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Journal of Multivariate Analysis 97 (2006) 274 – 294

Journal of Multivariate Analysis

www.elsevier.com/locate/jmva

Local efficiency of a Cramér–von Mises test of independence

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Received 13 September 2004
Available online 30 April 2005

Abstract

Deheuvels proposed a rank test of independence based on a Cramér–von Mises functional of the empirical copula process. Using a general result on the asymptotic distribution of this process under sequences of contiguous alternatives, the local power curve of Deheuvels' test is computed in the bivariate case and compared to that of competing procedures based on linear rank statistics. The Gil-Pelaez inversion formula is used to make additional comparisons in terms of a natural extension of Pitman's measure of asymptotic relative efficiency.

AMS 1991 subject classification: 62H15; 62H10; 62G10; 62G20; 62G30

Keywords: Asymptotic relative efficiency; Contiguous alternatives; Empirical copula process; Linear rank statistics

1. Introduction

Many procedures have been proposed to test whether two random characters X and Y are independent. The classical approach is based on Pearson's correlation coefficient, but

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0047-259X/$ - see front matter
doi:10.1016/j.jmva.2005.03.003

its lack of robustness to outliers and departures from normality eventually led researchers to consider alternative nonparametric procedures.

The most commonly used rank tests of independence—those of Savage, Spearman and van der Waerden in particular—rely on linear rank statistics, which may be conveniently written in the form

$$S_n^J = \frac{1}{n}\sum_{i=1}^{n} J\left(\frac{R_i}{n+1}, \frac{S_i}{n+1}\right) - \bar{J}_n, \tag{1}$$

where $J : (0,1)^2 \to \mathbb{R}$ is a score function,

$$\bar{J}_n = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} J\left(\frac{i}{n+1}, \frac{j}{n+1}\right)$$

is a centering factor, and $(R_1, S_1), \ldots, (R_n, S_n)$ are the pairs of ranks associated with a random sample $(X_1, Y_1), \ldots, (X_n, Y_n)$ from some population with bivariate cumulative distribution function $H(x, y)$ and continuous margins $F(x)$ and $G(y)$.

In fact, as shown by Behnen [2,3], essentially all statistics of the form (1) yield asymptotically optimal rank tests of independence for suitably selected local alternatives. See Genest and Verret [17] for a recent account of this literature, which includes major contributions by Bhuchongkul [4], Shirahata [25,26], and Ciesielska and Ledwina [7], among others.

In practice, however, it is rarely possible to identify with any precision the form of dependence characterized by a family of alternatives. For this reason, omnibus rank tests seem desirable. Because Sklar [28] showed that H admits a unique representation

$$H(x, y) = C\left\{F(x), G(y)\right\}, \quad x, y \in \mathbb{R}$$

in terms of a copula $C : [0,1]^2 \to [0,1]$, and given that independence between the continuous random variables X and Y occurs if and only if $C(u,v) = C_0(u,v) \equiv uv$ everywhere on its domain, a potentially fruitful rank-based approach to testing independence is rooted in the empirical copula

$$C_n(u, v) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\left\{F_n(X_i) \leqslant u, G_n(Y_i) \leqslant v\right\},$$

where

$$F_n(x) = \frac{1}{n+1}\sum_{i=1}^{n} \mathbf{1}(X_i \leqslant x) \quad \text{and} \quad G_n(y) = \frac{1}{n+1}\sum_{i=1}^{n} \mathbf{1}(Y_i \leqslant y)$$

are the re-scaled empirical versions of F and G, respectively. Observe that procedures based on C_n are rank-based, as $F_n(X_i) = R_i/(n+1)$ and $G_n(Y_i) = S_i/(n+1)$ for $i \in \{1, \ldots, n\}$.

Deheuvels [8–12] was the first to suggest tests of independence based on a continuous functional measuring the distance between C_n and C_0. This led him to study the weak convergence of the empirical copula process

$$\mathbb{C}_n(u, v) = n^{1/2}\left\{C_n(u, v) - uv\right\}$$

and its multivariate extension under the null hypothesis of independence. In particular, this made it possible for him to identify the limiting null distribution of the Cramér–von Mises test statistic based on $\mathbb{C}_n$, although he did not actually compare the performance of tests based on this statistic to any competitor.

In a recent extension of Deheuvels' work, Genest and Rémillard [16] report simulations which suggest that Cramér–von Mises statistics are generally more powerful than those based on the classical likelihood ratio statistic assuming normality; see Figs. 3–5 in their paper. Because it improves convergence and leads to a simpler formula for the test statistic, the version of the Cramér–von Mises functional they consider is actually based on the centered empirical copula process

$$\tilde{\mathbb{C}}_n(u, v) = n^{1/2} \{C_n(u, v) - C_n(u, 1)C_n(1, v)\},$$

where $C_n(u, 1) = C_n(1, u)$ is nothing but the distribution function of a uniform random variable on $\{1/(n+1), \dots, n/(n+1)\}$. The latter may be defined explicitly by

$$C_n(u, 1) = C_n(1, u) = \frac{1}{n} \min(n, \lfloor (n+1)u \rfloor), \quad 0 \leqslant u \leqslant 1,$$

where $\lfloor x \rfloor$ stands for the integer part of x.

Following Genest and Rémillard [16], therefore, a powerful nonparametric test of independence à la Deheuvels may thus be based on the Cramér–von Mises statistic

$$B_n = \int_{(0,1)^2} \left\{\tilde{\mathbb{C}}_n(u, v)\right\}^2 dv\, du = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} D_n(R_i, R_j) D_n(S_i, S_j),$$

where

$$D_n(s, t) = \frac{2n+1}{6n} + \frac{s(s-1)}{2n(n+1)} + \frac{t(t-1)}{2n(n+1)} - \frac{\max(s, t)}{n+1}.$$

In addition to being simple to compute, this statistic can be simulated easily in order to construct tables of critical values for any fixed sample size n through Monte Carlo methods. Asymptotic critical values for the standard levels may also be found in Table 1 of Genest and Rémillard [16].

The purpose of this paper is to compare the large-sample performance of standard rank tests of independence to the procedure based on B_n. To this end, the common asymptotic behavior of $\mathbb{C}_n$ and $\tilde{\mathbb{C}}_n$ under contiguous sequences (C_{θ_n}) of parametric alternatives is considered in Section 2. The result is then used in Sections 3 and 4 to derive the asymptotic distribution of S_n^J and B_n under such sequences of alternatives. Examples of calculations are given in Section 5.

In Section 6, the local asymptotic power curve of the test based on B_n is computed and compared to that of the locally most powerful linear rank statistic, identified by Shirahata [25,26]; see also Genest and Verret [17]. A natural extension of Pitman's measure of asymptotic relative efficiency is then used in Section 7 to make numerical power comparisons under various families of copula models. Finally, some concluding remarks are made in Section 8.

