

Enhanced one-sided confidence regions for a multivariate location parameter

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Abstract

If a one-sided test for a multivariate location parameter is inverted, the resulting confidence region may have an unpleasant shape. In particular, if the null and alternative hypothesis are both composite and complementary, the confidence region usually does not resemble the alternative parameter region in shape, but rather a reflected version of the null parameter region.

We illustrate this effect and show one possibility of obtaining confidence regions for the location parameter that are smaller and have a more suitable shape for the type of problems investigated. This method is based on the closed testing principle applied to a family of nested hypotheses.

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1. Introduction

Statistical inference can be done by means of either hypothesis tests or confidence regions. Most existing results on confidence regions for multivariate location parameters correspond to tests for unrestricted alternatives, which especially means that these confidence regions usually have a finite diameter. When one-sided problems are investigated, it would be more attractive to have a multivariate analog e.g. of a univariate lower confidence bound. Such a one-sided

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confidence region should be as strict as possible in a specific part of its border, but can include entire rays into directions that need not be excluded.

We assume that we are concerned with a single sample from a (at least directionally) symmetric distribution. The symmetry center serves us as the location parameter.

In this article, we first give a definition of a confidence region for a meta-parameter and establish the connection with hypothesis tests (Section 2). We then show how confidence regions for the location parameter of interest can be derived directly from confidence regions for the meta-parameter (Section 3). We discuss the shape of these confidence regions both theoretically and with an example (Section 4). In order to obtain sharper confidence regions for the location parameter (under suitable conditions) with a more appropriate shape, we propose a procedure based on the closed testing principle, and we illustrate this procedure with the same data again (Section 5).

2. Confidence region for a meta-parameter

Since the connection between hypothesis tests and confidence regions is not as intuitive as in the univariate setting, it seems worthwhile to establish an accurate notational basis for confidence regions in a multivariate parameter space $\Theta \subset \mathbb{R}^p$. We use the following definition of a confidence region:

Definition 1. Let $(\mathcal{X}, \mathcal{A}, (P_{\boldsymbol{\vartheta}})_{\boldsymbol{\vartheta} \in \Theta})$ be a probability space, i.e. \mathcal{X} is the space of observations, \mathcal{A} is a σ -algebra on \mathcal{X} , and $(P_{\boldsymbol{\vartheta}})_{\boldsymbol{\vartheta} \in \Theta}$ is a family of probability measures. Let $\alpha \in (0, 1)$, and, for each $\boldsymbol{\gamma} \in \Theta$, let $\Theta_0(\boldsymbol{\gamma}) \subset \Theta$ be specified.

On the basis of the family $(\Theta_0(\boldsymbol{\gamma}))_{\boldsymbol{\gamma} \in \Theta}$, a *confidence region* for $\boldsymbol{\gamma}$ with confidence level $1 - \alpha$ is a map $\mathcal{C}_{1-\alpha} : \mathcal{X} \rightarrow \mathcal{P}(\Theta)$ (where \mathcal{P} is the power set) such that

$$A(\boldsymbol{\gamma}) := \{x \in \mathcal{X} : \mathcal{C}_{1-\alpha}(x) \ni \boldsymbol{\gamma}\} \in \mathcal{A} \quad \forall \boldsymbol{\gamma} \in \Theta$$

and

$$P_{\boldsymbol{\vartheta}}(\mathcal{C}_{1-\alpha}(X) \ni \boldsymbol{\gamma}) \geq 1 - \alpha \quad \forall \boldsymbol{\vartheta} \in \Theta_0(\boldsymbol{\gamma}) \quad \forall \boldsymbol{\gamma} \in \Theta.$$

The first condition in Definition 1 only ensures measurability. It is easy to prove that such a confidence region corresponds to a non-randomized level α test of $H_0 : \boldsymbol{\vartheta} \in \Theta_0(\boldsymbol{\gamma})$ that accepts H_0 if and only if $X \in A(\boldsymbol{\gamma})$.

A confidence region in the sense of Definition 1 yields therefore a statement about the meta-parameter $\boldsymbol{\gamma} \in \Theta$ that specifies the null parameter region $\Theta_0(\boldsymbol{\gamma})$ of the corresponding test, and not a direct statement about the distribution parameter $\boldsymbol{\vartheta}$ itself. (We assume here for simplicity that the meta-parameter $\boldsymbol{\gamma}$ lies in Θ ; one could also use a different meta-parameter space.)

3. Direct derivation of a confidence region for the location parameter

We assume now that $\boldsymbol{\gamma} \in \Theta_0(\boldsymbol{\gamma})$, $\forall \boldsymbol{\gamma} \in \Theta$ (imagine e.g. the closed region $\Theta_0(\boldsymbol{\gamma}) = \boldsymbol{\gamma} + (-\infty, 0]^p$). It follows immediately that a confidence region for $\boldsymbol{\gamma}$ in the sense of Definition 1 is also a confidence region for the location parameter $\boldsymbol{\vartheta}$ itself in the sense that

$$P_{\boldsymbol{\vartheta}}(\mathcal{C}_{1-\alpha}(X) \ni \boldsymbol{\vartheta}) \geq 1 - \alpha \quad \forall \boldsymbol{\vartheta} \in \Theta.$$

This result implies that for the purpose of deriving a $1 - \alpha$ confidence region for $\boldsymbol{\vartheta}$ directly from a test, we only have to ensure that the test respects the level α at the simple null hypothesis

$H_0 : \boldsymbol{\vartheta} = \boldsymbol{\gamma}$. Within the class of level α tests for this simple null hypothesis, we can choose a test against any alternative such that a confidence region of the desired shape results.

