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Improved estimation of regression parameters in measurement error models

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Abstract

The problem of simultaneous estimation of the regression parameters in a multiple regression model with measurement errors is considered when it is suspected that the regression parameter vector may be the null-vector with some degree of uncertainty. In this regard, we propose two sets of four estimators, namely, (i) the unrestricted estimator, (ii) the preliminary test estimator, (iii) the Stein-type estimator and (iv) the postive-rule Stein-type estimator. In an asymptotic setup, properties of these estimators are studied based on asymptotic distributional bias, MSE matrices, and risks under a quadratic loss function. In addition to the asymptotic dominance of the Stein-type estimators, the paper contains discussion of dominating confidence sets based on the Stein-type estimation. Asymptotic analysis is considered based on a sequence of local alternatives to obtain the desired results.

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1. Introduction

Consider the multiple regression model with measurement errors, namely,

$$\left.\begin{aligned} Y_t &= \beta_0 + \mathbf{x}_t'\boldsymbol{\beta} + e_t, \\ \mathbf{X}_t &= \mathbf{x}_t + \mathbf{u}_t, \end{aligned}\right\} \quad t = 1, \ldots, n, \tag{1.1}$$

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where β_0 is the intercept and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ is the regression parameters while $\mathbf{x}_t = (x_{1t}, \ldots, x_{pt})'$, $\mathbf{u}_t = (u_{1t}, \ldots, u_{pt})'$, $\mathbf{X}_t = (X_{1t}, \ldots, X_{pt})'$ and e_t is the response error in the study variable and u_{it} is the measurement error in the ith regression variable x_{it}. Note that x_{it} is unobservable and X_{it} is the corresponding observed value. Similarly, Y_t is the observed response. We assume that

$$(\mathbf{x}_t', e_t, \mathbf{u}_t')' \sim \mathcal{N}_{2p+1}\Big\{(\boldsymbol{\mu}_x', 0, \mathbf{0}')'; \mathrm{Blockdiag}\Big(\boldsymbol{\Sigma}_{xx}, \sigma_{ee}, \boldsymbol{\Sigma}_{uu}\Big)\Big\}, \tag{1.2}$$

where $\boldsymbol{\mu}_x' = (\mu_{x_1}, \ldots, \mu_{x_p})'$.

Clearly, $(Y_t, \mathbf{X}_t')'$ follows a $(p+1)$-variate normal distribution with mean-vector $(\beta_0 + \boldsymbol{\beta}'\boldsymbol{\mu}_x, \boldsymbol{\mu}_x')'$ and covariance matrix

$$\begin{pmatrix} \sigma_{Y_tY_t} & \boldsymbol{\Sigma}_{Y_tX_t} \\ \boldsymbol{\Sigma}_{X_tY_t} & \boldsymbol{\Sigma}_{X_tX_t} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\beta}'\boldsymbol{\Sigma}_{xx}\boldsymbol{\beta} + \sigma_{ee} & \boldsymbol{\beta}'\boldsymbol{\Sigma}_{xx} \\ \boldsymbol{\Sigma}_{xx}\boldsymbol{\beta} & \boldsymbol{\Sigma}_{xx} + \boldsymbol{\Sigma}_{uu} \end{pmatrix}. \tag{1.3}$$

Thus, the distribution of $(\bar{Y}, \bar{\mathbf{X}}')'$ is a $(p+1)$-variate normal with mean-vector $(\beta_0 + \boldsymbol{\beta}'\boldsymbol{\mu}_x, \boldsymbol{\mu}_x')'$ and covariance matrix $\frac{1}{n}\boldsymbol{\Sigma}$. Now, the conditional distribution of Y_t given $\mathbf{X}_t = (X_{1t}, \ldots, X_{pt})'$ is normal with conditional mean and variance given by

$$E\Big[Y_t|\mathbf{X}_t\Big] = v_0 + \boldsymbol{v}'\mathbf{X}_t, \tag{1.4}$$

$$\mathrm{Var}\Big[Y_t|\mathbf{X}_t\Big] = \boldsymbol{\beta}'\boldsymbol{\Sigma}_{xx}\Big(\mathbf{I} - \mathbf{K}_{xx}\Big)\boldsymbol{\beta} + \sigma_{ee} = \sigma_{zz}(\text{say}), \tag{1.5}$$

where $\mathbf{K}_{xx}$ is the matrix of ratios of $\boldsymbol{\Sigma}_{XX}$ and $\boldsymbol{\Sigma}_{xx}$ defined by

$$\mathbf{K}_{xx} = \Big(\boldsymbol{\Sigma}_{xx} + \boldsymbol{\Sigma}_{uu}\Big)^{-1}\boldsymbol{\Sigma}_{xx} = \boldsymbol{\Sigma}_{XX}^{-1}\boldsymbol{\Sigma}_{xx},\ \boldsymbol{\Sigma}_{XX} = \boldsymbol{\Sigma}_{xx} + \boldsymbol{\Sigma}_{uu}. \tag{1.6}$$

Gleser [5] designates $\mathbf{K}_{xx}$ as the reliability matrix of $\mathbf{X}$.

Further we have

$$v_0 = \beta_0 + \boldsymbol{\beta}'\Big(\mathbf{I}_p - \mathbf{K}_{xx}'\Big)\boldsymbol{\mu}_x,\ \boldsymbol{v} = \mathbf{K}_{xx}\boldsymbol{\beta} \text{ and } \boldsymbol{\beta} = \mathbf{K}_{xx}^{-1}\boldsymbol{v}. \tag{1.7}$$

The model (1.1) as described is a structural linear errors-in-variable regression model.

Our basic problem is the estimation of the regression vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ when it is suspected but one is not sure that $\boldsymbol{\beta}$ may be the null-vector, i.e. $\boldsymbol{\beta} = 0$. Towards this goal we propose two sets of four estimators of $\boldsymbol{\beta}$ each, namely, (i) the unrestricted estimators (UE), (ii) the preliminary test estimator (PTE), (iii) the Stein-type estimator (SE) and (iv) the positive-rule Stein-type estimator (PRSE) and show that PRSE dominates SE as well as UE uniformly when $p \geqslant 3$. For $p \leqslant 2$, PTE is a preferable choice when $\boldsymbol{\beta}$ is near the origin, $\mathbf{0}$, otherwise UE is preferable. Further,it is shown that neither the PTE nor SE (PRSE) dominate each other uniformly. However, when $p \geqslant 3$, PRSE is the preferred choice for application. The same conclusion holds when we consider the confidence sets based on PRSE.

These types of estimations of $\boldsymbol{\beta}$ have been studied by Saleh and Han [14], Judge and Bock [10] among others for models without the measurement errors. Preliminary test estimators were introduced by Bancroft [1] and expanded by Saleh and Sen [15] in a nonparametric setup. Stein [24] and James and Stein [8] introduced the Stein-type estimators, which were expanded by Saleh and Sen [15,16] and Sen and Saleh [18] in the nonparametric setup. For the multiple regression model with measurement errors, see Fuller [4] and Cheng and Van Ness [3] for details and Schneeweiss [17] on consistency. Kim and Saleh [11] introduced the preliminary test estimation in a simple linear model with measurement errors.

It is interesting to note that Stein-type estimation eliminates the inconsistency of the traditional least-squares estimators (see [20,22,23]). Thus, Shalabh [21] studied the properties of Stein-type estimator when $\mathbf{\Sigma}_{uu}$ is known. Our study includes a broader class of estimators, such as the preliminary test and the positive-rule Stein-type estimator in addition to the usual Stein-type estimator studied by Shalabh [21] when $\mathbf{\Sigma}_{uu}$ is known.

We organize the paper as follows. In Section 2, we provide the proposed four estimators and motivate the estimators in various ways starting from the unrestricted estimator. Section 3 contains the asymptotic distributional properties of the estimators.

In Section 4, we obtain the asymptotic distributional bias, quadratic bias, MSE-matrices and risk (under a quadratic loss function) expressions. In Section 5, we provide the comparison of the estimators based on the asymptotic distributional bias, MSE matrix as well as risk analysis. We conclude the paper in Section 6 with a discussion of the asymptotic properties of the recentered confidence sets.

2. Estimation of regression coefficients

Our basic problem is the estimation of $\boldsymbol{\beta}$ when it is suspected but one is not sure that $\boldsymbol{\beta}$ may be equal to $\mathbf{0}$. For this purpose we assume that the variance–covariance matrix, $\mathbf{\Sigma}_{uu}$ of the measurement errors of regressors is known in order to obtain a consistent estimator of $\boldsymbol{\beta}$ (see [17]) while $\mathbf{K}_{xx}$ is unknown.

Let

$$\mathbf{S} = \begin{pmatrix} S_{YY} & \mathbf{S}_{YX} \\ \mathbf{S}_{XY} & \mathbf{S}_{XX} \end{pmatrix}, \tag{2.1}$$

where

(i) $S_{YY} = \sum_{i=1}^{n}(Y_t - \bar{Y})^2$,
(ii) $\mathbf{S}_{XX} = ((S_{X_iX_j}))$,
(iii) $S_{X_iX_j} = \sum_{t=1}^{n}(X_{it} - \bar{X}_i)(X_{jt} - \bar{X}_j),\ i, j = 1, \ldots, p$,
(iv) $S_{X_iY} = \sum_{t=1}^{n}(X_{it} - \bar{X}_i)(Y_t - \bar{Y})$,
(v) $\bar{X}_i = \frac{1}{n}\sum_{t=1}^{n} X_{it}$ and $\bar{Y} = \frac{1}{n}\sum_{t=1}^{n} Y_t$. (2.2)

Thus $\mathbf{S} \overset{\mathcal{D}}{\simeq} W_{p+1}(\mathbf{\Sigma}; n-1)$ where $W_{p+1}(\cdot;\cdot)$ stands for the Wishart distribution with $n-1$ degrees of freedom (DF). Clearly, $\frac{1}{n}\mathbf{S}$ is the MLE of $\mathbf{\Sigma}$ and $\frac{1}{n-1}\mathbf{S}$ is an unbiased estimator of $\mathbf{\Sigma}$. Consequently, $\frac{1}{n-1}\mathbf{S}_{XX}$ is an unbiased and consistent estimator of $\mathbf{\Sigma}_{XX}$ and $\mathbf{S}_{XX}$ is independent of $(\bar{Y}, \bar{\mathbf{X}}')'$. Gleser [5] showed that the MLE of v_0, $\boldsymbol{v}$ and σ_{zz} are just the naive least squares estimators (OLS), namely,

$$\tilde{v}_{0n} = \bar{Y} - \tilde{\boldsymbol{v}}_n'\bar{\mathbf{X}}, \quad \tilde{\boldsymbol{v}}_n = \mathbf{S}_{XX}^{-1}\mathbf{S}_{XY} \tag{2.3}$$

and

$$\tilde{\sigma}_{zz} = \frac{1}{n}\sum_{t=1}^{n}(Y_t - \tilde{v}_{0n} - \tilde{\boldsymbol{v}}_n'\bar{\mathbf{X}})^2 \tag{2.4}$$

provided

$$\tilde{\sigma}_{ee} = \tilde{\sigma}_{zz} - \tilde{\boldsymbol{v}}_n' \mathbf{K}_{xx}^{-1} \boldsymbol{\Sigma}_{uu} \tilde{\boldsymbol{v}}_n \geqslant 0. \tag{2.5}$$

Further, when $\boldsymbol{\Sigma}_{uu}$ is known and $\mathbf{K}_{xx}$ unknown then the reliability matrix is estimated consistently by

$$\hat{\mathbf{K}}_{xx} = \mathbf{S}_{XX}^{-1} \hat{\boldsymbol{\Sigma}}_{xx} = \mathbf{S}_{XX}^{-1} (\mathbf{S}_{XX} - n\boldsymbol{\Sigma}_{uu}). \tag{2.6}$$

Thus, the MLE of β_0, $\boldsymbol{\beta}$ and σ_{ee} are given by

$$\tilde{\beta}_{0n} = \tilde{v}_{0n} - \tilde{\boldsymbol{\beta}}_n' (\mathbf{I}_p - \hat{\mathbf{K}}_{xx}') \bar{\mathbf{X}}, \quad \tilde{\boldsymbol{\beta}}_n = \hat{\mathbf{K}}_{xx}^{-1} \tilde{\boldsymbol{v}}_n \tag{2.7}$$

and

$$\tilde{\sigma}_{ee} = \tilde{\sigma}_{zz} - \tilde{\boldsymbol{\beta}}_n' \boldsymbol{\Sigma}_{uu} \hat{\mathbf{K}}_{xx} \tilde{\boldsymbol{\beta}}_n. \tag{2.8}$$

Further, note that $\hat{\mathbf{K}}_{xx} \xrightarrow{P} \mathbf{K}_{xx}$ as $n \to \infty$. Finally, the explicit forms of the MLE of β_0 and $\boldsymbol{\beta}$ are given by

$$\tilde{\beta}_{0n} = \bar{Y} - \tilde{\boldsymbol{\beta}}_n' \bar{\mathbf{X}}, \quad \tilde{\boldsymbol{\beta}}_n = (\mathbf{S}_{XX} - n\boldsymbol{\Sigma}_{uu})^{-1} \mathbf{S}_{XY} \tag{2.9}$$

provided $\tilde{\sigma}_{ee} \geqslant 0$ as in (2.5). The estimators given at (2.3), (2.7) and/or (2.9) will be designated as the *unrestricted estimator* (UE) of $\boldsymbol{\beta}$.

Then, by Theorem 2.2.1. in Fuller [4] we have as $n \to \infty$

(i) $\sqrt{n}(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta})$ is normally distributed p-variate random vector with mean $\mathbf{0}$ and covariance matrix, $\mathbf{G}$ given by

$$\mathbf{G} = \boldsymbol{\Sigma}_{xx}^{-1} \Big[\Big(\boldsymbol{\beta}' \boldsymbol{\Sigma}_{uu} \boldsymbol{\beta} + \sigma_{ee} \Big) \boldsymbol{\Sigma}_{XX} + \boldsymbol{\Sigma}_{uu} \boldsymbol{\beta}\boldsymbol{\beta}' \boldsymbol{\Sigma}_{uu} \Big] \boldsymbol{\Sigma}_{xx}^{-1} \tag{2.10}$$

provided $\sigma_{ee} > 0$, $\boldsymbol{\Sigma}_{uu}$ is known and $\boldsymbol{\Sigma}_{xx}$ is a positive-definite matrix.

(ii) $\sqrt{n}(\tilde{\boldsymbol{v}}_n - \boldsymbol{v})$ is normally distributed p-variate random vector with mean vector $\mathbf{0}$ and covariance matrix $\sigma_{zz} \boldsymbol{\Sigma}_{XX}^{-1}$.

