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Quasi-arithmetic means of covariance functions with potential applications to space–time data

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ABSTRACT

The theory of quasi-arithmetic means represents a powerful tool in the study of covariance functions across space–time. In the present study we use quasi-arithmetic functionals to make inferences about the permissibility of averages of functions that are not, in general, permissible covariance functions. This is the case, e.g., of the geometric and harmonic averages, for which we obtain permissibility criteria. Also, some important inequalities involving covariance functions and preference relations as well as algebraic properties can be derived by means of the proposed approach. In particular, quasi-arithmetic covariances allow for ordering and preference relations, for a Jensen-type inequality and for a minimal and maximal element of their class. The general results shown in this paper are then applied to the study of spatial and spatio-temporal random fields. In particular, we discuss the representation and smoothness properties of a weakly stationary random field with a quasi-arithmetic covariance function. Also, we show that the generator of the quasi-arithmetic means can be used as a link function in order to build a space–time nonseparable structure starting from the spatial and temporal margins, a procedure that is technically sound for those working with copulas. Several examples of new families of stationary covariances obtainable with this procedure are shown. Finally, we use quasi-arithmetic functionals to generalise existing results concerning the construction of nonstationary spatial covariances, and discuss the applicability and limits of this generalisation.

1. Introduction

The importance of quasi-arithmetic means has been well understood at least since the 1930s, and a number of writers have since then contributed to their characterisation and to the study of their properties. In particular, Kolmogorov [1] and Nagumo [2] derived, independently of each other, necessary and sufficient conditions for the quasi-arithmeticity of the mean, that is, for the existence of a continuous strictly monotonic function f such that, for $x_1, \ldots, x_n$ in some real interval, the function $(x_1, \ldots, x_n) \mapsto M_n(x_1, \ldots, x_n) = f^{-1}(\frac{1}{n}\sum_1^n f(x_i))$ is a mean. Using this result, they partially modified the classical Cauchy [3] internality and Chisini's [4] invariance properties. As pointed out by Marichal [5], the Kolmogorov reflexive property is equivalent to the Cauchy internality, and both are accepted by statisticians as requisites for means.

Early works on the concept of mean include [6–10]. More recent contributions are the works of Wimp [11], Hutník [12], Matkowski [13,14], Jarczyk and Matkowski [15], Marichal [5], Daróczy and Hajdu [16], and Abrahamovic et al. [52]. Quasi-arithmetic means, in particular, have been applied in several disciplines. A special case of this class has been used in the

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theory of copulas under the name of Archimedean copulas [17] and a rich literature can be found under this name. In the theory of aggregation operators and fuzzy measures, a growing literature related to the use of quasi-arithmetics includes the works of Frank [18], Hajék [19], Kolesaróvá [20], Klement et al. [21], Grabish [22] and Calvo and Mesiar [23].

Despite the extensive quasi-arithmetic means literature, to the best of our knowledge, there is no published work relating quasi-arithmetic means with covariance functions, whose properties have been extensively studied both in mathematical analysis and statistical fields. In particular, the study of covariance functions is intimately related to that of positive definite functions, the latter being the subject of a considerable literature in a variety of fields, such as mathematical analysis [24], abelian semigroup theory, spatial statistics and geostatistics. For basic facts about positive definite functions, we refer to [25] and to [26]. The importance of positive definite functions in the determination of permissible covariance functions (ordinary and generalised) in spatial statistics was studied in detail by Christakos [27]. Subsequent considerations, in a spatial and a spatio-temporal context, include the works of Christakos [28–30], Sasvari [31] and Gneiting [32], among others.

Fundamental properties of covariance functions may be inferred by studying collections of them considered as convex cones closed in the topology of point-wise convergence. In this paper, quasi-arithmetic averages and positive definite functions are combined to gain valuable insight concerning certain covariance properties. Also, we apply the general results obtained by our analysis based on the concept of quasi-arithmeticity to build new classes of stationary and nonstationary space–time covariance functions. In such a context, we seek answers to the following questions:

(1) Consider an arbitrary number $n \in \mathbb{N}$ of covariance functions, not necessarily defined in the same space. Their arithmetic average and product (*i.e., the geometric average raised to the nth power*) are valid covariance functions, and so is the kth power average of covariance functions (k is a natural positive number). But, what about other types of averages? Since quasi-arithmetic means constitute a general group that includes the arithmetic, geometric, power and logarithmic means as special cases, it seems natural to use quasi-arithmetic representations in order to derive positive definiteness conditions for other classes of averages of covariance functions.
(2) Can we use the properties of quasi-arithmetic means to establish important inequalities, ordering and preference relations, and minimal and maximal elements within the class of covariance functions?
(3) Is it possible to find a class of link functions that, when applied to k covariances, can generate valid nonseparable covariances? If this is the case, other potentially desirable properties should be examined, e.g., this approach should be as general as possible and include *famous* constructions and separability as special cases; and it should preserve the margins.