If we just wanted to use this method to obtain confidence regions for $\boldsymbol{\vartheta}$, it would therefore suffice to consider confidence regions for $\boldsymbol{\vartheta}$ itself from the beginning, and we could simplify Definition 1 by using $\Theta_0(\boldsymbol{\gamma}) = \{\boldsymbol{\gamma}\}, \forall \boldsymbol{\gamma} \in \Theta$. However, we will see in Section 5 that we can take advantage of our more general definition of confidence regions for $\boldsymbol{\gamma}$.

4. Shape of the confidence region for the meta-parameter

Due to the correspondence between confidence regions with confidence level $1 - \alpha$ and non-randomized level α tests, we can characterize the shape of the confidence region by using properties of the corresponding test, especially the shape of its acceptance region.

One of the simplest cases is the following one, where the parameter $\boldsymbol{\vartheta}$, the observations, and the test statistic are all of the same dimension, and the decision rule is particularly simple:

Theorem 2. Let $\mathbf{X}_1, \dots, \mathbf{X}_n$ be i.i.d. random vectors in \mathbb{R}^p having a distribution $F_{\boldsymbol{\vartheta}}$, with $\boldsymbol{\vartheta} \in \Theta \subset \mathbb{R}^p$. Let a non-randomized level α test for $H_0 : \boldsymbol{\vartheta} \in \Theta_0(\boldsymbol{\gamma})$ vs. $H_1 : \boldsymbol{\vartheta} \in \Theta \setminus \Theta_0(\boldsymbol{\gamma})$ be given that accepts H_0 if and only if some test statistic $T(\mathbf{X}_1, \dots, \mathbf{X}_n) \in \mathbb{R}^p$ is within the acceptance region $A(\boldsymbol{\gamma})$, for each $\boldsymbol{\gamma} \in \Theta$.

If the acceptance region can be written as

$$A(\boldsymbol{\gamma}) = \boldsymbol{\gamma} + A$$

for some $A \subset \mathbb{R}^p$, then a confidence region for $\boldsymbol{\gamma}$ with confidence level $1 - \alpha$ is given by

$$\mathcal{C}_{1-\alpha}(T(\mathbf{X}_1, \dots, \mathbf{X}_n)) = T(\mathbf{X}_1, \dots, \mathbf{X}_n) - A.$$

Proof. The confidence region corresponding to the given test is

$$\begin{aligned} \mathcal{C}_{1-\alpha}(T(\mathbf{X}_1, \dots, \mathbf{X}_n)) &= \{\boldsymbol{\gamma} : A(\boldsymbol{\gamma}) \ni T(\mathbf{X}_1, \dots, \mathbf{X}_n)\} \\ &= \{\boldsymbol{\gamma} : \boldsymbol{\gamma} + A \ni T(\mathbf{X}_1, \dots, \mathbf{X}_n)\} \\ &= \{\boldsymbol{\gamma} : \boldsymbol{\gamma} \in T(\mathbf{X}_1, \dots, \mathbf{X}_n) - A\} \\ &= T(\mathbf{X}_1, \dots, \mathbf{X}_n) - A. \quad \square \end{aligned}$$

The assumptions in Theorem 2 are rather restrictive. However, we can use it for determining the shape of confidence regions corresponding to tests from two basic classes of multivariate tests. We define these tests first:

Definition 3. For $j = 1, \dots, p$, let a family of (univariate) non-randomized tests be given based on random variables X_{1j}, \dots, X_{nj} with some common distribution $F_{j, \boldsymbol{\vartheta}_j}$, where $\boldsymbol{\vartheta}_j$ is an unknown location parameter. For arbitrary $\alpha \in (0, 1)$, let $\varphi_{j, \alpha}((X_{1j}, \dots, X_{nj}), \gamma_j) = 1$ denote rejection of $H_{0j} : \boldsymbol{\vartheta}_j \leq \gamma_j$ in favor of $H_{1j} : \boldsymbol{\vartheta}_j > \gamma_j$ at the level α , and let $\varphi_{j, \alpha}((X_{1j}, \dots, X_{nj}), \gamma_j) = 0$ denote acceptance of H_{0j} .

Now let $\mathbf{X}_1, \dots, \mathbf{X}_n$ be p -variate random vectors with some common distribution $F_{\boldsymbol{\vartheta}}$, where $\boldsymbol{\vartheta} = (\boldsymbol{\vartheta}_1, \dots, \boldsymbol{\vartheta}_p)^T$ is an unknown location parameter. Let X_{ij} denote the j th component of \mathbf{X}_i . Based on the above univariate tests, the following multivariate tests can be defined:

(a) The *min test* rejects $H_0 : \exists j \in \{1, \dots, p\} : \boldsymbol{\vartheta}_j \leq \gamma_j$ in favor of $H_1 : \boldsymbol{\vartheta} > \boldsymbol{\gamma}$ at the level α if and only if

$$\varphi_{j, \alpha}((X_{1j}, \dots, X_{nj}), \gamma_j) = 1, \quad \forall j \in 1, \dots, p.$$

(b) The Bonferroni max test rejects $H_0 : \boldsymbol{\vartheta} \leq \boldsymbol{\gamma}$ in favor of $H_1 : \exists j \in \{1, \dots, p\} : \vartheta_j > \gamma_j$ at the level α if and only if

$$\exists j \in 1, \dots, p : \varphi_{j, \alpha/p}((X_{1j}, \dots, X_{nj}), \gamma_j) = 1.$$

While the term *min test* is commonly used in the literature (e.g. Sen and Silvapulle [1]), the term *max test* is used here in analogy and does not seem to be that common. Both terms are based on the case that the test statistic for each univariate test is the same and tends to take larger values for large values of ϑ_j . Then the minimum or maximum, respectively, of the componentwise test statistics can be used for the construction of a min or max test. Note that in the case of the max test, the componentwise tests have to be evaluated at a corrected significance level (we use the simple Bonferroni correction) in order to keep the multivariate test at the desired level α , while for the min test, such a correction is not necessary.

