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Majorants and Minorants for Elliptical Measures on $\mathbb{R}^k$

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Let $\mu(\cdot;\boldsymbol{\Sigma}, G_1)$ and $\mu(\cdot;\boldsymbol{\Omega}, G_2)$ be elliptically contoured measures on $\mathbb{R}^k$ centered at $\mathbf{0}$, having scale parameters $(\boldsymbol{\Sigma}, \boldsymbol{\Omega})$ and radial cdf's (G_1, G_2). Elliptical measures $\nu_{\mathrm{m}}(\cdot)$ and $\nu_{\mathrm{M}}(\cdot)$, depending on $(\boldsymbol{\Sigma}, \boldsymbol{\Omega}, G_1, G_2)$, are constructed such that $\nu_{\mathrm{m}}(C) \leqslant \{\mu(C;\boldsymbol{\Sigma}, G_1), \mu(C;\boldsymbol{\Omega}, G_2)\} \leqslant \nu_{\mathrm{M}}(C)$ for every symmetric convex set $C \subset \mathbb{R}^k$, with equality for certain sets. These in turn rely on the construction of spectral lower and upper matrix bounds for $(\boldsymbol{\Sigma}, \boldsymbol{\Omega})$. Extensions include bounds for certain ensembles and mixtures, including versions having star-shaped contours. The findings specialize to give envelopes for some nonstandard distributions of quadratic forms, with applications to stochastic characteristics of ballistic systems.

1. INTRODUCTION

Elliptically contoured measures on $\mathbb{R}^k$ are of note in statistics and applied probability. Especially in linear inference, multivariate statistical inference, and time series analysis, many normal-theory results carry over to apply in the larger class. Let $EC_k(\boldsymbol{\theta}, \boldsymbol{\Sigma}, G)$ be an elliptical distribution on $\mathbb{R}^k$ having location parameters $\boldsymbol{\theta} \in \mathbb{R}^k$, scale parameters $\boldsymbol{\Sigma}$, and radial cdf $G(\cdot)$ on $[0, \infty)$, and let $\mu(\cdot;\boldsymbol{\Sigma}, G)$ be the corresponding measure when $\boldsymbol{\theta} = \mathbf{0}$. Basic properties of these are developed in Cambanis *et al.* (1981) and in Jensen (1984), for example. See also Fang, Kotz, and Ng (1990), Fang and Zhang (1990), and Kariya and Sinha (1989).

Stochastic orderings are basic. Following Sherman (1955), a probability measure $\mu(\cdot)$ on $\mathbb{R}^k$ is said to be *more peaked* about $\mathbf{0} \in \mathbb{R}^k$ than $\nu(\cdot)$ if $\mu(C) \geqslant \nu(C)$ for every set C in the class $\mathbf{C}_0^k$ consisting of compact convex sets in $\mathbb{R}^k$ symmetric under reflection through $\mathbf{0}$, i.e., $\mathbf{x} \in C$ implies $-\mathbf{x} \in C$. Denote this ordering by $\mu \succcurlyeq_{\mathrm{p}} \nu$. We have the following.

DEFINITION 1. A probability measure ν_m is called a *symmetric minorant* for (μ_1, μ_2) on $\mathbb{R}^k$, and ν_{M} is a *symmetric majorant*, if $\nu_{\mathrm{m}}(C) \leqslant \{\mu_1(C), \mu_2(C)\} \leqslant \nu_{\mathrm{M}}(C)$ for every $C \in \mathbf{C}_0^k$.

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In this paper we construct symmetric minorants and majorants for nonsingular elliptical measures $\{\mu(\cdot\,;\mathbf{\Sigma}, G_1), \mu(\cdot\,;\mathbf{\Omega}, G_2)\}$ on $\mathbb{R}^k$. A basic link between peakedness and scale orderings for elliptical measures on $\mathbb{R}^k$ is that $\mu(\cdot\,;\mathbf{\Omega}, G)\succcurlyeq_{\mathrm{p}}\mu(\cdot\,;\mathbf{\Xi}, G)$ if and only if $\mathbf{\Xi}-\mathbf{\Omega}$ is positive semidefinite. Sufficiency is shown for nonsingular Gaussian measures in Anderson (1955), for nonsingular elliptical measures in Das Gupta *et al.* (1971) and Fefferman *et al.* (1972), and for singular elliptical measures in Jensen (1984). Necessity is shown in Jensen (1984). To bound $\{\mu(\cdot\,;\mathbf{\Sigma}, G), \mu(\cdot\,;\mathbf{\Omega}, G)\}$, it thus suffices to find matrices $\mathbf{\Xi}_{\mathrm{m}}$ and $\mathbf{\Xi}_{\mathrm{M}}$ such that $\mathbf{\Xi}_{\mathrm{m}}\preccurlyeq_{\mathrm{L}}\{\mathbf{\Sigma},\mathbf{\Omega}\}\preccurlyeq_{\mathrm{L}}\mathbf{\Xi}_{\mathrm{M}}$ in the positive semidefinite ordering $\preccurlyeq_{\mathrm{L}}$. An outline of the paper follows.

Section 2 comprises preliminary developments. In Section 3, which is of independent interest, we characterize all lower and upper bounds for positive definite matrices $(\mathbf{A}, \mathbf{B})$ under the ordering $\succcurlyeq_{\mathrm{L}}$. Of these, the spectral least upper and greatest lower matrix bounds for $(\mathbf{\Sigma}, \mathbf{\Omega})$ carry over in Section 4 to give greatest stochastic minorants and least stochastic majorants, in a sense to be described, for any elliptical measures $\{\mu(\cdot\,;\mathbf{\Sigma}, G_1), \mu(\cdot\,;\mathbf{\Omega}, G_2)\}$ on $\mathbb{R}^k$. These findings in turn support bounds for certain ensembles and mixtures of measures on $\mathbb{R}^k$, including certain star-unimodal distributions. Section 5 specializes earlier findings to give envelopes for certain nonstandard distributions of quadratic forms, with applications to the performance characteristics of ballistic systems.

2. Preliminaries

We first establish conventions for notation; we next review basic properties of ordering and monotonicity on spaces of interest; and we then describe classes of distributions to be considered subsequently.

2.1. *Notation*

Spaces of note include Euclidean n-space $\mathbb{R}^n$ and its positive orthant $\mathbb{R}^n_+$, the real $(n\times k)$ matrices $F_{n\times k}$, the real symmetric $(k\times k)$ matrices S_k, and their positive semidefinite (S_k^0), positive definite (S_k^+), and diagonal (D_k) varieties. The transpose and inverse of $\mathbf{A}$ are designated by $\mathbf{A}'$ and $\mathbf{A}^{-1}$ as appropriate. Special arrays include the unit matrix $\mathbf{I}_n$, and a typical diagonal matrix $\mathbf{D}_\alpha=\mathrm{Diag}(\alpha_1, ..., \alpha_k)\in D_k$. Groups of transformations acting on $\mathbb{R}^n$ include the general linear group $Gl(n)$ and the real orthogonal group $O(n)$. The *spectral decomposition* $\mathbf{A}=\sum_{i=1}^k \alpha_i\mathbf{q}_i\mathbf{q}_i'$ of $\mathbf{A}\in S_k^+$ yields its *symmetric square root* $\mathbf{A}^{1/2}=\sum_{i=1}^k \alpha_i^{1/2}\mathbf{q}_i\mathbf{q}_i'$.

Standard usage refers to independent, identically distributed (iid) variates and their cumulative distribution function (cdf). $\mathscr{L}(\mathbf{X})$ designates the law of distribution of $\mathbf{X}$, with $N_k(\boldsymbol{\mu}, \mathbf{\Sigma})$ as the Gaussian law on $\mathbb{R}^k$

having some mean $\boldsymbol{\mu}$ and dispersion matrix $\boldsymbol{\Sigma}$. $F_0 = F_0[0, \infty)$ is the class of cdf's on $[0, \infty)$, with $F_0[0, 1]$ as the subclass of cdf's on $[0, 1]$. Since probability measures on $\mathbb{R}^k$ are tight, in what follows we drop compactness from $\mathbf{C}_0^k$ and consider equivalently the class $\mathbf{C}^k$ consisting of symmetric convex sets in $\mathbb{R}^k$.