2. Asymptotic behavior of $\mathbb{C}_n$

Consider a family (C_θ) of absolutely continuous bivariate copulas indexed by a real parameter $\theta \in \Theta$ in such a way that $C_\theta(u, v)$ is monotone in θ and $C_{\theta_0}(u, v) = uv$ for all $u, v \in (0, 1)$. Let $\delta \in \mathbb{R}$ be such that $\theta_n = \theta_0 + \delta n^{-1/2} \in \Theta$ for n sufficiently large, and suppose that

(i) the density $\partial^2 C_\theta(u, v)/\partial u \partial v = c_\theta(u, v)$ admits a square-integrable, right derivative $\dot{c}_\theta$ at $\theta = \theta_0$ for every fixed $u, v \in (0, 1)$, and

$$\lim_{n\to\infty} \int_{(0,1)^2} \left[n^{1/2} \left\{ c_{\theta_n}^{1/2}(u, v) - 1 \right\} - \frac{\delta}{2}\, \dot{c}_{\theta_0}(u, v) \right]^2 dv\, du = 0;$$

(ii) for every $u, v \in (0, 1)$, the following identity holds:

$$\dot{C}_{\theta_0}(u, v) = \lim_{\theta\to\theta_0} \frac{\partial}{\partial\theta} C_\theta(u, v) = \int_0^u \int_0^v \dot{c}_{\theta_0}(s, t) dt\, ds.$$

Let also Q_n denote the joint distribution of a random sample $(X_{n1}, Y_{n1}), \ldots, (X_{nn}, Y_{nn})$ from distribution $C_{\theta_n}\{F(x), G(y)\}$, and denote by P_n the joint distribution of the same sample under independence. As can be deduced from Lemma 3.10.11 of van der Vaart and Wellner [29], Condition (i) is sufficient to ensure the contiguity of Q_n with respect to P_n. More precisely, if (A_n) is any sequence of sample-based events such that $P_n(A_n) \to 0$ as $n \to \infty$, then $Q_n(A_n) \to 0$, as $n \to \infty$.

Under these assumptions, the asymptotic behavior of the process $\mathbb{C}_n$ may be characterized as follows.

Proposition 1. *Under Conditions* (i)–(ii), *the sequence of empirical rank processes* $\mathbb{C}_n = n^{1/2}\left(C_n - C_{\theta_0}\right)$ *converges weakly in* $\mathbf{D}([0, 1]^2)$, *under* Q_n, *to a continuous Gaussian limit* $\mathbb{C}+\delta\dot{C}_{\theta_0}$, *where* $\mathbb{C}$ *is a continuous centered normal process such that* $\mathrm{cov}\{\mathbb{C}(u, v), \mathbb{C}(u', v')\} = \gamma(u, u')\gamma(v, v')$, with $\gamma(s, t) = \min(s, t) - st$.

Proof. Write $U_{ni} = F(X_{ni})$, $V_{ni} = G(Y_{ni})$, and introduce

$$\Phi_n(u) = \frac{1}{n+1} \sum_{i=1}^n \mathbf{1}\left(U_{ni} \leqslant u\right) \quad \text{and} \quad \Psi_n(v) = \frac{1}{n+1} \sum_{i=1}^n \mathbf{1}\left(V_{ni} \leqslant v\right).$$

Let also

$$\mathbb{A}_n(u, v) = n^{-1/2} \sum_{i=1}^n \left\{ \mathbf{1}\left(U_{ni} \leqslant u, V_{ni} \leqslant v\right) - uv \right\}.$$

Then

$$\mathbb{C}_n(u, v) = \mathbb{A}_n\left\{\Phi_n^{-1}(u), \Psi_n^{-1}(v)\right\} + n^{1/2}\left\{\Phi_n^{-1}(u)\Psi_n^{-1}(v) - uv\right\}. \tag{2}$$

Under Condition (i), it follows from Theorem 3.10.12 of van der Vaart and Wellner [29] that, under Q_n, the sequence $(\mathbb{A}_n)$ of processes converges in $\mathbf{D}\left([0,1]^2\right)$ to a continuous Gaussian limit of the form $\mathbb{A} + \delta \dot{C}_{\theta_0}$, where $\dot{C}_{\theta_0}$ is defined as in Condition (ii).

In particular, under Q_n, $\mathbb{A}_n(u,1) = n^{1/2}\{\Phi_n(u) - u\}$ converges in $\mathbf{D}([0,1])$ to $\mathbb{A}(u,1) + \delta \dot{C}_{\theta_0}(u,1)$, and the latter reduces to $\mathbb{A}(u,1)$, since

$$\dot{C}_{\theta_0}(u,1) = \lim_{\theta \to \theta_0} \frac{C_\theta(u,1) - C_{\theta_0}(u,1)}{\theta - \theta_0} = \lim_{\theta \to \theta_0} \frac{u-u}{\theta - \theta_0} = 0.$$

Thus, using identities (11) and (12) in Chapter 3 of Shorack and Wellner [27], one may deduce that

$$\sup_{u \in [0,1]} |\Phi_n(u) - u| = \sup_{u \in [0,1]} \left|\Phi_n^{-1}(u) - u\right|$$

tends to zero in probability, whence it follows that $n^{1/2}\left\{\Phi_n^{-1}(u) - u\right\}$ converges in $\mathbf{D}([0,1])$ to $-\mathbb{A}(u,1)$.

Likewise, $\sup_{v \in [0,1]} \left|\Psi_n^{-1}(v) - v\right|$ tends to zero in probability, and $n^{1/2}\left\{\Psi_n^{-1}(v) - v\right\}$ converges in $\mathbf{D}([0,1])$ to $-\mathbb{A}(1,v)$. Writing the second summand in (2) in the alternative form

$$n^{1/2}\left\{\Phi_n^{-1}(u) - u\right\}\Psi_n^{-1}(v) + u n^{1/2}\left\{\Psi_n^{-1}(v) - v\right\},$$

one may thus conclude that under Q_n, $\mathbb{C}_n$ converges in $\mathbf{D}\left([0,1]^2\right)$ to $\mathbb{C} + \delta \dot{C}_{\theta_0}$, where

$$\mathbb{C}(u,v) = \mathbb{A}(u,v) - v\mathbb{A}(u,1) - u\mathbb{A}(1,v),$$

whose covariance structure is as given in the statement of the proposition. $\square$

3. Asymptotic behavior of S_n^J

Henceforth, $J : (0,1)^2 \to \mathbb{R}$ is called a score function if it is right-continuous, square-integrable and quasi-monotone, i.e., $J(u',v') - J(u',v) - J(u,v') + J(u,v) \geqslant 0$ for all $u \leqslant u'$ and $v \leqslant v'$. Under these standard conditions, which are met in all classical cases, Quesada-Molina [24] showed that if (U_i, V_i) is distributed as copula C_i, then

$$\mathrm{E}\{J(U_1,V_1) - J(U_2,V_2)\} = \int_{(0,1)^2} \{C_1(s,t) - C_2(s,t)\}\, dJ(s,t),$$

provided $\mathrm{E}\{|J(U_i,V_i)|\} < \infty$ for $i = 1, 2$. Using this result, one may then reexpress the linear rank statistic S_n^J, defined by (1), as

$$n^{1/2} S_n^J = \int_{(0,1)^2} \tilde{\mathbb{C}}_n(u,v) dJ(u,v).$$

Since

$$\sup_{u \in [0,1]} |C_n(u,1) - u| \leqslant \frac{1}{n},$$

$\mathbb{C}_n$ and $\tilde{\mathbb{C}}_n$ obviously have the same limiting behavior under the conditions of Proposition 1. Thus for any closed interval $K \subset (0, 1)^2$, one has

$$\int_K \tilde{\mathbb{C}}_n(u, v) dJ(u, v) \rightsquigarrow \int_K \mathbb{C}(u, v) dJ(u, v) + \delta \int_K \dot{C}_{\theta_0}(u, v) dJ(u, v),$$

where $\rightsquigarrow$ denotes convergence in law. A technical argument described in the Appendix then implies that $n^{1/2} S_n^J$ converges in law to

$$\mathbb{S}^J = \int_{(0,1)^2} \mathbb{C}(u, v) dJ(u, v) + \delta \int_{(0,1)^2} \dot{C}_{\theta_0}(u, v) dJ(u, v),$$

under the additional condition

(iii) $\int_{(0,1)^2} \left| \dot{C}_{\theta_0}(u, v) \right| \mathrm{d}J(u, v) < \infty.$