We can now formally prove that the confidence region corresponding to a min test consists of all points that are above the (univariate) lower confidence bound with respect to at least one component:

Corollary 4. For $j = 1, \dots, p$, let the $1 - \alpha$ lower confidence bound for γ_j corresponding to a univariate test for $H_{0j} : \vartheta_j \leq \gamma_j$ vs. $H_{1j} : \vartheta_j > \gamma_j$ be given by $\ell_{j, 1-\alpha}(X_{1j}, \dots, X_{nj})$. Let a min test for $H_0 : \exists j \in \{1, \dots, p\} : \vartheta_j \leq \gamma_j$ vs. $H_1 : \boldsymbol{\vartheta} > \boldsymbol{\gamma}$ be based on these univariate tests.

Then the $1 - \alpha$ confidence region for $\boldsymbol{\gamma}$ corresponding to this min test is

$$(\ell_{1, 1-\alpha}(X_{11}, \dots, X_{n1}), \dots, \ell_{p, 1-\alpha}(X_{1p}, \dots, X_{np}))^T + (\mathbb{R}^p \setminus (-\infty, 0)^p).$$

Proof. Define

$$T(\mathbf{X}_1, \dots, \mathbf{X}_n) := (\ell_{1, 1-\alpha}(X_{11}, \dots, X_{n1}), \dots, \ell_{p, 1-\alpha}(X_{1p}, \dots, X_{np}))^T.$$

The min test rejects H_0 if and only if $\gamma_j < \ell_{j, 1-\alpha}(X_{1j}, \dots, X_{nj}), \forall j \in \{1, \dots, p\}$, which is equivalent to $T(\mathbf{X}_1, \dots, \mathbf{X}_n) \in \boldsymbol{\gamma} + (0, \infty)^p$. The acceptance region for H_0 is therefore $A(\boldsymbol{\gamma}) = \mathbb{R}^p \setminus (\boldsymbol{\gamma} + (0, \infty)^p) = \boldsymbol{\gamma} + (\mathbb{R}^p \setminus (0, \infty)^p)$, such that we can apply Theorem 2 with $A = \mathbb{R}^p \setminus (0, \infty)^p$, yielding that the $1 - \alpha$ confidence region for $\boldsymbol{\gamma}$ is

$$T(\mathbf{X}_1, \dots, \mathbf{X}_n) - (\mathbb{R}^p \setminus (0, \infty)^p) = T(\mathbf{X}_1, \dots, \mathbf{X}_n) + (\mathbb{R}^p \setminus (-\infty, 0)^p). \quad \square$$

This result may be astonishing at first glance: While a min test can e.g. be used to show that a bivariate location parameter is in the first quadrant, the corresponding confidence region is a translated version of the first, second, and fourth quadrant, and it therefore always contains parts of the second and fourth quadrant. However, we should remember that our confidence regions are for the meta-parameter $\boldsymbol{\gamma}$ and that excluding $\boldsymbol{\gamma}$ from the confidence region corresponds to rejecting the null hypothesis that $\boldsymbol{\vartheta} \in \Theta_0(\boldsymbol{\gamma}) = \boldsymbol{\gamma} + (\mathbb{R}^2 \setminus (0, \infty)^2)$; it does not seem sensible that any min test could reject this null hypothesis if e.g. γ_1 is above every X_{i1} , even if γ_2 is very low.

A similar result can be derived for max tests:

Corollary 5. Let the $1 - \alpha/p$ lower confidence bounds for γ_j corresponding to univariate tests for $H_{0j} : \vartheta_j \leq \gamma_j$ vs. $H_{1j} : \vartheta_j > \gamma_j$ be given by $\ell_{j, 1-\alpha/p}(X_{1j}, \dots, X_{nj})$. Let a Bonferroni max test for $H_0 : \boldsymbol{\vartheta} \leq \boldsymbol{\gamma}$ vs. $H_1 : \exists j \in \{1, \dots, p\} : \vartheta_j > \gamma_j$ be based on these univariate tests.

Then the $1 - \alpha$ confidence region for $\boldsymbol{\gamma}$ corresponding to this Bonferroni max test is

$$(\ell_{1, 1-\alpha/p}(X_{11}, \dots, X_{n1}), \dots, \ell_{p, 1-\alpha/p}(X_{1p}, \dots, X_{np}))^T + [0, \infty)^p.$$

Proof. Analogous to the proof of Corollary 4 — defining

$$T(\mathbf{X}_1, \dots, \mathbf{X}_n) := (\ell_{1,1-\alpha/p}(X_{11}, \dots, X_{n1}), \dots, \ell_{p,1-\alpha/p}(X_{1p}, \dots, X_{np}))^T,$$

the acceptance region for H_0 is $A(\boldsymbol{\gamma}) = \boldsymbol{\gamma} + (-\infty, 0]^p$, and the confidence region resulting from Theorem 2 is $T(\mathbf{X}_1, \dots, \mathbf{X}_n) - (-\infty, 0]^p$. \square

For min and max tests, we have shown that the corresponding confidence region can be written as $T(\mathbf{X}_1, \dots, \mathbf{X}_n) - \Theta_0(\mathbf{0})$, where $\Theta_0(\mathbf{0})$ is the null parameter region specified by $\boldsymbol{\gamma} = \mathbf{0}$. The shape of the confidence region is therefore a reflected version of that of the null parameter region.