2.2. *Ordered Spaces*

We adopt the terminology of Marshall and Olkin (1979). A set $\mathscr{H}$ together with a binary relation $\succcurlyeq_0$ is said to be *linearly ordered* if the relation is reflexive, transitive, antisymmetric, and complete. A *partial ordering* is reflexive, transitive, and antisymmetric, and a *preordering* is reflexive and transitive. Moreover, a partially ordered set $(\mathscr{H}, \succcurlyeq_0)$ is a *lower semi-lattice* if for any two elements x, y in $\mathscr{H}$, there is a greatest lower bound (glb) $x \wedge y$ in $\mathscr{H}$; an *upper semi-lattice* if there is a least upper bound (lub) $x \vee y$ in $\mathscr{H}$; and a *lattice* if it is both a lower and an upper semi-lattice.

In particular, $(\mathbb{R}^k, \geqslant_k)$ is ordered such that $\mathbf{x} \geqslant_k \mathbf{y}$ in $\mathbb{R}^k$ if and only if $\{x_i \geqslant y_i;\ 1 \leqslant i \leqslant k\}$. The space $(S_k, \succcurlyeq_{\mathrm{L}})$ is ordered as in Loewner (1934) such that $\mathbf{A} \succcurlyeq_{\mathrm{L}} \mathbf{B}$ in S_k if and only if $\mathbf{A} - \mathbf{B} \in S_k^0$, with $\mathbf{A} \succ_{\mathrm{L}} \mathbf{B}$ whenever $\mathbf{A} - \mathbf{B} \in S_k^+$. The space $(F_0, \geqslant_{\mathrm{df}})$ is partially ordered under point-wise ordering of cdf's such that $F \geqslant_{\mathrm{df}} G$ in $(F_0, \geqslant_{\mathrm{df}})$ if and only if $F(t) \geqslant G(t)$ for every $t \in \mathbb{R}_+^1$. If the survival functions are ordered point-wise as $1 - G(t) \geqslant 1 - F(t)$, then G is said to be *stochastically larger* than F in the usual terminology. With regard to lower and upper bounds, $(\mathbb{R}^k, \geqslant_k)$ is a lattice with $\mathbf{a} \wedge \mathbf{b} = [a_1 \wedge b_1, \ldots, a_k \wedge b_k]'$ and $\mathbf{a} \vee \mathbf{b} = [a_1 \vee b_1, \ldots, a_k \vee b_k]'$, where $\{a_i \wedge b_i = \min(a_i, b_i)$ and $a_i \vee b_i = \max(a_i, b_i);\ 1 \leqslant i \leqslant k\}$; see Vulikh (1967), for example. The space $(S_k, \succcurlyeq_{\mathrm{L}})$ is not a lattice (cf. Halmos, 1958, p. 142). Nonetheless, $(S_k^+, \succcurlyeq_{\mathrm{L}})$ is shown subsequently to have lower and upper bounds that are tight. Moreover, $(F_0, \geqslant_{\mathrm{df}})$ is seen to be a lattice with $(F \wedge G)(t) = \min[F(t), G(t)]$ and $(F \vee G)(t) = \max[F(t), G(t)]$ pointwise for each $t \in \mathbb{R}_+^1$.

A real-valued function $f(\cdot)$ on $(\mathscr{H}, \succcurlyeq_0)$ is said to be *order-preserving* if $x \succcurlyeq_0 y$ on $\mathscr{H}$ implies $f(x) \geqslant f(y)$ on $\mathbb{R}^1$, and to be *order-reversing* if $x \succcurlyeq_0 y$ on $\mathscr{H}$ implies $f(x) \leqslant f(y)$ or $\mathbb{R}^1$. Denote by $\boldsymbol{\Phi}(\mathscr{H}, \succcurlyeq_0)$ the class of order-preserving functions, and by $\boldsymbol{\Phi}^-(\mathscr{H}, \succcurlyeq_0)$ the order-reversing functions, on $(\mathscr{H}, \succ_0)$. Specifically, $\boldsymbol{\Phi}(\mathbb{R}^k, \geqslant_k)$ consists of functions $f(x_1, \ldots, x_k)$ non-decreasing in each argument. The class $\boldsymbol{\Phi}(S_k, \succcurlyeq_{\mathrm{L}})$ is characterized in Marshall, Walkup, an Wets (1967). Similarly, $\boldsymbol{\Phi}(F_0, \geqslant_{\mathrm{df}})$ and $\boldsymbol{\Phi}^-(F_0, \geqslant_{\mathrm{df}})$ consist of order-preserving and order-reversing functionals defined on $(F_0, \geqslant_{\mathrm{df}})$.

2.3. *Special Distributions*

A set $S \subset \mathbb{R}^k$ containing $\mathbf{0} \in \mathbb{R}^k$ is said to be *star-shaped* about $\mathbf{0} \in S$ if, for every $\mathbf{x} \in S$, the line segment joining $\mathbf{0}$ to $\mathbf{x}$ is in S. A distribution $P(\cdot)$ on

$\mathbb{R}^k$ is said to be *symmetric star-unimodal* about $\mathbf{0} \in \mathbb{R}^k$ if it belongs to the closed convex hull of the set of all uniform measures on symmetric sets in $\mathbb{R}^k$ that are star-shaped about $\mathbf{0} \in \mathbb{R}^k$. Let $\mathbb{P}_k(\mathbf{0})$ be the collection of all such distributions on $\mathbb{R}^k$. If $P(\cdot)$ has a continuous density $f(\cdot)$ on $\mathbb{R}^k$, then $P(\cdot) \in \mathbb{P}_k(\mathbf{0})$ if and only if, for $t > 0$, the level sets $B_t = \{\mathbf{x} \in \mathbb{R}^k : f(\mathbf{x}) > t\}$ are either symmetric star-shaped about $\mathbf{0} \in \mathbb{R}^k$ or they are empty. Kanter (1977) studied mixtures of measures on $\mathbb{R}^k$ including the class $K_k(\mathbf{0})$ generated as the closed convex hull of the set of all uniform measures on convex bodies in $\mathbb{R}^k$ symmetric about $\mathbf{0} \in \mathbb{R}^k$. Further details are given by Dharmadhikari and Joag-Dev (1988, p. 38ff.), who show that the classes $\mathbb{P}_k(\mathbf{0})$ and $K_k(\mathbf{0})$ coincide. These facts are used subsequently.

We consider elliptical measures on $\mathbb{R}^k$ having further structure as follows. If unimodal about $\boldsymbol{\theta} \in \mathbb{R}^k$, we designate $EC_k(\boldsymbol{\theta}, \boldsymbol{\Sigma}, G)$ instead as $UC_k(\boldsymbol{\theta}, \boldsymbol{\Sigma}, G)$, reserving the notation $EC_k(\boldsymbol{\Sigma}, G)$ and $UC_k(\boldsymbol{\Sigma}, G)$ for distributions centered at $\mathbf{0} \in \mathbb{R}^k$, with $\mu(\cdot; \boldsymbol{\Sigma}, G)$ as the corresponding measure as before. Subsequently $E_k(\mathbf{0}) = \{EC_k(\boldsymbol{\Sigma}, G); \boldsymbol{\Sigma} \in S_k^+, G \in F_0\}$ designates the class of all elliptical distributions on $\mathbb{R}^k$ centered at $\mathbf{0} \in \mathbb{R}^k$, with $U_k(\mathbf{0}) \subset E_k(\mathbf{0})$ as the subclass of unimodal distributions.

We consider ensembles of distribution in $E_k(\mathbf{0})$ and $U_k(\mathbf{0})$ as follows. Let $T \subset \mathbb{R}_+^1$ be an index set and let $\{G_\tau; \tau \in T\}$ be a collection of cdf's in F_0. Anticipating later applications, we first fix $(\boldsymbol{\Sigma}, \boldsymbol{\Omega})$ and suppose that the scale parameters $\boldsymbol{\Xi}(\alpha)$ of $\mu(\cdot; \boldsymbol{\Xi}(\alpha), G)$ lie on the line segment $\{\boldsymbol{\Xi}(\alpha) = \alpha\boldsymbol{\Sigma} + \bar{\alpha}\boldsymbol{\Omega}; \alpha \in [0, 1]\}$ connecting $\boldsymbol{\Sigma}$ and $\boldsymbol{\Omega}$ in S_k^+, with $\bar{\alpha} = 1 - \alpha$. Letting (α, G) vary generates the ensemble

$$E_k(\boldsymbol{\Sigma}, \boldsymbol{\Omega}, T) = \{EC_k(\boldsymbol{\Xi}(\alpha), G_\tau); \alpha \in [0, 1], \tau \in T\} \tag{2.1}$$

and similarly for the ensemble $U_k(\boldsymbol{\Sigma}, \boldsymbol{\Omega}, T)$ of unimodal distributions.