This finding may be summarized as follows:

Proposition 2. *Under Conditions* (i)–(iii), $n^{1/2} S_n^J$ *is asymptotically normal*, *under* Q_n. *Its mean and variance are*, *respectively*, *given by* $\mathrm{E}(\mathbb{S}^J) = \delta \mu_J$ *and* $\mathrm{var}(\mathbb{S}^J) = \sigma_J^2$, *where*

$$\mu_J = \int_{(0,1)^2} \dot{C}_{\theta_0}(u, v) dJ(u, v)$$

and

$$\sigma_J^2 = \int_{(0,1)^4} \gamma(u, u')\gamma(v, v') dJ(u, v) dJ(u', v') = \int_{(0,1)^2} \left\{ \tilde{J}(u, v) \right\}^2 dv\, du,$$

with

$$\tilde{J}(u, v) = J(u, v) - \int_{(0,1)} J(u, t)\, dt - \int_{(0,1)} J(s, v)\, ds + \int_{(0,1)^2} J(s, t)\, ds\, dt.$$

This is consistent with the results already reported by Genest and Verret [17] under a different set of conditions.

Remark. As can be seen from Table 1 below, many classical linear rank statistics have score functions of the form $J(u, v) = K_1^{-1}(u) K_2^{-1}(v)$, where, for $i = 1, 2$, K_i is a cumulative distribution function with zero mean and finite variance σ_i^2. In that case, it follows from Proposition 2 that

$$n^{1/2} S_n^J = \int_{\mathbb{R}^2} \tilde{C}_n \left\{ K_1(x), K_2(y) \right\} dy\, dx,$$

whence $J = \tilde{J}$ and $\sigma_J^2 = \sigma_1^2 \sigma_2^2$, as already reported in Proposition 3.1 of Genest and Rémillard [16].

4. Asymptotic behavior of B_n

Under the conditions of Proposition 1, $\dot{C}_{\theta_0}$ is continuous and bounded on $[0, 1]^2$, so that the limiting distribution of B_n under the contiguous sequence (Q_n) is given by

$$\mathbb{B} = \int_{(0,1)^2} \left\{\mathbb{C}(u, v) + \delta\dot{C}_{\theta_0}(u, v)\right\}^2 dv\, du.$$

Now it is well known (see, e.g., Shorack and Wellner [27], p. 213) that $\mathbb{C}$ admits a Karhunen–Loève expansion

$$\mathbb{C}(u, v) = \sum_{k,\ell\in\mathbb{N}} \lambda_{k\ell}^{1/2} f_{k\ell}(u, v) Z_{k\ell},$$

where the $Z_{k\ell}$ are mutually independent $\mathcal{N}(0, 1)$ random variables, and for all integers $k, \ell \in \mathbb{N} = \{1, 2, \ldots\}$,

$$\lambda_{k\ell} = \frac{1}{k^2\ell^2\pi^4} \quad \text{and} \quad f_{k\ell}(u, v) = 2\sin(k\pi u)\sin(\ell\pi v), \quad u, v \in (0, 1).$$

Accordingly, one has

$$\mathbb{B}_0 = \int_{(0,1)^2} \{\mathbb{C}(u, v)\}^2 dv\, du = \sum_{k,\ell\in\mathbb{N}} \lambda_{k\ell} Z_{k\ell}^2$$

and hence

$$\mathbb{B} = \sum_{k,\ell\in\mathbb{N}} \lambda_{k\ell} Z_{k\ell}^2 + 2\delta \sum_{k,\ell\in\mathbb{N}} \lambda_{k\ell} I_{k\ell} Z_{k\ell} + \delta^2 I,$$

where

$$I = \int_{(0,1)^2} \left\{\dot{C}_{\theta_0}(u, v)\right\}^2 dv\, du \quad \text{and} \quad I_{k\ell} = \lambda_{k\ell}^{-1/2} \int_{(0,1)^2} f_{k\ell}(u, v)\dot{C}_{\theta_0}(u, v) dv\, du.$$

Letting $\chi_1^2(v)$ denote a chi-square random variable with one degree of freedom and non-centrality parameter v, one may then state the following result:

Proposition 3. *Under Conditions* (i)–(ii), *the limiting distribution of B_n, under Q_n, is given by the weighted sum*

$$\mathbb{B} = \sum_{k,\ell\in\mathbb{N}} \lambda_{k\ell} \left(Z_{k\ell} + \delta I_{k\ell}\right)^2 = \sum_{k,\ell\in\mathbb{N}} \lambda_{k\ell} \chi_1^2\left(\delta^2 I_{k\ell}^2\right)$$

of noncentral χ_1^2 random variables which depends on the underlying contiguous family (C_{θ_n}) of copula alternatives only through $\dot{C}_{\theta_0}$ via the formula

$$I_{k\ell} = 2k\ell\pi^2 \int_{(0,1)^2} \sin(k\pi u)\sin(\ell\pi v)\dot{C}_{\theta_0}(u, v) dv\, du.$$

Proof. From direct substitution into the integral representation of $\mathbb{B}$ of Parseval's identity, $I = \sum_{k,\ell \in \mathbb{N}} \lambda_{k\ell} I_{k\ell}^2$. $\square$

5. Examples

Several commonly used families of bivariate copulas satisfy Conditions (i)–(ii). Interestingly, many of them yield the same value for $\dot{C}_{\theta_0}$, up to a constant. The copula models listed, e.g., in the books of Joe [20], Nelsen [21] or Drouet-Mari and Kotz [13] may thus be clustered into classes whose members all lead to essentially the same asymptotic distribution for B_n. Here are three examples.

Class 1: A simple calculation shows that $\dot{C}_{\theta_0}(u, v) \propto uv(1-u)(1-v)$ for the Ali-Mikhail-Haq, Dabrowska [23], Farlie-Gumbel-Morgenstern, Frank, and Plackett families of copulas. Note that Condition (iii) holds for any score function J, and that

$$\begin{aligned} \mu_J &\propto \int_{(0,1)^2} (1-2u)(1-2v)\tilde{J}(u, v)dv\,du \\ &= \int_{(0,1)^2} (1-2u)(1-2v)J(u, v)dv\,du, \end{aligned}$$

while

$$I_{k\ell} \propto \begin{cases} \dfrac{32}{k^2\ell^2\pi^4} & \text{if } k \text{ and } \ell \text{ are odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Class 2: For the Clayton and Gumbel-Barnett families, as well as for Model 4.2.10 of Nelsen [21], one has $\dot{C}_{\theta_0}(u, v) \propto \pm uv \log(u) \log(v)$, and hence

$$I_{k\ell} \propto \pm \frac{2}{k\ell\pi^2} SI(k\pi)SI(\ell\pi), \quad \text{where } SI(x) = \int_0^x t^{-1} \sin(t)dt.$$

Class 3: If C_θ is the Gaussian copula and N denotes the cumulative distribution function of a $\mathcal{N}(0, 1)$ random variable, then

$$\dot{C}_{\theta_0}(u, v) = N'\left\{N^{-1}(u)\right\} N'\left\{N^{-1}(v)\right\}$$

with $N' = dN(t)/dt$, so that $I_{k\ell} = 2k\ell\pi^2\, g(k)g(\ell)$, where

$$g(m) = \int_{\mathbb{R}} \{N'(t)\}^2 \sin\{m\pi N(t)\}dt.$$

Because of their connection with frailty models, bivariate Archimedean copula models [21, Chapter 4] are particularly common in practice. They can be expressed in the form

$$C_\theta(u, v) = \psi_\theta^{-1}\left\{\psi_\theta(u) + \psi_\theta(v)\right\}$$

in terms of a generator $\psi_\theta : (0, 1] \to [0, \infty)$ which is convex, decreasing, and such that $\psi_\theta(1) = 0$. A simple formula for $\dot{C}_{\theta_0}$ is given next for such models, under the assumption that $\dot{\psi}_\theta(t) = \partial\psi_\theta(t)/\partial\theta$ exists and is continuous in a neighborhood of θ_0. The result extends readily to the multivariate case.