A similar phenomenon occurs for many other one-sided location tests. However, the shape of the confidence region often only corresponds to the reflected null parameter region in an asymptotic sense, i.e. for parameter values that are distant enough from the observations — typically, the border of the confidence region does not exactly reproduce the non-smooth parts of the border of the (e.g. cone-shaped) null parameter region. Such an asymptotic result (under rather restrictive assumptions) is formulated in Vock [2].

Example

We use a data set from the literature to illustrate the shape of the confidence regions corresponding to three different tests: The pulmonary function data set for the example is taken from Table 3 in Randles [3] (which is a slightly modified version of a data set in Merchant et al. [4]). We only use two of the three variables: the change in forced vital capacity (FVC) and the change in forced expiratory volume (FEV_3) of twelve persons during exposure to cotton dust.

The conjecture is that the lung function deteriorates under cotton dust exposure, i.e. that the differences in FVC and FEV_3 tend to be negative. It is therefore appropriate to use one-sided location tests and the corresponding confidence regions. Different formulations of the exact hypotheses are possible in this example: We could try to show a deterioration in at least one variable, in both variables simultaneously, or in some measure combining both the variables.

Since one-sided alternative hypotheses are usually formulated such that they cover large parameter values, we change the signs of both the variables. We are therefore interested in the location parameter for $(-\text{FVC}, -\text{FEV}_3)^T$. These data are visualized in Fig. 1.

For this bivariate data set, we invert the following three tests in order to obtain 95% confidence regions:

- The min test resulting from univariate Wilcoxon signed rank tests,
- the Bonferroni max test resulting from univariate Wilcoxon signed rank tests, and
- the conditionally distribution-free sign test proposed by Larocque and Labarre [5].

The hypotheses (in terms of a meta-parameter $\boldsymbol{\gamma}$) for the min and max tests have been given above. For the test by Larocque and Labarre [5], the authors indicate the point null hypothesis $\boldsymbol{\vartheta} = \mathbf{0}$ and two contradictory definitions of the alternative hypothesis. The test seems to be appropriate for the composite null hypothesis $H_0 : \boldsymbol{\vartheta} \leq \mathbf{0}$ against $H_1 : \exists j \in \{1, \dots, p\} : \vartheta_j > 0$, and by subtracting $\boldsymbol{\gamma}$ from each data point, the test can easily be adapted for $H_0 : \boldsymbol{\vartheta} \leq \boldsymbol{\gamma}$ vs. $H_1 : \exists j \in \{1, \dots, p\} : \vartheta_j > \gamma_j$.

The lower/left borders of the resulting 95% confidence regions for $\boldsymbol{\gamma}$ are also given in Fig. 1; according to Section 3, these can directly be interpreted as 95% confidence regions for the location parameter $\boldsymbol{\vartheta}$. When we keep in mind the shape of the alternative region of each of the corresponding tests, we see that the shape of all three confidence regions is not adequate for the respective problem. In the following section, we will obtain a more adequate and even smaller confidence region for the case of the Wilcoxon min test.

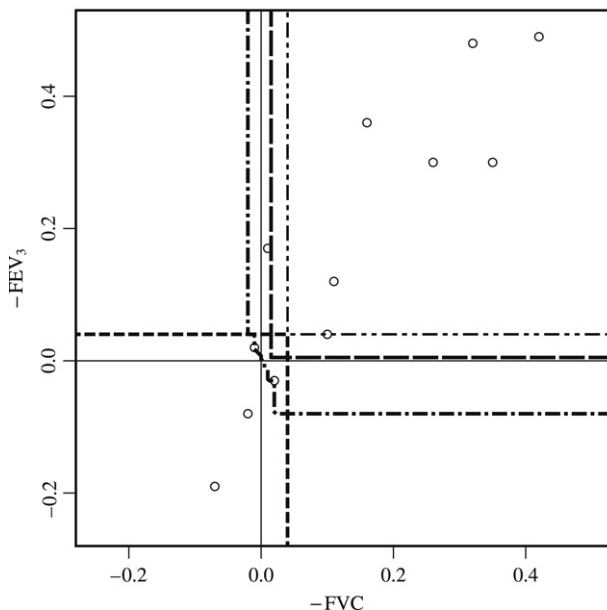


Fig. 1. Pulmonary function data for the example (circles). 95% confidence regions for $\boldsymbol{\vartheta}$ according to Section 3: — Wilcoxon Bonferroni max test; ---- Wilcoxon min test; -.-.- Larocque/Labarre [5]. 95% confidence region for $\boldsymbol{\vartheta}$ according to Section 5: Wilcoxon min test. The upper right corner of the figure is within each of the confidence regions.

5. A sharpened confidence region for the location parameter

In Section 3, we obtained confidence regions for $\boldsymbol{\vartheta}$ by simply ignoring the shape of the null parameter region of the underlying test. Of course, we can try to obtain sharper confidence regions by incorporating this information. (If we can exclude certain values of $\boldsymbol{\vartheta}$ due to a priori restrictions on the parameter space, we can trivially sharpen the confidence region by intersecting it with the values that are possible.)