A consistent estimator of $\mathbf{G}$ may be obtained by substituting in $\mathbf{G}$, the consistent estimators of $\boldsymbol{\beta}$ and σ_{ee} given by (2.7) and (2.8). Let us denote the consistent estimator of $\mathbf{G}$ by $\hat{\mathbf{G}}_n$ as given below.

$$\begin{aligned} \hat{\mathbf{G}}_n = & \left(\frac{1}{n} \mathbf{S}_{XX} - \boldsymbol{\Sigma}_{uu} \right)^{-1} \Big[\Big(\tilde{\boldsymbol{\beta}}_n' \boldsymbol{\Sigma}_{uu} \tilde{\boldsymbol{\beta}}_n + \tilde{\sigma}_{ee} \Big) \frac{1}{n} \mathbf{S}_{XX} \\ & + \boldsymbol{\Sigma}_{uu} \tilde{\boldsymbol{\beta}}_n \tilde{\boldsymbol{\beta}}_n' \boldsymbol{\Sigma}_{uu} \Big] \left(\frac{1}{n} \mathbf{S}_{XX} - \boldsymbol{\Sigma}_{uu} \right)^{-1}. \end{aligned} \tag{2.11}$$

Thus, we may write

$$\hat{\mathbf{G}}_n = \mathbf{G} + o_p(1). \tag{2.12}$$

Since we suspect that $\boldsymbol{\beta}$ may be equal to $\mathbf{0}$, we consider the Wald-type test-statistic $\mathcal{L}_n$ to test the null-hypothesis $\mathrm{H}_0 : \boldsymbol{\beta} = \mathbf{0}$ against $\mathrm{H}_A : \boldsymbol{\beta} \neq \mathbf{0}$, defined by

$$\mathcal{L}_n = n \Big(\tilde{\boldsymbol{\beta}}_n' \hat{\mathbf{G}}_n^{-1} \tilde{\boldsymbol{\beta}}_n \Big), \tag{2.13}$$

where the asymptotic distribution of $\mathcal{L}_n$ under H_0 follows a central chi-square distribution with p DF by Theorem 2.2.1 of Fuller [4].

Now, we consider the following four estimators which may improve over $\tilde{\boldsymbol{\beta}}_n$.

(i) Preliminary test estimator (PTE), $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$

$$\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}} = \tilde{\boldsymbol{\beta}}_n I\Big(\mathcal{L}_n > \chi_\alpha^2\Big) = \tilde{\boldsymbol{\beta}}_n - \tilde{\boldsymbol{\beta}}_n I\Big(\mathcal{L}_n < \chi_\alpha^2\Big), \tag{2.13a}$$

where $\chi^2_{p,\alpha}$ is the upper α-level critical value from a central chi-square distribution with p DF and $I(A)$ is the indicator function of set A.

(ii) Stein-type estimator (SE), $\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$

$$\hat{\boldsymbol{\beta}}_n^{\mathrm{S}} = \Big(1-(p-2)\mathcal{L}_n^{-1}\Big)\tilde{\boldsymbol{\beta}}_n, \quad p>2, \tag{2.13b}$$

where $(p-2)$ is the mode of the central chi-square distribution with p DF.

(iii) Positive-rule shrinkage estimator (PRSE), $\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}$

$$\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}} = \Big(1-(p-2)\mathcal{L}_n^{-1}\Big) I\Big(\mathcal{L}_n > p-2\Big)\tilde{\boldsymbol{\beta}}_n = \hat{\boldsymbol{\beta}}_n^{\mathrm{S}} I(\mathcal{L}_n > p-2), \quad \text{where } p>2. \tag{2.13c}$$

We may compactly write the four estimators as

$$\boldsymbol{\beta}_n^* = (1-g(\mathcal{L}_n))\tilde{\boldsymbol{\beta}}_n, \tag{2.14}$$

where

$$\begin{aligned} g(\mathcal{L}_n) &= 0 \\ &= I(\mathcal{L}_n < \chi^2_{p,\alpha}) \\ &= [1-(p-2)\mathcal{L}_n^{-1}] \\ &= [1-(p-2)\mathcal{L}_n^{-1}]I(\mathcal{L}_n > p-2) \end{aligned}$$

gives $\boldsymbol{\beta}_n^* = \tilde{\boldsymbol{\beta}}_n, \hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}, \hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$ and $\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}$, respectively.

Note that PTE is a discontinuous estimator and takes only two values, namely, $\mathbf{0}$ and $\tilde{\boldsymbol{\beta}}_n$ depending on the result of the test which heavily depend on the size α of the test. Now, replacing $I(\mathcal{L}_n < \chi^2_{p,\alpha})$, by a smooth version $(p-2)\mathcal{L}_n^{-1}$, we obtain the values between $\mathbf{0}$ and $\tilde{\boldsymbol{\beta}}_n$ depending on the sample value of $\mathcal{L}_n$ (and not on α). Also notice that $\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}$ is a PTE of $\boldsymbol{\beta}$ with fixed critical value $(p-2)$.

Similarly, consider the class of estimators of $\boldsymbol{\beta}$ (based on $\tilde{\boldsymbol{v}}_n$) defined by

$$\boldsymbol{v}_n^* = (1-g(\mathcal{L}_n))\tilde{\boldsymbol{v}}_n. \tag{2.15}$$

Accordingly, the four estimators $\tilde{\boldsymbol{v}}_n$, $\hat{\boldsymbol{v}}_n^{\mathrm{PT}}$, $\hat{\boldsymbol{v}}_n^{\mathrm{S}}$ and $\hat{\boldsymbol{v}}_n^{\mathrm{S+}}$ are obtained by choosing $g(\mathcal{L}_n) = 0,\ I(\mathcal{L}_n < \chi^2_\alpha),\ (p-2)\mathcal{L}_n^{-1}$ and $1-[1-(p-2)\mathcal{L}_n^{-1}]I(\mathcal{L}_n > p-2)$, respectively.

3. Asymptotic properties of the estimators

In this section we consider the asymptotic distribution of the estimators, $\boldsymbol{\beta}_n^*$ and $\boldsymbol{v}_n^*$. First, we consider the asymptotic distribution of the estimators under the *fixed alternatives* $\mathrm{H}_\delta : \boldsymbol{\beta} = \boldsymbol{\delta}(\boldsymbol{\delta} \neq 0)$ given by the following theorem.

Theorem 3.1. *Under fixed alternatives* $\mathrm{H}_{\boldsymbol{\delta}}: \boldsymbol{\beta} = \boldsymbol{\delta}(\neq \mathbf{0})$.

(1) $\mathcal{L}_n \to \infty$ *as* $n \to \infty$,
(2) $\sqrt{n}(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}) = \sqrt{n}(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) + o_p(1)$,
(3) $\sqrt{n}(\boldsymbol{v}_n^* - \boldsymbol{v}) = \sqrt{n}(\tilde{\boldsymbol{v}}_n - \boldsymbol{v}) + o_p(1)$.

Proof. To prove (1) consider

$$\sqrt{n}\tilde{\boldsymbol{\beta}}_n = \sqrt{n}(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) + \sqrt{n}\boldsymbol{\delta}. \tag{3.1}$$

Then,

$$\mathcal{L}_n = n(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta})'\hat{\mathbf{G}}_n^{-1}(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) + n\boldsymbol{\delta}'\hat{\mathbf{G}}_n^{-1}\boldsymbol{\delta} + 2n(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta})'\hat{\mathbf{G}}_n^{-1}\boldsymbol{\delta}. \tag{3.2}$$

Now, $\sqrt{n}\hat{\mathbf{G}}_n^{-1/2}(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) \xrightarrow{\mathcal{D}} \mathbf{Z}$ as $n \to \infty$, where $\mathbf{Z} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I}_p)$ and $n\boldsymbol{\delta}'\hat{\mathbf{G}}_n^{-1}\boldsymbol{\delta} \to \infty$ as $n \to \infty$, it follows that $\mathcal{L}_n \to \infty$ as $n \to \infty$.

Consequently,

$$\lim_{n\to\infty} P_\beta(\mathcal{L}_n > k) = 1 \quad \text{for all } k \in \mathcal{R}_1^+. \tag{3.3}$$

To prove (2), consider the quadratic differences

$$n||\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_n^*||^2_{\hat{\mathbf{G}}_n^{-1}}, \text{ where } \boldsymbol{\beta}_n^* = (1 - g(\mathcal{L}_n))\tilde{\boldsymbol{\beta}}_n. \tag{3.4}$$

Then,

$$\begin{aligned} n||\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_n^*||^2_{\hat{\mathbf{G}}_n^{-1}} &= (n\tilde{\boldsymbol{\beta}}_n'\hat{\mathbf{G}}_n^{-1}\tilde{\boldsymbol{\beta}}_n)g^2(\mathcal{L}_n) \\ &= \mathcal{L}_n g^2(\mathcal{L}_n). \end{aligned} \tag{3.5}$$

It may be verified that

$$\begin{aligned} &\lim_{n\to\infty} E[\mathcal{L}_n g^2(\mathcal{L}_n)] = 0 \quad \text{for} \\ &g(\mathcal{L}_n) = I(\mathcal{L}_n < \chi_\alpha^2), \\ &g(\mathcal{L}_n) = (p-2)\mathcal{L}_n^{-1} \end{aligned} \tag{3.6}$$

and

$$g(\mathcal{L}_n) = 1 - [1 - (p-2)\mathcal{L}_n^{-1}]I(\mathcal{L}_n > p-2).$$

Hence,

$$\sqrt{n}(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}) = \sqrt{n}(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) + o_p(1). \tag{3.7}$$

Thus, under fixed alternative, all the estimators are asymptotically equivalent in distribution which is $\mathcal{N}_p(\mathbf{0}, \mathbf{G})$.

Similarly,

$$\lim_{n\to\infty} E\Big[n||\tilde{\boldsymbol{v}}_n - \boldsymbol{v}_n^*||^2_{(\hat{\mathbf{K}}_{xx}'\hat{\mathbf{G}}_n\hat{\mathbf{K}}_{xx})^{-1}}\Big] = \lim_{n\to\infty} E\Big[\mathcal{L}_n g^2(\mathcal{L}_n)\Big] = 0. \tag{3.8}$$

Hence

$$\sqrt{n}(\boldsymbol{v}_n^* - \boldsymbol{v}) = \sqrt{n}(\tilde{\boldsymbol{v}}_n - \boldsymbol{v}) + o_p(1) \tag{3.9}$$

and all the estimators $\boldsymbol{v}_n^*$ are asymptotically equivalent in distribution which is $N_p(\mathbf{0}, \sigma_{zz}\boldsymbol{\Sigma}_{XX}^{-1})$.

In order to obtain a reasonable comparison, we need the asymptotic distributions of the estimators to be different. To accomplish this goal, we consider the asymptotic distribution of the estimators under the sequence of local alternatives $\left\{K_{(n)}\right\}$ defined by

$$K_{(n)} : \boldsymbol{\beta}_{(n)} = n^{-\frac{1}{2}}\boldsymbol{\delta}, \quad \boldsymbol{\delta} \text{ fixed finite vector.} \qquad \square \tag{3.10}$$

Now following Chapter 7 of Sen and Singer [19] together with Theorem 2.2.1 of Fuller [4] we note that under $K_{(n)}$, we have the following theorem.

Theorem 3.2. *Let the model* (1.1) *hold along with* $\boldsymbol{\Sigma}_{uu}$ *known,* $\sigma_{ee} > 0$*, and* $\boldsymbol{\Sigma}_{xx}$ *positive definite. Then under* $K_{(n)}$ *as* $n \to \infty$*, we have*

(i) $\sqrt{n}\left(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_{(n)}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(\mathbf{0}, \mathbf{G}^*), \quad \mathbf{G}^* = \sigma_{ee}(\mathbf{K}_{xx}'\boldsymbol{\Sigma}_{XX}\mathbf{K}_{xx})^{-1}$,

(ii) $\sqrt{n}\left(\tilde{\boldsymbol{v}}_n - \boldsymbol{\beta}_{(n)}\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(-(\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\delta}, \sigma_{ee}\boldsymbol{\Sigma}_{XX}^{-1}\right)$,

(iii) $P\left\{\mathcal{L}_n \leqslant x | K_{(n)}\right\} = H_p(x, \triangle^2)$,
where $H_\nu(\cdot; \triangle^2)$ *is the cdf of a non-central chi-square distribution with* ν *DF and non-centrality parameter* $\frac{1}{2}\triangle^2$ *with* $\triangle^2 = \boldsymbol{\delta}'(\mathbf{G}^*)^{-1}\boldsymbol{\delta} = ||\boldsymbol{\delta}^*||^2$.

(iv) $\sqrt{n}\hat{\mathbf{G}}_n^{-1/2}(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}_{(n)}) \xrightarrow{\mathcal{D}} \mathbf{Z} - (\mathbf{Z} + \boldsymbol{\delta}^*)g(||\mathbf{Z} + \boldsymbol{\delta}^*||^2)$, *where* $\boldsymbol{\delta}^* = \mathbf{G}^{*-\frac{1}{2}}\boldsymbol{\delta}$,

(v) $\sqrt{n}\hat{\boldsymbol{\Gamma}}_n^{-1/2}(\boldsymbol{v}_n^* - \boldsymbol{\beta}_{(n)}) \xrightarrow{\mathcal{D}} \mathbf{Z} - (\mathbf{Z} + \boldsymbol{\gamma}^*)g(||\mathbf{Z} + \boldsymbol{\gamma}^*||^2) - (\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\delta}$,
where $\boldsymbol{\Gamma} = \sigma_{zz}\boldsymbol{\Sigma}_{XX}^{-1}$ *and* $\hat{\boldsymbol{\Gamma}}_n$ *is a consistent estimator of* $\boldsymbol{\Gamma}$ *while* $\boldsymbol{\gamma}^* = \boldsymbol{\Gamma}^{*-1/2}\boldsymbol{\delta}$ *and* $\boldsymbol{\Gamma}^* = \sigma_{ee}\boldsymbol{\Sigma}_{XX}^{-1}$.
In (iv) *and* (v), $\mathbf{Z} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I}_p)$.