Proposition 4. *If (C_θ) is a parametric family of Archimedean copulas whose generators ψ_θ are normalized in such a way that $\psi_\theta(t) \to -\log(t)$ and $\psi'_\theta(t) \to -1/t$ as $\theta \to \theta_0$, then*

$$\dot{C}_{\theta_0}(u, v) = uv\left\{\dot{\psi}_{\theta_0}(uv) - \dot{\psi}_{\theta_0}(u) - \dot{\psi}_{\theta_0}(v)\right\}.$$

Proof. The conclusion obtains by letting $\theta \to \theta_0$ in the expression

$$\begin{aligned}\dot{C}_\theta(u, v) &= \frac{\partial}{\partial\theta}\psi_\theta^{-1}(t)\bigg|_{t=\psi_\theta(u)+\psi_\theta(v)} + \left\{\dot{\psi}_\theta(u) + \dot{\psi}_\theta(v)\right\}\frac{\partial}{\partial t}\psi_\theta^{-1}(t)\bigg|_{t=\psi_\theta(u)+\psi_\theta(v)} \\ &= -\frac{\dot{\psi}_\theta\{C_\theta(u, v)\}}{\psi'_\theta\{C_\theta(u, v)\}} + \frac{\dot{\psi}_\theta(u) + \dot{\psi}_\theta(v)}{\psi'_\theta\{C_\theta(u, v)\}},\end{aligned}$$

which results from straightforward applications of the Chain Rule and the Inverse Function Theorem. □

6. Comparisons between tests based on B_n and S_n^J

In addition to characterizing the asymptotic behavior of tests of independence based on B_n or S_n^J, Propositions 2 and 3 help to delineate the circumstances under which these various procedures might perform best.

6.1. Consistency

An advantage of basing a test of independence on B_n is that it is always consistent. Such is not necessarily the case for procedures involving S_n^J. Assume, for instance, that the data arise from the family (C_r) of Student copulas indexed by their "correlation coefficient" r, as is often assumed in financial applications (see [6] and references therein). Note that in this case, C_0 is *not* the independence copula.

Now suppose that J is a score function such that

$$J(u, v) + J(u, 1 - v) + J(1 - u, v) + J(1 - u, 1 - v) = 0$$

for all $u, v \in (0, 1)$. Under the latter condition, which is met for several of the classical score functions listed in Table 1, one finds $\bar{J}_n = 0$ and

$$\int_{(0,1)^2} J(u, v)\, dC_0(u, v) = 0 \tag{3}$$

whenever this integral exists.

The main result in Chapter 5 of [14] coupled with Quesada-Molina's identity, implies that

$$n^{1/2} S_n^J \rightsquigarrow \tilde{\mathbb{S}}^J = \int_{(0,1)^2} \tilde{\mathbb{C}}(u, v)\, dJ(u, v),$$

where

$$\tilde{C}(u, v) = \tilde{\mathbb{A}}(u, v) - u\tilde{\mathbb{A}}(1, v) - v\tilde{\mathbb{A}}(u, 1)$$

and $\tilde{\mathbb{A}}$ is the limiting distribution of the process $n^{1/2}\{C_n(u, v) - C_0(u, v)\}$. In view of (3), $\mathbb{S}^J$ is Gaussian with zero mean, so that the test based on this particular S_n^J would be inconsistent, while B_n/n would still converge in probability to

$$\int_{(0,1)^2} \{C_0(u, v) - uv\}^2 dv\, du > 0.$$

Note, incidentally, that the same inconsistent behavior of S_n^J would hold true for any non-Gaussian, meta-elliptical copula with $r = 0$. See Abdous et al. [1] for related properties of this large class of copulas.

6.2. *Asymptotic local power*

Additional comparisons between procedures based on B_n and S_n^J can be made through the notion of asymptotic local power function for tests of size α based on these statistics. Letting $z_{\alpha/2} = N^{-1}(1 - \alpha/2)$ represent the quantile of order $1 - \alpha/2$ of a standard normal random variable Z, and assuming the conditions of Proposition 2, one can see that the asymptotic local power of the test based on S_n^J along the sequence (Q_n) of contiguous alternatives is given by

$$\beta_{S^J}(\delta, \alpha) = \lim_{n\to\infty} Q_n\left(\left|n^{1/2} S_n^J\right| > \sigma_J z_{\alpha/2}\right) = \mathrm{P}\left(\left|Z + \delta\mu_J/\sigma_J\right| > z_{\alpha/2}\right).$$

Note that since the mapping $a \mapsto \mathrm{P}(-z_{\alpha/2} - a \leqslant Z \leqslant z_{\alpha/2} - a)$ is decreasing in a on $[0, \infty)$, a rank test of size α based on score function J will be preferable to another rank test of the same size based on score function K whenever $|\mu_J/\sigma_J| > |\mu_K/\sigma_K|$. Moreover,

$$\mathrm{ARE}\left(S^J, S^K\right) = \left(\frac{\mu_J/\sigma_J}{\mu_K/\sigma_K}\right)^2,$$

known as Pitman's asymptotic relative efficiency, may be interpreted as the ratio of sample sizes required for the two test statistics to maintain the same level and power along the contiguous sequence (C_{θ_n}) of copula alternatives. Obviously, the index $\mathrm{ARE}(S^J, S^K)$ is the same for any two families (C_θ) and (D_λ) in the same class, i.e., whenever $\dot{C}_{\theta_0} \propto \dot{D}_{\lambda_0}$.

Listed in Table 1 are the score functions J of some linear rank statistics S_n^J that satisfy the conditions of Proposition 2. Except for two, they are all products of quantile functions, and hence the remark at the end of Section 3 applies to them. The exceptions are the symmetrized versions both of the Wilcoxon and of the Blest [5] statistic, obtained by taking $J^*(u, v) = J(u, v) + J(v, u)$. (See [15] for additional details.)

Table 1
Score function of some linear rank statistics S_n^J whose expectation vanishes under the null hypothesis of independence

Test statistic	$J(u, v)$
Blest	$\left\{1-3(1-u)^2\right\}(2v-1)$
Symmetrized Blest	$(3-u-v)\{3(2u-1)(2v-1)-1\}+2$
Exponential	$\{1+\log(1-u)\}\{1+\log(1-v)\}$
Laplace	$\xi(u)\xi(v)$
Savage	$(1+\log u)(1+\log v)$
Spearman	$(2u-1)(2v-1)$
van der Waerden	$N^{-1}(u)N^{-1}(v)$
Wilcoxon	$(2u-1)\log\left(\dfrac{v}{1-v}\right)$
Symmetrized Wilcoxon	$(2u-1)\log\left(\dfrac{v}{1-v}\right)+(2v-1)\log\left(\dfrac{u}{1-u}\right)$

with $\xi(u) = 0.5\ \mathrm{sign}(1/2-u)\log\{2\min(u, 1-u)\}$.