In the case of a non-convex $\Theta_0(\boldsymbol{\gamma})$ leading to a non-convex confidence region $\mathcal{C}_{1-\alpha}(X)$ for $\boldsymbol{\gamma}$ (e.g. in the case of a min test), it is tempting to sharpen the confidence region for $\boldsymbol{\vartheta}$ by excluding all points that are contained in any $\Theta_0(\boldsymbol{\gamma})$ with $\boldsymbol{\gamma} \notin \mathcal{C}_{1-\alpha}(X)$. However, this usually means that we apply multiple tests to each point, and therefore, this procedure does not guarantee the specified confidence level to hold.

To overcome this multiple testing problem, we can restrict ourselves to a suitable set of meta-parameters $\boldsymbol{\gamma}$ specified in advance and apply the closed testing principle by Marcus, Peritz, and Gabriel [6] to the resulting nested family of hypotheses:

Theorem 6. Let $\mathcal{C}_{1-\alpha} : \mathcal{X} \rightarrow \mathcal{P}(\Theta)$ be a confidence region for $\boldsymbol{\gamma}$ with confidence level $1 - \alpha$ based on the family $(\Theta_0(\boldsymbol{\gamma}))_{\boldsymbol{\gamma} \in \Theta}$. Further, let $(\boldsymbol{\gamma}_i)_{i \in I}$, $I \subset \mathbb{R}$, be a subset of Θ such that

$$\Theta_0(\boldsymbol{\gamma}_{i_1}) \subset \Theta_0(\boldsymbol{\gamma}_{i_2}) \quad \forall i_1, i_2 \in I : i_1 < i_2.$$

As a technical condition, assume that for each subset $\tilde{I} \subset I$, there exists an $i_0 \in I$ such that

$$\bigcap_{i \in \tilde{I}} \Theta_0(\boldsymbol{\gamma}_i) = \Theta_0(\boldsymbol{\gamma}_{i_0}).$$

Define

$$\tilde{\mathcal{C}}_{1-\alpha}(X) := \Theta \setminus \bigcup_{i \in I: \boldsymbol{\gamma}_{i'} \notin \mathcal{C}_{1-\alpha}(X) \ \forall i' \leq i} \Theta_0(\boldsymbol{\gamma}_i).$$

Then

$$\mathbb{P}_{\boldsymbol{\vartheta}}(\tilde{\mathcal{C}}_{1-\alpha}(X) \ni \boldsymbol{\vartheta}) \geq 1 - \alpha \quad \forall \boldsymbol{\vartheta} \in \Theta.$$

Proof. Let $(\varphi_i)_{i \in I}$ be the non-randomized tests for $H_{0i} : \boldsymbol{\vartheta} \in \Theta_0(\boldsymbol{\gamma}_i)$ corresponding to $\mathcal{C}_{1-\alpha}$, i.e. $\varphi_i(X) = 1(\mathcal{C}_{1-\alpha}(X) \not\ni \boldsymbol{\gamma}_i)$. By the definition of a confidence region, each φ_i is of level α .

For every $i \in I$, define a new test $\tilde{\varphi}_i(X) := \prod_{i' \leq i} \varphi_{i'}(X)$, which rejects H_{0i} if and only if all $\varphi_{i'}(X)$ with $i' \leq i$ reject $H_{0i'}$. The family $(H_{0i})_{i \in I}$ is closed under intersections due to the technical condition, and the family $(\tilde{\varphi}_i)_{i \in I}$ forms a closed testing procedure for $(H_{0i})_{i \in I}$ as proposed in Marcus, Peritz, and Gabriel [6]. Therefore,

$$\mathbb{P}_{\boldsymbol{\vartheta}}(\tilde{\varphi}_i(X) = 0 \ \forall i \in I : \Theta_0(\boldsymbol{\gamma}_i) \ni \boldsymbol{\vartheta}) \geq 1 - \alpha \quad \forall \boldsymbol{\vartheta} \in \Theta$$

(i.e. these tests respect the familywise error rate as defined in Hochberg and Tamhane [7]). Note that while the closed testing principle is most often used for finite families of hypotheses, such an assumption is not needed.

For arbitrary $\boldsymbol{\vartheta} \in \Theta$, it therefore follows that

$$\begin{aligned} \mathbb{P}_{\boldsymbol{\vartheta}}(\tilde{\mathcal{C}}_{1-\alpha}(X) \ni \boldsymbol{\vartheta}) &= \mathbb{P}_{\boldsymbol{\vartheta}}\left(\Theta \setminus \bigcup_{i \in I: \boldsymbol{\gamma}_{i'} \notin \mathcal{C}_{1-\alpha}(X) \ \forall i' \leq i} \Theta_0(\boldsymbol{\gamma}_i) \ni \boldsymbol{\vartheta}\right) \\ &= 1 - \mathbb{P}_{\boldsymbol{\vartheta}}\left(\bigcup_{i \in I: \boldsymbol{\gamma}_{i'} \notin \mathcal{C}_{1-\alpha}(X) \ \forall i' \leq i} \Theta_0(\boldsymbol{\gamma}_i) \ni \boldsymbol{\vartheta}\right) \\ &= 1 - \mathbb{P}_{\boldsymbol{\vartheta}}(\exists i \in I : \boldsymbol{\gamma}_{i'} \notin \mathcal{C}_{1-\alpha}(X) \ \forall i' \leq i, \Theta_0(\boldsymbol{\gamma}_i) \ni \boldsymbol{\vartheta}) \\ &= 1 - \mathbb{P}_{\boldsymbol{\vartheta}}(\exists i \in I : \tilde{\varphi}_i(X) = 1, \Theta_0(\boldsymbol{\gamma}_i) \ni \boldsymbol{\vartheta}) \\ &= \mathbb{P}_{\boldsymbol{\vartheta}}(\tilde{\varphi}_i(X) = 0 \ \forall i \in I : \Theta_0(\boldsymbol{\gamma}_i) \ni \boldsymbol{\vartheta}) \\ &\geq 1 - \alpha. \quad \square \end{aligned}$$