4. Asymptotic distributional bias, mean-squares error and quadratic risk of the estimators of slope parameter $\boldsymbol{\beta}$

In this section, we consider the asymptotic distributional bias (ADB), the asymptotic distributional quadratic bias (ADQB), the asymptotic distributional mean-square error matrix (ADMSE) and the asymptotic distributional quadratic risks (ADQR) of the four estimators of $\boldsymbol{\beta}$ defined in Section 2 using Theorem 3.2 under $\{K_{(n)}\}$ and calculating $E[\mathbf{U}_g^{(1)}]$, $E[\mathbf{U}_g^{(1)}\mathbf{U}_g^{(1)'}]$ and $E[\mathbf{U}_g^{(1)'}\mathbf{Q}\mathbf{U}_g^{(1)}]$ where

$$\mathbf{U}_g^{(1)} = \mathbf{Z} - (\mathbf{Z} + \boldsymbol{\delta}^*)g(||\mathbf{Z} + \boldsymbol{\delta}^*||^2). \tag{4.1}$$

Similarly, the ADB, the ADQB, the ADMSE and the ADQR of the four estimators of $\boldsymbol{v}$ by calculating $E[\mathbf{U}_g^{(2)}]$, $E[\mathbf{U}_g^{(2)}\mathbf{U}_g^{(2)'}]$ and $E[\mathbf{U}_g^{(2)'}\mathbf{Q}\mathbf{U}_g^{(2)}]$ where

$$\mathbf{U}_g^{(2)} = \mathbf{Z} - (\mathbf{Z} + \boldsymbol{\gamma}^*)g(||\mathbf{Z} + \boldsymbol{\gamma}^*||^2) - (\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\delta}. \tag{4.2}$$

We only present the expressions for the estimators of $\boldsymbol{\beta}$ leaving the expressions of the estimators of $\boldsymbol{v}$ which may be obtained similarly. Then the ADB and ADQB of $\tilde{\boldsymbol{\beta}}_n$, $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$, $\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$ and $\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}$ are given by the $E[\mathbf{U}_g^{(1)}]$ for the four estimators as follows.

(i) $\mathbf{b}_1(\tilde{\boldsymbol{\beta}}_n) = \mathbf{0}$ and $B_1(\tilde{\boldsymbol{\beta}}_n) = \mathbf{0}$, (4.3a)

(ii) $\mathbf{b}_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}) = -\boldsymbol{\delta} H_{p+2}(\chi^2_\alpha; \triangle^2)$ and $B_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}) = \triangle^2 \Big\{ H_{p+2}(\chi^2_\alpha; \triangle^2) \Big\}^2$, (4.3b)

(iii) $\mathbf{b}_3(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}) = -(p-2)\boldsymbol{\delta} E\Big[\chi^{-2}_{p+2}(\triangle^2)\Big]$

and

$$B_3(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}) = (p-2)^2 \triangle^2 \Big\{ E\Big[\chi^{-2}_{p+2}(\triangle^2)\Big]\Big\}^2, \tag{4.3c}$$

(iv) $$\mathbf{b}_4(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}) = -(p-2)\boldsymbol{\delta}\Big\{ E\Big[\chi^{-2}_{p+2}(\triangle^2)\Big] - E\Big[\chi^{-2}_{p+2}(\triangle^2) I\Big(\chi^2_{p+2}(\triangle^2) < p-2\Big)\Big] + \tfrac{1}{p-2} H_{p+2}((p-2); \triangle^2)\Big\},$$

and

$$\begin{aligned} B_4(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}) = {} & (p-2)^2 \triangle^2 \Big\{ E\Big[\chi^{-2}_{p+2}(\triangle^2)\Big] \\ & - E\Big[\chi^{-2}_{p+2}(\triangle^2) I\Big(\chi^2_{p+2}(\triangle^2) < p-2\Big)\Big] + \frac{1}{p-2} H_{p+2}((p-2); \triangle^2)\Big\}^2, \end{aligned} \tag{4.3d}$$

respectively.

Similarly, ADMSE and ADQR of $\tilde{\boldsymbol{\beta}}_n$, $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$, $\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$ and $\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}$ using the loss function $n\Big(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}_{(n)}\Big)' \mathbf{Q}\Big(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}_{(n)}\Big)$ by considering the computation of $E[\mathbf{U}_g \mathbf{U}_g']$ and transforming back using $\mathbf{G}^*$. The final result is given below.

(i) $\mathbf{M}_1(\tilde{\boldsymbol{\beta}}_n) = \mathbf{G}^*$ and $R_1(\tilde{\boldsymbol{\beta}}_n; \mathbf{Q}) = \mathrm{tr}[\mathbf{G}^*\mathbf{Q}]$, (4.4a)

(ii) $$\begin{aligned} \mathbf{M}_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}) = {} & \mathbf{G}^*\Big\{1 - H_{p+2}(\chi^2_\alpha; \triangle^2)\Big\} \\ & + \boldsymbol{\delta}\boldsymbol{\delta}'\Big\{2H_{p+2}(\chi^2_\alpha; \triangle^2) - H_{p+4}(\chi^2_\alpha; \triangle^2)\Big\} \end{aligned} \tag{4.4b}$$

and

$$\begin{aligned} R_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}; \mathbf{Q}) = {} & \mathrm{tr}[\mathbf{G}^*\mathbf{Q}]\Big\{1 - H_{p+2}(\chi^2_\alpha; \triangle^2)\Big\} \\ & + \boldsymbol{\delta}'\mathbf{Q}\boldsymbol{\delta}\Big\{2H_{p+2}(\chi^2_\alpha; \triangle^2) - H_{p+4}(\chi^2_\alpha; \triangle^2)\Big\}, \end{aligned}$$

(iii) $$\begin{aligned} \mathbf{M}_3(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}) = {} & \mathbf{G}^*\Big\{1 - (p-2)\Big[2E\Big[\chi^{-2}_{p+2}(\triangle^2)\Big] \\ & - (p-2)E\Big[\chi^{-4}_{p+2}(\triangle^2)\Big]\Big]\Big\} + (p-2)\boldsymbol{\delta}\boldsymbol{\delta}'\Big\{2E\Big[\chi^{-2}_{p+2}(\triangle^2)\Big] \\ & - 2E\Big[\chi^{-2}_{p+4}(\triangle^2)\Big] + (p-2)E\Big[\chi^{-4}_{p+4}(\triangle^2)\Big]\Big\} \end{aligned} \tag{4.4c}$$

and

$$\begin{aligned}R_3(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}};\mathbf{Q}) = \operatorname{tr}[\mathbf{G}^*\mathbf{Q}]\Big\{1-(p-2)\Big\{2E\Big[\chi_{p+2}^{-2}(\triangle^2)\Big]\\ -(p-2)E\Big[\chi_{p+2}^{-4}(\triangle^2)\Big]\Big\}\Big\}+(p-2)\boldsymbol{\delta}'\mathbf{Q}\boldsymbol{\delta}\Big\{2E\Big[\chi_{p+2}^{-2}(\triangle^2)\Big]\\ -2E\Big[\chi_{p+4}^{-2}(\triangle^2)\Big]+(p-2)E\Big[\chi_{p+4}^{-4}(\triangle^2)\Big]\Big\}\end{aligned}$$

(iv) $$\begin{aligned}\mathbf{M}_4(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}) = \mathbf{M}_3(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}})-\mathbf{G}^*\Big[H_{p+2}((p-2);\triangle^2)\\ -(p-2)\Big\{2E\Big[\chi_{p+2}^{-2}(\triangle^2)I\Big(\chi_{p+2}^{2}(\triangle^2)<p-2\Big)\Big]\\ -(p-2)E\Big[\chi_{p+2}^{-4}(\triangle^2)I\Big(\chi_{p+2}^{2}(\triangle^2)<p-2\Big)\Big]\Big\}\Big]\\ -\boldsymbol{\delta}\boldsymbol{\delta}'\Big[H_{p+4}((p-2);\triangle^2)-2H_{p+2}((p-2);\triangle^2)\\ +(p-2)\Big\{2E\Big[\chi_{p+2}^{-2}(\triangle^2)I\Big(\chi_{p+2}^{2}(\triangle^2)<p-2\Big)\Big]\\ -2E\Big[\chi_{p+4}^{-2}(\triangle^2)I\Big(\chi_{p+4}^{2}(\triangle^2)<p-2\Big)\Big]\\ +(p-2)E\Big[\chi_{p+4}^{-4}(\triangle^2)I\Big(\chi_{p+4}^{2}(\triangle^2)<p-2\Big)\Big\}\Big]\end{aligned} \tag{4.4d}$$

and

$$\begin{aligned}R_4(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}};\mathbf{Q}) = R_3(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}})-\operatorname{tr}[\mathbf{G}^*\mathbf{Q}]\Big\{H_{p+2}((p-2);\triangle^2)\\ -(p-2)\Big\{2E\Big[\chi_{p+2}^{-2}(\triangle^2)I\Big(\chi_{p+2}^{2}(\triangle^2)<p-2\Big)\Big]\\ -(p-2)E\Big[\chi_{p+2}^{-4}(\triangle^2)I\Big(\chi_{p+2}^{2}(\triangle^2)<p-2\Big)\Big]\Big\}\Big\}\\ -\boldsymbol{\delta}'\mathbf{Q}\boldsymbol{\delta}\Big\{H_{p+4}((p-2);\triangle^2)-2H_{p+2}((p-2);\triangle^2)\\ +(p-2)\Big\{2E\Big[\chi_{p+2}^{-2}(\triangle^2)I\Big(\chi_{p+2}^{2}(\triangle^2)<p-2\Big)\Big]\\ -2E\Big[\chi_{p+4}^{-2}(\triangle^2)I\Big(\chi_{p+4}^{2}(\triangle^2)<p-2\Big)\Big]\\ +(p-2)E\Big[\chi_{p+4}^{-4}(\triangle^2)I\Big(\chi_{p+4}^{2}(\triangle^2)<p-2\Big)\Big\}\Big\},\end{aligned}$$

respectively.

5. ADB, MSE-matrix and risk comparisons

In this section, we compare the ADB, MSE-matrix and risks of the four estimators of $\boldsymbol{\beta}$ based on Theorem 3.2 under $\{K_{(n)}\}$.

5.1. ADB comparison

Clearly, $\tilde{\boldsymbol{\beta}}_n$ is asymptotically unbiased and $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$, $\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$ and $\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}$ are biased. Under H_0, they are all unbiased. Also, as $\triangle^2\to\infty$, the bias reduces to zero for all estimators. For the PTE, as $\alpha\to 0$, the quadratic bias goes to zero.

In general as $\triangle^2$ moves away from the origin, the quadratic bias function increases to a maximum then decreases towards zero as $\triangle^2$ tends to infinity. First note that for all $\triangle^2$,

$$\begin{aligned} B_3(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}) - B_4(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}) = & (p-2)^2\triangle^2\Big\{2E\Big[\chi_{p+2}^{-2}(\triangle^2)\Big] \\ & + \frac{1}{p-2}H_{p+2}(p-2;\triangle^2) \\ & - E\Big[\chi_{p+2}^{-2}(\triangle^2)I\Big(\chi_{p+2}^{2}(\triangle^2) < p-2\Big)\Big]\Big\} \\ & \times\Big\{E\Big[\chi_{p+2}^{-2}(\triangle^2)I\Big(\chi_{p+2}^{2}(\triangle^2) < p-2\Big)\Big] \\ & - \frac{1}{p-2}H_{p+2}(p-2;\triangle^2)\Big\} \geqslant 0. \end{aligned} \tag{5.1}$$

Thus, the graph of the ADQB of PRSE remains below the graph of the ADQB of SE, that is to say, the bias of PRSE is always smaller than that of SE, we may order the estimators according to the quadratic bias as follows:

$$\tilde{\boldsymbol{\beta}}_n < \hat{\boldsymbol{\beta}}_n^{\mathrm{S+}} < \hat{\boldsymbol{\beta}}_n^{\mathrm{S}} \quad \text{for all } \Delta^2. \tag{5.2}$$

In the case of PTE and SE, the ADQB difference shows that

$$\begin{aligned} B_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}) - B_3(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}) = \triangle^2\Big\{H_{p+2}(\chi_\alpha^2;\triangle^2) + E[\chi_{p+2}^{-2}(\triangle^2)]\Big\}\Big\{H_{p+2}(\chi_\alpha^2;\triangle^2) \\ - E[\chi_{p+2}^{-2}(\triangle^2)]\Big\}. \end{aligned}$$

Thus, the graph of ADQB of SE remains below the graph of ADQB of PTE whenever

$$H_{p+2}(\chi_\alpha^2;\triangle^2) \geqslant E\Big[\chi_{p+2}^{-2}(\triangle^2)\Big] \quad \text{for all } (\alpha, \triangle^2), \tag{5.3}$$

otherwise, the graph of ADQB of PTE remains below the graph of ADQB of SE. In general, the graph of ADQB of PTE and SE intersects at some point $\triangle^2$ for fixed α and the bias of PTE is worse than that of the SE.

Similarly, the ADQB-difference of PTE and PRSE is given by

$$\begin{aligned} B_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}) - B_4(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}) = & \triangle^2\Big\{H_{p+2}(\chi_\alpha^2;\triangle^2) \\ & + (p-2)E\Big[\chi_{p+2}^{-2}I\Big(\chi_{p+2}^{2}(\triangle^2) > p-2\Big)\Big]\Big\} \\ & + H_{p+2}(p-2;\triangle^2)\Big\}\Big\{H_{p+2}(\chi_\alpha^2;\triangle^2) - H_{p+2}(p-2;\triangle^2) \\ & - (p-2)E\Big[\chi_{p+2}^{-2}(\triangle^2)I\Big(\chi_{p+2}^{2}(\triangle^2) > p-2\Big)\Big]\Big\} \gtreqless 0, \end{aligned} \tag{5.4}$$

which may be positive or negative whenever

$$H_{p+2}(\chi_\alpha^2;\Delta^2) \gtreqless E\Big[\Big(1-(p-2)\chi_{p+2}^{-2}(\Delta^2)\Big)I\Big(\chi_{p+2}^{2}(\Delta^2) > p-2\Big)\Big].$$

Hence, for fixed α,

$$B_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}) < B_4(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}) \text{ or } B_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}) > B_4(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}) \text{ for some } \triangle^2 < \Delta_*^2(\alpha), \tag{5.5}$$

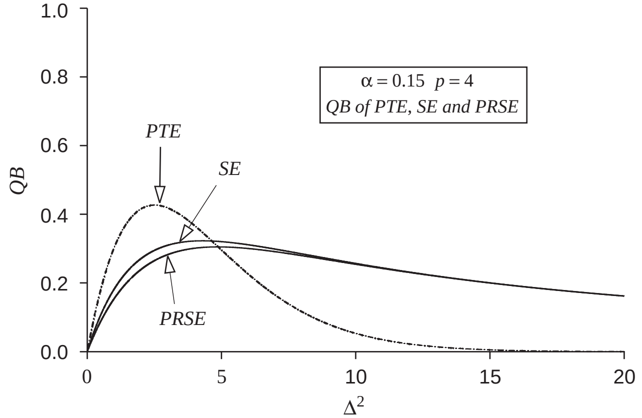

Fig. 1. ADQB of PTE, SE, PRSE with $p = 4$.