More generally, with $M \subset S_k^+$ and $T \subset \mathbb{R}_+^1$ both considered as index sets, we consider the corresponding ensemble

$$E_k(M, T) = \{EC_k(\boldsymbol{\Xi}, G_\tau); \boldsymbol{\Xi} \in M, \tau \in T\} \tag{2.2}$$

in $E_k(\mathbf{0})$, and similarly the ensemble $U_k(M, T)$ consisting of unimodal distributions in $U_k(\mathbf{0})$. We subsequently seek elliptical majorants and minorants for such ensembles, as well as for certain mixtures over these ensembles to be described next.

If we now take $E_k(\boldsymbol{\Sigma}, \boldsymbol{\Omega}, T)$ to be a mixing family with mixing parameters (α, t), then under measurability conditions we consider mixtures of the type

$$\nu(\cdot; \boldsymbol{\Sigma}, \boldsymbol{\Omega}, H, F) = \int_0^1 \int_T \mu(\cdot; \boldsymbol{\Xi}(\alpha), G_\tau)\, dH(\alpha)\, dF(\tau) \tag{2.3}$$

with $\Xi(\alpha) = \alpha\Sigma + \bar{\alpha}\Omega$ and with $H(\cdot)$ and $F(\cdot)$ as cdf's defined on $[0, 1]$ and T, respectively. More generally, with $\Gamma(\cdot)$ as a probability measure on $M \subset S_k^+$, a typical mixture over the mixing ensemble (2.2) under measurability conditions takes the form

$$\nu(\cdot; \Gamma, F) = \int_M \int_T \mu(\cdot; \Xi, G_\tau)\, d\Gamma(\Xi)\, dF(\tau). \tag{2.4}$$

On fixing one, then another, of the mixing parameters (Ξ, t), we construct the partial mixtures

$$\nu_1(\cdot; \Xi, F) = \int_T \mu(\cdot; \Xi, G_\tau)\, dF(\tau) \tag{2.5}$$

and

$$\nu_2(\cdot; \Gamma, G_\tau) = \int_M \mu(\cdot; \Xi, G_\tau)\, d\Gamma(\Xi). \tag{2.6}$$

We return to these subsequently. Basic connections between these mixtures and the class $\mathbb{P}_k(\mathbf{0})$ of symmetric star-unimodal distributions on $\mathbb{R}^k$ are given in the following.

LEMMA 1. *Consider mixtures of the types* (2.3)–(2.6) *over ensembles in* $U_k(\mathbf{0})$ *consisting of elliptical distributions unimodal about* $\mathbf{0} \in \mathbb{R}^k$. *Then all such mixtures belong to the class* $\mathbb{P}_k(\mathbf{0})$ *of symmetric star-unimodal distributions.*

Proof. It is well known that each distribution in the class $U_k(\mathbf{0})$ can be represented as a mixture of uniform measures over convex bodies of the type $B(r; \Sigma) = \{\mathbf{x} \in \mathbb{R}^k : \mathbf{x}'\Sigma^{-1}\mathbf{x} \leqslant r\}$ as r varies with some mixing distribution $G_0 \in F_0$. Thus the generators are in $\mathbb{P}_k(\mathbf{0})$; see also Kanter (1977) and Dharmadhikari and Joag-Dev (1988). Since mixtures are convex and since $\mathbb{P}_k(\mathbf{0})$ comprises the closed convex hull of uniform measures generating the class, it follows that mixtures of the types (2.3)–(2.6) all belong to $\mathbb{P}_k(\mathbf{0})$. ∎

3. MATRIX EXTREMES IN S_k^+

We next characterize the sets of lower and upper bounds for pairs of matrices in $(S_k^+, \succcurlyeq_L)$.

3.1. *Lower and Upper Matrix Bounds*

Given $(\mathbf{A}, \mathbf{B})$ in $(S_k^+, \geqslant_{\mathrm{L}})$, we study first the class $\mathrm{H}_{\mathrm{L}}(\mathbf{A}, \mathbf{B}) = \{\mathbf{S} \in S_k^+ : \mathbf{S} \leqslant_{\mathrm{L}} \mathbf{A} \text{ and } \mathbf{S} \leqslant_{\mathrm{L}} \mathbf{B}\}$ consisting of lower bounds, and then the class $\mathrm{H}_{\mathrm{U}}(\mathbf{A}, \mathbf{B}) = \{\mathbf{T} \in S_k^+ : \mathbf{T} \geqslant_{\mathrm{L}} \mathbf{A} \text{ and } \mathbf{T} \geqslant_{\mathrm{L}} \mathbf{B}\}$ consisting of upper bounds. The ordering $\mathbf{L} \leqslant_{\mathrm{L}} \{\mathbf{A}, \mathbf{B}\} \leqslant_{\mathrm{L}} \mathbf{U}$ always holds with $\mathbf{L} = \mathbf{0}$ and $\mathbf{U} = \mathbf{A} + \mathbf{B}$, and if $\mathbf{A} \leqslant_{\mathrm{L}} \mathbf{B}$, then $\mathbf{L} = \mathbf{A}$ and $\mathbf{U} = \mathbf{B}$. Since $\mathbf{A} \geqslant_{\mathrm{L}} \mathbf{S}$ if and only if $\mathbf{GAG}' \geqslant_{\mathrm{L}} \mathbf{GSG}'$ for any $\mathbf{G} \in Gl(k)$, it suffices to consider a canonical form in which $(\mathbf{A}, \mathbf{B}) \to (\mathbf{GAG}', \mathbf{GBG}') \to (\mathbf{B}^{-1/2}\mathbf{AB}^{-1/2}, \mathbf{I}_k) \to (\mathbf{D}_\gamma, \mathbf{I}_k)$, where $\mathbf{B}^{-1/2}\mathbf{AB}^{-1/2} = \sum_{i=1}^k \gamma_i \mathbf{q}_i \mathbf{q}_i'$ is its spectral decomposition and $\mathbf{D}_\gamma = \mathrm{Diag}(\gamma_1, ..., \gamma_k)$ contains the ordered roots of $|\mathbf{A} - \gamma \mathbf{B}| = 0$. We thus seek $\mathbf{E} = \mathbf{GLG}'$ and $\mathbf{F} = \mathbf{GUG}'$ such that $\mathbf{E} \leqslant_{\mathrm{L}} \{\mathbf{D}_\gamma, \mathbf{I}_k\} \leqslant_{\mathrm{L}} \mathbf{F}$ or, equivalently, the classes $\mathrm{H}_{\mathrm{L}}(\mathbf{D}_\gamma, \mathbf{I}_k)$ and $\mathrm{H}_{\mathrm{U}}(\mathbf{D}_\gamma, \mathbf{I}_k)$. Since S_k^+ is open, we may require that $\mathrm{H}_{\mathrm{L}}(\mathbf{A}, \mathbf{B}) \subset S_k^+$ as stipulated earlier.

First note that $\mathbf{A} \geqslant_{\mathrm{L}} \mathbf{B}$ on S_k^+ if and only if $\{\gamma_1 \geqslant \cdots \geqslant \gamma_k \geqslant 1\}$, whereas $\mathbf{A} >_{\mathrm{L}} \mathbf{B}$ corresponds to $\gamma_k > 1$. If neither $\mathbf{A} \geqslant_{\mathrm{L}} \mathbf{B}$ nor $\mathbf{B} \geqslant_{\mathrm{L}} \mathbf{A}$, then at least one of two integers (r, s) can be found such that

$$\{\gamma_1 \geqslant \cdots \geqslant \gamma_r > \gamma_{r+1} = 1 = \cdots = \gamma_{r+s} > \gamma_{r+s+1} \geqslant \cdots \geqslant \gamma_k > 0\}. \quad (3.1)$$

Now let $t = k - r - s$, and let $\{\varepsilon_1 \geqslant \cdots \geqslant \varepsilon_k > 0\}$ be the ordered eigenvalues of $\mathbf{E} \in S_k^+$. Essential properties of the lower bounds $\mathrm{H}_{\mathrm{L}}(\mathbf{D}_\gamma, \mathbf{I}_k)$ are summarized in the following lemma.