If I is finite, the technical condition of the above theorem follows from the assumed inclusion of the null parameter regions, and the modified confidence region simplifies to

$$\tilde{\mathcal{C}}_{1-\alpha}(X) = \Theta \setminus \Theta_0(\boldsymbol{\gamma}_{i^*}) \quad \text{with } i^* = \max\{i \in I : \boldsymbol{\gamma}_{i'} \notin \mathcal{C}_{1-\alpha}(X), \ \forall i' \leq i\}$$

(or $\tilde{\mathcal{C}}_{1-\alpha}(X) = \Theta$ if an $i \in I$ with the desired property does not exist).

A possible application of [Theorem 6](#) is obtained by using (essentially) the straight line $((r, \dots, r)^T)_{r \in \mathbb{R}} \subset \mathbb{R}^p$ as the meta-parameters to be examined:

Corollary 7. Let $\mathcal{C}_{1-\alpha} : \mathcal{X} \rightarrow \mathcal{P}(\mathbb{R}^p)$ be a confidence region for $\boldsymbol{\gamma}$ with confidence level $1 - \alpha$ based on the family $(\Theta_0(\boldsymbol{\gamma}))_{\boldsymbol{\gamma} \in \mathbb{R}^p}$. Assume that $\Theta_0(\boldsymbol{\gamma}) = \boldsymbol{\gamma} + \Theta_0(\mathbf{0})$, $\forall \boldsymbol{\gamma} \in \mathbb{R}^p$, that $\Theta_0(\mathbf{0})$ is closed, and that

$$\Theta_0(\boldsymbol{\gamma}) \subset \Theta_0(\boldsymbol{\gamma} + (\delta, \dots, \delta)^T) \quad \forall \boldsymbol{\gamma} \in \mathbb{R}^p, \delta > 0.$$

Let ℓ be any real number. Use $\boldsymbol{\gamma}_i = (i, \dots, i)^T \in \mathbb{R}^p$, $\forall i \in I = [\ell, \infty)$.

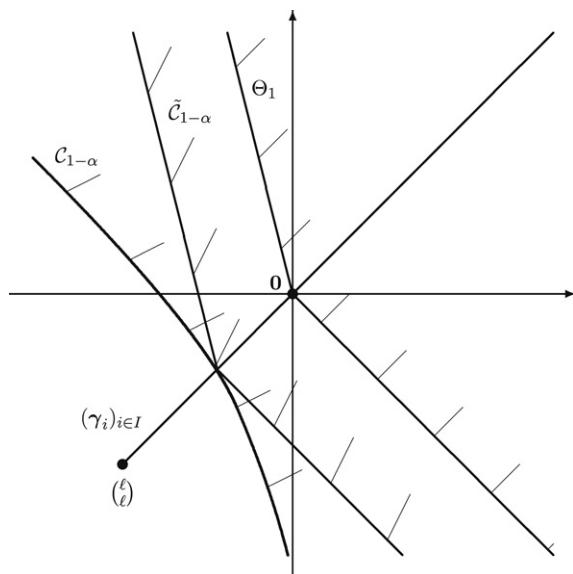


Fig. 2. Illustration of [Corollary 7](#): Alternative region $\Theta_1 = \Theta_1(\mathbf{0}) = \mathbb{R}^2 \setminus \Theta_0(\mathbf{0})$, confidence region $C_{1-\alpha}$ based on the inversion of a test for $H_0 : \boldsymbol{\vartheta} \in \Theta_0(\boldsymbol{\gamma})$ vs. $H_1 : \boldsymbol{\vartheta} \in \mathbb{R}^2 \setminus \Theta_0(\boldsymbol{\gamma})$, and the sharpened confidence region $\tilde{C}_{1-\alpha}$ obtained from [Corollary 7](#).

For $\tilde{C}_{1-\alpha}$ as defined in [Theorem 6](#) (with $\Theta = \mathbb{R}^p$),

$$P_{\boldsymbol{\vartheta}}(\tilde{C}_{1-\alpha}(X) \ni \boldsymbol{\vartheta}) \geq 1 - \alpha \quad \forall \boldsymbol{\vartheta} \in \mathbb{R}^p.$$

Proof. The inclusion of the null parameter regions that is needed for the application of [Theorem 6](#) is obviously fulfilled.

For the technical condition of the theorem, let $\tilde{I} \subset I$. Since $\Theta_0(\boldsymbol{\gamma}) = \boldsymbol{\gamma} + \Theta_0(\mathbf{0})$ and because these regions are closed, we can write

$$\begin{aligned} \bigcap_{i \in \tilde{I}} \Theta_0(\boldsymbol{\gamma}_i) &= \bigcap_{i \in \tilde{I}} (\boldsymbol{\gamma}_i + \Theta_0(\mathbf{0})) \\ &= \boldsymbol{\gamma}_{i_0} + \Theta_0(\mathbf{0}) \\ &= \Theta_0(\boldsymbol{\gamma}_{i_0}), \end{aligned}$$

with $i_0 = \inf \tilde{I}$. Since I is closed at the lower end, $i_0 \in I$.