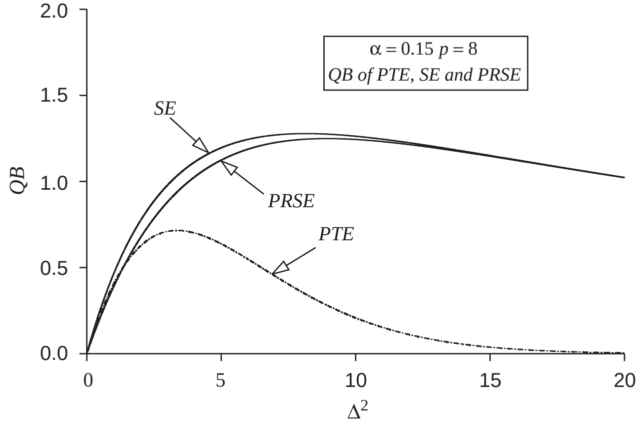

Fig. 2. ADQB of PTE, SE and PRSE with $p = 8$.

where $\Delta_*^2(\alpha)$ is the point of intersection of the graph of $B_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}})$ and $B_4(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}})$. The graphical representation of ADQB depicts these findings in Figs. 1 and 2.

5.2. ADQR analysis

5.2.1. Comparison of $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$ and $\tilde{\boldsymbol{\beta}}_n$

In this case the risk-difference $R_1(\tilde{\boldsymbol{\beta}}_n : \mathbf{Q}) - R_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}} : \mathbf{Q}) \geqslant 0$ whenever

$$\triangle^2 \leqslant \frac{\mathrm{tr}(\mathbf{G}^*\mathbf{Q})}{\mathrm{Ch}_{\max}(\mathbf{G}^*\mathbf{Q})} \frac{H_{p+2}(\chi_\alpha^2 : \triangle^2)}{\{2H_{p+2}(\chi_\alpha^2 : \triangle^2) - H_{p+4}(\chi_\alpha^2 : \triangle^2)\}}, \tag{5.6}$$

where $\mathrm{Ch}_{\max}(\mathbf{A})$ is the maximum characteristic value of the matrix $\mathbf{A}$.

Then, $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$ performs better than $\tilde{\boldsymbol{\beta}}_n$ in this range of $\triangle^2$. On the other hand, $\tilde{\boldsymbol{\beta}}_n$ performs better than $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$ if

$$\triangle^2 > \frac{\mathrm{tr}(\mathbf{G}^*\mathbf{Q})}{\mathrm{Ch}_{\min}(\mathbf{G}^*\mathbf{Q})} \frac{H_{p+2}(\chi_\alpha^2 : \triangle^2)}{\{2H_{p+2}(\chi_\alpha^2 : \triangle^2) - H_{p+4}(\chi_\alpha^2 : \triangle^2)\}}, \tag{5.7}$$

where $\mathrm{Ch}_{\min}(\mathbf{A})$ is the minimum characteristic value of the matrix $\mathbf{A}$.

Under H_0, the risk-difference becomes $\mathrm{tr}[\mathbf{G}^*\mathbf{Q}]H_{p+2}(\chi_\alpha^2 : 0)$. Hence, the relative gain in using $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$ against $\tilde{\boldsymbol{\beta}}_n$ is $100\mathrm{tr}(\mathbf{G}^*\mathbf{Q})H_{p+2}(\chi_\alpha^2 : 0)\%$ at α-level of significance test. An optimum PTE with minimum guaranteed efficiency E_0 may be obtained at α^*-level of significance by solving the equality,

$$\min_{\triangle^2} E(\alpha, \triangle^2) = E(\alpha, \Delta^2_{\min}(\alpha)) = E_0, \tag{5.8}$$

where

$$E(\alpha, \triangle^2) = \frac{R_1(\tilde{\boldsymbol{\beta}}_n : \mathbf{Q})}{R_2(\tilde{\boldsymbol{\beta}}_n^{\mathrm{PT}} : \mathbf{Q})}.$$

Now setting $\mathbf{Q} = (\mathbf{G}^*)^{-1}$, we have

$$E(\alpha, \triangle^2) = \Big[1 - H_{p+2}(\chi_\alpha^2 : \Delta^2) + \frac{1}{p}\Delta^2\Big\{2H_{p+2}(\chi_\alpha^2 : \Delta^2) - H_{p+4}(\chi_\alpha^2 : \Delta^2)\Big\}\Big]^{-1}. \tag{5.9}$$

Clearly, $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$ is better than $\tilde{\boldsymbol{\beta}}_n$ whenever

$$\Delta < \frac{pH_{p+2}(\chi_\alpha^2; \Delta^2)}{2H_{p+2}(\chi_\alpha^2; \Delta^2) - H_{p+4}(\chi_\alpha^2; \Delta^2)}.$$

Otherwise, $\tilde{\boldsymbol{\beta}}_n$ is better than $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$.

Table 1 gives the maximum and minimum asymptotic efficiency, $E(\alpha, \Delta^2)$ of $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$ relative to $\tilde{\boldsymbol{\beta}}_n$ as a function of Δ^2 for $\mathbf{Q} = \mathbf{G}^{*-1}$ and for chosen α-values and $p = 4(2)14$ using (5.9). For $p = 8$, if PTE of $\boldsymbol{\beta}$ is to be chosen with at least .80 efficiency, we choose $\alpha^* = 0.05$ as the level of the test.

5.2.2. *Comparison of $\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$ and $\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}$ with $\tilde{\boldsymbol{\beta}}_n$*

The risk of $\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$ may be rewritten as

$$\mathrm{tr}(\mathbf{G}^*\mathbf{Q}) - (p-2)\mathrm{tr}(\mathbf{G}^*\mathbf{Q})\Big\{(p-2)E[\chi_{p+2}^{-4}(\Delta^2)] + 2\Big(1 - \frac{(p+2)\boldsymbol{\delta}'\mathbf{Q}\boldsymbol{\delta}}{2\triangle^2\mathrm{tr}(\mathbf{G}^*\mathbf{Q})}\Big)\triangle^2 E[\chi_{p+4}^{-4}(\Delta^2)]\Big\} \tag{5.10}$$

Table 1
Maximum and minimum of risk-based efficiencies

α	p	4	6	8	10	12	14
0.05	$E_{\max}$	6.75908	7.89292	8.72360	9.37375	9.90415	10.34988
	$E_{\max}$	0.67741	0.76147	0.81707	0.85639	0.88538	0.90738
	$\Delta^2_{\min}$	8.70073	10.82463	12.79251	14.66773	16.48109	18.24932
0.1	$E_{\max}$	3.92601	4.49118	4.89829	5.21311	5.46767	5.67991
	$E_{\max}$	0.75506	0.82544	0.87014	0.90071	0.92260	0.93879
	$\Delta^2_{\min}$	7.73232	9.77526	11.68294	13.51002	15.28335	17.01821
0.15	$E_{\max}$	2.89783	3.26722	3.53047	3.73248	3.89487	4.02965
	$E_{\max}$	0.80429	0.86374	0.90061	0.92530	0.94267	0.95530
	$\Delta^2_{\min}$	7.16494	9.15150	11.01743	12.81104	14.55626	16.26681
0.2	$E_{\max}$	2.35590	2.62511	2.81549	2.96075	3.07701	3.17317
	$E_{\max}$	0.84032	0.89067	0.92141	0.94171	0.95579	0.96591
	$\Delta^2_{\min}$	6.76058	8.70106	10.53304	12.29940	14.02168	15.71231
0.25	$E_{\max}$	2.01845	2.22631	2.37244	2.48345	2.57199	2.64502
	$E_{\max}$	0.86844	0.91105	0.93681	0.95363	0.96518	0.97339
	$\Delta^2_{\min}$	6.44492	8.34493	10.14718	11.88969	13.59187	15.26500
0.3	$E_{\max}$	1.78732	1.95346	2.06973	2.15774	2.22775	2.28535
	$E_{\max}$	0.89120	0.92715	0.94874	0.96272	0.97225	0.97897
	$\Delta^2_{\min}$	6.18490	8.04786	9.82294	11.54366	13.22745	14.88455
0.35	$E_{\max}$	1.61900	1.75471	1.84936	1.92081	1.97751	2.02408
	$E_{\max}$	0.91005	0.94021	0.95827	0.96989	0.97776	0.98326
	$\Delta^2_{\min}$	5.96289	7.79095	9.54048	11.24068	12.90716	14.54916
0.4	$E_{\max}$	1.49112	1.60350	1.68169	1.74059	1.78724	1.82549
	$E_{\max}$	0.92589	0.95099	0.96604	0.97567	0.98215	0.98666
	$\Delta^2_{\min}$	5.76832	7.56281	9.28777	10.96826	12.61807	14.24553
0.45	$E_{\max}$	1.39096	1.48475	1.54992	1.59894	1.63770	1.66944
	$E_{\max}$	0.93933	0.96002	0.97245	0.98039	0.98571	0.98938
	$\Delta^2_{\min}$	5.59431	7.35598	9.05691	10.71809	12.35159	13.96478
0.5	$E_{\max}$	1.31074	1.38927	1.44382	1.48480	1.51718	1.54367
	$E_{\max}$	0.95079	0.96763	0.97781	0.98430	0.98862	0.99160
	$\Delta^2_{\min}$	5.43611	7.16522	8.84227	10.48427	12.10153	13.70053

The risk-difference $R_1(\tilde{\boldsymbol{\beta}}_n : \mathbf{Q}) - R_3(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}} : \mathbf{Q}) \geqslant 0$ for all $\triangle^2$ if $\mathbf{Q}$ satisfies the condition

$$\frac{\mathrm{tr}(\mathbf{G}^*\mathbf{Q})}{\mathrm{Ch}_{\max}(\mathbf{G}^*\mathbf{Q})} \geqslant \frac{p+2}{2}. \tag{5.11}$$

Thus, $\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$ uniformly dominates $\tilde{\boldsymbol{\beta}}_n$ under (5.11). As $\triangle^2 \to \infty$, the risk-difference goes to zero.

Now consider the risk-difference $R_3(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}} : \mathbf{Q}) - R_4(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}} : \mathbf{Q})$. It is nonnegative for all $(\triangle^2, \mathbf{Q})$ since the risk-difference is

$$\begin{aligned} &\Big\{\mathrm{tr}(\mathbf{G}^*\mathbf{Q})E\Big[\Big(1-(p-2)\chi_{p+2}^{-2}(\triangle^2)\Big)^2 I\Big(\chi_{p+2}^2(\triangle^2) < p-2\Big)\Big] \\ &\quad +\boldsymbol{\delta}'\mathbf{Q}\boldsymbol{\delta}E\Big[\Big(1-(p-2)\chi_{p+4}^{-2}(\triangle^2)\Big)^2 I\Big(\chi_{p+4}^2(\triangle^2) < p-2\Big)\Big]\Big\} \\ &\quad +2\boldsymbol{\delta}'\mathbf{Q}\boldsymbol{\delta}E\Big[\Big((p-2)\chi_{p+2}^{-2}(\triangle^2)-1\Big)I\Big(\chi_{p+2}^2(\triangle^2) < p-2\Big)\Big] \geqslant 0. \end{aligned} \tag{5.12}$$

Hence, $\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}$ dominates $\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$ for all $(\mathbf{Q}, \Delta^2)$. This leads to the conclusion that

$$R_1(\tilde{\boldsymbol{\beta}}_n : \mathbf{Q}) \geqslant R_3(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}} : \mathbf{Q}) \geqslant R_4(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}} : \mathbf{Q})$$

uniformly in $\triangle^2$ under (5.11).

5.2.3. *Comparison of $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$ with $\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$ or $\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}$*

Consider the risk-difference of $R_3(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}} : \mathbf{Q}) - R_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}} : \mathbf{Q})$ under H_0 given by

$$\mathrm{tr}(\mathbf{G}^*\mathbf{Q})\Big[H_{p+2}(\chi_\alpha^2 : 0) - \frac{p-2}{p}\Big] \quad \text{for } p > 2. \tag{5.13}$$

It is nonnegative if $H_{p+2}(\chi_\alpha^2 : 0) > \frac{p-2}{p}$. Thus, $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$ dominates $\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$ for a set of the level of significance α for which

$$H_{p+2}(\chi_\alpha^2 : 0) > \frac{p-2}{p} = 1 - \frac{2}{p} \quad \text{for } p > 2. \tag{5.14}$$

We have used the results $E[\chi_p^{-2}(0)] = (p-2)^{-1}$ and $E[\chi_p^{-4}(0)] = (p-2)^{-1}(p-4)^{-1}$ in obtaining (5.13) and (5.14). Thus, neither $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$ nor $\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$ dominate each other uniformly. In general, $\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}$ should be used whenever $p \geqslant 3$ but for $p \leqslant 2$, $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$ is preferable in case $\boldsymbol{\beta}$ is close to $\mathbf{0}$.

Now, the risk-difference of $R_4(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}} : \mathbf{Q}) - R_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}} : \mathbf{Q})$ under H_0 is given by

$$\begin{aligned} &\mathrm{tr}(\mathbf{G}^*\mathbf{Q})\Big[H_{p+2}(\chi_\alpha^2; 0) + (p-2)\Big\{2E[\chi_{p+2}^{-2} I(\chi_{p+2}^2 < p-2)] \\ &\quad -(p-2)E[\chi_{p+2}^{-4} I(\chi_{p+2}^2 < p-2)]\Big\} - \Big(\frac{p-2}{p} + H_{p+2}(p-2; 0)\Big)\Big] \\ &\quad \text{for } p > 2. \end{aligned} \tag{5.15}$$

Hence, $\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}$ has lesser risk than $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$ when the level of significance satisfies the relation

$$\begin{aligned} H_{p+2}(\chi_\alpha^2; 0) \geqslant &\Big\{\Big(1 - \frac{2}{p}\Big) \\ &+H_{p+2}(p-2; 0) - (p-2)E\Big[\chi_{p+2}^{-2} I(\chi_{p+2}^2 < p-2)\Big]\Big\}. \end{aligned} \tag{5.16}$$

Figs. 3 and 4 provide the graphical representation of the risks and efficiencies of the various estimators when $\mathbf{Q} = \mathbf{G}^{*-1}$.

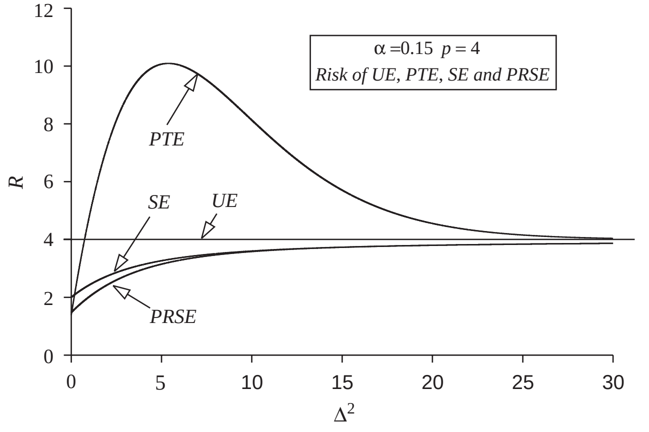

Fig. 3. Risk of UE, PTE, SE and PRSE.