Table 2 gives the value of $\mathrm{ARE}(S^J, S^{J_{\mathrm{opt}}})$ for the various choices of J listed in Table 1 and $J_{\mathrm{opt}} \propto \dot{c}_{\theta_0}$ for the three families of copulas considered in Section 5. As shown by Genest and Verret [17], this choice of J_{opt} is equivalent in the limit to the locally most powerful rank test statistic for the family of alternatives under consideration. The calculation of the ARE for the symmetrized statistics is facilitated by the fact that when $J(u, v) = K_1^{-1}(u)K_2^{-1}(v)$ is a product of quantile functions with mean zero and finite variance, Proposition 2 implies that

$$\mathrm{ARE}\left(S^{J^*}, S^J\right) = \frac{2}{1+\rho^2} \geqslant 1,$$

where $\rho = \mathrm{corr}\{K_1^{-1}(U), K_2^{-1}(U)\}$.

As clearly illustrated in Table 2, the performance of a linear rank statistic can vary substantially when it is compared to the locally most powerful rank test of independence within a given class. It is as low as 41.59% for the exponential rank statistic when alternatives belong to the Clayton family, for example, but it reaches 99.07% for the symmetrized version of Wilcoxon's rank statistic in the normal copula model.

In a sense, however, the AREs reported in Table 2 are deceptively low. For, it should be borne in mind that while no linear rank test can ever be more efficient that the locally most powerful procedure, identification of the latter is contingent on the exact knowledge of the direction in which departures from independence occur.

To make comparisons with the Cramér–von Mises statistic B_n, one must resort to the following formula of Gil-Pelaez [18], which states that if X is a random variable with continuous distribution function F and characteristic function $\hat{f}$, then

$$1-F(x) = \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Im}\left\{t^{-1}e^{-ixt}\hat{f}(t)\right\} dt,$$

where $\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.

Table 2
Pitman asymptotic relative efficiency ARE(S^J, $S^{J_{\rm opt}}$) of test statistic S_n^J versus the locally most powerful rank test $S_n^{J_{\rm opt}}$ of independence for local alternatives in the three classes of copulas considered in Section 5

Test statistic	Copula families		
	Class 1	Class 2	Class 3
Blest	$\frac{15}{16} = 0.9375$	$\frac{125}{192} \approx 0.6510$	$\frac{135}{16\pi^2} \approx 0.8549$
Symmetrized Blest	$\frac{30}{31} \approx 0.9677$	$\frac{125}{186} \approx 0.6720$	$\frac{270}{31\pi^2} \approx 0.8825$
Exponential	$\frac{9}{16} = 0.5625$	$\frac{(\pi^2-6)^2}{36} \approx 0.4159$	0.6655
Laplace	$\frac{729}{1024} \approx 0.7119$	0.6615	0.9274
Savage	$\frac{9}{16} = 0.5625$	1.0000	0.6655
Spearman	1.000	$\frac{9}{16} = 0.5625$	$\frac{9}{\pi^2} \approx 0.9119$
van der Waerden	$\frac{9}{\pi^2} \approx 0.9119$	0.6655	1.0000
Wilcoxon	$\frac{9}{\pi^2} \approx 0.9119$	$\frac{\pi^2}{16} \approx 0.6169$	0.9471
Symmetrized Wilcoxon	$\frac{18}{9+\pi^2} \approx 0.9539$	$\frac{\pi^4}{8(\pi^2+9)} \approx 0.6453$	0.9907

To use this identity in the present context, proceed as in Imhof [19] and write

$$\hat{\eta}(t,\mu) = \frac{1}{(1-2it)^{1/2}}\, e^{\left(\frac{i\mu t}{1-2it}\right)} = \frac{1}{(1+4t^2)^{1/4}}\, e^{-\frac{2t\mu^2}{1+4t^2}}\, e^{it\frac{\mu}{1+4t^2}}\, e^{i\arctan(2t)/2}.$$

Then call on Proposition 3 to see that

$$\hat{f}(t,\delta) = \mathrm{E}\left(e^{it\mathbb{B}}\right) = \prod_{k,\,\ell\,\in\,\mathbb{N}} \hat{\eta}\left(\lambda_{k\ell}\,t,\, \delta^2 I_{k\ell}^2\right) = \xi(t)\, e^{-2\delta^2 t^2 \kappa_1(t)}\, e^{i\kappa_2(t) + i\delta^2\kappa_3(t)},$$

where

$$\xi(t) = \prod_{k,\,\ell\,\in\,\mathbb{N}} (1+4t^2\lambda_{k\ell}^2)^{-1/4}, \quad \kappa_1(t) = \sum_{k,\,\ell\,\in\,\mathbb{N}} \lambda_{k\ell}^2 I_{k\ell}^2/(1+4t^2\lambda_{k\ell}^2),$$
$$\kappa_2(t) = \frac{1}{2}\sum_{k,\,\ell\,\in\,\mathbb{N}} \arctan(2t\lambda_{k\ell}), \quad \kappa_3(t) = t\sum_{k,\,\ell\,\in\,\mathbb{N}} \lambda_{k\ell} I_{k\ell}^2/(1+4t^2\lambda_{k\ell}^2).$$

Note that $\xi(t)$ and $t^2\xi(t)$ are integrable, that κ_1 is bounded, that $\kappa_i(t)/t$ is bounded for $i = 2, 3$, and that $\kappa_2(t)/t \to 1/36$ and $\kappa_3(t)/t \to I^2$, as $t \to 0$.

In the light of the Gil-Pelaez formula, one may deduce that

$$\mathrm{P}(\mathbb{B} > x) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin\{\kappa(x,t)\}}{t\zeta(t)}\, dt, \tag{4}$$

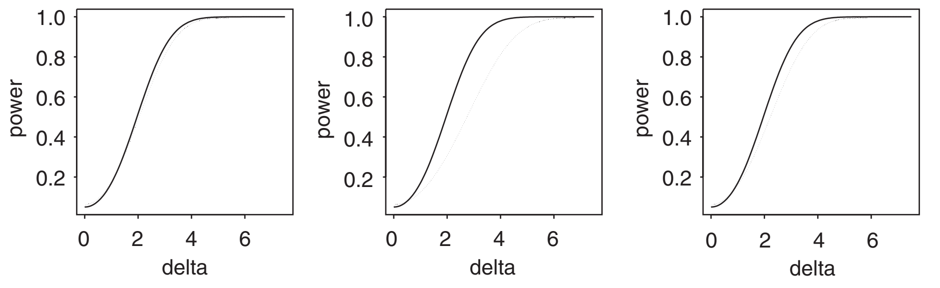

Fig. 1. Comparative power of two rank statistics used to test independence for alternatives from three different classes of copulas: broken line, Cramér–von Mises statistic; solid line, locally most powerful procedure.

where

$$\kappa(x,t) = -\frac{xt}{2} + \frac{1}{2}\sum_{k,\ell\in\mathbb{N}}\left\{\arctan(\lambda_{k\ell}\, t) + \delta^2\,\frac{\lambda_{k\ell} I_{k\ell}^2 t}{1+\lambda_{k\ell}^2\, t^2}\right\}$$

and

$$\zeta(t) = \exp\left(\frac{\delta^2 t^2}{2}\sum_{k,\ell\in\mathbb{N}}\frac{\lambda_{k\ell}^2 I_{k\ell}^2}{1+\lambda_{k\ell}^2\, t^2}\right)\prod_{k,\ell\in\mathbb{N}}\left(1+\lambda_{k\ell}^2 t^2\right)^{1/4}.$$

Accordingly, numerical approximation routines can be used to compute the local power function $\beta_B(\delta,\alpha) = \mathrm{P}(\mathbb{B} > p_\alpha)$ of B_n. The critical values $p_\alpha = 0.0469$, 0.0592 and 0.0869 correspond to the traditional levels $\alpha = 0.1$, 0.05, and 0.01, respectively.