We can therefore apply [Theorem 6](#). \square

Note that, in the case of complementary null and alternative hypotheses (i.e. $\Theta_1(\boldsymbol{\gamma}) = \mathbb{R}^p \setminus \Theta_0(\boldsymbol{\gamma})$), confidence regions for $\boldsymbol{\vartheta}$ based on [Corollary 7](#) have the same shape as the alternative parameter region $\Theta_1(\mathbf{0})$ of the corresponding test; cf. [Fig. 2](#). In this figure, $\tilde{C}_{1-\alpha}(X)$ is obviously less conservative because it is a subset of $C_{1-\alpha}(X)$. We need the following definition to state a sufficient condition for this inclusion property:

Definition 8. Let $C \subset \mathbb{R}^p$ be a convex cone. A function $f : (\mathbb{R}^p)^n \rightarrow \mathbb{R}$ is *cone order monotone* in the sample with respect to C if

$$f(\mathbf{x}_1, \dots, \mathbf{x}_n) \leq f(\mathbf{x}_1 + \boldsymbol{\delta}, \dots, \mathbf{x}_n + \boldsymbol{\delta}) \quad \forall \mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^p, \boldsymbol{\delta} \in C.$$

Cohen and Sackrowitz [8] propose that if the alternative region Θ_1 is a cone, tests should satisfy the cone order monotonicity property with respect to Θ_1 and/or its positive dual $\Theta_1^* = \{\mathbf{a} : \mathbf{a}^T \boldsymbol{\vartheta} \geq 0 \ \forall \ \boldsymbol{\vartheta} \in \Theta_1\}$. (For a discussion of the adequacy of requiring cone order monotonicity in different situations, see e.g. Perlman and Chaudhuri [9] and Cohen and Sackrowitz [10].) Their definition of cone order monotonicity only applies to a test based on a single p -variate statistic. When we look at a general test based on a sample of n p -variate observations, the above definition is one possible generalization. A second one (yielding a stronger property) would be to require *cone order monotonicity in each observation*, where each observation could be moved by an individual $\boldsymbol{\delta}_i \in C$.

Theorem 9. Let a non-randomized level α test for $H_0 : \boldsymbol{\vartheta} \in \Theta_0(\boldsymbol{\gamma})$ vs. $H_1 : \boldsymbol{\vartheta} \in \Theta_1(\boldsymbol{\gamma}) = \mathbb{R}^p \setminus \Theta_0(\boldsymbol{\gamma})$ be given, where $\Theta_0(\boldsymbol{\gamma}) = \boldsymbol{\gamma} + \Theta_0(\mathbf{0}) \subset \mathbb{R}^p$ is closed, $\forall \ \boldsymbol{\gamma} \in \mathbb{R}^p$, the test being based on random vectors $\mathbf{X}_1, \dots, \mathbf{X}_n \in \mathbb{R}^p$. Let the test be translation invariant and cone order monotone in the sample with respect to the convex cone $\Theta_1(\mathbf{0})$. Let the $1 - \alpha$ confidence region $\mathcal{C}_{1-\alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ corresponding to the test be closed.

Then the sharpened confidence region $\tilde{\mathcal{C}}_{1-\alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ according to Corollary 7 satisfies

$$\tilde{\mathcal{C}}_{1-\alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n) \subset \mathcal{C}_{1-\alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n).$$

Proof. Cone order monotonicity in the sample for a non-randomized test φ_α means that, for all $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^p$, $\boldsymbol{\gamma} \in \mathbb{R}^p$, $\boldsymbol{\delta} \in \Theta_1(\mathbf{0})$,

$$\varphi_\alpha((\mathbf{x}_1, \dots, \mathbf{x}_n), \boldsymbol{\gamma}) = 1 \Rightarrow \varphi_\alpha((\mathbf{x}_1 + \boldsymbol{\delta}, \dots, \mathbf{x}_n + \boldsymbol{\delta}), \boldsymbol{\gamma}) = 1,$$

where $\varphi_\alpha((\mathbf{x}_1, \dots, \mathbf{x}_n), \boldsymbol{\gamma}) = 1$ denotes the rejection of $H_0 : \boldsymbol{\vartheta} \in \Theta_0(\boldsymbol{\gamma})$ at the level α . By contraposition, the implication can also be written as

$$\varphi_\alpha((\mathbf{x}_1 + \boldsymbol{\delta}, \dots, \mathbf{x}_n + \boldsymbol{\delta}), \boldsymbol{\gamma}) = 0 \Rightarrow \varphi_\alpha((\mathbf{x}_1, \dots, \mathbf{x}_n), \boldsymbol{\gamma}) = 0. \quad (*)$$

Take now any $\boldsymbol{\delta} \in \Theta_1(\mathbf{0})$. Using (*) and the translation invariance (t. i.), we obtain

$$\begin{aligned} \boldsymbol{\gamma} \in \mathcal{C}_{1-\alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n) &\Leftrightarrow \varphi_\alpha((\mathbf{x}_1, \dots, \mathbf{x}_n), \boldsymbol{\gamma}) = 0 \\ &\stackrel{\text{t.i.}}{\Leftrightarrow} \varphi_\alpha((\mathbf{x}_1 + \boldsymbol{\delta}, \dots, \mathbf{x}_n + \boldsymbol{\delta}), \boldsymbol{\gamma} + \boldsymbol{\delta}) = 0 \\ &\stackrel{(*)}{\Rightarrow} \varphi_\alpha((\mathbf{x}_1, \dots, \mathbf{x}_n), \boldsymbol{\gamma} + \boldsymbol{\delta}) = 0 \\ &\Leftrightarrow \boldsymbol{\gamma} + \boldsymbol{\delta} \in \mathcal{C}_{1-\alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n). \end{aligned}$$

Therefore, $\mathcal{C}_{1-\alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n) + \Theta_1(\mathbf{0}) \subset \mathcal{C}_{1-\alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n)$, which implies also

$$\mathcal{C}_{1-\alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n) + \overline{\Theta_1(\mathbf{0})} \subset \overline{\mathcal{C}_{1-\alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n)} = \mathcal{C}_{1-\alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n), \quad (**)$$

where \overline{A} is the closure of A .