LEMMA 2. *Let* $\mathbf{E} = [e_{ij}]$ *have eigenvalues* $\{\varepsilon_1 \geqslant \cdots \geqslant \varepsilon_k > 0\}$, *and consider the class* $\mathrm{H}_{\mathrm{L}}(\mathbf{D}_\gamma, \mathbf{I}_k)$ *with* $\mathbf{D}_\gamma$ *fixed. In order that* $\mathbf{E} \in \mathrm{H}_{\mathrm{L}}(\mathbf{D}_\gamma, \mathbf{I}_k)$, *it is necessary that*

(i) $\{\varepsilon_i \leqslant 1;\ 1 \leqslant i \leqslant k\}$, *and that*

(ii) $\{e_{ii} \leqslant 1;\ 1 \leqslant i \leqslant r+s\}$ *and* $\{e_{ii} \leqslant \gamma_i;\ r+s+1 \leqslant i \leqslant k\}$.

Given that $\{e_{ii};\ 1 \leqslant i \leqslant k\}$ *are assigned their maximal values, a necessary and sufficient condition that* $\mathbf{E} \in \mathrm{H}_{\mathrm{L}}(\mathbf{D}_\gamma, \mathbf{I}_k)$ *is that* $\mathbf{E}$ *take the form*

(iii) $\mathbf{E}_{\mathrm{M}} = \mathrm{Diag}(\mathbf{I}_r, \mathbf{I}_s, \gamma_{r+s+1}, ..., \gamma_k)$.

Proof. Conclusion (i) follows from the equivalence of $\mathbf{I}_k \geqslant_{\mathrm{L}} \mathbf{E}$ and $\mathbf{I}_k \geqslant_{\mathrm{L}} \mathrm{Diag}(\varepsilon_1, ..., \varepsilon_k)$. Conclusion (ii) follows on noting that if $\mathbf{D}_\gamma - \mathbf{E}$ and $\mathbf{I}_k - \mathbf{E}$ are to be positive semidefinite, then their diagonal elements are necessarily nonnegative. To see necessity in conclusion (iii), assume first that $\mathbf{D}_\gamma - \mathbf{E} \in S_k^0$, so that $\mathbf{E}$ when assigned its maximal diagonal elements takes the form $\mathbf{E}_0 = \mathrm{Diag}(\mathbf{H}, \mathbf{I}_s, \gamma_{r+s+1}, ..., \gamma_k)$ such that $\mathbf{H} = [h_{ij}]$ with $\{h_{ii} = 1;\ 1 \leqslant i \leqslant r\}$, and $\mathrm{Diag}(\gamma_1, ..., \gamma_r) - \mathbf{H} \in S_k^0$. Other off-diagonal elements vanish since the corresponding diagonal elements of $\mathbf{D}_\gamma - \mathbf{E} \in S_k^0$ vanish. Now recalling additionally that $\mathbf{I}_k - \mathbf{E}_0 \in S_k^0$, we conclude that $\mathbf{I}_r - \mathbf{H} \in S_k^0$, hence the off-diagonal elements of $\mathbf{H}$ must vanish also, giving

$\mathbf{E}_{\mathrm{M}}$ as in conclusion (iii). Sufficiency of the form (iii) follows since the diagonals of $\mathbf{E}_{\mathrm{M}}$ take their maximal values, and both $\mathbf{D}_{\gamma}-\mathbf{E}_{\mathrm{M}}$ and $\mathbf{I}_k-\mathbf{E}_{\mathrm{M}}$ are positive semidefinite by construction. ▮

Turning to upper bounds for the pair $(\mathbf{D}_{\gamma}, \mathbf{I}_k)$, and thereby for $(\mathbf{A}, \mathbf{B})$, essential properties of $\mathrm{H}_{\mathrm{U}}(\mathbf{D}_{\gamma}, \mathbf{I}_k)$ are as summarized without further proof in the following lemma.

LEMMA 3. *Let* $\mathbf{F}=[f_{ij}]$ *have eigenvalues* $\{\eta_1 \geqslant \cdots \geqslant \eta_k > 0\}$, *and consider the class* $\mathrm{H}_{\mathrm{U}}(\mathbf{D}_{\gamma}, \mathbf{I}_k)$ *with* $\mathbf{D}_{\gamma}$ *fixed. In order that* $\mathbf{F} \in \mathrm{H}_{\mathrm{U}}(\mathbf{D}_{\gamma}, \mathbf{I}_k)$, *it is necessary that*

(i) $(1 \leqslant \eta_i < \infty;\ 1 \leqslant i \leqslant k\}$, *and that*

(ii) $\{f_{ii} \geqslant \gamma_i;\ 1 \leqslant i \leqslant r\}$ *and* $\{f_{ii} \geqslant 1;\ r+1 \leqslant i \leqslant k\}$.

Given that $\{f_{ii};\ 1 \leqslant i \leqslant k\}$ *are assigned their minimal values, a necessary and sufficient condition that* $\mathbf{F} \in \mathrm{H}_{\mathrm{U}}(\mathbf{D}_{\gamma}, \mathbf{I}_k)$ *is that* $\mathbf{F}$ *take the form*

(iii) $\mathbf{F}_{\mathrm{m}}=\mathrm{Diag}(\gamma_1, \ldots, \gamma_r, \mathbf{I}_s, \mathbf{I}_t)$ *with* $t=k-r-s$.

All lower and upper bounds for $(\mathbf{A}, \mathbf{B})$ in $(S_k^+, \succcurlyeq_{\mathrm{L}})$ follow on mapping back to the original space. Since $\mathbf{A}=\mathbf{B}^{1/2}\mathbf{Q}\mathbf{D}_{\gamma}\mathbf{Q}'\mathbf{B}^{1/2}$ and $\mathbf{B}=\mathbf{B}^{1/2}\mathbf{Q}\mathbf{I}_k\mathbf{Q}'\mathbf{B}^{1/2}$, we conclude that $\mathbf{L} \in \mathrm{H}_{\mathrm{L}}(\mathbf{A}, \mathbf{B})$ if and only if $\mathbf{L}=\mathbf{B}^{1/2}\mathbf{Q}\mathbf{E}\mathbf{Q}'\mathbf{B}^{1/2}$ for some $\mathbf{E} \in \mathrm{H}_{\mathrm{L}}(\mathbf{D}_{\gamma}, \mathbf{I}_k)$. Similarly, $\mathbf{U} \in \mathrm{H}_{\mathrm{U}}(\mathbf{A}, \mathbf{B})$ if and only if $\mathbf{U}=\mathbf{B}^{1/2}\mathbf{Q}\mathbf{F}\mathbf{Q}'\mathbf{B}^{1/2}$ for some $\mathbf{F} \in \mathrm{H}_{\mathrm{U}}(\mathbf{D}_{\gamma}, \mathbf{I}_k)$.

3.2. *Spectral Bounds*

If we require that lower and upper bounds for diagonal matrices be diagonal, then $(D_k, \succcurlyeq_{\mathrm{L}})$ is seen to be a lattice on imbedding it in $(\mathbb{R}^k, \geqslant_k)$. Then the glb and lub of $(\mathbf{D}_{\gamma}, \mathbf{I}_k)$ are $\mathbf{D}_{\gamma} \wedge \mathbf{I}_k=\mathbf{E}_{\mathrm{M}}$ and $\mathbf{D}_{\gamma} \vee \mathbf{I}_k=\mathbf{F}_{\mathrm{m}}$, precisely as defined in Lemmas 1 and 2. This is in turn prompts the following.

DEFINITION 2. The matrices given by $\mathbf{A} \wedge \mathbf{B}=\mathbf{B}^{1/2}\mathbf{Q}(\mathbf{D}_{\gamma} \wedge \mathbf{I}_k)\,\mathbf{Q}'\mathbf{B}^{1/2}$ and $\mathbf{A} \vee \mathbf{B}=\mathbf{B}^{1/2}\mathbf{Q}(\mathbf{D}_{\gamma} \vee \mathbf{I}_k)\,\mathbf{Q}'\mathbf{B}^{1/2}$ are called the *spectral glb* and the *spectral lub* for $(\mathbf{A}, \mathbf{B})$ in $(S_k^+, \succcurlyeq_{\mathrm{L}})$.

Properties of these spectral extremes are studied next. The main issues include the possible interchangeability of $\mathbf{A}$ and $\mathbf{B}$, and whether the spectral bounds are tight. Both are answered affirmatively in developments culminating in Theorem 1.