Fig. 1 compares graphically the power of the 5%-level rank tests of independence based on the Cramér–von Mises statistic (broken line) and the locally most powerful procedure (solid line) for the three classes of parametric copula alternatives considered in Section 5. Panels 1–3 (from left to right) correspond to Classes 1–3, for which the optimal rank tests are based on the Spearman, Savage and van der Waerden statistics, respectively.

The plotted curves are based on a numerical approximation of (4) obtained by integrating on [0, 100] and restricting the sum and integral to integers $k, \ell \leqslant 10$, which guaranteed numerical stability within computer accuracy. As the picture highlights, the power of the test based on B_n is generally close to that of the asymptotically optimal rank statistic S_n^J with $J = \dot{c}_{\theta_0}$. The statistic B_n does best for Class 1 alternatives in the neighborhood of independence; its performance is least impressive for moderate values of δ in Class 2, i.e., dependence models of the Clayton or Gumbel-Barnett variety.

7. Asymptotic relative efficiency calculations

For score functions J and K, the asymptotic relative efficiency

$$\mathrm{ARE}\left(S^J, S^K\right) = \left(\frac{\mu_J/\sigma_J}{\mu_K/\sigma_K}\right)^2$$

is a natural measure of local power comparison because the asymptotic behavior of the related statistics S_n^J and S_n^K is Gaussian, under the assumptions of Proposition 2. However, a more general definition of asymptotic relative efficiency is needed if comparisons must be extended to statistics such as B_n, whose limiting distribution is not normal. Several options exist; see, e.g., Nyblom and Mäkeläinen [22] and references therein.

The approach pursued here for comparing tests based on statistics T_n and T_n' involves a ratio of the slopes of the power curves in a neighborhood of $\delta = 0$, viz.

$$e\left(T, T'\right) = \lim_{\delta \to 0} \frac{\beta_T(\delta, \alpha) - \alpha}{\beta_{T'}(\delta, \alpha) - \alpha}.$$

This ratio, which is superior to 1 for all T' whenever T is locally most powerful, provides a natural extension of Pitman's efficiency beyond the case of normal statistics. For, suppose that under Q_n, the limiting power function $\beta_T(\delta, \alpha)$ of tests of size α based on T_n is given by

$$\beta_T(\delta, \alpha) = 1 - N(z_{\alpha/2} - \delta\mu_T/\sigma_T) + N(-z_{\alpha/2} - \delta\mu_T/\sigma_T),$$

where N is the distribution function of the standard Gaussian and N' is the corresponding density. Then

$$\lim_{\delta \to 0} \delta^{-2} \left\{\beta_T(\delta, \alpha) - \alpha\right\} = z_{\alpha/2} N'(z_{\alpha/2}) \left(\mu_T/\sigma_T\right)^2$$

and hence

$$e\left(T, T'\right) = \lim_{\delta \to 0} \frac{\beta_T(\delta, \alpha) - \alpha}{\beta_{T'}(\delta, \alpha) - \alpha} = \mathrm{ARE}\left(T, T'\right).$$

The following proposition characterizes the local behavior of $\beta_B(\delta, \alpha) - \alpha$ at $\delta = 0$ for the Cramér–von Mises statistic B_n. The proof of this result, which uses the Gil-Pelaez representation, is given in the Appendix.

Proposition 5. *Under Conditions* (i)–(ii), *one has*

$$\lim_{\delta \to 0} \delta^{-2}\{\beta_B(\delta, \alpha) - \alpha\} = \sum_{k,\, \ell \in \mathbb{N}} \lambda_{k\ell} I_{k\ell}^2 \, h_{k\ell}(p_\alpha),$$

where $h_{k\ell}$ is a density whose associated characteristic function

$$\frac{\hat{f}(t, 0)}{1 - 2i\lambda_{k\ell}\, t} = (1 - 2i\lambda_{k\ell}\, t)^{-1} \prod_{q,\, r \in \mathbb{N}} \left(1 - 2i\lambda_{qr}\, t\right)^{-1/2}$$

is that of $\mathbb{B}_0 + \lambda_{k\ell}\, \chi_2^2$, in which the summands are taken to be independent.

Finally, note that

$$h_{k\ell}(x) = \frac{1}{\pi} \int_0^\infty (1 + 4\lambda_{k\ell}^2\, t^2)^{-1/2} \xi(t) \cos\left\{\kappa_2(t) + \arctan(\lambda_{k\ell}) - tx\right\} dt$$

Table 3
Local asymptotic relative efficiency of B_n with respect to the locally most powerful statistic $S^{J_{\text{opt}}}$ for three classes of copulas

Level	Copula families		
	Class 1	Class 2	Class 3
1%	0.8337	0.4229	0.6961
5%	0.8122	0.4181	0.6791
10%	0.8380	0.4386	0.7019

from which it is possible to conclude that

$$e\left(B,\ S^J\right) = \lim_{\delta \to 0} \delta^{-2} \left\{\beta_B(\delta, \alpha) - \alpha\right\} \Big/ \lim_{\delta \to 0} \delta^{-2} \left\{\beta_{S^J}(\delta, \alpha) - \alpha\right\}$$
$$= \frac{1}{z_{\alpha/2} N'(z_{\alpha/2}) \, \left(\mu_J / \sigma_J\right)^2} \sum_{k,\, \ell \in \mathbb{N}} \lambda_{k\ell} I_{k\ell}^2 \, h_{k\ell}(p_\alpha)\,,$$

whenever the score function also satisfies Condition (iii).

The local asymptotic relative efficiencies $e\left(B, S^{J_{\text{opt}}}\right)$ of B_n with respect to the locally most powerful statistic $S_n^{J_{\text{opt}}}$ for copulas from Classes 1–3 are presented in Table 3. These numerical approximations were obtained by integrating on [0, 1500] by the trapezoidal rule (with a mesh of 1/2500) and restricting the sum and product to terms with integers $k, \ell \leqslant 10$ in the Gil-Pelaez formula. Comparisons involving any other linear rank statistic S_n^J in Table 1 may be made easily since

$$e\left(B, S^J\right) = \frac{e\left(B,\ S^{J_{\text{opt}}}\right)}{e\left(S^J, S^{J_{\text{opt}}}\right)}\,.$$

In conformance with Fig. 1, B_n is seen to do quite well against the locally most powerful nonparametric test of independence for Class 1 alternatives. Its performance is somewhat worse for Class 3 Gaussian alternatives, and more questionable for Class 2 alternatives, namely the Clayton and Gumbel-Barnett copulas. A rationale for this phenomenon is still lacking.

8. Conclusion

Because they allow analysts to model dependence separately from the margins, copulas provide a handy (and increasingly popular) way of constructing alternatives to independence in multivariate contexts. This paper identifies conditions under which a family of copulas gives rise to a contiguous sequence of alternatives. The asymptotic behavior of the empirical copula process is characterized under alternatives of this sort. This leads to a computable expression for the limiting local power of a bivariate Cramér–von Mises statistic originally suggested by Deheuvels, and to meaningful asymptotic relative efficiency comparisons with various linear rank tests of independence.

