikis [1].

For estimating the intensity λ at a point s, in [4] we suggested the following estimator:

$$\hat{\lambda}_{n,K}(s) := \frac{\hat{\tau}_n}{|W_n|} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} K\left(\frac{x - (s + k\hat{\tau}_n)}{h_n}\right) X(dx). \tag{1.3}$$

In order to demonstrate that $\hat{\lambda}_{n,K}(s)$ is a consistent estimator of $\lambda(s)$, we need to impose several assumptions. Namely, let s be a Lebesgue point of the intensity function λ. Furthermore, let $h_1, h_2, \ldots$ be (strictly) positive real numbers such that

$$h_n \to 0, \tag{1.4}$$

$$h_n |W_n| \to \infty. \tag{1.5}$$

Finally, let the kernel function K be a bounded probability density function with (closed) support $\mathrm{supp}(K) \subseteq [-1, 1]$. If it is not stated otherwise (cf., e.g., Section 5 below), we also assume that K has only a finite number of discontinuities. Under the assumptions above, in [4] we proved consistency of the estimator $\hat{\lambda}_{n,K}(s)$, as well as obtained a rate of consistency. In particular, we proved the following theorem.

Theorem 1.1 (Helmers et al. [4]). *Let the following assumption*

$$\mathbf{P}\left\{\frac{|W_n|}{h_n}|\hat{\tau}_n - \tau| \geqslant \delta\right\} = o(1) \tag{1.6}$$

hold for any $\delta > 0$. *Then the estimator* $\hat{\lambda}_{n,K}(s)$ *is* (*weakly*) *consistent.*

In the present paper we focus on further statistical properties of the estimator $\hat{\lambda}_{n,K}(s)$: asymptotic unbiasedness (cf. Section 2 below), asymptotic behaviour of the variance and the mean-squared error (cf. Section 3 below). In fact, in our considerations below we shall use the following modification of the estimator $\hat{\lambda}_{n,K}(s)$:

$$\hat{\lambda}^{\diamond}_{n,K}(s) := \mathbf{I}\{\hat{\lambda}_{n,K}(s) \leqslant D_n\}\hat{\lambda}_{n,K}(s) + \mathbf{I}\{\hat{\lambda}_{n,K}(s) > D_n\}D_n, \tag{1.7}$$

where "truncating" constants D_n are deterministic and converge to infinity when $n \to \infty$. We shall see from the main results below that the choice of D_n depends on how well the estimator $\hat{\tau}_n$ estimates τ. Specifically, the closer the estimator $\hat{\tau}_n$ is to τ, the large the value of D_n can be taken. This is natural since errors made when estimating the period τ are accumulated and enlarged a number of times when estimating the intensity function $\lambda(s)$ itself. In the extreme case when τ is known, we can certainly choose $\hat{\tau}_n := \tau$ and thus, in turn and somewhat formally, $D_n = \infty$ for any n. The latter reduces the estimator $\hat{\lambda}^{\diamond}_{n,K}(s)$ to $\hat{\lambda}_{n,K}(s)$ considered in [4]. The asymptotic unbiasedness of $\hat{\lambda}_{n,K}(s)$ is easy:

$$\begin{aligned}
\mathbf{E}\hat{\lambda}_{n,K}(s) &= \frac{\tau}{|W_n|}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)\,dx \\
&\approx \int_{\mathbf{R}} K(x)\lambda(h_n x + s)\,dx \\
&\to \lambda(s).
\end{aligned} \tag{1.8}$$

We note that the convergence to $\lambda(s)$ in (1.8) is due to the assumptions that K is a probability density function and s is a Lebesgue point of λ. For more details on the case when the period τ is known (but in more complicated than purely periodic situations) we refer to Helmers and Zitikis [5].

2. Results: asymptotic unbiasedness

In this section we present two main results: Theorems 2.1 and 2.2. In the first theorem we prove the asymptotic unbiasedness of $\hat{\lambda}^{\diamond}_{n,K}(s)$, whereas in Theorem 2.2 we consider the rate of convergence of $\mathbf{E}\hat{\lambda}^{\diamond}_{n,K}(s)$ to $\lambda(s)$. Naturally, the performance of $\hat{\lambda}^{\diamond}_{n,K}(s)$ depends on the performance of $\hat{\tau}_n$, the fact that is reflected by assumptions (2.1) and (2.4).

Theorem 2.1. *Assuming that, for any* $\delta > 0$,

$$\mathbf{P}\left\{\frac{|W_n|}{h_n}|\hat{\tau}_n - \tau| \geqslant \delta\right\} = o\left(\frac{1}{D_n}\right), \tag{2.1}$$

we have that

$$\mathbf{E}\hat{\lambda}^{\diamond}_{n,K}(s) \to \lambda(s). \tag{2.2}$$

Assumption (2.1) connects the truncation level D_n in the definition of $\hat{\lambda}^{\diamond}_{n,K}$ with the rate of convergence of $\hat{\tau}_n$ to τ. We note in this regard that if the construction of $\hat{\tau}_n$ allows one to calculate, or estimate, the second moment $\mathbf{E}(\hat{\tau}_n - \tau)^2$, then the verification of (2.1) can be carried out with the help of the following (somewhat stronger) condition:

$$\mathbf{E}(\hat{\tau}_n - \tau)^2 = o\left(\frac{h_n^2}{D_n|W_n|^2}\right). \tag{2.3}$$

Using (2.3), we can now describe a class of possible "truncation levels"' D_n, depending on h_n, W_n, and the rate of convergence of $\mathbf{E}(\hat{\tau}_n - \tau)^2$ to 0.

Theorem 2.2. *Let the second derivative* $\lambda''(s)$ *exist and be finite. Let the kernel K be symmetric around* 0 *and satisfy the Lipschitz condition between the (finite number of) discontinuity points. Furthermore, let the sequence* D_n *be such that, for some* $c > 0$ *and* $\varepsilon > 0$, *the bound* $D_n \geqslant ch_n^{-\varepsilon}$ *holds for all sufficiently large n, and let* $h_n^2|W_n| \to \infty$. *Assuming that, for any* $\delta > 0$,

$$\mathbf{P}\left\{\frac{|W_n|}{h_n^3}|\hat{\tau}_n - \tau| \geqslant \delta\right\} = o\left(\frac{h_n^2}{D_n}\right), \tag{2.4}$$

we have that

$$\mathbf{E}\hat{\lambda}^{\diamond}_{n,K}(s) = \lambda(s) + \frac{1}{2}\lambda''(s)h_n^2\int_{-1}^{1}x^2K(x)\,dx + o(h_n^2). \tag{2.5}$$

Note that contrary to Theorem 2.1, in Theorem 2.2 we require that the truncation level D_n should not be too low, that is, $D_n \geqslant ch_n^{-\varepsilon}$. This is so in order to be able to extract the term $\frac{1}{2}\lambda''(s)h_n^2\int_{-1}^{1}x^2K(x)\,dx$ out of the estimator $\hat{\lambda}^{\diamond}_{n,K}(s)$ with the desired error $o(h_n^2)$. Note also that, given the constraints of Theorem 2.2, if we take the lowest truncation level $D_n = ch_n^{-\varepsilon}$, then we shall get the weakest assumption (2.4), that is,

$$\mathbf{P}\left\{\frac{|W_n|}{h_n^3}|\hat{\tau}_n - \tau| \geqslant \delta\right\} = o(h_n^{2+\varepsilon}).$$

The main reason for formulating a result like Theorem 2.2 with general D_n is to allow some needed flexibility when combining results with different sequences D_n. We employ this observation, for example, in deriving (3.5) below, which is a consequence of two results: Theorems 2.2 and 3.2.

In Theorem 2.2 we assume that $h_n^2|W_n| \to \infty$, which is a stronger assumption than (1.5). In fact, without assuming $h_n^2|W_n| \to \infty$, we can only prove that the remainder term on the right-hand side of (2.5) is of the order $o(h_n^2) + O(|W_n|^{-1})$. Since the second term on the right-hand side of (2.5) is exactly of the order $O(h_n^2)$, it is therefore natural to have $|W_n|^{-1} = o(h_n^2)$, which is the assumption $h_n^2|W_n| \to \infty$ in Theorem 2.2.