With regard to the reduction $(\mathbf{A}, \mathbf{B}) \rightarrow (\mathbf{D}_{\gamma}, \mathbf{I}_k)$, we may take instead $(\mathbf{A}, \mathbf{B}) \rightarrow (\mathbf{I}_k, \mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2}) \rightarrow (\mathbf{I}_k, \mathbf{D}_{\theta})$. Here $\mathbf{D}_{\theta}$ is the diagonal matrix $\mathbf{D}_{\theta}=\mathrm{Diag}(\theta_1, \ldots, \theta_k)$ with $\{0<\theta_1 \leqslant \cdots \leqslant \theta_k\}$ as the reverse-ordered roots of $|\mathbf{B}-\theta\mathbf{A}|=0$, where $\mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2}=\sum_{i=1}^{k}\theta_i\mathbf{p}_i\mathbf{p}_i'$ is the spectral decomposition with $\mathbf{P} = [\mathbf{p}_1, \ldots, \mathbf{p}_k] \in O(k)$. Proceeding as before, define $\mathbf{B} \wedge \mathbf{A} = \mathbf{A}^{1/2}\mathbf{P}(\mathbf{I}_k \wedge \mathbf{D}_\theta)\,\mathbf{P}'\mathbf{A}^{1/2}$ and $\mathbf{B} \vee \mathbf{A} = \mathbf{A}^{1/2}\mathbf{P}(\mathbf{I}_k \vee \mathbf{D}_\theta)\,\mathbf{P}'\mathbf{A}^{1/2}$. To investigate whether the spectral bounds are invariant with regard to decomposition, i.e., whether $\mathbf{A} \wedge \mathbf{B} = \mathbf{B} \wedge \mathbf{A}$ and $\mathbf{A} \vee \mathbf{B} = \mathbf{B} \vee \mathbf{A}$, we first note several duality relations. These are $\mathbf{D}_\theta = \mathbf{D}_\gamma^{-1}$, $\mathbf{I}_k \wedge \mathbf{D}_\theta = (\mathbf{D}_\gamma \vee \mathbf{I}_k)^{-1}$, $\mathbf{I}_k \vee \mathbf{D}_\theta = (\mathbf{D}_\gamma \wedge \mathbf{I}_k)^{-1}$, $(\mathbf{D}_\gamma \wedge \mathbf{I}_k)(\mathbf{D}_\gamma \vee \mathbf{I}_k) = \mathbf{D}_\gamma$, and $\mathbf{B}^{1/2}\mathbf{Q} = \mathbf{A}^{1/2}\mathbf{P}\mathbf{D}_\gamma^{-1/2}$ and $\mathbf{A}^{-1/2}\mathbf{P}\mathbf{D}_\gamma^{1/2} = \mathbf{B}^{-1/2}\mathbf{Q}$. The latter expressions follow on establishing relationships between the normalized eigenvectors of $\{\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}\mathbf{q}_i = \gamma_i\mathbf{q}_i;\ 1 \leqslant i \leqslant k\}$ and $\{\mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2}\mathbf{p}_i = \theta_i\mathbf{p}_i;\ 1 \leqslant i \leqslant k\}$. Our principal findings with regard to the lower and upper spectral bounds are as follows.

LEMMA 4. *Let* $\{\mathbf{A} \wedge \mathbf{B}, \mathbf{A} \vee \mathbf{B}\}$ *and* $\{\mathbf{B} \wedge \mathbf{A}, \mathbf{B} \vee \mathbf{A}\}$ *be spectral glb's and lub's as defined, and let* $\mathbf{C}(\alpha) = \alpha\mathbf{A} + \bar{\alpha}\mathbf{B}$ *with* $\alpha = 1 - \bar{\alpha} \in [0, 1]$. *Then for any* $(\mathbf{A}, \mathbf{B})$ *in* $(S_k^+, \succcurlyeq_{\mathrm{L}})$,

(i) $\mathbf{A} \wedge \mathbf{B} \preccurlyeq_{\mathrm{L}} (\alpha\mathbf{A} + \bar{\alpha}\mathbf{B}) \preccurlyeq_{\mathrm{L}} \mathbf{A} \vee \mathbf{B}$ *for each* $\alpha \in [0, 1]$;

(ii) $\phi(\mathbf{A} \wedge \mathbf{B}) \leqslant \{\phi(\mathbf{C}(\alpha));\ \alpha \in [0, 1]\} \leqslant \phi(\mathbf{A} \vee \mathbf{B})$ *for each* $\phi \in \mathbf{\Phi}(S_k^+, \succcurlyeq_{\mathrm{L}})$; *and*

(iii) $\mathbf{A} \wedge \mathbf{B} = \mathbf{B} \wedge \mathbf{A}$ *and* $\mathbf{A} \vee \mathbf{B} = \mathbf{B} \vee \mathbf{A}$.

Moreover, the bounds are tight in the sense that

(iv) *if* $\{\mathbf{A}, \mathbf{B}\} \preccurlyeq_{\mathrm{L}} \mathbf{T}$ *and* $\mathbf{T} \preccurlyeq_{\mathrm{L}} \mathbf{A} \vee \mathbf{B}$, *then* $\mathbf{T} = \mathbf{A} \vee \mathbf{B}$, *and*

(v) *if* $\{\mathbf{A}, \mathbf{B}\} \succcurlyeq_{\mathrm{L}} \mathbf{S}$ *and* $\mathbf{S} \succcurlyeq_{\mathrm{L}} \mathbf{A} \wedge \mathbf{B}$, *then* $\mathbf{S} = \mathbf{A} \wedge \mathbf{B}$.

Proof. Conclusion (i) in the form $\mathbf{A} \wedge \mathbf{B} \preccurlyeq_{\mathrm{L}} \{\mathbf{A}, \mathbf{B}\} \preccurlyeq_{\mathrm{L}} \mathbf{A} \vee \mathbf{B}$ follows from Lemmas 1 and 2, so that $\mathbf{A} \wedge \mathbf{B} \preccurlyeq_{\mathrm{L}} (\alpha\mathbf{A} + \bar{\alpha}\mathbf{B}) \preccurlyeq_{\mathrm{L}} \mathbf{A} \vee \mathbf{B}$ for each $\alpha \in [0, 1]$. This in turn implies (ii) in view of the monotonicity of $\mathbf{\Phi}(S_k^+, \succcurlyeq_{\mathrm{L}})$. To see conclusion (iii), note that if $\mathbf{A} \preccurlyeq_{\mathrm{L}} \mathbf{B}$, then $\mathbf{A} \wedge \mathbf{B} = \mathbf{A} = \mathbf{B} \wedge \mathbf{A}$ and $\mathbf{A} \vee \mathbf{B} = \mathbf{B} = \mathbf{B} \vee \mathbf{A}$. Otherwise let $\mathbf{B}_1' = \mathbf{B}^{1/2}\mathbf{Q}$ and $\mathbf{B}_2' = \mathbf{A}^{1/2}\mathbf{P}\mathbf{D}_\gamma^{-1/2}$; observe that $\mathbf{A} \wedge \mathbf{B} = \mathbf{B}_1'(\mathbf{D}_\gamma \wedge \mathbf{I}_k)\mathbf{B}_1$; and recall that $\mathbf{B} \wedge \mathbf{A} = \mathbf{A}^{1/2}\mathbf{P}(\mathbf{I}_k \wedge \mathbf{D}_\theta)\,\mathbf{P}'\mathbf{A}^{1/2}$. From the duality relations cited earlier it follows that $\mathbf{I}_k \wedge \mathbf{D}_\theta = (\mathbf{D}_\gamma \vee \mathbf{I}_k)^{-1} = (\mathbf{D}_\gamma \wedge \mathbf{I}_k)\,\mathbf{D}_\gamma^{-1}$, so that $\mathbf{B} \wedge \mathbf{A} = \mathbf{A}^{1/2}\mathbf{P}\mathbf{D}_\gamma^{-1/2}(\mathbf{D}_\gamma \wedge \mathbf{I}_k)\,\mathbf{D}_\gamma^{-1/2}\mathbf{P}'\mathbf{A}^{1/2} = \mathbf{B}_2'(\mathbf{D}_\gamma \wedge \mathbf{I}_k)\mathbf{B}_2$. But another duality asserts that $\mathbf{B}_1' = \mathbf{B}_2'$, so that $\mathbf{A} \wedge \mathbf{B} = \mathbf{B} \wedge \mathbf{A}$ as claimed. The assertion $\mathbf{A} \vee \mathbf{B} = \mathbf{B} \vee \mathbf{A}$ follows similarly. To establish (iv), first suppose that $\{\mathbf{A}, \mathbf{B}\} \preccurlyeq_{\mathrm{L}} \mathbf{T}$ and $\mathbf{T} \preccurlyeq_{\mathrm{L}} \mathbf{A} \vee \mathbf{B}$. Then since $\mathbf{D}_\gamma = \mathbf{Q}'\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}\mathbf{Q}$ and $\mathbf{D}_\gamma \vee \mathbf{I}_k = \mathbf{Q}'\mathbf{B}^{-1/2}(\mathbf{A} \vee \mathbf{B})\,\mathbf{B}^{-1/2}\mathbf{Q}$, the ordering $\mathbf{A} \preccurlyeq_{\mathrm{L}} \mathbf{T} \preccurlyeq_{\mathrm{L}} \mathbf{A} \vee \mathbf{B}$ implies