In addition to being easy to implement, Deheuvels' test based on B_n is always consistent. The numerical comparisons reported in Fig. 1 and Table 3 also show that as an *omnibus* procedure, it generally holds up its power reasonably well against the *model-specific* locally most powerful rank-based test. Considering that the latter test may not be consistent if the alternatives have not been specified correctly, the test based on B_n certainly represents a viable solution, if not an ideal one. Its mitigated success in reproducing the optimal power is obviously a function of the type of departure from independence embodied in the family of local alternatives. Just what aspect of association is at stakes seems hard to pin down, however.

9. Acknowledgments

Funding in partial support of this work was provided by the Natural Sciences and Engineering Research Council of Canada, the Fonds québécois de la recherche sur la nature et les technologies, and the Institut de finance mathématique de Montréal.

Appendix

A.1. Proof of Proposition 2

First, one needs to show that the expression for σ_J^2 is correct. To this end, set

$$A = \int_{(0,1)^4} \gamma(u, u')\gamma(v, v')dJ(u, v)dJ(u', v') = A_1 + A_2 + A_3 + A_4,$$

where

$$\begin{aligned}
A_1 &= \int_{0<u\leqslant u'<1,\ 0<v\leqslant v'<1} \gamma(u, u')\gamma(v, v')dJ(u, v)dJ(u', v'),\\
A_2 &= \int_{0<u\leqslant u'<1,\ 0<v'<v<1} \gamma(u, u')\gamma(v, v')dJ(u, v)dJ(u', v'),\\
A_3 &= \int_{0<u'<u<1,\ 0<v\leqslant v'<1} \gamma(u, u')\gamma(v, v')dJ(u, v)dJ(u', v'),\\
A_4 &= \int_{0<u'<u<1,\ 0<v'<v<1} \gamma(u, u')\gamma(v, v')dJ(u, v)dJ(u', v').
\end{aligned}$$

Using Tonelli's Theorem, one may write

$$\begin{aligned}
A_1 &= \int_{0<u\leqslant u'<1,\ 0<v\leqslant v'<1} u(1-u')v(1-v')dJ(u, v)dJ(u', v')\\
&= \int_{(0,1)^4}\int_{0<u\leqslant u'<1,\ 0<v\leqslant v'<1} \mathbf{1}(x<u)\mathbf{1}(u'\leqslant y)\mathbf{1}(z<v)\mathbf{1}(v'\leqslant w)\\
&\quad \times dJ(u, v)dJ(u', v')dx\,dy\,dz\,dw\\
&= \int_{(0,1)^8} \mathbf{1}(x<u\leqslant u')\mathbf{1}(x<u'<y)\mathbf{1}(z<v\leqslant v')\mathbf{1}(z<v'\leqslant w)\\
&\quad \times \mathbf{1}(x<y)\mathbf{1}(z<w)\mathrm{d}J(u, v)\mathrm{d}J(u', v')dx\,dy\,dz\,dw
\end{aligned}$$

$$= \int_{(0,1)^8} \mathbf{1}(x < u \leqslant u')\mathbf{1}(x < u' \leqslant y)\mathbf{1}(z < v \leqslant v')\mathbf{1}(z < v' \leqslant w) \times \mathbf{1}(x < y)\mathbf{1}(z < w)\mathrm{d}J(u,v)\mathrm{d}J(u',v')dx\,dy\,dz\,dw,$$

where the last equality follows from the absolute continuity of Lebesgue's measure.

Similarly,

$$\begin{aligned}
A_2 &= \int_{0<u\leqslant u'<1,\ 0<v'<v<1} u(1-u')v'(1-v)dJ(u,v)dJ(u',v') \\
&= \int_{(0,1)^4}\int_{0<u\leqslant u'<1,\ 0<v'<v<1} \mathbf{1}(x<u)\mathbf{1}(u'\leqslant y)\mathbf{1}(z<v')\mathbf{1}(v\leqslant w) \\
&\quad \times dJ(u,v)dJ(u',v')dx\,dy\,dz\,dw \\
&= \int_{(0,1)^8} \mathbf{1}(x<u\leqslant u')\mathbf{1}(x<u'\leqslant y)\mathbf{1}(v'<v\leqslant w)\mathbf{1}(z<v'<w) \\
&\quad \times \mathbf{1}(x<y)\mathbf{1}(z<w)dJ(u,v)dJ(u',v')dx\,dy\,dz\,dw \\
&= \int_{(0,1)^8} \mathbf{1}(x<u\leqslant u')\mathbf{1}(x<u'\leqslant y)\mathbf{1}(v'<v\leqslant w)\mathbf{1}(z<v'\leqslant w) \\
&\quad \times \mathbf{1}(x<y)\mathbf{1}(z<w)dJ(u,v)dJ(u',v')dx\,dy\,dz\,dw.
\end{aligned}$$

Hence

$$A_1 + A_2 = \int_{(0,1)^8} \mathbf{1}(x<u\leqslant u')\mathbf{1}(x<u'\leqslant y)\mathbf{1}(z<v\leqslant w)\mathbf{1}(z<v'\leqslant w) \times \mathbf{1}(x<y)\mathbf{1}(z<w)dJ(u,v)dJ(u',v')dx\,dy\,dz\,dw.$$

Using the same technique, one also gets

$$A_3 + A_4 = \int_{(0,1)^8} \mathbf{1}(u'<u\leqslant y)\mathbf{1}(x<u'\leqslant y)\mathbf{1}(z<v\leqslant w)\mathbf{1}(z<v'\leqslant w) \times \mathbf{1}(x<y)\mathbf{1}(z<w)dJ(u,v)dJ(u',v')dx\,dy\,dz\,dw$$

from which one may conclude that

$$\begin{aligned}
A &= \int_{(0,1)^8} \mathbf{1}(x<u\leqslant y)\mathbf{1}(x<u'\leqslant y)\mathbf{1}(z<v\leqslant w)\mathbf{1}(z<v'\leqslant w) \\
&\quad \times \mathbf{1}(x<y)\mathbf{1}(z<w)dJ(u,v)dJ(u',v')dx\,dy\,dz\,dw \\
&= \int_{(0,1)^4} \{J(x,z)+J(y,w)-J(y,z)-J(x,w)\}^2\,\mathbf{1}(x<y)\mathbf{1}(z<w)dx\,dy\,dz\,dw \\
&= \frac{1}{4}\int_{(0,1)^4} \{J(x,z)+J(y,w)-J(y,z)-J(x,w)\}^2\,dx\,dy\,dz\,dw \\
&= \frac{1}{4}\int_{(0,1)^4} \left\{\tilde{J}(x,z)+\tilde{J}(y,w)-\tilde{J}(y,z)-\tilde{J}(x,w)\right\}^2 dx\,dy\,dz\,dw \\
&= \int_{(0,1)^2} \left\{\tilde{J}(u,v)\right\}^2 du\,dv = \sigma_J^2.
\end{aligned}$$

Next, observe that under P_n, it follows by construction that

$$\mathrm{E}\left\{\tilde{\mathbb{C}}_n(u,v)\right\} = 0$$

for any $(u, v) \in [0, 1]^2$. Furthermore, for any $(u, v, u', v') \in [0, 1]^4$ and $n \geqslant 2$, one has

$$\mathrm{E}\left\{\tilde{\mathbb{C}}_n(u,v)\tilde{\mathbb{C}}_n(u',v')\right\} = \frac{n}{n-1}\,\gamma_n(u,u')\gamma_n(v,v') \leqslant \frac{9}{2}\,\gamma(u,u')\gamma(v,v'),$$

where

$$\gamma_n(u,v) = \gamma\left\{C_n(u,1), C_n(v,1)\right\} \leqslant \frac{n+1}{n}\,\gamma(u,v)$$

for arbitrary $(u, v) \in [0, 1]^2$.