In view of the above considerations, the paper is organised as follows: Section 2 discusses the background, notation and proposed methodology; it also provides a very brief introduction to positive definite functions. In Section 3, the main theoretical results are presented. In particular, through straightforward arguments, we show permissibility criteria for the quasi-arithmetic mean of covariance functions. Of particular importance are two corollaries of this result, where we obtain permissibility criteria for the (weighted) geometric average of covariance functions, as well as to the harmonic average, and these results are absolutely novel for the literature.

Furthermore, we derive important covariance inequalities of Jensen type as well as ordering and preference relations between covariance functions. Finally, minimal and maximal elements of the quasi-arithmetic covariance class are identified, and an associativity property of this class is provided. In Section 4, we apply our results to the construction of new families of space–time nonseparable stationary covariance functions. Also, we extend Stein [33] result to a more general class of spatial nonstationary covariance functions. Several examples of stationary and nonstationary covariances are proposed and their mathematical properties discussed. Finally, we study the properties of quasi-arithmetic random fields (defined later in the paper) in terms of mean square differentiability and variance. Section 5 concludes with a critical discussion of the preceding analysis. All the proofs of the results derived in this paper can be found in the Appendix.

2. Background and methodology

2.1. Covariance functions: Characterisation and basic properties

In this section, we shall assume the real mapping $\mathrm{C} : \mathbb{X} \times \mathbb{X} \subseteq \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ to be continuous and Lebesgue measurable on the domain $\mathbb{X} \times \mathbb{X}$, where $\mathbb{X}$ can be either a compact space or the entire d-dimensional Euclidean space, $d \in \mathbb{N}$. C is a covariance function of a Gaussian random field if and only if it is positive definite, that is

$$\sum_{i=1}^{n} \sum_{i=1}^{n} c_i c_j \mathrm{C}(\mathbf{x}_i, \mathbf{x}_j) \geq 0 \tag{1}$$

for any finite set of real coefficients $\{c_i\}_{i=1}^{n}$, and for $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{X}$. Christakos [27] calls the covariance condition (1) *permissibility*, and throughout the paper we shall use both this term and that of positive definiteness to characterise valid covariance functions. A subclass of positive definite functions, called *stationary*, is obtained if

$$\mathrm{C}(\mathbf{x}_i, \mathbf{x}_j) := \tilde{\mathrm{C}}_0(\mathbf{x}),$$

with $\mathbf{x} = \mathbf{x}_i - \mathbf{x}_j$ and $\tilde{\mathrm{C}}_0 : \mathbb{X} \to \mathbb{R}$ such that $\tilde{\mathrm{C}}_0(\mathbf{0}) < \infty$.

By Bochner's theorem, condition (1) is then equivalent to the requirement that $\tilde{\mathrm{C}}_0$ is the Fourier transform $\mathcal{F}$ of a positive bounded measure $\hat{\mathrm{C}}_0$ with support in $\mathbb{R}^d$, that is

$$\tilde{\mathrm{C}}_0(\mathbf{x}) := \mathcal{F}[\hat{\mathrm{C}}_0](\mathbf{x}) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\omega'\mathbf{x}} \mathrm{d}\hat{\mathrm{C}}_0(\boldsymbol{\omega}).$$

Additionally, if $\hat{\mathrm{C}}_0$ is absolutely continuous with respect to the Lebesgue measure (ensured if $\tilde{\mathrm{C}}_0 \in L_1(\mathbb{R}^d)$), then the expression above can be written as a function of $\mathrm{d}\hat{\mathrm{C}}_0(\boldsymbol{\omega}) = \hat{\mathrm{c}}_0(\boldsymbol{\omega})\mathrm{d}\boldsymbol{\omega}$, where $\hat{\mathrm{c}}_0$ is called the spectral density of $\tilde{\mathrm{C}}_0$. For a detailed mathematical discussion of the Fourier representation in a spatial–temporal statistics context, we refer to volumes by Yaglom [34] and Christakos [30].