$$\mathbf{D}_\gamma \preccurlyeq_{\mathrm{L}} \mathbf{Q}'\mathbf{B}^{-1/2}\mathbf{T}\mathbf{B}^{-1/2}\mathbf{Q} \preccurlyeq_{\mathrm{L}} \mathbf{D}_\gamma \vee \mathbf{I}_k \tag{3.2}$$

whereas $\mathbf{B} \preccurlyeq_{\mathrm{L}} \mathbf{T} \preccurlyeq_{\mathrm{L}} \mathbf{A} \vee \mathbf{B}$ gives

$$\mathbf{I}_k \preccurlyeq_{\mathrm{L}} \mathbf{Q}'\mathbf{B}^{-1/2}\mathbf{T}\mathbf{B}^{-1/2}\mathbf{Q} \preccurlyeq_{\mathrm{L}} \mathbf{D}_\gamma \vee \mathbf{I}_k. \tag{3.3}$$

Letting $\mathbf{c}_i' = [0, ..., 0, 1, 0, ..., 0]$ have unity in the ith coordinate and zeros elsewhere, we infer from (3.2) and (3.3) that

$$\gamma_i \leqslant \mathbf{c}_i'\mathbf{Q}'\mathbf{B}^{-1/2}\mathbf{T}\mathbf{B}^{-1/2}\mathbf{Q}\mathbf{c}_i \leqslant \gamma_i, \qquad 1 \leqslant i \leqslant r \tag{3.4}$$

$$1 \leqslant \mathbf{c}_i'\mathbf{Q}'\mathbf{B}^{-1/2}\mathbf{T}\mathbf{B}^{-1/2}\mathbf{Q}\mathbf{c}_i \leqslant 1, \qquad r+1 \leqslant i \leqslant k. \tag{3.5}$$

Combining these and letting $\mathbf{W} = [(\mathbf{D}_\gamma \vee \mathbf{I}_k) - \mathbf{Q}'\mathbf{B}^{-1/2}\mathbf{T}\mathbf{B}^{-1/2}\mathbf{Q}]$, we find that $\mathbf{c}_i'\mathbf{W}\mathbf{c}_i = 0$, so that the diagonal elements of $\mathbf{W}$ are zero. But since $\mathbf{W} \succcurlyeq_{\mathrm{L}} \mathbf{0}$, this implies that $\mathbf{W} = \mathbf{0}$ and thus $\mathbf{T} = \mathbf{B}^{1/2}\mathbf{Q}(\mathbf{D}_\gamma \vee \mathbf{I}_k)\mathbf{Q}'\mathbf{B}^{1/2} = \mathbf{A} \vee \mathbf{B}$ as claimed in conclusion (iv). The proof for (v) proceeds similarly, to complete the proof. ∎

4. Symmetric Majorants and Minorants

Elliptically contoured majorants and minorants for $\{\mu(\cdot; \mathbf{\Sigma}, G), \mu(\cdot; \mathbf{\Omega}, G)\}$ follow directly from the foregoing developments. In addition, with G fixed, these bounds are seen to apply to every such elliptical measure whose scale parameters lie on a line segment connecting $\mathbf{\Sigma}$ and $\mathbf{\Omega}$ as in the ensemble (2.1). Details are supplied in the following, where $\mathbf{\Sigma}$ and $\mathbf{\Omega}$ correspond respectively to $\mathbf{A}$ and $\mathbf{B}$ in the notation of Section 3.

Theorem 1. *Let $\{\mu(\cdot; \mathbf{\Sigma}, G), \mu(\cdot; \mathbf{\Omega}, G)\}$ be any elliptically contoured measures on $\mathbb{R}^k$; let $\nu_{\mathrm{m}}(\cdot; G) = \mu(\cdot; \mathbf{\Sigma} \vee \mathbf{\Omega}, G)$ and $\nu_{\mathrm{M}}(\cdot; G) = \mu(\cdot; \mathbf{\Sigma} \wedge \mathbf{\Omega}, G)$; and write $\Xi(\alpha) = \alpha\mathbf{\Sigma} + \bar{\alpha}\mathbf{\Omega}$ with $\alpha = 1 - \bar{\alpha} \in [0, 1]$.*

(i) *All elliptically symmetric minorants and majorants for $\{\mu(\cdot; \mathbf{\Sigma}, G), \mu(\cdot; \mathbf{\Omega}, G)\}$, with radial cdf G, are generated by*

$$\mu(\cdot; \mathbf{L}, G) \preccurlyeq_{\mathrm{p}} \{\mu(\cdot; \mathbf{\Sigma}, G), \mu(\cdot; \mathbf{\Omega}, G)\} \preccurlyeq_{\mathrm{p}} \mu(\cdot; \mathbf{U}, G) \tag{4.1}$$

as $\mathbf{L}$ ranges over $\mathrm{H}_{\mathrm{L}}(\mathbf{\Sigma}, \mathbf{\Omega})$ and as $\mathbf{U}$ ranges over $\mathrm{H}_{\mathrm{U}}(\mathbf{\Sigma}, \mathbf{\Omega})$.

(ii) *The bounds $\nu_{\mathrm{m}}(C; G) \leqslant \{\mu(C; \Xi(\alpha), G); \alpha \in [0, 1]\} \leqslant \nu_{\mathrm{M}}(C; G)$ hold for every $C \in \mathbf{C}^k$.*

(iii) *The bounds in* (ii) *are sharp in the sense that $\nu_{\mathrm{m}}(C_1; G) = \mu(C_1; \mathbf{\Sigma}, G)$ and $\mu(C_2; \mathbf{\Omega}, G) = \nu_{\mathrm{M}}(C_2; G)$ for some sets (C_1, C_2) in $\mathbf{C}^k$.*

(iv) *If $s \geqslant 1$ in expression* (3.1), *then equality holds in* (ii) *for some $C \in \mathbf{C}^k$, giving*

$$\nu_{\mathrm{m}}(C; G) = \mu(C; \mathbf{\Sigma}, G) = \mu(C; \mathbf{\Omega}, G) = \nu_{\mathrm{M}}(C; G). \tag{4.2}$$

Proof. Conclusion (i) follows from Lemmas 2 and 3 and the equivalence between peakedness and scale orderings for elliptical measures on $\mathbb{R}^k$ with G fixed, as noted earlier. Conclusion (ii) follows by construction from

Lemma 4. The remaining conclusions follow on examining the problem in the canonical form of Lemma 4. To these ends let $\mathbf{D}_\gamma(r) = \text{Diag}(\gamma_1, ..., \gamma_r)$ and $\mathbf{M} = \mathbf{D}_\gamma \vee \mathbf{I}_k$, where $\mathbf{D}_\gamma = \mathbf{P}'\mathbf{\Omega}^{-1/2}\mathbf{\Sigma}\mathbf{\Omega}^{-1/2}\mathbf{P}$ as in Section 3. It is clear that the joint marginal measures of the first r elements corresponding to $\mu(\cdot; \mathbf{D}_\gamma, G)$ and $\nu_m(\cdot; G)$ are both equal to $\mu(\cdot; \mathbf{D}_\gamma(r), G)$ on $\mathbb{R}^r$. Equivalently, we have that $\mu(C_0; \mathbf{D}_\gamma, G) = \nu_m(C_0; G)$ for every cylinder set $C_0 \subset \mathbb{R}^k$ having sections in $L_1 = \text{Sp}(\mathbf{p}_1, ..., \mathbf{p}_r)$. Transforming back to the original space shows that $\mu(C_1; \mathbf{\Sigma}, G) = \nu_m(C_1; G)$ with C_1 as the image of C_0, and the set C_2 of conclusion (iii) may be identified similarly. A parallel development identifies C in conclusion (iv) as the image of a cylinder set having symmetric convex sections in $L_2 = \text{Sp}(\mathbf{p}_{r+1}, ..., \mathbf{p}_{r+s})$, since $\mathbf{C}^k$ is closed under nonsingular linear transformations. This completes our proof. ∎