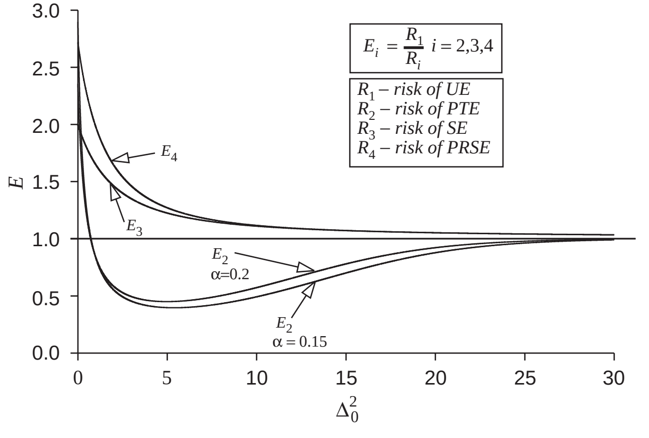

Fig. 4. Graph of efficiencies of different estimators.

5.3. ADMSE comparisons

First we compare $\tilde{\boldsymbol{\beta}}_n$ and $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$. In this case MSE matrix difference $\mathbf{M}_1(\tilde{\boldsymbol{\beta}}_n) - \mathbf{M}_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}})$ is given by

$$\mathbf{M}_1(\tilde{\boldsymbol{\beta}}_n) - \mathbf{M}_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}) = \mathbf{G}^* H_{p+2}(\chi^2_\alpha; \triangle^2) - \boldsymbol{\delta\delta}' \Big\{ 2H_{p+2}(\chi^2_\alpha; \triangle^2) - H_{p+4}(\chi^2_\alpha; \triangle^2) \Big\}. \quad (5.17)$$

Thus, for a given nonzero vector ℓ, we have

$$\boldsymbol{\ell}'(\mathbf{M}_1(\tilde{\boldsymbol{\beta}}_n) - \mathbf{M}_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}))\boldsymbol{\ell} = \boldsymbol{\ell}'\mathbf{G}^*\boldsymbol{\ell} H_{p+2}(\chi^2_\alpha; \triangle^2) - \boldsymbol{\ell}'\boldsymbol{\delta\delta}'\boldsymbol{\ell} \Big\{ 2H_{p+2}(\chi^2_\alpha; \triangle^2) - H_{p+4}(\chi^2_\alpha; \triangle^2) \Big\}. \quad (5.18)$$

RHS is nonnegative for all nonzero $\boldsymbol{\ell}$ if and only if

$$\max_{\boldsymbol{\ell}} \frac{\boldsymbol{\ell}'\boldsymbol{\delta}\boldsymbol{\delta}'\boldsymbol{\ell}}{\boldsymbol{\ell}'\mathbf{G}^*\boldsymbol{\ell}}\left\{2H_{p+2}(\chi^2_\alpha;\triangle^2) - H_{p+4}(\chi^2_\alpha;\triangle^2)\right\} \leqslant H_{p+2}(\chi^2_\alpha;\triangle^2) \tag{5.19}$$

or

$$\triangle^2 \leqslant \frac{H_{p+2}(\chi^2_\alpha;\triangle^2)}{2H_{p+2}(\chi^2_\alpha;\triangle^2) - H_{p+4}(\chi^2_\alpha;\triangle^2)}, \tag{5.20}$$

since $\max_{\boldsymbol{\ell}} \frac{\boldsymbol{\ell}'\boldsymbol{\delta}\boldsymbol{\delta}'\boldsymbol{\ell}}{\boldsymbol{\ell}'\mathbf{G}^*\boldsymbol{\ell}} = \triangle^2$.

Now, $\tilde{\boldsymbol{\beta}}_n$ would be better than $\hat{\boldsymbol{\beta}}_n^{\rm PT}$ if and only if (5.18) $\leqslant 0$ for all nonzero vector $\boldsymbol{\ell}$ with, say, $\boldsymbol{\ell}'\boldsymbol{\ell} = 1$. But this can never be so because $\boldsymbol{\ell}'\boldsymbol{\delta}\boldsymbol{\delta}'\boldsymbol{\ell}$ can be made arbitrarily small whereas $\boldsymbol{\ell}'\mathbf{G}^*\boldsymbol{\ell}$ stays away from zero. Thus, if (5.20) is not satisfied, neither $\hat{\boldsymbol{\beta}}_n^{\rm PT}$ nor $\tilde{\boldsymbol{\beta}}_n$ be better than the other one.

Next, we consider the MSE matrix differences $\mathbf{M}_1(\tilde{\boldsymbol{\beta}}_n) - \mathbf{M}_3(\hat{\boldsymbol{\beta}}_n^{\rm S})$ and $\mathbf{M}_2(\hat{\boldsymbol{\beta}}_n^{\rm PT}) - \mathbf{M}_3(\hat{\boldsymbol{\beta}}_n^{\rm S})$. Now,

$$\begin{aligned}\mathbf{M}_1(\tilde{\boldsymbol{\beta}}_n) - \mathbf{M}_3(\hat{\boldsymbol{\beta}}_n^{\rm S}) &= (p-2)\mathbf{G}^*\left\{2E[\chi^{-2}_{p+2}(\triangle^2)] - (p-2)E[\chi^{-4}_{p+2}(\triangle^2)]\right\} \\ &\quad -(p^2-4)\boldsymbol{\delta}\boldsymbol{\delta}' E[\chi^{-4}_{p+4}(\triangle^2)].\end{aligned} \tag{5.21}$$

The RHS is not negative whenever for any given nonzero vector ℓ we have

$$(p+2)\frac{\boldsymbol{\ell}'\boldsymbol{\delta}\boldsymbol{\delta}'\boldsymbol{\ell}}{\boldsymbol{\ell}'\mathbf{G}^*\boldsymbol{\ell}} E[\chi^{-4}_{p+4}(\triangle^2)] \leqslant E[\chi^{-2}_{p+2}(\triangle^2)] + \triangle^2 E[\chi^{-4}_{p+4}(\triangle^2)] \tag{5.22}$$

Since, $\max_{\boldsymbol{\ell}} \frac{\boldsymbol{\ell}'\boldsymbol{\delta}\boldsymbol{\delta}'\boldsymbol{\ell}}{\boldsymbol{\ell}'\mathbf{G}^*\boldsymbol{\ell}} = \Delta^2$, (5.22) implies

$$(p+2)\triangle^2 E[\chi^{-4}_{p+4}(\triangle^2)] \leqslant E[\chi^{-2}_{p+2}(\triangle^2)] + \triangle^2 E[\chi^{-4}_{p+4}(\triangle^2)]. \tag{5.23}$$

Using the identity $E[\chi^{-2}_{p+2}(\Delta^2)] - (p-2)E[\chi^{-4}_{p+2}(\Delta^2)] = \Delta^2 E[\chi^{-4}_{p+4}(\Delta^2)]$, we obtain

$$pE[\chi^{-2}_{p+2}(\triangle^2)] \leqslant (p-2)E[\chi^{-4}_{p+2}(\triangle^2)], \tag{5.24}$$

which is a contradiction (for example, $\Delta^2 = 0$ implies $1 \leqslant \frac{1}{p}$). Hence, $\hat{\boldsymbol{\beta}}_n^{\rm S}$ does not dominate $\tilde{\boldsymbol{\beta}}_n$ uniformly.

Next we observe that

$$\begin{aligned}&\mathbf{M}_3(\hat{\boldsymbol{\beta}}_n^{\rm S}) - \mathbf{M}_4(\hat{\boldsymbol{\beta}}_n^{\rm S+}) \\ &\quad = \mathbf{G}^*\Big\{E\Big[\Big(1-(p-2)\chi^{-2}_{p+2}(\Delta^2)\Big)^2 I\Big(\chi^2_{p+2}(\Delta^2) < p-2\Big)\Big] \\ &\qquad +\boldsymbol{\delta}\boldsymbol{\delta}'\Big\{2E\Big[\Big(1-(1-p)\chi^{-2}_{p+2}(\Delta^2)\Big) I\Big(\chi^2_{p+2}(\Delta^2) < p-2\Big)\Big] \\ &\qquad -E\Big[\Big(1-(p-2)\chi^{-2}_{p+4}(\Delta^2)\Big)^2 I\Big(\chi^2_{p+2}(\Delta^2) < p-2\Big)\Big]\Big\} \geqslant \mathbf{0}\end{aligned} \tag{5.25}$$

for all $\boldsymbol{\delta}$ since $\chi^2_{p+t}(\Delta^2) < p-2$ for $t = 2$ and 4. Hence, $\hat{\boldsymbol{\beta}}_n^{\rm S+}$ dominates $\hat{\boldsymbol{\beta}}_n^{\rm S}$ uniformly with respect to the MSE-matrices.

The comparison of $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$ and $\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$ is obtained by considering

$$\begin{aligned}\mathbf{M}_3(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}) - \mathbf{M}_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}) = \mathbf{G}^*\Big\{ & H_{p+2}(\chi^2_\alpha; \triangle^2) \\ & -(p-2)\Big[2E[\chi^{-2}_{p+2}(\triangle^2)] - (p-2)E[\chi^{-4}_{p+2}(\triangle^2)]\Big]\Big\} \\ & -\boldsymbol{\delta}\boldsymbol{\delta}'\Big\{\Big[2H_{p+2}(\chi^2_\alpha; \triangle^2) - H_{p+4}(\chi^2_\alpha; \triangle^2)\Big] \\ & -(p^2-4)E[\chi^{-4}_{p+4}(\triangle^2)]\Big\}. \end{aligned} \tag{5.26}$$

Thus, the difference $\mathbf{M}_3(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}) - \mathbf{M}_2(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}})$ is nonnegative semi-definite iff for any level of significance, α.

$$\triangle^2 \leqslant \frac{H_{p+2}(\chi^2_\alpha; \triangle^2) - (p-2)\{2E[\chi^{-2}_{p+2}(\triangle^2)] - (p-2)E[\chi^{-4}_{p+2}(\triangle^2)]\}}{(p^2-4)E[\chi^{-4}_{p+4}(\triangle^2)] - \{2H_{p+2}(\chi^2_\alpha; \triangle^2) - H_{p+4}(\chi^2_\alpha; \triangle^2)\}}. \tag{5.27}$$

Thus, $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$ is better than $\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$ if and only if $\triangle^2$ satisfies (5.27). By similar argument as before (see after (5.20)), we find that neither $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$ nor $\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$ dominate each other uniformly. Similar conclusion holds for $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$ and $\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}$ and neither $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$ nor $\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}$ dominate each other uniformly for any level of significance, α. Similar conclusion holds for $\tilde{\boldsymbol{v}}_n$, $\hat{\boldsymbol{v}}_n^{\mathrm{PT}}$, $\hat{\boldsymbol{v}}_n^{\mathrm{S}}$ and $\hat{\boldsymbol{v}}_n^{\mathrm{S+}}$ and is not repeated.

5.4. *Comparison of $\boldsymbol{v}_n^*$ with $\boldsymbol{\beta}_n^*$*

We have two set of estimators of $\boldsymbol{\beta}$, namely, $\boldsymbol{v}_n^*$ and $\boldsymbol{\beta}_n^*$. In this section, we compare them under $\{K_{(n)}\}$. In order to carry out valid comparison, we first obtain the canonical bias, MSE matrices and risk expressions for $\boldsymbol{v}_n^*$ and $\boldsymbol{\beta}_n^*$.

5.4.1. *Canonical bias and MSE Expression*

There exists a non-singular $p \times p$ matrix $\mathbf{T}$ that simultaneously diagonalizes $\boldsymbol{\Sigma}_{XX}$ and $\boldsymbol{\Sigma}_{xx}$ (see for example [5]). Thus,

$$\mathbf{T}'\boldsymbol{\Sigma}_{XX}\mathbf{T} = \mathbf{I}_p, \quad \mathbf{T}'\boldsymbol{\Sigma}_{xx}\mathbf{T} = \mathbf{D} = \mathrm{diag}(\delta_1, \ldots, \delta_p) \tag{5.28}$$

where $\delta_1 \leqslant \cdots \leqslant \delta_p$ are the ordered eigenvalues of $\boldsymbol{\Sigma}^{-1}_{XX}\boldsymbol{\Sigma}_{xx} = \mathbf{K}_{xx}$.

It is easily seen that $\mathbf{K}_{xx} = \mathbf{TDT}^{-1}$ so that the ith column of $\mathbf{T}$ is the right eigenvector of $\mathbf{K}_{xx}$ corresponding to the eigenvalue δ_i, $i = 1, \ldots, p$.

Let $\boldsymbol{\theta} = \mathbf{T}^{-1}\boldsymbol{\beta}$. Then, $\tilde{\boldsymbol{\theta}}_n = \mathbf{T}^{-1}\tilde{\boldsymbol{\beta}}_n$ and $\hat{\boldsymbol{\theta}}_n = \mathbf{T}^{-1}\tilde{\boldsymbol{v}}_n$.

Now consider the estimators $\mathbf{T}^{-1}\boldsymbol{v}_n^*$ and $\mathbf{T}^{-1}\boldsymbol{\beta}_n^*$ of $\mathbf{T}^{-1}\boldsymbol{\beta} = \boldsymbol{\theta}$.

Then the expressions of ADB and ADMSE of $\mathbf{T}^{-1}\boldsymbol{v}_n^*$ are given, respectively, by

$$\mathbf{b}^{(1)} = \sqrt{n}\mathbf{T}^{-1}\lim_{n\to\infty} E[\boldsymbol{v}_n^* - \boldsymbol{\beta}_{(n)}] = -(\mathbf{I}-\mathbf{D})\boldsymbol{\gamma} - \mathbf{D}\boldsymbol{\gamma}E\Big[g(\chi^2_{p+2}(\Delta^2))\Big] \tag{5.29}$$

and

$$\begin{aligned}\mathbf{M}^{(1)} &= n\mathbf{T}^{-1}\lim_{n\to\infty} E[(\boldsymbol{v}_n^* - \boldsymbol{\beta}_{(n)})(\boldsymbol{v}_n^* - \boldsymbol{\beta}_{(n)})']\mathbf{T}^{-1'} \\ &= \sigma_{ee}\mathbf{I} + (\mathbf{I}_p - \mathbf{D})\boldsymbol{\gamma\gamma}'(\mathbf{I}-\mathbf{D}) - \sigma_{ee}\mathbf{I}\Big\{2E\Big[g(\chi^2_{p+2}(\Delta^2))\Big] \\ &\quad -E\Big[g^2(\chi^2_{p+2}(\Delta^2))\Big]\Big\} + \mathbf{D}\boldsymbol{\gamma\gamma}'\mathbf{D}\Big\{2E\Big[g(\chi^2_{p+2}(\Delta^2))\Big] \\ &\quad -2E\Big[g(\chi^2_{p+4}(\Delta^2))\Big] + E\Big[g^2(\chi^2_{p+4}(\Delta^2))\Big]\Big\} \\ &\quad +(\mathbf{I}-\mathbf{D})\boldsymbol{\gamma\gamma}'\mathbf{D}E\Big[g(\chi^2_{p+2}(\Delta^2))\Big] + \mathbf{D}\boldsymbol{\gamma\gamma}'(\mathbf{I}-\mathbf{D})E\Big[g(\chi^2_{p+2}(\Delta^2))\Big], \end{aligned} \tag{5.30}$$

where $\boldsymbol{\gamma} = \sqrt{n}\mathbf{T}^{-1}\boldsymbol{\beta}_{(n)} = \mathbf{T}^{-1}\boldsymbol{\delta}$.