3. Results: asymptotic variance and mean-squared error

In the following two theorems we consider the convergence of the variance $\mathbf{Var}(\hat{\lambda}^{\diamond}_{n,K}(s))$ to 0, as well as the rate of convergence. Combining these two results with those in the previous section, we in turn obtain the corresponding results about the asymptotic behaviour of the mean-squared error of $\hat{\lambda}^{\diamond}_{n,K}(s)$.

Theorem 3.1. *Assuming that, for any* $\delta > 0$,

$$\mathbf{P}\left\{\frac{|W_n|}{h_n}|\hat{\tau}_n - \tau| \geqslant \delta\right\} = o\left(\frac{1}{D_n^2}\right), \tag{3.1}$$

we have that

$$\boldsymbol{Var}(\hat{\lambda}^{\diamond}_{n,K}(s)) \to 0. \tag{3.2}$$

Using Theorems 2.1 and 3.1, we immediately obtain that under assumption (3.1) the mean-squared error of $\hat{\lambda}^{\diamond}_{n,K}(s)$ converges to 0.

In view of the discussion immediately after Theorem 2.1, it should not be surprising that the rate $o(D_n^{-2})$ is assumed in Theorem 3.1, if compared to $o(D_n^{-1})$ in Theorem 2.1. Indeed, even moderate errors when estimating τ may enlarge the variance of $\hat{\lambda}^{\diamond}_{n,K}(s)$ in a more profound way than in the case of the mean $\mathbf{E}\hat{\lambda}^{\diamond}_{n,K}(s)$.

In Theorem 3.2 below we derive the first asymptotic term of the variance $\mathbf{Var}(\hat{\lambda}^{\diamond}_{n,K}(s))$ and in this way demonstrate that the variance is of the order $O(1/(|W_n|h_n))$. Naturally, the result requires stronger assumptions than those in Theorem 3.1.

Theorem 3.2. *Let the kernel K satisfy the Lipschitz condition between the (finite number of) discontinuity points. Furthermore, let the sequence* D_n *be such that, for some* $c > 0$ *and* $\varepsilon > 0$, *the bound* $D_n \geqslant c(h_n|W_n|)^{\varepsilon}$ *holds for all sufficiently large n.*

Assuming that, for any $\delta > 0$,

$$\mathbf{P}\left\{\frac{|W_n|^{3/2}}{h_n^{1/2}}|\hat{\tau}_n - \tau| \geqslant \delta\right\} = o\left(\frac{1}{D_n^2|W_n|h_n}\right), \tag{3.3}$$

we have that

$$\mathbf{Var}(\hat{\lambda}^{\diamond}_{n,K}(s)) = \frac{\tau\lambda(s)}{|W_n|h_n}\int_{-1}^{1} K^2(x)\,dx + o\left(\frac{1}{|W_n|h_n}\right). \tag{3.4}$$

Using Theorems 2.2 and 3.2, we derive the following asymptotic formula for the mean-squared error of $\hat{\lambda}^{\diamond}_{n,K}(s)$:

$$\frac{\tau\lambda(s)}{|W_n|h_n}\int_{-1}^{1} K^2(x)\,dx + \frac{1}{4}\left(\lambda''(s)\int_{-1}^{1} x^2K(x)\,dx\right)^2 h_n^4 + R_n, \tag{3.5}$$

where the remainder term R_n is of the order $o((|W_n|h_n)^{-1}) + o(h_n^4)$. Minimizing the sum of the two main terms in (3.5), we obtain the following (optimal) choice for the bandwidth h_n:

$$h_n = (c_0/|W_n|)^{1/5}, \tag{3.6}$$

where the constant c_0 is defined by the formula

$$c_0 := \tau\lambda(s)\int_{-1}^{1} K^2(x)\,dx \bigg/ \left(\lambda''(s)\int_{-1}^{1} x^2K(x)\,dx\right)^2.$$

Using the just obtained bandwidth in formula (3.5), we obtain that the mean-squared error of $\hat{\lambda}^{\diamond}_{n,K}(s)$ is of the order $O(|W_n|^{-4/5})$.

4. A comparison of the current and classical results

The main results in the previous two sections closely resemble the corresponding ones in the classical kernel-type density estimation. To demonstrate this we now construct an artificial density function f as follows:

$$f(s) := \begin{cases} \frac{1}{\theta\tau}\lambda(s), & s \in [0, \tau], \\ 0 & \text{otherwise}, \end{cases}$$

where $\theta := \tau^{-1}\int_0^\tau \lambda(s)\,ds$. For the sake of argument, assume that both the period τ and the parameter θ are known; this is an unrealistic but convenient assumption to demonstrate the connection between the results of this paper and those in the classical area of kernel-type density estimation. Under the assumptions above, the quantity

$$\hat{f}_{n,K}(s) := \frac{1}{\theta\tau}\hat{\lambda}^{\diamond}_{n,K}(s)$$

can be viewed as an estimator of $f(s)$.

Applying (2.5) in the situation described above, we obtain

$$\begin{aligned}\mathbf{E}\hat{f}_{n,K}(s) &= \frac{1}{\theta\tau}\lambda(s) + \frac{f''(s)\theta\tau}{2\theta\tau}h_n^2\int_{-1}^{1}x^2K(x)\,dx + o(h_n^2) + O\left(\frac{1}{|W_n|}\right)\\ &= f(s) + \left[\frac{f''(s)}{2}h_n^2\int_{-1}^{1}x^2K(x)\,dx\right] + o(h_n^2) + O\left(\frac{1}{|W_n|}\right). \end{aligned}\tag{4.1}$$

Note that the term in brackets $[\cdot]$ on the right-hand side of (4.1) is the same as the well-known formula for the asymptotic bias in the classical kernel-type density estimation.

Applying (3.4) in the situation described above, we obtain the following formula:

$$\begin{aligned}\mathbf{Var}\,(\hat{f}_{n,K}(s)) &= \frac{1}{(\theta\tau)^2}\frac{\tau f(s)(\theta\tau)}{|W_n|h_n}\int_{-1}^{1}K^2(x)\,dx + o\left(\frac{1}{|W_n|h_n}\right)\\ &= \frac{f(s)}{\theta|W_n|h_n}\int_{-1}^{1}K^2(x)\,dx + o\left(\frac{1}{|W_n|h_n}\right). \end{aligned}\tag{4.2}$$

Note that since λ is periodic, $\mathbf{E}X(W_n)$ is approximately $\theta|W_n|$. Hence, it is appropriate to compare $\theta|W_n|$ in the context of the current paper with the sample size N in the context of kernel-type density estimation. Therefore, replacing $\theta|W_n|$ on the right-hand side of (4.2) by N, we reduce the right-hand side of (4.2) to the following well-known expression for the variance in the kernel density estimation:

$$\mathbf{Var}(\hat{f}_{n,K}(s)) = \frac{1}{Nh_n}f(s)\int_{-1}^{1}K^2(x)\,dx + o\left(\frac{1}{Nh_n}\right).\tag{4.3}$$

Combining (4.1) and (4.3), we obtain the corresponding formulas for the mean-squared error of $\hat{f}_{n,K}(s)$, which are in parallel to the corresponding ones in the classical area of the kernel density estimation.

5. Assumptions on the kernel K

When formulating the results in the previous sections we assumed that the kernel K has only a finite number of discontinuities. This assumption is needed to control the fluctuations of the function

$$x \mapsto K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)\tag{5.1}$$

depending on the fluctuations of $\hat{\tau}_n$ around τ. In fact, the assumption concerning the finite number of discontinuities of K can be weakened even further. Namely, straightforward calculations show that the results of the present paper hold under the following assumption.

Assumption 5.1. For any $\alpha > 0$, there exists a finite collection of disjoint compact intervals $B_1, \ldots, B_{M_\alpha}$ and a continuous function $K_\alpha : \mathbf{R} \to \mathbf{R}$ such that

(i) the Lebesgue measure of the set $[-1, 1] \setminus \bigcup_{i=1}^{M_\alpha} B_i$ does not exceed α, and
(ii) $|K(u) - K_\alpha(u)| \leqslant \alpha$ for all $u \in \bigcup_{i=1}^{M_\alpha} B_i$.

Furthermore, note that by the classical Weierstrass theorem, the continuous function K_α in Assumption 5.1 can be replaced by a Lipschitz function L_α. Thus, without loss of generality, assumption (ii) can be replaced by the following one:

(iii) $|K(u) - L_\alpha(u)| \leqslant \alpha$ for all $u \in \bigcup_{i=1}^{M_\alpha} B_i$.