We next consider elliptical measures having different radial distributions. A basic connection between peakedness ordering on $\mathbb{R}^k$, and the stochastic ordering of radial distributions on $\mathbb{R}^1_+$, is that $\mu(\cdot; \mathbf{\Sigma}, G_1) \succcurlyeq_{\text{p}} \mu(\cdot; \mathbf{\Sigma}, G_2)$ if and only if $G_1(t) \geqslant G_2(t)$ for every $t \in \mathbb{R}^1_+$; see Jensen (1984). That is, for fixed C and $\mathbf{\Sigma}$, the functional $\mu(C; \mathbf{\Sigma}, G)$ as G varies is in $\mathbf{\Phi}(F_0, \geqslant_{\text{df}})$. If $G_1 \geqslant_{\text{df}} G_2$, then elliptically symmetric minorants and majorants follow directly as

$$\mu(\cdot; \mathbf{\Sigma} \vee \mathbf{\Omega}, G_2) \preccurlyeq_{\text{p}} \{\mu(\cdot; \mathbf{\Sigma}, G_1), \mu(\cdot; \mathbf{\Omega}, G_2)\} \preccurlyeq_{\text{p}} \mu(\cdot; \mathbf{\Sigma} \wedge \mathbf{\Omega}, G_1). \quad (4.3)$$

Our principal findings under boundedness and measurability assumptions are the following.

THEOREM 2. *Let $E_k(M, T)$ be an ensemble as in* (2.2); *suppose that M is bounded such that $\mathbf{\Xi}_{\text{m}} \preccurlyeq_{\text{L}} \mathbf{\Xi} \preccurlyeq_{\text{L}} \mathbf{\Xi}_{\text{M}}$ for every $\mathbf{\Xi} \in M$; and define $F_{\text{M}}(t) = \sup_{\tau \in T} G_\tau(t)$ and $F_{\text{m}}(t) = \inf_{\tau \in T} G_\tau(t)$.*

(i) *For the ensemble* (2.2) *bounds of the type*

$$\mu(\cdot; \mathbf{\Xi}_{\text{M}}, F_{\text{m}}) \preccurlyeq_{\text{p}} \mu(\cdot; \mathbf{\Xi}, F_{\text{m}}) \preccurlyeq_{\text{p}} \mu(\cdot; \mathbf{\Xi}, G_\tau) \preccurlyeq_{\text{p}} \mu(\cdot; \mathbf{\Xi}, F_{\text{M}}) \preccurlyeq_{p} \mu(\cdot; \mathbf{\Xi}_{\text{m}}, F_{\text{M}})$$

and

$$\mu(\cdot; \mathbf{\Xi}_{\text{M}}, F_{\text{m}}) \preccurlyeq_{\text{p}} \mu(\cdot; \mathbf{\Xi}_{\text{M}}, G_\tau) \preccurlyeq_{\text{p}} \mu(\cdot; \mathbf{\Xi}, G_\tau) \preccurlyeq_{\text{p}} \mu(\cdot; \mathbf{\Xi}_{\text{m}}, G_\tau) \preccurlyeq_{\text{p}} \mu(\cdot; \mathbf{\Xi}_{\text{m}}, F_{\text{M}})$$

hold for every elliptical measure $\mu(\cdot; \mathbf{\Xi}, G_\tau)$ in the ensemble $E_k(M, T)$.

(ii) *For mixtures of the type* (2.4) *the bounds*

$$\begin{aligned}\mu(\cdot; \mathbf{\Xi}_{\text{M}}, F_{\text{m}}) &\preccurlyeq_{\text{p}} \nu_1(\cdot; \mathbf{\Xi}_{\text{M}}, F) \preccurlyeq_{\text{p}} \nu(\cdot; \mathbf{\Gamma}, F) \\ &\preccurlyeq_{\text{p}} \nu_1(\cdot; \mathbf{\Xi}_{\text{m}}, F) \preccurlyeq_{\text{p}} \mu(\cdot; \mathbf{\Xi}_{\text{m}}, F_{\text{M}})\end{aligned}$$

hold uniformly for every such measure.

(iii) *Alternatively, the bounds*

$$\mu(\cdot;\Xi_{\mathrm{M}},F_{\mathrm{m}})\preccurlyeq_{\mathrm{p}}\nu_2(\cdot;\Gamma,F_{\mathrm{m}})\preccurlyeq_{\mathrm{p}}\nu(\cdot;\Gamma,F)\preccurlyeq_{\mathrm{p}}\nu_2(\cdot;\Gamma,F_{\mathrm{M}})\preccurlyeq_{\mathrm{p}}\mu(\cdot;\Xi_{\mathrm{m}},F_{\mathrm{M}})$$

hold uniformly for every mixture of the type (2.4).

Proof. Conclusion (i) follows from Theorem 1, from properties of Ξ_{m} and Ξ_{M}, and from the fact that $F_{\mathrm{m}}(t)$ and $F_{\mathrm{M}}(t)$ are themselves cdf's. Conclusions (ii) and (iii) follow from conclusion (i) and the convexity of mixtures, to conclude our proof. ∎

In summary, it is seen that elliptical measures may serve as symmetric majorants and minorants for pairs (Theorem 1) and ensembles (Theorem 2(i)) of elliptical measures on $\mathbb{R}^k$. Less intuitive is the fact that elliptical measures also may serve as symmetric majorants and minorants for certain star-unimodal mixtures in $\mathbb{P}_k(\mathbf{0})$. Specifically, the inner and outer bounds of Theorem 2(ii) and the outer bounds of Theorem 2(iii) constitute pairs of elliptical measures. In contrast, the inner bounds of Theorem 2(iii) show that symmetric star-unimodal measures may serve as symmetric majorants and minorants for certain star-unimodal mixtures in $\mathbb{P}_k(\mathbf{0})$. While tighter, these inner bounds do require evaluation of the indicated partial mixtures from (2.6). Although little is known about the latter and other distributions in $\mathbb{P}_k(\mathbf{0})$ generally, much is now known about elliptical measures and their properties, and research continues apace. Recent treatises on this topic include the works of Fang, Kotz, and Ng (1990), Fang and Zhang (1990), and Kariya and Sinha (1989), for example.

5. Some Applications

Applications of the foregoing results are developed in the context of distributions of quadratic forms and normal-theory confidence sets in linear models.

5.1. *Distributions of Quadratic Forms*

Consider the distribution of the definite quadratic form $Q_{\mathrm{M}}(\mathbf{Y})=\mathbf{Y}'\mathbf{M}'\mathbf{M}\mathbf{Y}$ in one of two Gaussian laws having zero means, dispersion matrices $(\mathbf{\Lambda},\mathbf{\Delta})$, and corresponding measures $\mu(\cdot;\mathbf{\Lambda})$ and $\mu(\cdot;\mathbf{\Delta})$ on $\mathbb{R}^k$. In canonical form with $\mathbf{Y}\to\mathbf{Z}=\mathbf{M}\mathbf{Y}$, we consider equivalently the form $Q(\mathbf{Z})=\mathbf{Z}'\mathbf{Z}$ together with measures $\mu(\cdot;\mathbf{\Sigma})$ and $\mu(\cdot;\mathbf{\Omega})$ such that $\mathbf{\Sigma}=\mathbf{M}\mathbf{\Lambda}\mathbf{M}'$ and $\mathbf{\Omega}=\mathbf{M}\mathbf{\Delta}\mathbf{M}'$. The cdf's for $Q(\mathbf{Z})$ under the two models have the structure

$$G_{\Sigma}(t)=\mu(B(t);\mathbf{\Sigma})=G(t;\sigma_1,\ldots,\sigma_k) \tag{5.1}$$

and

$$G_{\mathbf{\Omega}}(t)=\mu(B(t);\mathbf{\Omega})=G(t;\omega_1,...,\omega_k), \tag{5.2}$$