Similarly, the expressions of ADB and ADMSE of $\mathbf{T}^{-1}\boldsymbol{\beta}_n^*$ are given, respectively by

$$\mathbf{b}^{(2)} = \sqrt{n}\mathbf{T}^{-1}\lim_{n\to\infty} E[\boldsymbol{\beta}_n^* - \boldsymbol{\beta}_{(n)}] = -\boldsymbol{\gamma}E\Big[g(\chi^2_{p+2}(\Delta^2))\Big] \tag{5.31}$$

and

$$\begin{aligned}\mathbf{M}^{(2)} &= n\mathbf{T}^{-1}\lim_{n\to\infty} E[(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}_{(n)})(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}_{(n)})']\mathbf{T}^{-1'} \\ &= \sigma_{ee}\mathbf{D}^{-2} - \sigma_{ee}\mathbf{D}^{-2}\Big\{2E\Big[g(\chi^2_{p+2}(\Delta^2))\Big] \\ &\quad -E\Big[g^2(\chi^2_{p+2}(\Delta^2))\Big]\Big\} + \boldsymbol{\gamma\gamma}'\Big\{2E\Big[g(\chi^2_{p+2}(\Delta^2))\Big] \\ &\quad -2E\Big[g(\chi^2_{p+4}(\Delta^2))\Big] + E\Big[g^2(\chi^2_{p+4}(\Delta^2))\Big]\Big\}. \end{aligned} \tag{5.32}$$

The expressions of (5.29)–(5.32) are the canonical forms of the corresponding expressions of $\boldsymbol{v}_n^*$ and $\boldsymbol{\beta}_n^*$, respectively, based on the limiting distributions by Theorem 3.2 under $\{K_{(n)}\}$.

5.4.2. Canonical risk expressions for $\tilde{\mathbf{v}}_n$ and $\tilde{\boldsymbol{\beta}}_n$

$$\begin{aligned} R_1^{(1)}(\hat{\boldsymbol{\theta}}_n) &= \sigma_{ee}\mathrm{tr}[\mathbf{I}] + \boldsymbol{\gamma}'(\mathbf{I}-\mathbf{D})^2\boldsymbol{\gamma}, \quad \mathrm{tr}[\mathbf{I}] = p, \\ R_1^{(2)}(\tilde{\boldsymbol{\theta}}_n) &= \sigma_{ee}\mathrm{tr}[\mathbf{D}^{-2}]. \end{aligned} \tag{5.33}$$

The difference between the two risks is given by

$$R_1^{(1)}(\hat{\boldsymbol{\theta}}_n) - R_1^{(2)}(\tilde{\boldsymbol{\theta}}_n) = \sigma_{ee}\mathrm{tr}[\mathbf{I}-\mathbf{D}^{-2}] + \boldsymbol{\gamma}'(\mathbf{I}-\mathbf{D})^2\boldsymbol{\gamma}. \tag{5.34}$$

The risk of $\hat{\boldsymbol{\theta}}_n$ is smaller than the risk of $\tilde{\boldsymbol{\beta}}_n$ if

$$\Delta^2 \leqslant \frac{\sigma_{ee}\mathrm{tr}[\mathbf{D}^{-2}-\mathbf{I}]}{\mathrm{Ch}_{\min}[\mathbf{A}_1\mathbf{D}^{-2}]}, \tag{5.35}$$

where $\mathbf{A}_1 = (\mathbf{I}-\mathbf{D})^2$ and the risk of $\tilde{\boldsymbol{\beta}}_n$ is smaller than $\hat{\boldsymbol{\theta}}_n$ if

$$\Delta^2 > \frac{\sigma_{ee}\mathrm{tr}[\mathbf{D}^{-2}-\mathbf{I}]}{\mathrm{Ch}_{\max}[\mathbf{A}_1\mathbf{D}^{-2}]}. \tag{5.36}$$

Clearly, we find that $\hat{\boldsymbol{\theta}}_n$ dominates $\tilde{\boldsymbol{\theta}}_n$ under the null-hypotheses. The Fig. 5 shows the entire dominance picture of $\hat{\boldsymbol{\theta}}_n$ and $\tilde{\boldsymbol{\theta}}_n$ thereby of $\tilde{\mathbf{v}}_n$ and $\tilde{\boldsymbol{\beta}}_n$.

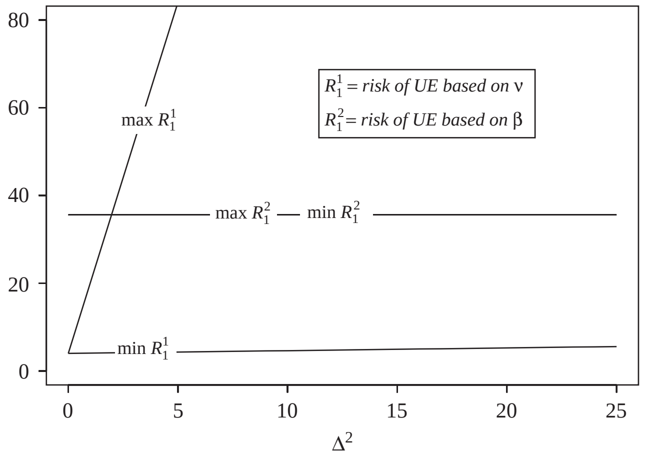

Fig. 5. Risk bounds as a function of Δ^2 for the $\hat{\boldsymbol{\theta}}_n$ and the $\tilde{\boldsymbol{\theta}}_n$ for $p = 4$ and $\mathbf{D} = \mathrm{diag}(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$, $\sigma_{ee} = 1$.

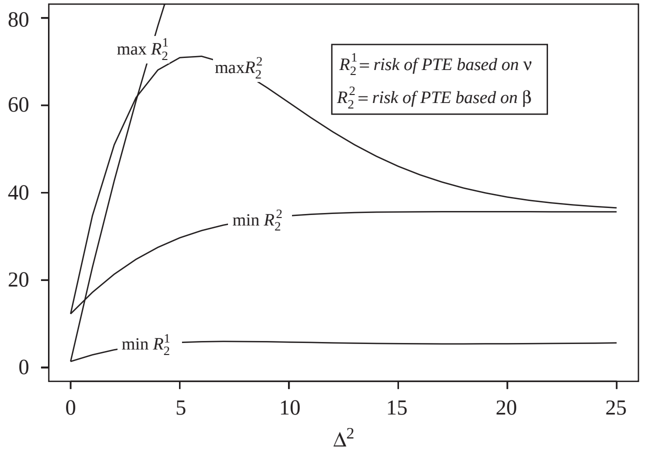

Fig. 6. Risk bounds as a function of Δ^2 for the $\hat{\boldsymbol{\theta}}_n^{\mathrm{PT}}$ and the $\tilde{\boldsymbol{\theta}}_n^{\mathrm{PT}}$ for $p = 4$ and $\mathbf{D} = \mathrm{diag}(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$, $\sigma_{ee} = 1$.

5.4.3. *Canonical risk expression of $\hat{\mathbf{v}}_n^{\mathrm{PT}}$ and $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$*

$$\begin{aligned}
R_2^{(1)}(\hat{\boldsymbol{\theta}}_n^{\mathrm{PT}}) &= R_1^{(1)}(\hat{\boldsymbol{\theta}}_n) - \sigma_{ee}\mathrm{tr}[\mathbf{I}]H_{p+2}(\chi_\alpha^2 : \triangle^2) \\
&\quad + \boldsymbol{\gamma}'\mathbf{D}^2\boldsymbol{\gamma}\Big\{2H_{p+2}(\chi_\alpha^2 : \triangle^2) - H_{p+4}(\chi_\alpha^2 : \triangle^2)\Big\} \\
&\quad + 2\boldsymbol{\gamma}'\mathbf{D}(\mathbf{I} - \mathbf{D})\boldsymbol{\gamma}H_{p+2}(\chi_\alpha^2 : \triangle^2), \quad \mathrm{tr}[\mathbf{I}] = p, \\
R_2^{(2)}(\tilde{\boldsymbol{\theta}}_n^{\mathrm{PT}}) &= R_1^{(2)}(\tilde{\boldsymbol{\theta}}_n) - \sigma_{ee}\mathrm{tr}[\mathbf{D}^{-2}]H_{p+2}(\chi_\alpha^2 : \triangle^2) \\
&\quad + \boldsymbol{\gamma}'\boldsymbol{\gamma}\Big\{2H_{p+2}(\chi_\alpha^2 : \triangle^2) - H_{p+4}(\chi_\alpha^2 : \triangle^2)\Big\}. \qquad (5.37)
\end{aligned}$$

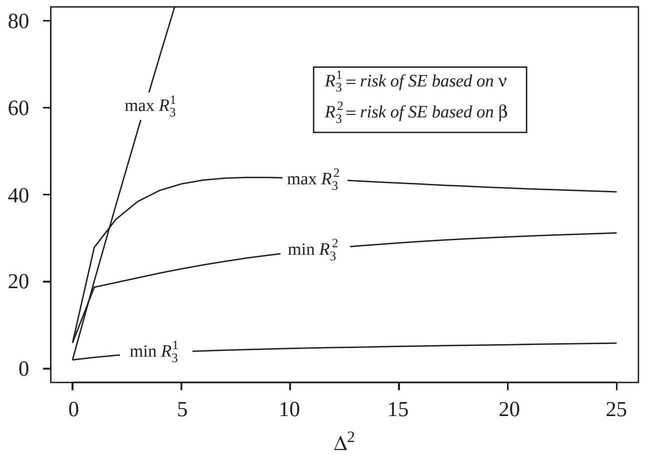

Fig. 7. Risk bounds as a function of Δ^2 for the $\hat{\boldsymbol{\theta}}_n^{\mathrm{S}}$ and the $\tilde{\boldsymbol{\theta}}_n^{\mathrm{S}}$ for $p = 4$ and $\mathbf{D} = \mathrm{diag}(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$, $\sigma_{ee} = 1$.

Clearly, we find that $\hat{\boldsymbol{\theta}}_n^{\mathrm{PT}}$ dominates $\tilde{\boldsymbol{\theta}}_n^{\mathrm{PT}}$ under the null-hypotheses. The Fig. 6 shows the entire dominance picture of $\hat{\boldsymbol{\theta}}_n^{\mathrm{PT}}$ and $\tilde{\boldsymbol{\theta}}_n^{\mathrm{PT}}$ thereby of $\hat{\boldsymbol{\nu}}_n^{\mathrm{PT}}$ and $\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}$.

5.4.4. *Canonical risk expressions of $\hat{\boldsymbol{\nu}}_n^{\mathrm{S}}$ and $\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$*

$$\begin{aligned} R_3^{(1)}(\hat{\boldsymbol{\theta}}_n^{\mathrm{S}}) = {} & R_1^{(1)}(\hat{\boldsymbol{\theta}}_n) \\ & -(p-2)\sigma_{ee}\mathrm{tr}[\mathbf{I}]\Big\{\Big[2E\Big[\chi_{p+2}^{-2}(\triangle^2)\Big] - (p-2)E\Big[\chi_{p+2}^{-4}(\triangle^2)\Big]\Big]\Big\} \\ & +(p-2)\boldsymbol{\gamma}'\mathbf{D}^2\boldsymbol{\gamma}\Big\{2E\Big[\chi_{p+2}^{-2}(\triangle^2)\Big] - 2E\Big[\chi_{p+4}^{-2}(\triangle^2)\Big] \\ & +(p-2)E\Big[\chi_{p+4}^{-4}(\triangle^2)\Big]\Big\} + 2(p-2)\boldsymbol{\gamma}'\mathbf{D}(\mathbf{I}-\mathbf{D})\boldsymbol{\gamma}E\Big[\chi_{p+2}^{-2}(\triangle^2)\Big], \\ & \text{where } \mathrm{tr}[\mathbf{I}] = p. \end{aligned} \tag{5.38}$$

$$\begin{aligned} R_3^{(2)}(\tilde{\boldsymbol{\theta}}_n^{\mathrm{S}}) = {} & R_1^{(2)}(\tilde{\boldsymbol{\theta}}_n) - (p-2)\sigma_{ee}\mathrm{tr}[\mathbf{D}^{-2}]\Big\{2E\Big[\chi_{p+2}^{-2}(\triangle^2)\Big] \\ & -(p-2)E\Big[\chi_{p+2}^{-4}(\triangle^2)\Big]\Big\} \\ & +(p-2)\boldsymbol{\gamma}'\boldsymbol{\gamma}\Big\{2E\Big[\chi_{p+2}^{-2}(\triangle^2)\Big] \\ & -2E\Big[\chi_{p+4}^{-2}(\triangle^2)\Big] + (p-2)E\Big[\chi_{p+4}^{-4}(\triangle^2)\Big]\Big\}. \end{aligned} \tag{5.39}$$

Clearly, we find that $\hat{\boldsymbol{\theta}}_n^{\mathrm{S}}$ dominates $\tilde{\boldsymbol{\theta}}_n^{\mathrm{S}}$ under the null-hypotheses. The Fig. 7 shows the entire dominance picture of $\hat{\boldsymbol{\theta}}_n^{\mathrm{S}}$ and $\tilde{\boldsymbol{\theta}}_n^{\mathrm{S}}$ thereby of $\hat{\boldsymbol{\nu}}_n^{\mathrm{S}}$ and $\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}$.