However, in order to make the proofs shorter and more transparent, in the following sections we present them only in the case when

$$\text{the kernel function } K \text{ is a Lipschitz function on the whole real line } \mathbf{R}, \tag{5.2}$$

which is a stronger requirement than Assumption 5.1. However, generalizing the proofs to piecewise Lipschitzian kernel functions K—which would mean presenting the proofs under Assumption 5.1—is straightforward and thus omitted from the current paper.

We conclude this section with a notation that we use in the proofs below:

$$\Delta K_{k,n}(x) := K\left(\frac{x - (s + k\hat{\tau}_n)}{h_n}\right) - K\left(\frac{x - (s + k\tau)}{h_n}\right).$$

6. Proof of Theorem 2.1

Denote

$$A_n := \left\{ |\hat{\tau}_n - \tau| \leqslant \frac{\delta h_n}{|W_n|} \right\}. \tag{6.1}$$

With this notation, we have the representation

$$\mathbf{E}\hat{\lambda}^{\diamond}_{n,K}(s) = \Gamma_n(1) - \Gamma_n(2) + \Gamma_n(3) + \Gamma_n(4) + \Gamma_n(5), \tag{6.2}$$

where

$$\Gamma_n(1) := \mathbf{E}(\mathbf{I}\{A_n^c\}\mathbf{I}\{\hat{\lambda}_{n,K}(s) \leqslant D_n\}\hat{\lambda}_{n,K}(s)),$$

$$\Gamma_n(2) := \mathbf{E}(\mathbf{I}\{A_n\}\mathbf{I}\{\hat{\lambda}_{n,K}(s) > D_n\}\hat{\lambda}_{n,K}(s)),$$

$$\Gamma_n(3) := \mathbf{E}(\mathbf{I}\{A_n\}\hat{\lambda}_{n,K}(s)),$$

$$\Gamma_n(4) := D_n\mathbf{E}(\mathbf{I}\{A_n^c\}\mathbf{I}\{\hat{\lambda}_{n,K}(s) > D_n\}),$$

$$\Gamma_n(5) := D_n\mathbf{E}(\mathbf{I}\{A_n\}\mathbf{I}\{\hat{\lambda}_{n,K}(s) > D_n\}).$$

We shall demonstrate below that by taking n sufficiently large and/or $\delta > 0$ sufficiently small we can make the quantities $\Gamma_n(k)$, $k = 1, 2, 4, 5$, as small as desired, and the quantity $\Gamma_n(3)$ as close to $\lambda(s)$ as desired. These statements together with (6.2) will then complete the proof of Theorem 2.1.

The quantities $\Gamma_n(1)$ and $\Gamma_n(4)$ do not exceed $D_n \mathbf{P}(A_n^c)$. Due to assumption (2.1), the quantity $D_n \mathbf{P}(A_n^c)$ converges to 0 for any fixed $\delta > 0$. This proves the desired smallness of $\Gamma_n(1)$ and $\Gamma_n(4)$.

The quantities $\Gamma_n(2)$ and $\Gamma_n(5)$ do not exceed $D_n^{-1}\mathbf{E}(\mathbf{I}\{A_n\}\hat{\lambda}_{n,K}^2(s))$. Since $D_n \to \infty$, the desired smallness of $\Gamma_n(2)$ and $\Gamma_n(5)$ follows if the expectation $\mathbf{E}(\mathbf{I}\{A_n\}\hat{\lambda}_{n,K}^2(s))$ is asymptotically bounded. The latter statement follows from statement (8.2) below. We ought to note in this regard that the proof of (8.2) is a part of the proof of Theorem 3.1 which is formulated under stronger assumption (3.1) than (2.1). However, we shall see below that the proof of (8.2) does not require (3.1) and can be carried out under (2.1) only. With these notes we conclude the proof of the desired smallness of $\Gamma_n(2)$ and $\Gamma_n(5)$.

We shall now prove that $\limsup_{n\to\infty} |\Gamma_n(3) - \lambda(s)|$ can be made as small as desired by taking $\delta > 0$ sufficiently small. We start with the representation

$$\Gamma_n(3) = \Lambda_n(1) + \Lambda_n(2) + \Lambda_n(3), \tag{6.3}$$

where

$$\Lambda_n(1) := \mathbf{E}\left(\mathbf{I}\{A_n\}\left(1 - \frac{\tau}{\hat{\tau}_n}\right)\hat{\lambda}_{n,K}(s)\right),$$

$$\Lambda_n(2) := \frac{\tau}{|W_n| h_n}\mathbf{E}\left(\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n} \Delta K_{k,n}(x) X(dx)\right),$$

$$\Lambda_n(3) := \frac{\tau}{|W_n| h_n}\mathbf{E}\left(\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\frac{x - (s + k\tau)}{h_n}\right) X(dx)\right).$$

In Lemmas 6.1–6.3 below we prove that by taking n sufficiently large and $\delta > 0$ sufficiently small we can make the quantities $\Lambda_n(1)$ and $\Lambda_n(2)$ as small as desired and the quantity $\Lambda_n(3)$ as close to $\lambda(s)$ as desired.

Lemma 6.1. *For any fixed $\delta > 0$, we have $\lim_{n\to\infty}\Lambda_n(1) = 0$.*

Proof. We start the proof with the note that if $\hat{\tau}_n = 0$, then $\hat{\lambda}_{n,K}(s) = 0$. Thus, we can and thus do restrict ourselves to the case $\hat{\tau}_n > 0$ only. Since the kernel K is bounded and has support in $[-1, 1]$, we obtain that

$$\begin{aligned}
\hat{\lambda}_{n,K}(s) &\leqslant c\frac{\hat{\tau}_n}{|W_n| h_n}\int_{W_n}\sum_{k=-\infty}^{\infty}\mathbf{I}\left\{\frac{x - s}{\hat{\tau}_n} + k \in \frac{h_n}{\hat{\tau}_n}[-1, 1])\right\} X(dx) \\
&\leqslant c\frac{\hat{\tau}_n}{|W_n| h_n}\sup_{z\in\mathbf{R}}\left(\sum_{k=-\infty}^{\infty}\mathbf{I}\left\{z + k \in \frac{h_n}{\hat{\tau}_n}[-1, 1])\right\}\right) X(W_n).
\end{aligned} \tag{6.4}$$

For any real ρ we have the bound

$$\sup_{z\in\mathbf{R}}\left(\sum_{k=-\infty}^{\infty}\mathbf{I}\{z+k\in\rho[-1,1])\}\right)\leqslant 2|\rho|+1. \tag{6.5}$$

Applying (6.5) on the right-hand side of (6.4), we obtain that, for all sufficiently large n,

$$\begin{aligned}\hat{\lambda}_{n,K}(s)&\leqslant c\left\{\frac{\hat{\tau}_n}{h_n}+1\right\}\frac{X(W_n)}{|W_n|}\\&\leqslant\frac{c}{h_n}\frac{X(W_n)}{|W_n|},\end{aligned} \tag{6.6}$$

where the right-most bound in (6.6) was obtained using the fact that we have restricted ourselves to the event A_n when calculating the expectation in the definition of $\Lambda_n(1)$. Using (6.6), we in turn obtain that, for all sufficiently large n,

$$\begin{aligned}|\Lambda_n(1)|&\leqslant\frac{c}{h_n}\mathbf{E}\left(\mathbf{I}\{A_n\}\left|1-\frac{\tau}{\hat{\tau}_n}\right|\frac{X(W_n)}{|W_n|}\right)\\&\leqslant c\frac{\delta}{|W_n|}\mathbf{E}\left(\frac{X(W_n)}{|W_n|}\right).\end{aligned} \tag{6.7}$$

It is easy to check that, for any $p\geqslant 1$,

$$\lim_{n\to\infty}\ \mathbf{E}(X(W_n)/|W_n|)^p<\infty. \tag{6.8}$$

Using (6.8) with $p=1$ and since $|W_n|\to\infty$ by assumption, the right-hand side of (6.7) converges to 0. This completes the proof of Lemma 6.1. □

Lemma 6.2. *There is a constant $c<\infty$ such that for all sufficiently large n we have the bound $\Lambda_n(2)\leqslant c\delta$ for all $\delta>0$. Thus, by choosing $\delta>0$ sufficiently small, we can make the quantity $\limsup_{n\to\infty}\Lambda_n(2)$ as small as desired.*