In what follows, we shall consider some interesting restrictions on the general class of stationary covariance functions. But first, we need to introduce some standard notation for arbitrary partitions and operations between vectors. In order to manipulate arbitrary decompositions of nonnegative integer numbers, let us consider the set $\exp(\mathbb{Z}_+) = \emptyset \cup \mathbb{Z}_+ \cup \mathbb{Z}_+^2 \cup \mathbb{Z}_+^3 \cup \cdots$ (disjoint union). An element $\mathbf{d}$ of $\exp(\mathbb{Z}_+)$ can be expressed either as $\mathbf{d} = \emptyset$ or as $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ if $\mathbf{d} \in \mathbb{Z}_+^n$ with $n \geq 1$. In the latter case we denote by $n(\mathbf{d}) = n$ the dimension of $\mathbf{d}$ and $|\mathbf{d}| = \sum_{i=1}^n d_i$ the length of $\mathbf{d}$. Both values are taken to be 0 whenever $\mathbf{d} = \emptyset$. For $\mathbf{d}, \mathbf{d}' \in \exp(\mathbb{Z}_+)$ we say that $\mathbf{d} \leq \mathbf{d}'$ if and only if $n(\mathbf{d}) = n(\mathbf{d}')$ and $d_i \leq d_i'$ for all $i = 1, 2, \ldots, n(\mathbf{d})$. Usual vector operations are possible only between elements of the same dimension. Vectors with all components equal are denoted in bold symbols, such as $\mathbf{0}$, $\mathbf{1}$.

Now, the restrictions considered in this paper are of the following type:

1. The **isotropic** case:

$$\tilde{\mathrm{C}}_0(\mathbf{x}) := \tilde{\mathrm{C}}_1(\|\mathbf{x}\|), \quad \mathbf{x} \in \mathbb{X}, \tag{2}$$

that is, C is said to be represented by the function $\tilde{\mathrm{C}}_1 : \mathbb{R}_+ \to \mathbb{R}$, that is rotation–translation invariant (or radially symmetric), and where $\|.\|$ denotes the Euclidean norm. This is the most popular case in spatial and spatio-temporal statistics [35,34,36].

2. The **component-wise isotropic** case: Let us consider the d-dimensional space $\mathbb{R}^d$, and let $\mathbf{d}$ be an element of $\exp(\mathbb{Z}_+)$ such that $|\mathbf{d}| = d$ and $\mathbf{1} \leq \mathbf{d}$. Thus, one can create opportune partitions of the spatial lag vector $\mathbf{x} \in \mathbb{R}^d$ in the following way. If $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ and $\mathbf{x} \in \mathbb{R}^d$ we can always write

$$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_n}$$

so that:

(i) $\tilde{\mathrm{C}}_0(\mathbf{x}) = \tilde{\mathrm{C}}_0(\mathbf{k})$ for any $\mathbf{x}, \mathbf{k} \in \mathbb{R}^d$ if and only if $\|\mathbf{x}_i\| = \|\mathbf{k}_i\|$ for all $i = 1, 2, \ldots, n$.

(ii) The resulting covariance admits the representation

$$\begin{aligned}\tilde{\mathrm{C}}_0(\mathbf{x}) &:= \tilde{\mathrm{C}}_1(\|\mathbf{x}_1\|, \ldots, \|\mathbf{x}_n\|) \\ &= \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n \Omega_{d_i}(\|\mathbf{x}_i\| r_i) \mathrm{d}F(r_1, \ldots, r_n)\end{aligned} \tag{3}$$

with $\Omega_d(t) = \Gamma(d/2)\left(\frac{2}{t}\right) \mathcal{J}_{(d-2)/2}(t)$, $\mathcal{J}_d(.)$ denoting the Bessel function of the first kind of order d [37], F an n-variate distribution function and $\tilde{\mathrm{C}}_1 : \mathbb{R}^n \to \mathbb{R}$. Thus, Eq. (2) is a special case of (3) ($|\mathbf{d}| = d$ and $\mathbf{d} := d$, a scalar) and its corresponding integral representation can be readily obtained. The special case $\mathbf{d} = \mathbf{1}$ and $|\mathbf{d}| = d$ is particularly interesting in the subsequent sections of this paper, as the function $(x_1, \ldots, x_n) \mapsto \tilde{\mathrm{C}}_1(|x_1|, \ldots, |x_n|)$, does not depend on the Euclidean norm, but on the Manhattan or city block distance, with important implications in spatial and spatio-temporal statistics as pointed out by Christakos [38] and Banerjee [39].