say, with $B(t)=\{\mathbf{z}\in\mathbb{R}^k:\mathbf{z}'\mathbf{z}\leqslant t\}$. Here $\{\sigma_1\geqslant\cdots\geqslant\sigma_k\}$ and $\{\omega_1\geqslant\cdots\geqslant\omega_k\}$ are the ordered eigenvalues of $\mathbf{\Sigma}$ and $\mathbf{\Omega}$, respectively. Now letting $\mathbf{\Xi}_{\mathrm{m}}=\mathbf{\Sigma}\wedge\mathbf{\Omega}$ and $\mathbf{\Xi}_{\mathrm{M}}=\mathbf{\Sigma}\vee\mathbf{\Omega}$ have eigenvalues $\{\xi_1\geqslant\cdots\geqslant\xi_k\}$ and $\{\gamma_1\geqslant\cdots\geqslant\gamma_k\}$, respectively, we apply results from Section 4 to obtain the bounds

$$G(t;\gamma_1,...,\gamma_k)\leqslant\{\mu(B(t);\alpha\mathbf{\Sigma}+\bar{\alpha}\mathbf{\Omega});\alpha\in[0,1]\}\leqslant G(t;\xi_1,...,\xi_k) \tag{5.3}$$

for all $t\in\mathbb{R}^1_+$. Moreover, as t varies this provides an envelope containing all cdf's $G_{\mathbf{\Xi}}(t)$ corresponding to a Gaussian ensemble $\{\mu(\cdot;\mathbf{\Xi});\mathbf{\Xi}\in M\}$, such that $\mathbf{\Xi}_{\mathrm{m}}\preccurlyeq_{\mathrm{L}}\mathbf{\Xi}\preccurlyeq_{\mathrm{L}}\mathbf{\Xi}_{\mathrm{M}}$, as well as mixtures of these over M. Series expansions for $G(t;\gamma_1,...,\gamma_k)$ and $G(t;\xi_1,...,\xi_k)$ are developed in Johnson and Kotz (1970) together with further references.

Quadratic forms are germane to many developments in applied probability and statistics, including performance characteristics of certain ballistic systems and the detection of signals from noise. In ballistics let $\mathbf{X}\in\mathbb{R}^3$ be the point of impact of a missile subject to chance disturbances attributable to local turbulence; precipitation; variations in air pressure, currents, and velocities; and other extraneous circumstances. Often $\mathbf{X}$ is modeled stochastically as a Gaussian vector having some mean point of impact $\boldsymbol{\mu}\in\mathbb{R}^3$ and dispersion matrix $\mathbf{\Xi}$. If r is the effective radius of the weapon on impact and if $B(r)$ is the ball of radius r centered at $\boldsymbol{\mu}$, then the probability of a "kill" is given by $\mu(\beta(r);\mathbf{\Xi})$. Details are given in Eckler and Burr (1972), for example. In practice such models are often overly simplistic. Gilliland (1968) considered more general distributions unimodal in the sense of Anderson (1955) having convex level sets. We suppose instead that dispersion characteristics of the trajectory vary with weather conditions such that $\mathbf{\Omega}$ is appropriate under mild conditions, whereas $\mathbf{\Sigma}$ applies in severe weather. If dispersion parameters can be modeled reasonably for intermediate cases as $\mathbf{\Xi}(\alpha)=\alpha\mathbf{\Sigma}+\bar{\alpha}\mathbf{\Omega}$, with $\alpha\in[0,1]$ reflecting the weather intensity, then the bounds (5.3) apply directly to give envelopes for hit probabilities as r varies. These bounds now bracket the actual probabilities, whatever may be the actual weather pattern on a given occasion. Identical conclusions apply to mixtures of these distributions as the weather intensity α varies randomly according to some cdf $H(\cdot)$ in $F[0,1]$ as in (2.3), giving bounds on hit probabilities for distributions of trajectories having star-shaped contours.

Similar considerations apply in signal detection using a square law envelope detector. Here an incoming input is either processed as a signal or suppressed as noise according to whether or not its amplitude exceeds a threshold value c. Under Gaussian noise the amplitude has a generalized Rayleigh distribution and its square has a distribution of the type (5.1); see Miller (1975), for example. The bounds (5.3) apply to distributions of the squared amplitudes generated by the Gaussian ensemble, as well as to mixtures over the ensemble, to give an envelope for distributions of squared amplitudes arising from mixtures of Gaussian noise processes having symmetric star-unimodal projections. Properties of such distributions are unknown at present and no doubt the distributions themselves are quire complicated. Nonetheless, the stochastic bounds (5.3) apply to all such mixtures.

5.2. *Confidence Sets*

Consider linear models of full rank having independent Gaussian errors with zero means and unit variances. Under designs $(\mathbf{X}, \mathbf{Z})$ in $F_{n \times k}$, these models are $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$ and $\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \mathbf{e}$, and the corresponding Gauss–Markov estimators are $\hat{\boldsymbol{\beta}}(\mathbf{X}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ and $\hat{\boldsymbol{\beta}}(\mathbf{Z}) = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y}$, respectively. Induced measures for $[\hat{\boldsymbol{\beta}}(\mathbf{X}) - \boldsymbol{\beta}]$ and $[\hat{\boldsymbol{\beta}}(\mathbf{Z}) - \boldsymbol{\beta}]$ are $\mu(\cdot; \boldsymbol{\Sigma})$ and $\mu(\cdot; \boldsymbol{\Omega})$, respectively, with $\boldsymbol{\Sigma} = (\mathbf{X}'\mathbf{X})^{-1}$ and $\boldsymbol{\Omega} = (\mathbf{Z}'\mathbf{Z})^{-1}$. In addition, with c_α as the $100(1-\alpha)$ percentile of the chi-squared distribution having k degrees of freedom, the confidence regions

$$A(t; \boldsymbol{\Sigma}) = \{\boldsymbol{\beta} \in \mathbb{R}^k : (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})' \boldsymbol{\Sigma}^{-1} (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \leqslant t\} \tag{5.4}$$

and

$$A(t; \boldsymbol{\Omega}) = \{\boldsymbol{\beta} \in \mathbb{R}^k : (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})' \boldsymbol{\Omega}^{-1} (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \leqslant t\} \tag{5.5}$$

both have confidence coefficient $1-\alpha$ at $t = c_\alpha$ under their respective measures $\mu(\cdot; \boldsymbol{\Sigma})$ and $\mu(\cdot; \boldsymbol{\Omega})$.

Applying developments from Section 4, again let $\boldsymbol{\Xi}_{\mathrm{m}} = \boldsymbol{\Sigma} \wedge \boldsymbol{\Omega}$ and $\boldsymbol{\Xi}_{\mathrm{M}} = \boldsymbol{\Sigma} \vee \boldsymbol{\Omega}$, and define $A(t; \boldsymbol{\Xi}_{\mathrm{m}})$ and $A(t; \boldsymbol{\Xi}_{\mathrm{M}})$ as in (5.4) and (5.5) together with their respective Gaussian measures $\mu(\cdot; \boldsymbol{\Xi}_{\mathrm{m}})$ and $\mu(\cdot; \boldsymbol{\Xi}_{\mathrm{M}})$. Again all regions have confidence coefficient $1-\alpha$ at $t = c_\alpha$ under their respective measures. Moreover, we see that these regions are nested by inclusion according to

$$A(t; \boldsymbol{\Xi}_{\mathrm{m}}) \subset \{A(t; \boldsymbol{\Sigma}), A(t; \boldsymbol{\Omega})\} \subset A(t; \boldsymbol{\Xi}_{\mathrm{M}}) \tag{5.6}$$

for every $t \in \mathbb{R}^1_+$ and thus for every α. In addition, these inclusion relations extend to include any bounded ensemble $X(\boldsymbol{\Xi}_{\mathrm{m}}, \boldsymbol{\Xi}_{\mathrm{M}})$ of designs such that $X(\boldsymbol{\Xi}_{\mathrm{m}}, \boldsymbol{\Xi}_{\mathrm{M}}) = \{\mathbf{X} \in F_{n \times k} : \boldsymbol{\Xi}_{\mathrm{m}} \preccurlyeq_{\mathrm{L}} (\mathbf{X}'\mathbf{X})^{-1} \preccurlyeq_{\mathrm{L}} \boldsymbol{\Xi}_{\mathrm{M}}\}$, as well as mixtures over $X(\boldsymbol{\Xi}_{\mathrm{m}}, \boldsymbol{\Xi}_{\mathrm{M}})$ in the case of random designs. Finally, note that parallel results carry over to include linear models having any spherical error distribution.

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