Proof. Let

$$\hat{u}:=\frac{x-(s+k\hat{\tau}_n)}{h_n},\quad u:=\frac{x-(s+k\tau)}{h_n}.$$

Since the support of the kernel K is in the interval $[-1,1]$, the difference $K(\hat{u})-K(u)$ can be decomposed in the following way:

$$\begin{aligned}K(\hat{u})-K(u)&=(K(\hat{u})-K(u))\mathbf{I}\{\hat{u}\in[-1,1]\}\\&\quad+K(u)(\mathbf{I}\{\hat{u}\in[-1,1]\}-\mathbf{I}\{u\in[-1,1]\}).\end{aligned} \tag{6.9}$$

Using decomposition (6.9), we write $\Lambda_n(2)$ as follows:

$$\Lambda_n(2)=\Theta_n(1)+\Theta_n(2), \tag{6.10}$$

where

$$\Theta_n(1) := \frac{\tau}{|W_n| h_n} \times \mathbf{E}\left(\mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} \int_{W_n} \Delta K_{k,n}(x) \mathbf{I}\left\{\frac{x-(s+k\hat{\tau}_n)}{h_n} \in [-1,1]\right\} X(dx)\right),$$

$$\Theta_n(2) := \frac{\tau}{|W_n| h_n} \mathbf{E}\left(\mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) \times \left(\mathbf{I}\left\{\frac{x-(s+k\hat{\tau}_n)}{h_n} \in [-1,1]\right\} - \mathbf{I}\left\{\frac{x-(s+k\tau)}{h_n} \in [-1,1]\right\}\right) X(dx)\right).$$

We shall prove below that $\Theta_n(1)$ and $\Theta_n(2)$ can be made as small as desired by taking n sufficiently large and $\delta > 0$ sufficiently small.

We start with $\Theta_n(1)$. Since K is a Lipschitz function (cf. assumption (5.2)), we have that

$$\Theta_n(1) \leqslant \frac{c}{|W_n| h_n} \mathbf{E}\left(\mathbf{I}\{A_n\} \frac{|\hat{\tau}_n - \tau|}{h_n} \sum_{k=-\infty}^{\infty} k X(\{s + k\hat{\tau}_n + h_n[-1,1]\} \cap W_n)\right). \tag{6.11}$$

The infinite sum $\sum_{k=-\infty}^{\infty}$ on the right-hand side has only a finite number of nonzero summands. Thus, we replace $\sum_{k=-\infty}^{\infty}$ by $\sum_{k \in \mathscr{K}}$, where the set $\mathscr{K} \subset \mathbf{Z}$ is finite and such that the number κ_n of elements in $\mathscr{K}$ satisfies the asymptotic relationship $\kappa_n \approx c|W_n|$, where the constant c does not depend on n and δ. Next, we estimate k on the right-hand side of (6.11) by $c|W_n|$. Furthermore, we estimate $|\hat{\tau}_n - \tau|$ on the right-hand side of (6.11) by $\delta h_n/|W_n|$. Consequently, we have the bound

$$\Theta_n(1) \leqslant c\delta \Psi_n^{\circ}, \tag{6.12}$$

where

$$\Psi_n^{\circ} := \frac{1}{|W_n| h_n} \sum_{k=-\infty}^{\infty} \mathbf{E} X\left(\left\{s + k\tau + k\frac{\delta h_n}{|W_n|}[-1,1] + h_n[-1,1]\right\} \cap W_n\right). \tag{6.13}$$

Let $\mathscr{K}$ be now (possibly another) subset of $\mathbf{Z}$ such that the expectations on the right-hand side of (6.13) are nonzero for any $k \in \mathscr{K}$, and let κ_n be the number of elements in $\mathscr{K}$. We obtain

$$\begin{aligned}
\Psi_n^{\circ} &= \frac{1}{|W_n| h_n} \sum_{k \in \mathscr{K}} \mathbf{E} X\left(\left\{s + k\tau + k\frac{\delta h_n}{|W_n|}[-1,1] + h_n[-1,1]\right\} \cap W_n\right) \\
&\leqslant \frac{1}{|W_n| h_n} \sum_{k \in \mathscr{K}} \mathbf{E} X\left(s + k\frac{\delta h_n}{|W_n|}[-1,1] + h_n[-1,1]\right) \\
&\leqslant \frac{\kappa_n}{|W_n| h_n} \mathbf{E} X(s + c\delta h_n[-1,1] + h_n[-1,1]).
\end{aligned}$$

Thus, $\limsup_{n\to\infty} \Psi_n^\circ$ does not exceed a constant. In view of (6.12), the latter fact implies that $\lim_{n\to\infty} \Theta_n(1)$ can be made as small as desired by taking $\delta > 0$ sufficiently small.

We shall now prove that $\limsup_{n\to\infty} \Theta_n(2)$ can be made as small as desired by taking $\delta > 0$ sufficiently small. For this, we first estimate the difference of the two indicators in the definition of $\Theta_n(2)$ as follows. First, we rewrite $k\hat{\tau}_n$ as the sum of $k\tau$ and $k(\hat{\tau}_n - \tau)$. Then, we estimate $k(\hat{\tau}_n - \tau)$ by $k\delta h_n/|W_n|$. Since the number κ_n of nonzero summands in the definition of $\Theta_n(2)$ is of the order $\kappa_n \approx c|W_n|$, we estimate k in $k\delta h_n/|W_n|$ by $c|W_n|$. The notes above imply that the absolute value of the difference between the two indicators in the definition of $\Theta_n(2)$ does not exceed

$$\mathbf{I}\{x - (s + k\tau) \in h_n[-1 - c\delta, -1 + c\delta]\} + \mathbf{I}\{x - (s + k\tau) \in h_n[1 - c\delta, 1 + c\delta]\}.$$

This, in turn, implies the bound

$$\Theta_n(2) \leqslant \Theta_n^+(2) + \Theta_n^-(2), \tag{6.14}$$

where

$$\begin{aligned}\Theta_n^\pm(2) := \frac{\tau}{|W_n|h_n}\mathbf{E}\Bigg(\mathbf{I}\{A_n\}\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\frac{x - (s + k\tau)}{h_n}\right)\\ \times \mathbf{I}\{x - (s + k\tau) \in h_n[\pm 1 - c\delta, \pm 1 + c\delta]\}X(dx)\Bigg).\end{aligned}$$

Using the boundedness of the kernel K, we obtain

$$\begin{aligned}\Theta_n^\pm(2) &\leqslant \frac{c}{|W_n|h_n}\mathbf{E}\left(\sum_{k=-\infty}^{\infty}\int_{W_n}\mathbf{I}\{x - (s + k\tau) \in h_n[\pm 1 - c\delta, \pm 1 + c\delta]\}X(dx)\right)\\ &\leqslant \frac{c}{|W_n|h_n}\sum_{k\in\mathscr{K}}\mathbf{E}X(s + k\tau + h_n[\pm 1 - c\delta, \pm 1 + c\delta])\\ &\leqslant \frac{c}{h_n}\mathbf{E}X(s + h_n[\pm 1 - c\delta, \pm 1 + c\delta])\\ &\leqslant c\delta.\end{aligned} \tag{6.15}$$

Thus, taking $\delta > 0$ sufficiently small, we can make $\limsup_{n\to\infty} \Theta_n^\pm(2)$ as small as desired. This also completes the proof of the same claim concerning $\limsup_{n\to\infty} \Theta_n(2)$. The proof of Lemma 6.2 is finished. □

Lemma 6.3. *The statement* $\lim_{n\to\infty} \Lambda_n(3) = \lambda(s)$ *holds.*

Proof. We decompose $\Lambda_n(3)$ in the following way:

$$\Lambda_n(3) = \Xi_n^* + \Xi_n^{**}, \tag{6.16}$$

where

$$\Xi_n^* := \frac{\tau}{|W_n|}\mathbf{E}\left\{\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)\right\},$$

$$\Xi_n^{**} := \frac{\tau}{|W_n|}\mathbf{E}\left\{\mathbf{I}\{A_n^c\}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)\right\}.$$