5.4.5. Canonical risk expressions of $\hat{\boldsymbol{v}}_n^{\mathrm{S+}}$ and $\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}$

$$\begin{aligned}
R_4^{(1)}(\hat{\boldsymbol{\theta}}_n^{\mathrm{S+}}) = {} & R_3^{(1)}(\hat{\boldsymbol{\theta}}_n^{\mathrm{S}}) - \sigma_{ee}\mathrm{tr}[\mathbf{I}]\Big[H_{p+2}((p-2);\triangle^2) \\
& -(p-2)\Big\{2E\Big[\chi_{p+2}^{-2}(\triangle^2)I\Big(\chi_{p+2}^2(\triangle^2)<p-2\Big)\Big] \\
& -(p-2)E\Big[\chi_{p+2}^{-4}(\triangle^2)I\Big(\chi_{p+2}^2(\triangle^2)<p-2\Big)\Big]\Big\}\Big] \\
& -\boldsymbol{\gamma}'\mathbf{D}^2\boldsymbol{\gamma}\Big[H_{p+4}((p-2);\triangle^2)-2H_{p+2}((p-2);\triangle^2) \\
& +(p-2)\Big\{2E\Big[\chi_{p+2}^{-2}(\triangle^2)I\Big(\chi_{p+2}^2(\triangle^2)<p-2\Big)\Big] \\
& -2E\Big[\chi_{p+4}^{-2}(\triangle^2)I\Big(\chi_{p+4}^2(\triangle^2)<p-2\Big)\Big] \\
& +(p-2)E\Big[\chi_{p+4}^{-4}(\triangle^2)I\Big(\chi_{p+4}^2(\triangle^2)<p-2\Big)\Big\}\Big] \\
& +2(p-2)\boldsymbol{\gamma}'\mathbf{D}(\mathbf{I}-\mathbf{D})\boldsymbol{\gamma} \\
& \times\Big\{H_{p+2}((p-2);\triangle^2)+(p-2)E\Big[\chi_{p+2}^{-2}(\triangle^2)\Big] \\
& -E\Big[\chi_{p+2}^{-2}(\triangle^2)I\Big(\chi_{p+2}^2(\triangle^2)<p-2\Big)\Big]\Big\}, \quad \mathrm{tr}[\mathbf{I}]=p,
\end{aligned} \tag{5.40}$$

$$\begin{aligned}
R_4^{(2)}(\tilde{\boldsymbol{\theta}}_n^{\mathrm{S+}}) = {} & R_3^{(2)}(\tilde{\boldsymbol{\theta}}_n^{\mathrm{S}}) - \mathrm{tr}[\mathbf{D}^{-2}]\Big\{H_{p+2}((p-2);\triangle^2) \\
& -(p-2)\Big\{2E\Big[\chi_{p+2}^{-2}(\triangle^2)I\Big(\chi_{p+2}^2(\triangle^2)<p-2\Big)\Big] \\
& -(p-2)E\Big[\chi_{p+2}^{-4}(\triangle^2)I\Big(\chi_{p+2}^2(\triangle^2)<p-2\Big)\Big]\Big\}\Big\} \\
& -\boldsymbol{\gamma}'\boldsymbol{\gamma}\Big\{H_{p+4}((p-2);\triangle^2)-2H_{p+2}((p-2);\triangle^2) \\
& +(p-2)\Big\{2E\Big[\chi_{p+2}^{-2}(\triangle^2)I\Big(\chi_{p+2}^2(\triangle^2)<p-2\Big)\Big] \\
& -2E\Big[\chi_{p+4}^{-2}(\triangle^2)I\Big(\chi_{p+4}^2(\triangle^2)<p-2\Big)\Big] \\
& +(p-2)E\Big[\chi_{p+4}^{-4}(\triangle^2)I\Big(\chi_{p+4}^2(\triangle^2)<p-2\Big)\Big\}\Big\}.
\end{aligned} \tag{5.41}$$

Clearly, we find that $\hat{\boldsymbol{\theta}}_n^{\mathrm{S+}}$ dominates $\tilde{\boldsymbol{\theta}}_n^{\mathrm{S+}}$ under the null-hypotheses. The Fig. 8 shows the entire dominance picture of $\hat{\boldsymbol{\theta}}_n^{\mathrm{S+}}$ and $\tilde{\boldsymbol{\theta}}_n^{\mathrm{S+}}$ thereby of $\hat{\boldsymbol{v}}_n^{\mathrm{S+}}$ and $\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}$.

We note that $\tilde{\boldsymbol{v}}_n$ dominates $\tilde{\boldsymbol{\beta}}_n$ near the null-hypothesis. This is true for all the estimators in the class $\boldsymbol{v}_n^*$ and $\boldsymbol{\beta}_n^*$. Further, one may establish that under $K_{(n)}$,

$$\mathrm{cov}(\boldsymbol{\beta}_n^*) \geqslant \mathrm{cov}(\boldsymbol{v}_n^*) \tag{5.42}$$

Hence, with respect to covariance matrices, $\boldsymbol{v}_n^*$ is better than $\boldsymbol{\beta}_n^*$, a result similar to Gleser [5, p. 704].

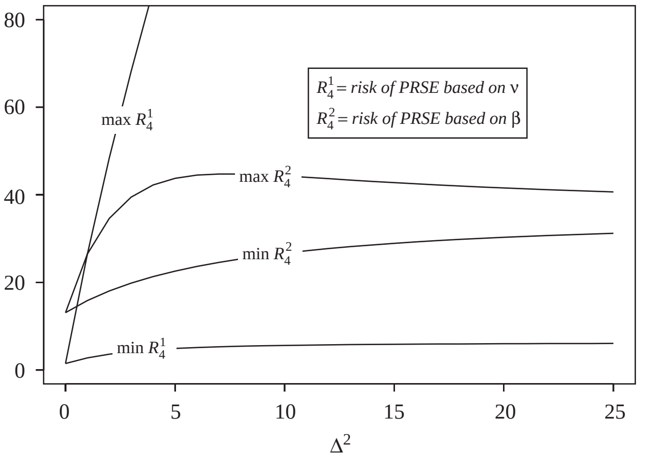

Fig. 8. Risk bounds as a function of Δ^2 for the $\hat{\boldsymbol{\theta}}_n^{\mathrm{S+}}$ and the $\tilde{\boldsymbol{\theta}}_n^{\mathrm{S+}}$ for $p = 4$ and $\mathbf{D} = \mathrm{diag}(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$, $\sigma_{ee} = 1$.

6. Asymptotic properties of recentered confidence sets

In this section, we discuss various confidence sets and their properties.

6.1. Recentered confidence sets

The basic confidence set for $\boldsymbol{\beta}$ is defined by

$$C_\gamma^0\left(\tilde{\boldsymbol{\beta}}_n\right) = \left\{\boldsymbol{\beta} : n\left(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}_n\right)' \hat{\mathbf{G}}_n^{-1}\left(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}_n\right) \leqslant \chi_\gamma^2\right\}, \tag{6.1}$$

where χ_γ^2 is the γ-level critical value based on the null-distribution, as a result

$$\lim_{n\to\infty} P_\beta\left\{C_\gamma^0\left(\tilde{\boldsymbol{\beta}}_n\right)\right\} = 1 - \gamma \quad \text{for all } \boldsymbol{\beta}. \tag{6.2}$$

In our case, we have three more confidence sets defined by

$$\begin{aligned}
&\text{(i) } C_\gamma^{\mathrm{PT}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}\right) = \left\{\boldsymbol{\beta} : n\left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}\right)' \hat{\mathbf{G}}_n^{-1}\left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}\right) \leqslant \chi_\gamma^2\right\},\\
&\text{(ii) } C_\gamma^{\mathrm{S}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}\right) = \left\{\boldsymbol{\beta} : n\left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_n^{\mathrm{S}}\right)' \hat{\mathbf{G}}_n^{-1}\left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_n^{\mathrm{S}}\right) \leqslant \chi_\gamma^2\right\},\\
&\text{(iii) } C_\gamma^{\mathrm{S+}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}(c)\right) = \left\{\boldsymbol{\beta} : n\left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}(c)\right)' \hat{\mathbf{G}}_n^{-1}\left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}(c)\right) \leqslant \chi_\gamma^2\right\},
\end{aligned} \tag{6.3}$$

where $\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}(c) = \left(1 - c\mathcal{L}_n^{-1}\right) I\left(\mathcal{L}_n > c\right)\tilde{\boldsymbol{\beta}}_n$.

Our basic problem is to show that $C_\gamma^{\mathrm{PT}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}\right)$ is not admissible while $C_\gamma^{\mathrm{S}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}\right)$ and $C_\gamma^{\mathrm{S+}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}(c)\right)$ uniformly dominate $C_\gamma^0\left(\tilde{\boldsymbol{\beta}}_n\right)$ as $n \to \infty$. To obtain such results we consider the following definition: A confidence set $C_\gamma^{(1)}\left(\tilde{\boldsymbol{\beta}}_n\right)$ is said to dominate locally $C_\gamma^{(2)}\left(\tilde{\boldsymbol{\beta}}_n\right)$

if $K_{(n)}$ holds (i.e. $\boldsymbol{\beta}_{(n)} = n^{-1/2}\boldsymbol{\delta}$ for all $\boldsymbol{\delta}$) and

(i) $\lim_{n\to\infty} P_{K_{(n)}}\left\{\boldsymbol{\beta}_{(n)} \in C_\gamma^{(1)}\left(\tilde{\boldsymbol{\beta}}_n\right)\right\} \geqslant \lim_{n\to\infty} P_{K_{(n)}}\left\{\boldsymbol{\beta}_{(n)} \in C_\gamma^{(2)}\left(\tilde{\boldsymbol{\beta}}_n\right)\right\}$
and
(ii) $\lim_{n\to\infty} \mathrm{Vol}_{K_{(n)}}\left[C_\gamma^{(1)}\left(\tilde{\boldsymbol{\beta}}_n\right)\right] \leqslant \lim_{n\to\infty}\left[\mathrm{Vol}_{K_{(n)}} C_\gamma^{(2)}\left(\tilde{\boldsymbol{\beta}}_n\right)\right]$ (6.4)

with strict inequality holding either for (i) or (ii) for all $\boldsymbol{\delta}$ with positive Lebesque measure. See [13]. To obtain the asymptotic coverage probabilities of the three sets we first note that under *fixed alternatives* the probability is $1-\gamma$ for the three sets as $n \to \infty$ according to Section 3. Thus, we only consider the probability contents of these sets under *local alternatives*. Accordingly, we have the following theorem for the expressions of the coverage probabilities based on Theorem 3.2. under $\{K_{(n)}\}$. In our discussions, we kept the volume of the sphere fixed, while the coverage probability vary.

Theorem 6.1. *Under the conditions of Theorem* 3.2 *with* $\boldsymbol{\delta}^*$ *and* $\mathbf{Z}$ *defined there,* $K_{(n)}$: $\boldsymbol{\beta}_{(n)} = n^{-\frac{1}{2}}\boldsymbol{\delta}$, $\boldsymbol{\delta}$ *fixed, we have*

$$\text{(i)}\quad \lim_{n\to\infty} P_{K_{(n)}}\left\{n||\boldsymbol{\beta}_{(n)} - \tilde{\boldsymbol{\beta}}_n||^2_{\hat{\mathbf{G}}_n^{-1}} \leqslant \chi^2_\gamma\right\} = 1-\gamma, \tag{6.5}$$

$$\begin{aligned}\text{(ii)}\quad & \lim_{n\to\infty} P_{K_{(n)}}\left\{n||\boldsymbol{\beta}_{(n)} - \hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}||^2_{\hat{\mathbf{G}}_n^{-1}} \leqslant \chi^2_\gamma\right\}\\ &= H_p\left(\chi^2_\alpha; \Delta^2\right) I\left(\triangle^2 < \chi^2_\gamma\right)\\ &\quad + P\left\{||\mathbf{Z}||^2 < \chi^2_\gamma;\, ||\mathbf{Z}+\boldsymbol{\delta}^*||^2 > \chi^2_\alpha\right\},\end{aligned} \tag{6.6}$$

$$\begin{aligned}\text{(iii)}\quad & \lim_{n\to\infty} P_{K_{(n)}}\left\{n||\boldsymbol{\beta}_{(n)} - \hat{\boldsymbol{\beta}}_n^{\mathrm{S}}||^2_{\hat{\mathbf{G}}_n^{-1}} \leqslant \chi^2_\gamma\right\}\\ &= P\left\{\left|\left|\mathbf{Z} - (p-2)(\mathbf{Z}+\boldsymbol{\delta}^*)||\mathbf{Z}+\boldsymbol{\delta}^*||^{-2}\right|\right|^2 \leqslant \chi^2_\gamma\right\},\end{aligned} \tag{6.7}$$

$$\begin{aligned}\text{(iv)}\quad & \lim_{n\to\infty} P_{K_{(n)}}\left\{n||\boldsymbol{\beta}_{(n)} - \hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}(c)||^2_{\hat{\mathbf{G}}_n^{-1}} \leqslant \chi^2_\gamma\right\}\\ &= P\left\{\left|\left|\mathbf{Z} - (\mathbf{Z}+\boldsymbol{\delta}^*) I\left(||\mathbf{Z}+\boldsymbol{\delta}^*||^2 < c\right)\right.\right.\right.\\ &\quad \left.\left.\left. - \frac{c(\mathbf{Z}+\boldsymbol{\delta}^*)}{||\mathbf{Z}+\boldsymbol{\delta}^*||^2} I\left(||\mathbf{Z}+\boldsymbol{\delta}^*||^2 > c\right)\right|\right|^2 < \chi^2_\gamma\right\}.\end{aligned} \tag{6.8}$$

See [12,13].

The expression (6.6)–(6.8) follows from Theorem 3.2 (iv), that is,

$$\begin{aligned}& \lim_{n\to\infty} P_{K_{(n)}}\left\{n||\boldsymbol{\beta}_{(n)} - \boldsymbol{\beta}_n^*||^2_{\hat{\mathbf{G}}_n^{-1}} \leqslant \chi^2_\gamma\right\}\\ &= P\left\{\left|\left|\mathbf{Z} - (\mathbf{Z}+\boldsymbol{\delta}^*) g\left(||\mathbf{Z}+\boldsymbol{\delta}^*||^2\right)\right|\right|^2 < \chi^2_\gamma\right\},\end{aligned} \tag{6.9}$$

where

$$\begin{aligned} g\left(||\mathbf{Z}+\boldsymbol{\delta}^*||^2\right) &= I(||\mathbf{Z}+\boldsymbol{\delta}^*||^2 < \chi^2_\alpha),\\ &= (p-2)||\mathbf{Z}+\boldsymbol{\delta}^*||^{-2},\\ &= 1 - \left\{1-(p-2)||\mathbf{Z}+\boldsymbol{\delta}^*||^{-2}\right\} I(||\mathbf{Z}+\boldsymbol{\delta}^*||^2 > p-2),\end{aligned} \tag{6.10}$$

respectively.

6.2. Asymptotic comparisons of the recentered confidence sets

First, note that the basic confidence set $C_\gamma^0\left(\tilde{\boldsymbol{\beta}}_n\right)$ has the probability content $1-\gamma$ as $n \to \infty$. Next, by Brown [2], Hwang and Casella [6,7], and Joshi [9] one knows that the probability content of $C_\gamma^{\mathrm{S}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}\right) \geqslant 1-\gamma$. i.e. $C_\gamma^{\mathrm{S}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{S}}\right)$ dominates $C_\gamma^0\left(\tilde{\boldsymbol{\beta}}_n\right)$ for all $\triangle^2$. Next, we show that $C_\gamma^{\mathrm{PT}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}\right)$ is inadmissible via Theorem 6.2 given below regarding the asymptotic coverage probability of $C_\gamma^{\mathrm{PT}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}\right)$ as function of Δ^2 keeping α, γ, and p fix.