For any $A \subset (0, 1)^2$, let

$$R_{A,n} = \int_A \tilde{\mathbb{C}}_n(u,v)\, dJ(u,v)$$

and define

$$\sigma^2_{A,J} = \int_{A\times A} \gamma(u,u')\gamma(v,v')dJ(u,v)dJ(u',v').$$

For arbitrary $n \geqslant 2$, one can then see that under P_n,

$$\begin{aligned}\operatorname{var}\left(R_{A,n}\right) &= \frac{n}{n-1}\int_{A\times A}\gamma_n(u,u')\gamma_n(v,v')\,dJ(u,v)\,dJ(u',v')\\ &\leqslant \frac{9}{2}\int_{A\times A}\gamma(u,u')\gamma(v,v')\,dJ(u,v)\,dJ(u',v') = \frac{9}{2}\,\sigma^2_{A,J}.\end{aligned}$$

It follows from the Dominated Convergence Theorem that for any $A \subset (0, 1)^2$,

$$\lim_{n\to\infty}\operatorname{var}\left(R_{A,n}\right) = \int_{A\times A}\gamma(u,u')\gamma(v,v')\,dJ(u,v)\,dJ(u',v') = \sigma^2_{A,J} \leqslant \sigma^2_J.$$

In particular, for any $m \geqslant 1$, one can find a closed interval $K_m \subset (0, 1)^2$ so that $K_m \uparrow (0,1)^2$, $\sigma^2_{K_m^c,\,J} < 1/m$ and $\sigma^2_{K_m,\,J} + 1/m > \sigma^2_J$. Hence, for any $\lambda > 0$ and any $n \geqslant 2$,

$$P_n\left(|R_{K_m^c,\,n}| > \lambda\right) \leqslant \frac{9}{2m\lambda^2}\,.$$

Since m can be chosen arbitrarily large, it follows from the contiguity of Q_n with respect to P_n that for fixed $\lambda > 0$, $\limsup_{n\to\infty} Q_n\left(|R_{K_m^c,\,n}| > \lambda\right)$ may be made arbitrarily small.

Finally, $S_n^J = R_{K_m,\,n} + R_{K_m^c,\,n}$. Moreover, under Q_n, one has

$$R_{K_m,\,n} \rightsquigarrow \int_{K_m}\mathbb{C}(u,v)\mathrm{d}J(u,v) + \delta\int_{K_m}\dot{C}_{\theta_0}(u,v)\mathrm{d}J(u,v),$$

which is Gaussian, with mean $\delta\mu_{K_m,J}$ and variance $\sigma^2_{K_m,\ J}$. In the light of Condition (iii), it follows that both $\mu_{K_m,J} \to \mu_J$ and $\sigma^2_{K_m,\ J} \to \sigma^2{}_J$, as $m \to \infty$. This completes the proof of Proposition 2.

Remark. Under additional assumptions, e.g., if

$$\tilde{J}(u,t)\partial\dot{C}(u,t)\Big/\partial u \quad \text{and} \quad \tilde{J}(t,v)\partial\dot{C}(t,v)\Big/\partial v$$

both converge boundedly to 0 as $t \to 1$, then one may conclude that

$$\mu_J = \int_0^1 \int_0^1 \tilde{J}(u,v)\dot{c}(u,v)\,du\,dv,$$

as obtained by Genest and Verret [17], under different assumptions on *J*.

A.2. Proof of Proposition 5

For simplicity, set $x = p_\alpha$. It follows from the Gil-Pelaez representation that

$$\beta_B(\delta,\alpha) - \alpha = \frac{1}{2\pi} \int_{-\infty}^{+\infty} t^{-1}\mathrm{Im}\left\{e^{-itx}\hat{f}(t,\delta) - e^{-itx}\hat{f}(t,0)\right\}\,dt.$$

From the definition of $\hat{f}$, one has

$$t^{-1}\mathrm{Im}\left\{e^{-itx}\hat{f}(t,\delta)\right\} = t^{-1}\xi(t)\,e^{-2\delta^2t^2\kappa_1(t)}\,\sin\left\{\kappa_2(t)+\delta^2\kappa_3(t)-tx\right\}$$

and it follows that

$$\left(\delta^2 t\right)^{-1}\mathrm{Im}\left\{e^{-itx}\hat{f}(t,\delta) - e^{-itx}\hat{f}(t,0)\right\}$$

can be decomposed as the sum of $A_1(t,\delta)t^2\xi(t) + A_2(t,\delta)\xi(t)$, where

$$A_1(t,\delta) = \left(\delta^2 t^3\right)^{-1}\left\{e^{-2\delta^2t^2\kappa_1(t)} - 1\right\}\,\sin\left\{\kappa_2(t)+\delta^2\kappa_3(t)-tx\right\}$$

and

$$A_2(t,\delta) = \left(\delta^2 t\right)^{-1}\left[\sin\left\{\kappa_2(t)+\delta^2\kappa_3(t)-tx\right\} - \sin\left\{\kappa_2(t)-tx\right\}\right].$$

Now, both terms are bounded and converge, respectively, as $\delta \to 0$, to

$$A_1(t,0) = -2t^{-1}\kappa_1(t)\,\sin\left\{\kappa_2(t)-tx\right\}$$

and

$$A_2(t,0) = t^{-1}\kappa_3(t)\,\cos\left\{\kappa_2(t)-tx\right\}.$$

An application of Lebesgue's Dominated Convergence Theorem thus yields

$$\lim_{\delta\to 0} \delta^{-2} \int_{-\infty}^{\infty} t^{-1}\mathrm{Im}\left\{e^{-itx}\hat{f}(t,\delta) - e^{-itx}\hat{f}(t,0)\right\} dt = \int_{-\infty}^{\infty} \psi(t,x)\, dt,$$

where

$$\psi(t,x) = \xi(t)\left[t^{-1}\kappa_3(t)\cos\{\kappa_2(t) - tx\} - 2t\kappa_1(t)\sin\{\kappa_2(t) - tx\}\right].$$

It is easy to check that ψ can also be expressed as

$$\psi(x,t) = \sum_{k,\ell\in\mathbb{N}} \lambda_{k\ell} I_{k\ell}^2 \,\mathrm{Re}\left\{e^{-itx}\hat{f}(t,0)(1-2it\lambda_{k\ell})^{-1}\right\}.$$

Since ξ is integrable, it follows that the characteristic function $\hat{f}(t,0)(1-2it\lambda_{k\ell})^{-1}$ is integrable, and hence

$$(2\pi)^{-1}\int_{-\infty}^{\infty} \psi(t,x)\, dt = \pi^{-1}\int_{0}^{\infty} \psi(t,x)\, dt = \sum_{k,\ell\in\mathbb{N}} \lambda_{k\ell} I_{k\ell}^2 h_{k\ell}(x),$$

where $h_{k\ell}$ is the density of $\mathbb{B}_0 + \lambda_{k\ell}\,\chi_2^2$, whose summands are taken to be independent. This completes the proof.

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