We shall demonstrate that

$$\Xi_n^* \to \lambda(s), \tag{6.17}$$

$$\Xi_n^{**} \to 0. \tag{6.18}$$

We start the proof of (6.17) with the following equalities:

$$\begin{aligned}\Xi_n^* &= \frac{\tau}{|W_n|}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{\mathbf{R}}\mathbf{I}\{x\in W_n\}K\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)\,dx\\ &= \frac{\tau}{|W_n|}\sum_{k=-\infty}^{\infty}\int_{\mathbf{R}}\mathbf{I}\{h_nx+s+k\tau\in W_n\}K(x)\lambda(h_nx+s+k\tau)\,dx\\ &= \frac{\tau}{|W_n|}\int_{\mathbf{R}}\left(\sum_{k=-\infty}^{\infty}\mathbf{I}\{h_nx+s+k\tau\in W_n\}\right)K(x)\lambda(h_nx+s)\,dx.\end{aligned} \tag{6.19}$$

Since W_n is an interval, we have that, for any $z\in\mathbf{R}$,

$$\sum_{k=-\infty}^{\infty}\mathbf{I}\{z+k\tau\in W_n\}\in\left[\frac{|W_n|}{\tau}-1,\frac{|W_n|}{\tau}+1\right]. \tag{6.20}$$

Therefore, when $n\to\infty$, the right-hand side of (6.19) asymptotically behaves like

$$\tilde{\Xi}_n^* := \int_{\mathbf{R}} K(x)\lambda(h_nx+s)\,dx.$$

Since the kernel K is a bounded probability density function and has support in $[-1,1]$, we obtain the representation

$$\tilde{\Xi}_n^* = \lambda(s)+\theta\frac{c}{h_n}\int_{-h_n}^{h_n}|\lambda(x+s)-\lambda(s)|\,dx \tag{6.21}$$

for some $|\theta|\leqslant 1$. Since s is a Lebesgue point of λ, the second summand on the right-hand side of (6.21) (with θ in front of it) converges to 0. This completes the proof of (6.17).

We shall now prove (6.18). Using the Cauchy–Schwarz inequality, we have that

$$(\Xi_n^{**})^2 \leqslant \mathbf{P}\left\{\frac{|W_n|}{\delta h_n}|\hat{\tau}_n-\tau|\geqslant 1\right\}\Pi_n, \tag{6.22}$$

where

$$\Pi_n := \frac{\tau^2}{|W_n|^2h_n^2}\mathbf{E}\left(\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)\right)^2.$$

By assumption (2.1), for any fixed $\delta > 0$ the probability on the right-hand side of (6.22) converges to 0 when $n \to \infty$. Therefore, in order to complete the proof of statement (6.18), we need to show that the quantity Π_n is asymptotically bounded. In fact, we shall demonstrate that

$$\Pi_n \to \lambda^2(s). \tag{6.23}$$

We start the proof of (6.23) with the note that, since $h_n \downarrow 0$ and the kernel K has support in $[-1, 1]$, the random variables ξ_k, $k \geqslant 1$, defined by the formula

$$\xi_k := \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx)$$

are independent for all sufficiently large n. Therefore,

$$\Pi_n = \Pi_n^* - \Pi_n^{**} + \Pi_n^{***}, \tag{6.24}$$

where

$$\Pi_n^* := \frac{\tau^2}{|W_n|^2 h_n^2} \left(\sum_{k=-\infty}^{\infty} \mathbf{E} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \right)^2,$$
$$\Pi_n^{**} := \frac{\tau^2}{|W_n|^2 h_n^2} \sum_{k=-\infty}^{\infty} \left(\mathbf{E} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \right)^2,$$
$$\Pi_n^{***} := \frac{\tau^2}{|W_n|^2 h_n^2} \sum_{k=-\infty}^{\infty} \mathbf{E} \left(\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \right)^2.$$

Note that $\lim_{n\to\infty} \Pi_n^* = \lambda^2(s)$. Indeed, we have that $\Pi_n^* = \{\Xi_n^*\}^2$, where Ξ_n^* is the same as in (6.16). But we have already proved that $\Xi_n^* \to \lambda(s)$. Thus, in order to complete the proof of (6.23), we need to show that $\lim_{n\to\infty} \Pi_n^{**} = 0$ and $\lim_{n\to\infty} \Pi_n^{***} = 0$. Since statement $\lim_{n\to\infty} \Pi_n^{**} = 0$ follows from $\lim_{n\to\infty} \Pi_n^{***} = 0$, we need to prove the latter only. Since K is bounded and has support in $[-1, 1]$, we have that

$$\Pi_n^{***} \leqslant c \frac{1}{|W_n|^2 h_n^2} \sum_{k=-\infty}^{\infty} \mathbf{E}(X(\{s + k\tau + h_n[-1, 1]\} \cap W_n))^2. \tag{6.25}$$

Replacing the infinite sum $\sum_{k=-\infty}^{\infty}$ on the right-hand side of (6.25) by a finite one $\sum_{k \in \mathscr{K}}$ as we have already done several times above, we obtain the bounds

$$\begin{aligned} \Pi_n^{***} &\leqslant c \frac{1}{|W_n|^2 h_n^2} \sum_{k \in \mathscr{K}} \mathbf{E}(X(\{s + k\tau + h_n[-1, 1]\} \cap W_n))^2 \\ &\leqslant c \frac{\kappa_n}{|W_n|^2 h_n^2} \mathbf{E}(X(\{s + h_n[-1, 1]\}))^2 \\ &\leqslant c \frac{1}{|W_n| h_n}. \end{aligned} \tag{6.26}$$

The right-hand side of (6.26) converges to 0 since we have $|W_n|h_n \to \infty$ by assumption. This completes the proof of (6.23) and, in turn, of (6.18). Lemma 6.3 is proved, and so is Theorem 2.1. $\square$

7. Proof of Theorem 2.2

We closely follow the proof of Theorem 2.1. Denote

$$B_n := \left\{ |\hat{\tau}_n - \tau| \leqslant \frac{\delta h_n^3}{|W_n|} \right\}. \tag{7.1}$$

As in the proof of Theorem 2.1, we use the following representation:

$$\mathbf{E}\hat{\lambda}_{n,K}^{\diamond}(s) = \Gamma_n(1) - \Gamma_n(2) + \Gamma_n(3) + \Gamma_n(4) + \Gamma_n(5), \tag{7.2}$$

where $\Gamma_n(1), \dots, \Gamma_n(5)$ are defined in (6.2) but now with the set B_n instead of A_n.

The quantities $\Gamma_n(1)$ and $\Gamma_n(4)$ do not exceed $D_n \mathbf{P}(B_n^c)$. Due to assumption (2.4), the latter quantity is of the order $o(h_n^2)$.

The quantities $\Gamma_n(2)$ and $\Gamma_n(5)$ do not exceed $D_n^{-r}\mathbf{E}(\mathbf{I}\{B_n\}\{\hat{\lambda}_{n,K}(s)\}^{r+1})$ for any $r \geqslant 0$. Since $D_n \geqslant c h_n^{-\varepsilon}$, we can find a large $r \geqslant 0$ such that $1/D_n^r \leqslant o(h_n^2)$. This implies that both $\Gamma_n(2)$ and $\Gamma_n(5)$ are of the order $o(h_n^2)$ provided that the expectation $\mathbf{E}(\mathbf{I}\{B_n\}\hat{\lambda}_{n,K}^{r+1}(s))$ is asymptotically bounded. In order to demonstrate this, we first replace the set B_n in the expectation $\mathbf{E}(\mathbf{I}\{B_n\}\hat{\lambda}_{n,K}^{r+1}(s))$ by the set A_n defined in (6.1). Then, with some obvious modifications, we follow the proof of (8.2) below (that we have already mentioned in the proof of Theorem 2.1 above) and demonstrate that the expectation $\mathbf{E}(\mathbf{I}\{B_n\}\hat{\lambda}_{n,K}^{r+1}(s))$ is asymptotically bounded.