Theorem 6.2. *Under $K_{(n)}$ and the conditions of Theorem* 3.2,

(i) *If $\triangle^2 < \chi_\gamma^2$, then*

$$\lim_{n\to\infty} P_{K_{(n)}}\left\{C_\gamma^{\mathrm{PT}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}\right)\right\} \geqslant 1-\gamma,$$

(ii) *If $\chi_\gamma^2 \leqslant \triangle^2 < \left(\chi_\gamma + \chi_\alpha\right)^2$, then*

$$\lim_{n\to\infty} P_{K_{(n)}}\left\{C_\gamma^{\mathrm{PT}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}\right)\right\} \leqslant 1-\gamma,$$

(iii) *If $\left(\chi_\gamma + \chi_\alpha\right)^2 \leqslant \triangle^2$, then*

$$\lim_{n\to\infty} P_{K_{(n)}}\left\{C_\gamma^{\mathrm{PT}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}\right)\right\} = 1-\gamma.$$

Proof. Consider the asymptotic probability of $C_\gamma^{\mathrm{PT}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}\right)$ given by (6.6). Then,

(i) If $\triangle^2 < \chi_\gamma^2$, the RHS of (6.6) is

$$\begin{aligned}
&\geqslant P\left\{||\mathbf{Z}||^2 < \chi_\gamma^2;\, ||\mathbf{Z}+\boldsymbol{\delta}^*||^2 \leqslant \chi_\alpha^2\right\} \\
&\quad + P\left\{||\mathbf{Z}||^2 < \chi_\gamma^2;\, ||\mathbf{Z}+\boldsymbol{\delta}^*||^2 > \chi_\alpha^2\right\} \\
&= P\left\{||\mathbf{Z}||^2 < \chi_\gamma^2\right\} = H_p(\chi_\gamma^2; 0) = 1-\gamma.
\end{aligned} \tag{6.11}$$

Hence, for $0 < \triangle^2 \leqslant \chi_\gamma^2$, the coverage probability of $C_\gamma^{\mathrm{PT}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}\right) \geqslant 1-\gamma$.

(ii) Now let $\chi_\gamma^2 \leqslant \triangle^2 < \left(\chi_\gamma + \chi_\alpha\right)^2$, then

$$\begin{aligned}
\lim_{n\to\infty} P_{K_{(n)}}\left\{C_\gamma^{\mathrm{PT}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}\right)\right\} &= P\left\{||\mathbf{Z}||^2 < \chi_\gamma^2;\, ||\mathbf{Z}+\boldsymbol{\delta}^*||^2 > \chi_\alpha^2\right\} \\
&\leqslant P\left\{||\mathbf{Z}||^2 < \chi_\gamma^2\right\} = 1-\gamma.
\end{aligned} \tag{6.12}$$

(iii) If $\triangle^2 \geqslant \left(\chi_\gamma + \chi_\alpha\right)^2$, then

$$\lim_{n\to\infty} P_{K_{(n)}}\left\{C_\gamma^{\mathrm{PT}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}\right)\right\} = P\left\{||\mathbf{Z}+\boldsymbol{\delta}^*||^2 > \chi_\alpha^2;\, ||\mathbf{Z}||^2 \leqslant \chi_\gamma^2\right\}. \tag{6.13}$$

Note that $\chi_\gamma + \chi_\alpha < \Delta = ||\boldsymbol{\delta}^*||$. Thus, $||\mathbf{Z}||^2 \leqslant \chi_\gamma^2$ implies that $||\mathbf{Z}|| \leqslant \chi_\gamma$ which in turn implies

$$\begin{aligned} &||\boldsymbol{\delta}^*|| - ||\mathbf{Z} + \boldsymbol{\delta}^*|| \leqslant ||\mathbf{Z}|| \leqslant \chi_\gamma \\ &\quad \Rightarrow \Delta - ||\mathbf{Z} + \boldsymbol{\delta}^*|| \leqslant ||\mathbf{Z}|| \leqslant \chi_\gamma \Rightarrow \Delta < ||\mathbf{Z} + \boldsymbol{\delta}^*|| + \chi_\gamma. \end{aligned} \tag{6.14}$$

Thus, from (6.11) we get $\chi_\gamma + \chi_\alpha < \Delta < ||\mathbf{Z} + \boldsymbol{\delta}^*|| + \chi_\gamma$. It follows that $\chi_\alpha \leqslant ||\mathbf{Z} + \boldsymbol{\delta}^*||$ which is equivalent to $||\mathbf{Z} + \boldsymbol{\delta}^*||^2 > \chi_\alpha^2$.

$$\lim_{n\to\infty} P_{K_{(n)}} \left\{ C_\gamma^{\mathrm{PT}} \left(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}} \right) \right\} = P\left\{ ||\mathbf{Z}||^2 \leqslant \chi_\gamma^2 \right\} = 1 - \gamma. \tag{6.15}$$

This completes the proof. □

It is evident from Theorem 6.2 that the asymptotic coverage probability of $C_\gamma^{\mathrm{PT}}(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}})$ as a function of Δ^2 (for fixed (α, γ, p)) decrease (from its maximum at $\Delta^2 = 0$) monotonically towards $(1-\gamma)$ near $\Delta^2 = \chi_\gamma^2$, then drops to a minimum, say $1-\gamma^*(\leqslant 1-\gamma)$ at $\Delta^2 = \chi_\gamma^2$ then increase towards $(1-\gamma)$. The coverage probability fluctuates around $1-\gamma$ depends on Δ^2 for fixed (α, γ, p) implying inadmissibility. Some values of coverage probabilities of $C_\gamma^{\mathrm{PT}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{PT}}\right)$ for different values of Δ^2 and p, when $\alpha = 0.05$ and $\gamma = 0.1$ are given in Table 2. The picture is similar to the efficiency graph (as a function of Δ^2) of the PTE versus UE using (5.4) which may be observed in Fig. 4.

Consider now the recentered confidence set $C_\gamma^{\mathrm{S+}}\left(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}(c)\right)$. In this case, the asymptotic coverage probability expression is given by (6.8) which may be rewritten as

$$\begin{aligned} &H_p(c; \triangle^2) I\left(\triangle^2 < \chi_\gamma^2\right) \\ &\quad + P\left\{ \left|\left| \mathbf{Z} - \frac{c(\mathbf{Z} + \boldsymbol{\delta}^*)}{||\mathbf{Z} + \boldsymbol{\delta}^*||^2} \right|\right|^2 < \chi_\gamma^2 ; ||\mathbf{Z} + \boldsymbol{\delta}^*||^2 > c \right\}. \end{aligned} \tag{6.16}$$

The expression (6.16) is greater than or equal to $1-\gamma$ by Hwang and Casella [6,7] for all $c \leqslant c_\gamma^*$. The number of c_γ^* is determined by solving the equation (for $p \geqslant 4$) for c_0 (see [6])

$$\left\{ \chi_\gamma + \sqrt{(\chi_\gamma^2 + c_0)} \right\}^{p-3} = c_0^{\frac{1}{2}(p-3)} \exp\left\{ \sqrt{c_0 \chi_\gamma^2} \right\}. \tag{6.17}$$

To include the case for $p = 3$, the number c_γ^* is chosen as the minimum of the two solutions of the equations, namely,

$$\left(\frac{1}{2}\chi_\gamma + \sqrt{\frac{1}{4}\chi_\gamma^2 + c_1} \right)^{p-2} = c_1^{\frac{1}{2}(p-2)} \exp\left\{ \frac{1}{2}\chi_\gamma c_1^{\frac{1}{2}} \right\} \tag{6.18}$$

and

$$\begin{aligned} &\left(\sqrt{(\chi_\gamma^2 + 4c_1)} - \chi_\gamma \right) \left(\chi_\gamma + \sqrt{\chi_\gamma^2 + 4c_2} \right)^{p-2} \\ &\quad = 2\, c_2^{\frac{1}{2}p} \exp\left\{ \sqrt{\chi_\gamma^2 c_2} \right\}. \end{aligned} \tag{6.19}$$

(See [7], Eqs. (2.26) and (2.27))

An approximate value of c is $0.8(p-2)$. However, using $c = p-2$ one may compute the coverage probability (6.17) by rewriting (6.17) as

$$\phi_1 + \phi_2,$$

where

$$\phi_1 = H_p(p-2; \triangle^2) I\left(\triangle^2 < \chi^2_\gamma\right) \tag{6.20a}$$

and

$$\phi_2 = P\left\{\left|\left|\mathbf{Z} - \frac{c(\mathbf{Z}+\boldsymbol{\delta}^*)}{||\mathbf{Z}+\boldsymbol{\delta}^*||^2}\right|\right|^2 < \chi^2_\gamma;\ ||\mathbf{Z}|| > c\right\}. \tag{6.20b}$$

Let $r = ||\mathbf{Z}||$ and $\boldsymbol{\theta}$ is the angle between $\mathbf{Z}$ and $\boldsymbol{\delta}^*$ then we may write (for $\Delta^2 < \chi^2_\gamma$)

$$C_\gamma^{\mathrm{S+}}(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}(c)) = \{(r, \theta) : r \leqslant r_+(\theta), \theta \in [-\pi, \pi]\}, \tag{6.21}$$

where

$$r_+(\theta) = \tfrac{1}{2}\left(r_+^0(\theta) + \sqrt{[r_+^0(\theta)]^2 + 4c}\right) \tag{6.22a}$$

and

$$r_+^0(\theta) = \Delta\cos\theta + \sqrt{\chi^2_\gamma - \Delta^2\cos^2\theta}. \tag{6.22b}$$

On the other hand, when $\Delta^2 > \chi^2_\gamma$

$$C_\gamma^{\mathrm{S+}}(\hat{\boldsymbol{\beta}}_n^{\mathrm{S+}}(c)) = \{(r, \theta) : r_-(\theta) \leqslant r \leqslant r_+(\theta), \theta \in [-\theta_0, \theta_0]\}, \tag{6.23a}$$

$$r_-(\theta) = \tfrac{1}{2}\left(r_-^0(\theta) + \sqrt{[r_-^0(\theta)]^2 + 4c}\right), \tag{6.23b}$$

$$r_-^0(\theta) = \Delta\cos\theta - \sqrt{\chi^2_\gamma - \Delta^2\sin^2\theta} \text{ and } \sin\theta_0 = \tfrac{\chi_\gamma}{\Delta}. \tag{6.23c}$$

Note that $r_-(\theta) > c$ and $r_+(\theta)$ is a decreasing function of θ. Now if $\Delta^2 < \chi^2_\gamma$, we have

$$\phi_2 = 2K \textstyle\int_0^\pi \int_0^{r_+(\theta)} h(r, \theta)\, dr d\theta \tag{6.24a}$$

where

$$h(r, \theta) = r^{p-1}(\sin\theta)^{p-2}\exp\left\{-\tfrac{1}{2}(r^2 - 2r\Delta\cos\theta + \Delta^2)\right\}, \tag{6.24b}$$

and

$$K = (2\pi)^{-\frac{1}{2}(p-2)}\prod_{j=1}^{p-3}\left\{\int_0^\pi \sin^j\theta\, d\theta\right\}. \tag{6.24c}$$

Further, as Δ^2 increases to χ^2_γ, $r_+(\theta)$ tends to $\chi_\gamma\left(\cos\theta + \sqrt{\cos\theta + \frac{c}{\chi^2_\gamma}}\right)$. On the other hand if $\Delta^2 > \chi^2_\gamma$

$$\phi_2 = 2K\int_0^{\theta_0}\int_{r_-(\theta)}^{r_+(\theta)} h(r, \theta)\, dr\, d\theta. \tag{6.25}$$

Table 2
Asymptotic coverage probabilities of sets with $\gamma = 0.05$ and $\alpha = 0.1$

Δ	p	5	7	9	11	13	15
0	S+	0.9879	0.9959	0.9985	0.9994	0.9998	0.9999
	PTE	0.9500	0.9500	0.9500	0.9500	0.9500	0.9500
2	S+	0.9809	0.9926	0.9972	0.9989	0.9995	0.9998
	PTE	0.9318	0.9304	0.9297	0.9293	0.9291	0.9289
4	S+	0.9343	0.9622	0.9808	0.9949	0.9977	0.9989
	PTE	0.9264	0.9345	0.9233	0.9226	0.9221	0.9218
6	S+	0.9162	0.9337	0.9510	0.9661	0.9780	0.9866
	PTE	0.9224	0.9202	0.9190	0.9181	0.9176	0.9171
8	S+	0.9093	0.9202	0.9323	0.9443	0.9556	0.9657
	PTE	0.9191	0.9169	0.9156	0.9147	0.9141	0.9137
10	S+	0.9060	0.9133	0.9218	0.9307	0.9397	0.9484
	PTE	0.5937	0.9141	0.9129	0.9121	0.9115	0.9110
15	S+	0.9027	0.9061	0.9102	0.9147	0.9196	0.9247
	PTE	0.7773	0.7296	0.6871	0.9074	0.9069	0.9066
20	S+	0.9015	0.9035	0.9059	0.9085	0.9114	0.9145
	PTE	0.8593	0.8336	0.8079	0.7826	0.7580	0.9039
25	S+	0.9010	0.9022	0.9038	0.9055	0.9075	0.9095
	PTE	0.8890	0.8777	0.8650	0.8514	0.8371	0.8223
50	S+	0.9002	0.9006	0.9010	0.9014	0.9019	0.9024
	PTE	0.9000	0.9000	0.9000	0.8999	0.8997	0.8995
100	S+	0.9001	0.9001	0.9002	0.9004	0.9005	0.9006
	PTE	0.9000	0.9000	0.9000	0.9000	0.9000	0.9000

As Δ^2 increases to χ^2_γ, $r_+(\theta)$ approaches $\chi_\gamma\left(\cos\theta + \sqrt{\cos\theta + \frac{c}{\chi^2_\gamma}}\right)$, $r_-(\theta)$ approaches $\sqrt{c}$ and $\theta_0 \to \frac{\pi}{2}$. Therefore, for $\Delta^2 \leqslant \chi^2_\gamma$, the limit of (6.24a) is given by

$$2K\int_0^{\pi}\int_{\sqrt{c}}^{r_+(\theta)} h(r,\theta)\,dr\,d\theta, \tag{6.26}$$

and for $\Delta^2 > \chi^2_\gamma$, the limit is

$$2K\int_0^{\frac{\pi}{2}}\int_{\sqrt{c}}^{r_+(\theta)} h(r,\theta)\,dr\,d\theta \tag{6.27}$$

(See [12]). Table 2 gives some numerical values of the coverage probabilities for $\gamma = 0.05$ and $\alpha = 0.1$.

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