In the situation of Corollary 7, we can write

$$\begin{aligned} \tilde{\mathcal{C}}_{1-\alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n) &= \bigcap_{i \in I : \boldsymbol{\gamma}_{i'} \notin \mathcal{C}_{1-\alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n) \ \forall \ i' \leq i} (\mathbb{R}^p \setminus \Theta_0(\boldsymbol{\gamma}_i)) \\ &\subset \overline{\Theta_1(\boldsymbol{\gamma}_{i^*})} = \boldsymbol{\gamma}_{i^*} + \overline{\Theta_1(\mathbf{0})}, \end{aligned}$$

where $i^* = \sup\{i \in I : \boldsymbol{\gamma}_{i'} \notin \mathcal{C}_{1-\alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n) \ \forall \ i' \leq i\}$. Since $\mathcal{C}_{1-\alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is closed, $\boldsymbol{\gamma}_{i^*} \in \mathcal{C}_{1-\alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n)$, which, together with (**), completes the proof. \square

Example (continued)

Finally, we look at the sharpened confidence region obtained from the application of [Corollary 7](#) to the 95% confidence region resulting from the inversion of the Wilcoxon min test. The lower/left border of this confidence region is given in [Fig. 1](#). Its shape corresponds to that of the alternative parameter region $(0, \infty)^2$, and the sharpened confidence region obtained from the Wilcoxon min test is uniformly better than the confidence regions directly obtained by inverting the test by Larocque and Labarre [5] or the Wilcoxon Bonferroni max test, which have a similar shape. In this example, the sharpened confidence region clearly benefits from the fact that the data points are situated rather close to the diagonal.

6. Summary and conclusion

We have examined the meaning of confidence regions obtained by the inversion of one-sided tests. It is important to realize that, in principle, these confidence regions have to be interpreted with respect to a meta-parameter. In many practically important cases, they will also be valid for the location parameter itself (see [Section 3](#)), but unnecessarily conservative, and their shape may be inappropriate for the problem considered. While we have only derived results about the shape of the confidence region in two very simple cases, similar properties can be observed in many other situations.

In the case of complementary null and alternative parameter regions and a convex alternative, the proposed method based on the closed testing principle yields confidence regions with a shape that corresponds to that of the alternative parameter region. Under suitable conditions, these confidence regions can also be shown to be less conservative (i.e. smaller) than the original ones. On the other hand, an unpleasant property is that the procedure reduces the multivariate confidence region problem to a univariate problem (a search on the diagonal) and that the set of possible resulting confidence regions is therefore rather restricted.

An anonymous referee pointed out that an application of our method to the problem of univariate or multivariate bioequivalence could be investigated because such a problem can be rewritten using the one-sided hypotheses of a min test. For the univariate case, our method yields a confidence interval corresponding to the interval I_S given in Hsu, Hwang, Liu, and Ruberg [11], where a smaller confidence interval for the same problem is also mentioned. For the multivariate case (e.g. Wang, Hwang, and Dasgupta [12]), the application of our method would require a more detailed examination.

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References

- [1] P.K. Sen, M.J. Silvapulle, An appraisal of some aspects of statistical inference under inequality constraints, *J. Statist. Plann. Inference* 107 (2002) 3–43.
- [2] M. Vock, “One-sided” statistical inference for a multivariate location parameter. Ph.D. Thesis, University of Bern, 2007. Available at: <http://www.imsv.unibe.ch/content/download/theses/Vock2007.pdf>.
- [3] R.H. Randles, A distribution-free multivariate sign test based on interdirections, *J. Amer. Statist. Assoc.* 84 (1989) 1045–1050.

- [4] J.A. Merchant, G.M. Halprin, A.R. Hudson, K.H. Kilburn, W.N. McKenzie, D.J. Hurst, P. Bermazohn, Responses to cotton dust, *Arch. Environ. Health* 30 (1975) 222–229.
- [5] D. Larocque, M. Labarre, A conditionally distribution-free multivariate sign test for one-sided alternatives, *J. Amer. Statist. Assoc.* 99 (2004) 499–509.
- [6] R. Marcus, E. Peritz, K.R. Gabriel, On closed testing procedures with special reference to ordered analysis of variance, *Biometrika* 63 (1976) 655–660.
- [7] Y. Hochberg, A.C. Tamhane, *Multiple Comparison Procedures*, Wiley, New York, 1987.
- [8] A. Cohen, H.B. Sackrowitz, Directional tests for one-sided alternatives in multivariate models, *Ann. Statist.* 26 (1998) 2321–2338.
- [9] M.D. Perlman, S. Chaudhuri, The role of reversals in order-restricted inference, *Canad. J. Statist.* 32 (2004) 193–198.
- [10] A. Cohen, H.B. Sackrowitz, A discussion of some inference issues in order restricted models, *Canad. J. Statist.* 32 (2004) 199–205.
- [11] J.C. Hsu, J.T.G. Hwang, H.-K. Liu, S.J. Ruberg, Confidence intervals associated with tests for bioequivalence, *Biometrika* 81 (1994) 103–114.
- [12] W. Wang, J.T.G. Hwang, A. Dasgupta, Statistical tests for multivariate bioequivalence, *Biometrika* 86 (1999) 395–402.