In view of the notes above, we complete the proof of Theorem 2.1 provided that

$$\Gamma_n(3) = \lambda(s) + \frac{1}{2}\lambda''(s)h_n^2 \int_{-1}^{1} x^2 K(x)\,dx + o(h_n^2). \tag{7.3}$$

Just like in (6.3), we decompose $\Gamma_n(3)$ into the sum of $\Lambda_n(1)$, $\Lambda_n(2)$ and $\Lambda_n(3)$ defined below (6.3) but now with B_n instead of A_n. Recall that when proving Lemma 6.1 we showed that $\Lambda_n(1) = O(|W_n|^{-1})$. Since in the current proof we assume $h_n^2|W_n| \to \infty$, we have $|W_n|^{-1} = o(h_n^2)$. Consequently, $\Lambda_n(1) = o(h_n^2)$. As to the quantity $\Lambda_n(2)$, we apply Lemma 6.2 with δ replaced there by δh_n^2. (Note that the latter replacement makes the set A_n into the set B_n.) This proves that there exists a constant $c < \infty$ such that for all sufficiently large n the bound $\Lambda_n(2) \leqslant c\delta h_n^2$ holds for all $\delta > 0$. This implies that $\limsup_{n\to\infty} h_n^{-2}\Lambda_n(2)$ can be made as small as desired by choosing $\delta > 0$ sufficiently small. In view of these notes, statement (7.3) follows from

$$\Lambda_n(3) = \lambda(s) + \frac{1}{2}\lambda''(s)h_n^2 \int_{-1}^{1} x^2 K(x)\,dx + o(h_n^2). \tag{7.4}$$

To prove (7.4), we write

$$\Lambda_n(3) = \Xi_n^* + \Xi_n^{**}, \tag{7.5}$$

where Ξ_n^* and Ξ_n^{**} are defined as in the proof of Lemma 6.3 but with B_n instead of A_n. We shall verify below that

$$\Xi_n^* = \lambda(s) + \frac{1}{2}\lambda''(s)h_n^2 \int_{-1}^{1} x^2 K(x)\,dx + o(h_n^2), \tag{7.6}$$

$$\Xi_n^{**} = o(h_n^2). \tag{7.7}$$

We start the proof of (7.6) with the equalities

$$\begin{aligned}
\Xi_n^* &= \frac{\tau}{|W_n|} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{\mathbf{R}} K\left(\frac{x-(s+k\tau)}{h_n}\right) \lambda(x) I(x \in W_n)\,dx \\
&= \frac{\tau}{|W_n| h_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \sum_{k=-\infty}^{\infty} \lambda(x+s+k\tau)\mathbf{I}(x+s+k\tau \in W_n)\,dx \\
&= \frac{\tau}{|W_n| h_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \lambda(x+s) \left(\sum_{k=-\infty}^{\infty} \mathbf{I}(x+s+k\tau \in W_n)\right) dx.
\end{aligned} \tag{7.8}$$

Bound (6.20) shows that the right-hand side of (7.8) equals

$$\left(1 + \theta \frac{1}{|W_n|}\right) \frac{1}{h_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \lambda(x+s)\,dx \tag{7.9}$$

with some $|\theta| \leqslant 1$. Using the Taylor theorem and the assumption that K is symmetric around zero (which implies that $\int_{-1}^{1} xK(x)\,dx = 0$), we have that

$$\begin{aligned}
\frac{1}{h_n} \int_{-h_n}^{h_n} K\left(\frac{x}{h_n}\right) \lambda(s+x)\,dx &= \int_{-1}^{1} K(x)\lambda(s+xh_n)\,dx \\
&= \lambda(s) + \frac{1}{2}\lambda''(s)h_n^2 \int_{-1}^{1} x^2 K(x)\,dx + o(h_n^2).
\end{aligned} \tag{7.10}$$

Using (7.10) in (7.9) together with $|W_n|^{-1} = o(h_n^2)$, we finish the proof of (7.6).

In order to prove (7.7), we start with the inequality

$$\Xi_n^{**} \leqslant \mathbf{P}(B_n^c)^{1/r} \nabla_n(q)^{1/q}, \tag{7.11}$$

where $r, q > 1$ are such that $r^{-1} + q^{-1} = 1$, and

$$\nabla_n(q) := \mathbf{E}\left(\frac{\tau}{|W_n|} \sum_{k=-\infty}^{\infty} \frac{1}{h_n} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx)\right)^q. \tag{7.12}$$

By assumption (2.4) and $D_n \geqslant ch_n^{-\varepsilon}$, we have that $\mathbf{P}(B_n^c) = o(h_n^{2+\varepsilon})$. Therefore, choosing $r > 1$ sufficiently close to 1, we obtain that $\mathbf{P}(B_n^c)^{1/r} = o(h_n^2)$. Consequently, in order to have (7.7), we need to verify that, for a sufficiently large $q > 1$, $\limsup_{n\to\infty} \nabla_n(q) < \infty$. In fact, we shall prove that this is true for any even number

$q \geqslant 2$. We start with the bound

$$\nabla_n(q) \leqslant c(Q_n(1) + Q_n(2)), \tag{7.13}$$

where

$$\begin{aligned} Q_n(1) := \mathbf{E}\Bigg(\sum_{k=-\infty}^{\infty} \frac{1}{|W_n| h_n} \Bigg(\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \\ - \mathbf{E} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \Bigg)\Bigg)^q, \end{aligned} \tag{7.14}$$

$$Q_n(2) := \left(\mathbf{E} \sum_{k=-\infty}^{\infty} \frac{1}{|W_n| h_n} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \right)^q.$$

Note that $Q_n(2) = (\Xi_n^*)^q$, where Ξ_n^* is the same as in (6.16). We have proved in (6.17) that Ξ_n^* is asymptotically bounded, and so is $Q_n(2)$. Consequently, we are left to demonstrate that $\limsup_{n\to\infty} Q_n(1) < \infty$. Note first that the sum $\sum_{k=-\infty}^{\infty}$ in the definition of $Q_n(1)$ has at most $\kappa_n \approx c|W_n|$ nonzero summands. Since h_n converges to 0, the summands are independent for all sufficiently large n. Furthermore, the summands have means zero. Thus, we conclude that

$$\begin{aligned} Q_n(1) &\leqslant c\kappa_n^{q/2} \sup_k \mathbf{E}\left(\frac{1}{|W_n| h_n} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \right)^q \\ &\leqslant c\left(\frac{1}{|W_n| h_n^2}\right)^{q/2} \sup_k \mathbf{E}\left(\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx) \right)^q \\ &\leqslant c\left(\frac{1}{|W_n| h_n^2}\right)^{q/2} \sup_k \mathbf{E}(X(\{s+k\tau+h_n[-1,1]\} \cap W_n))^q \\ &\leqslant c\left(\frac{1}{|W_n| h_n^2}\right)^{q/2} \mathbf{E}(X(\{s+h_n[-1,1]\} \cap W_n))^q. \end{aligned} \tag{7.15}$$

The expectation on the right-hand side of (7.15) is bounded (it even converges to 0). Since $h_n^2 |W_n| \to \infty$ by assumption, this completes the proof (7.7). This finishes the proof of Theorem 2.2. $\square$

8. Proof of Theorem 3.1

We start the proof by writing $\mathbf{Var}(\hat{\lambda}_{n,K}^{\diamond}(s))$ as the difference between $\mathbf{E}(\hat{\lambda}_{n,K}^{\diamond}(s))^2$ and $(\mathbf{E}\hat{\lambda}_{n,K}^{\diamond}(s))^2$. From Theorem 2.1 we know that the quantity $\mathbf{E}\hat{\lambda}_{n,K}^{\diamond}(s)$ equals $\lambda(s) + o(1)$. Thus, in order to prove Theorem 3.1 we need to show that $\mathbf{E}(\hat{\lambda}_{n,K}^{\diamond}(s))^2$ equals $\lambda^2(s) + o(1)$. We proceed with the representation:

$$\mathbf{E}(\hat{\lambda}_{n,K}^{\diamond}(s))^2 = \Upsilon_n(1) - \Upsilon_n(2) + \Upsilon_n(3) + \Upsilon_n(4) + \Upsilon_n(5), \tag{8.1}$$

where

$$\Upsilon_n(1) := \mathbf{E}(\mathbf{I}\{A_n^c\}\mathbf{I}\{\hat{\lambda}_{n,K}(s) \leqslant D_n\}\{\hat{\lambda}_{n,K}(s)\}^2),$$

$$\Upsilon_n(2) := \mathbf{E}(\mathbf{I}\{A_n\}\mathbf{I}\{\hat{\lambda}_{n,K}(s) > D_n\}\{\hat{\lambda}_{n,K}(s)\}^2),$$

$$\Upsilon_n(3) := \mathbf{E}(\mathbf{I}\{A_n\}\{\hat{\lambda}_{n,K}(s)\}^2),$$

$$\Upsilon_n(4) := D_n^2\mathbf{E}(\mathbf{I}\{A_n^c\}\mathbf{I}\{\hat{\lambda}_{n,K}(s) > D_n\}),$$

$$\Upsilon_n(5) := D_n^2\mathbf{E}(\mathbf{I}\{A_n\}\mathbf{I}\{\hat{\lambda}_{n,K}(s) > D_n\})$$

with the same set A_n as in the proof of Theorem 2.1. Theorem 3.1 follows if we verify that by choosing n sufficiently large and/or $\delta > 0$ sufficiently small we can make $\Upsilon_n(k)$, $k = 1, 2, 4, 5$ as small as desired and $\Upsilon_n(3)$ as close to $\lambda^2(s)$ as desired.

The proof that $\Upsilon_n(1)$ and $\Upsilon_n(4)$ converge to 0 for any fixed $\delta > 0$ follows from the fact that the two quantities do not exceed $D_n^2\mathbf{P}(A_n^c)$; assumption (3.1) completes the proof.

The proof that the two quantities $\Upsilon_n(2)$ and $\Upsilon_n(5)$ converge to 0 for any fixed $\delta > 0$ starts with the fact that both of them do not exceed $D_n^{-2}\mathbf{E}(\mathbf{I}\{A_n\}\hat{\lambda}_{n,K}^4(s))$. Now, we need to verify that the expectation $\mathbf{E}(\mathbf{I}\{A_n\}\hat{\lambda}_{n,K}^4(s))$ is asymptotically bounded. The proof of the latter statement closely resembles the proof that the quantity

$$\limsup_{n\to\infty} |\Upsilon_n(3) - \lambda^2(s)| \tag{8.2}$$

can be made as small as desired by taking $\delta > 0$ sufficiently small. We shall prove the latter statement. Before proceeding we note in passing that when proving Theorem 2.1 we referred to statement (8.2) and claimed that it holds under assumption (2.1), which is weaker than the in the current proof assumed (3.1). For this reason, throughout the rest of the current proof we only assume (2.1).

We start the proof of the aforementioned smallness of (8.2) with the equality

$$\Upsilon_n(3) = \Phi_n(1) + \Phi_n(2) + \Phi_n(3), \tag{8.3}$$

where

$$\Phi_n(1) := \frac{1}{|W_n|^2 h_n^2}\mathbf{E}\left(\mathbf{I}\{A_n\}(\hat{\tau}_n - \tau)^2\left\{\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx)\right\}^2\right),$$

$$\Phi_n(2) := \frac{1}{|W_n|^2 h_n^2}\mathbf{E}\left(\mathbf{I}\{A_n\}2\tau(\hat{\tau}_n - \tau)\left\{\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx)\right\}^2\right),$$

$$\Phi_n(3) := \frac{1}{|W_n|^2 h_n^2}\mathbf{E}\left(\mathbf{I}\{A_n\}\tau^2\left\{\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx)\right\}^2\right).$$

We shall demonstrate below that $\Phi_n(3)$ can be made as close to $\lambda^2(s)$ as desired. This will also imply that $\Phi_n(1) \to 0$ and $\Phi_n(2) \to 0$ since $|\hat{\tau}_n - \tau|$ does not exceed $\delta h_n/|W_n|$, which converges to 0. Thus, in order to verify (8.2), we need to prove that

$\limsup_{n\to\infty} |\Phi_n(3) - \lambda^2(s)|$ can be made as small as desired by taking $\delta > 0$ sufficiently small. We write

$$\Phi_n(3) = \Phi_n^*(3) + \Phi_n^{**}(3) + \Phi_n^{***}(3) + \Pi_n, \tag{8.4}$$

where Π_n is the same as in (8.2) and

$$\Phi_n^*(3) := \frac{\tau^2}{|W_n|^2 h_n^2} \mathbf{E}\left(\mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} \int_{W_n} \Delta K_{k,n}(x) X(dx)\right)^2,$$

$$\Phi_n^{**}(3) := \frac{2\tau^2}{|W_n|^2 h_n^2} \mathbf{E}\Bigg(\mathbf{I}\{A_n\} \sum_{k=-\infty}^{\infty} \int_{W_n} \Delta K_{k,n}(x) X(dx) \times \sum_{l=-\infty}^{\infty} \int_{W_n} K\left(\frac{x-(s+l\tau)}{h_n}\right) X(dx)\Bigg),$$

$$\Phi_n^{***}(3) := \frac{\tau^2}{|W_n|^2 h_n^2} \mathbf{E}\left(\mathbf{I}\{A_n^{\mathrm{c}}\} \sum_{k=-\infty}^{\infty} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx)\right)^2.$$

We have already proved in (6.23) that $\Pi_n \to \lambda^2(s)$. Consequently, we are left to verify that $\Phi_n^*(3)$, $\Phi_n^{**}(3)$ and $\Phi_n^{***}(3)$ can be made as small as desired. In fact, we only need to prove this for $\Phi_n^*(3)$ and $\Phi_n^{***}(3)$ since the desired smallness of $\Phi_n^{**}(3)$ follows from the smallness of $\Phi_n^*(3)$ due to the Cauchy–Schwarz inequality which implies that $\Phi_n^{**}(3)$ does not exceed $c\Phi_n^*(3)^{1/2}\Pi_n^{1/2}$. The smallness of $\Phi_n^{***}(3)$ follows from the fact that the expectation in the definition of $\Phi_n^{***}(3)$ does not exceed $\mathbf{P}(A_n^{\mathrm{c}})^{1/r}\{\nabla_n(q)\}^{1/q}$ with the same $\nabla_n(q)$ as in (7.11) and with $r, q > 1$ such that $r^{-1} + q^{-1} = 1$. By assumption (2.1), we have that $\mathbf{P}(A_n^{\mathrm{c}}) = O(1/D_n)$, which converges to 0. Furthermore, we have already proved that $\limsup_{n\to\infty} \nabla_n(q) < \infty$ for any (even) number $q \geqslant 2$. In view of the notes above, we need to verify that $\limsup_{n\to\infty} \Phi_n^*(3)$ can be made as small as desired by taking $\delta > 0$ sufficiently small. The proof of the latter fact resembles the proof of Lemma 6.2 and we therefore omit it. This completes the proof of (8.2) and, in turn, of Theorem 3.1. □

9. Proof of Theorem 3.2

Throughout this section we use the notation

$$C_n := \left\{\frac{|W_n|^{3/2}}{h_n^{1/2}} |\hat{\tau}_n - \tau| \leqslant \delta\right\}. \tag{9.1}$$

With the quantities Υ_n and Γ_n as in (8.1) and (6.2) but now with C_n instead of A_n, we have the representation

$$\mathbf{Var}(\hat{\lambda}_{n,K}^{\diamond}(s)) = \mathbf{Var}(\mathbf{I}\{C_n\}\hat{\lambda}_{n,K}(s)) + R_n, \tag{9.2}$$

where

$$\begin{aligned}R_n &:= \Upsilon_n(1) - \Upsilon_n(2) + \Upsilon_n(4) + \Upsilon_n(5)\\ &\quad - (\Gamma_n(1) - \Gamma_n(2) + \Gamma_n(4) + \Gamma_n(5))^2\\ &\quad - 2\Gamma_n(3)(\Gamma_n(1) - \Gamma_n(2) + \Gamma_n(4) + \Gamma_n(5)).\end{aligned}$$

We shall prove below that by choosing sufficiently small $\delta > 0$, the quantity

$$\limsup_{n\to\infty}\left(|W_n|h_n\left\{\mathbf{Var}(\mathbf{I}\{C_n\}\hat{\lambda}_{n,K}(s)) - \frac{\tau\lambda(s)}{|W_n|h_n}\int_{-1}^{1}K^2(x)\,dx\right\}\right) \tag{9.3}$$

can be made as small as desired. Statement (9.3) and (9.2) imply Theorem 3.2 provided that $R_n = o(1/(|W_n|h_n))$. The latter statement can be proved following the lines of similar proofs involving related quantities in the proof of Theorems 2.1 and 3.1. We therefore omit further details and now prove only that the quantity in (9.3) can be made as small as desired by choosing sufficiently small $\delta > 0$. We start with the equality

$$\mathbf{Var}(\mathbf{I}\{C_n\}\hat{\lambda}_{n,K}(s)) = \tau^2 V_n(1) + V_n(2) + \theta 2\tau\sqrt{V_n(1)V_n(2)}, \tag{9.4}$$

where $|\theta| \leqslant 1$ and

$$V_n(1) := \mathbf{Var}\left(\mathbf{I}\{C_n\}\frac{1}{|W_n|h_n}\sum_{k=-\infty}^{\infty}\int_{W_n}K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx)\right),$$

$$V_n(2) := \mathbf{Var}\left(\mathbf{I}\{C_n\}(\hat{\tau}_n - \tau)\frac{1}{|W_n|h_n}\sum_{k=-\infty}^{\infty}\int_{W_n}K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx)\right).$$

We have that $V_n(2) = o(1/(|W_n|h_n))$. Indeed,

$$\begin{aligned}V_n(2) &\leqslant \mathbf{E}\left(\mathbf{I}\{C_n\}(\hat{\tau}_n - \tau)\frac{1}{|W_n|h_n}\sum_{k=-\infty}^{\infty}\int_{W_n}K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx)\right)^2\\ &\leqslant \delta\frac{h_n}{|W_n|^3}\left\{\frac{1}{|W_n|^2h_n^2}\right.\\ &\quad\left.\times\mathbf{E}\left(\mathbf{I}\{C_n\}\left(\sum_{k\in\mathscr{K}}\int_{W_n}K\left(\frac{x-(s+k\hat{\tau}_n)}{h_n}\right)X(dx)\right)^2\right)\right\},\end{aligned} \tag{9.5}$$

where the number of elements in the set $\mathscr{K}$ is asymptotically of order $c|W_n|$. Thus, the quantity inside $\{\cdot\}$ on the right-hand side of (9.5) is asymptotically bounded, which can be proved following the lines of the proof that the quantity $\nabla_n(q)$ is asymptotically bounded. This finishes the proof of the above claimed statement that $V_n(2) = o(1/(|W_n|h_n))$.

We now consider $V_n(1)$. The equality

$$V_n(1) = R_n(1) + R_n(2) + \theta 2\sqrt{R_n(1)R_n(2)} \tag{9.6}$$

holds with some $|\theta|\leqslant 1$ and

$$R_n(1) := \mathbf{Var}\left(\mathbf{I}\{C_n\}\frac{1}{|W_n|h_n}\sum_{k=-\infty}^{\infty}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)\right),$$

$$R_n(2) := \mathbf{Var}\left(\mathbf{I}\{C_n\}\frac{1}{|W_n|h_n}\sum_{k=-\infty}^{\infty}\int_{W_n} \Delta K_{k,n}(x)X(dx)\right).$$

Choosing $\delta>0$ sufficiently small, we can make $\limsup_{n\to\infty}|W_n|h_nR_n(2)$ as small as desired. Indeed, this can be achieved by first using the inequality

$$R_n(2)\leqslant\frac{1}{|W_n|^2h_n^2}\mathbf{E}\left(\mathbf{I}\{C_n\}\sum_{k=-\infty}^{\infty}\int_{W_n}\Delta K_{k,n}(x)X(dx)\right)^2 \tag{9.7}$$

and then following the lines of the proof to Lemma 6.2 with some obvious changes.

We shall now consider $R_n(1)$. We start with the equality

$$R_n(1) = Y_n(1) + Y_n(2) + \theta 2\sqrt{Y_n(1)\,Y_n(2)}, \tag{9.8}$$

where $|\theta|\leqslant 1$ and

$$Y_n(1) := \mathbf{Var}\left(\sum_{k=-\infty}^{\infty}\frac{1}{|W_n|h_n}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)\right),$$

$$Y_n(2) := \mathbf{Var}\left(\mathbf{I}\{C_n^{\mathrm{c}}\}\sum_{k=-\infty}^{\infty}\frac{1}{|W_n|h_n}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)\right).$$

To prove that $Y_n(2)=o(1/\{|W_n|h_n\})$, we first use the bound $\mathbf{Var}(\xi\eta)\leqslant\mathbf{E}(\xi^2\eta^2)$ and then apply the Hölder's inequality. This gives us the bound

$$\begin{aligned} Y_n(2)\leqslant &\left(\mathbf{P}\left\{\frac{|W_n|^{3/2}}{h_n^{1/2}}|\hat\tau_n-\tau|\geqslant\delta\right\}\right)^{1/r} \\ &\times\left\{\mathbf{E}\left(\sum_{k=-\infty}^{\infty}\frac{1}{|W_n|h_n}\int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right)X(dx)\right)^{2q}\right\}^{1/q}, \end{aligned} \tag{9.9}$$

where $r,q>1$ are such that $r^{-1}+q^{-1}=1$. We have already proved above (cf. (7.12) and the calculations below it) that the expectation on the right-hand side of (9.9) is asymptotically bounded. As to the probability on the right-hand side of (9.9), we use assumption (3.3) together with $D_n\geqslant|W_n|^{\varepsilon}h_n^{\varepsilon}$ and obtain that, for $r>1$ sufficiently close to 1,

$$\left(\mathbf{P}\left\{\frac{|W_n|^{3/2}}{h_n^{1/2}}|\hat\tau_n-\tau|\geqslant\delta\right\}\right)^{1/r} = o\left(\frac{1}{|W_n|h_n}\right). \tag{9.10}$$

Consequently, from (9.9) we obtain that $Y_n(2) = o(1/\{|W_n|h_n\})$ holds.

We shall now prove the following statement:

$$\tau^2 Y_n(1) = \frac{\tau\lambda(s)}{|W_n|h_n} \int_{-1}^{1} K^2(x)\,dx + o\left(\frac{1}{|W_n|h_n}\right) \tag{9.11}$$

and in this way finish the proof of Theorem 3.2. Since the summands in the definition of $Y_n(1)$ are independent for sufficiently large n, we have that

$$Y_n(1) = \sum_{k=-\infty}^{\infty} \mathbf{Var}\left(\frac{1}{|W_n|h_n} \int_{W_n} K\left(\frac{x-(s+k\tau)}{h_n}\right) X(dx)\right). \tag{9.12}$$

We calculate the variances on the right-hand side of (9.12) using Lemma 1.1 on p. 18 of Kutoyants (1998) and obtain that

$$\begin{aligned} Y_n(1) &= \frac{1}{|W_n|^2 h_n^2} \sum_{k=-\infty}^{\infty} \int_{W_n} K^2\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)\,dx \\ &= \frac{1}{|W_n|^2 h_n^2} \int_{-\infty}^{\infty} K^2\left(\frac{x}{h_n}\right)\lambda(x+s) \sum_{k=-\infty}^{\infty} \mathbf{I}(x+s+k\tau \in W_n)\,dx. \end{aligned} \tag{9.13}$$

An application of (6.20) on the right-hand side of (6.13) yields, for some $\theta \in [-1, 1]$,

$$\begin{aligned} \tau^2 Y_n(1) &= \left(\frac{\tau}{|W_n|h_n^2} + \theta\frac{\tau^2}{|W_n|^2 h_n^2}\right) \int_{-\infty}^{\infty} K^2\left(\frac{x}{h_n}\right)\lambda(x+s)\,dx \\ &= \left(\frac{\tau}{|W_n|h_n^2} + \theta\frac{\tau^2}{|W_n|^2 h_n^2}\right) \int_{-\infty}^{\infty} K^2\left(\frac{x}{h_n}\right)(\lambda(x+s)-\lambda(s))\,dx \\ &\quad + \left(\frac{\tau}{|W_n|h_n^2} + \theta\frac{\tau^2}{|W_n|^2 h_n^2}\right) h_n\lambda(s) \int_{-1}^{1} K^2(x)\,dx. \end{aligned} \tag{9.14}$$

Since s is a Lebesgue point of λ and the kernel K is bounded with support in $[-1, 1]$, we have that

$$\begin{aligned} \int_{-\infty}^{\infty} K^2\left(\frac{x}{h_n}\right)|\lambda(x+s)-\lambda(s)|\,dx &= \int_{-h_n}^{h_n} K^2\left(\frac{x}{h_n}\right)|\lambda(x+s)-\lambda(s)|\,dx \\ &= o(h_n). \end{aligned} \tag{9.15}$$

Applying (9.15) on the right-hand side of (9.14), we arrive at (9.11). This also completes the proof of Theorem 3.2. □

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