It is worth noticing that covariance functions of the type (3) have a property called reflection symmetry [40], or full symmetry [41]. This means that $\mathrm{C}(\mathbf{x}_1, \ldots, \mathbf{x}_i, \ldots, \mathbf{x}_n) = \mathrm{C}(\mathbf{x}_1, \ldots, -\mathbf{x}_i, \ldots, \mathbf{x}_n) = \cdots = \mathrm{C}(-\mathbf{x}_1, \ldots, -\mathbf{x}_i, \ldots, -\mathbf{x}_n)$. Whenever no confusion arises, in the remainder of the paper we shall drop the under- and super-script denoting a stationary or stationary and isotropic covariance function, respectively.

As far as the basic properties of covariance functions are concerned, assume that $\mathrm{C}_i : \mathbb{R}^{d_i} \to \mathbb{R}_+$ ($i = 1, \ldots, n$) are positive, continuous and integrable stationary covariance functions with $d_i \in \mathbb{Z}_+$ and let $\mathbf{d} = (d_1, \ldots, d_n)$ such that $|\mathbf{d}| = d$. It is well known that some mean operators preserve the permissibility of the resulting structure. In particular, if we assume, without loss of generality, that the θ_i are nonnegative weights, the following are then permissible on $\mathbb{R}^d$:

1. The arithmetic average

$$\mathrm{C}_A(\mathbf{x}) = \sum_{i=1}^n \theta_i \mathrm{C}_i(\mathbf{x}_i),$$

2. the nonweighted geometric average up to a power n,

$$\mathrm{C}_{\mathrm{G}}(\mathbf{x}) = \prod_{i=1}^{n} \mathrm{C}_i(\mathbf{x}_i),$$

3. the k-power average ($k \in \mathbb{Z}_+$), up to a power k,

$$\mathrm{C}_{\mu^k}(\mathbf{x}) = \sum_{i=1}^{n} \theta_i C_i^k(\mathbf{x}_i).$$

4. Scale and power mixtures of covariance functions [30],

$$\mathrm{C}(\mathbf{x}) = \int_{\Theta} \mathrm{C}(\mathbf{x}; \theta)\mathrm{d}\mu(\theta),$$

for μ a positive measure and $\theta \in \Theta \subseteq \mathbb{R}^p$, $p \in \mathbb{N}$.

2.2. The methodology. Quasi-arithmetic multivariate compositions

Quasi-arithmetic averages have been extensively treated by Hardy et al. [42]. Our methodology generalises the concept of quasi-arithmetic averages and introduces formalisms and notation, since our aim is to find a class of compositions of covariance functions that satisfies desirable properties.

Let Φ be the class of real-valued functions φ defined in some domain $D(\varphi) \subset \mathbb{R}$, admitting a proper inverse φ^{-1} defined in $D(\varphi^{-1}) \subset \mathbb{R}$ and such that $\varphi(\varphi^{-1}(t)) = t$ for all $t \in D(\varphi^{-1})$. Also, let Φ_{c} and Φ_{cm} be the subclasses of Φ obtained by restricting φ to be, respectively, convex or completely monotone on the positive real line. Let us call a *quasi-arithmetic class of functionals* the class

$$\mathfrak{Q} := \left\{ \psi : D(\varphi^{-1}) \times \cdots \times D(\varphi^{-1}) \to \mathbb{R} : \psi(\mathbf{u}) = \varphi\left(\sum_{i=1}^{n} \theta_i \varphi^{-1}(u_i)\right),\ \varphi \in \Phi \right\}, \tag{4}$$

where θ_i are nonnegative weights and $\mathbf{u} = (u_1, \ldots, u_n)'$, for $n \geq 2$ positive integer. Also, we shall call $\mathfrak{Q}_{\mathrm{c}}$ and $\mathfrak{Q}_{\mathrm{cm}}$ the subclasses of $\mathfrak{Q}$ when restricting φ to belong, respectively, to Φ_{c} and Φ_{cm}.

If $\psi \in \mathfrak{Q}$, then we should write φ_ψ as the function such that: for any nonnegative vector $\mathbf{u}$, $\psi(\mathbf{u}) = \varphi_\psi\left(\sum_{i=1}^{n} \theta_i \varphi_\psi^{-1}(u_i)\right)$. For ease of notation, we simply write φ instead of φ_ψ, whenever no confusion arises.

Next, we introduce a new class of functionals that will be used extensively throughout the paper.

Definition 1 (*Quasi-arithmetic compositions*). If $f_i : \mathbb{R}^{d_i} \to \mathbb{R}_+$ such that $\cup_i^n f_i(\mathbb{R}^{d_i}) \subset D(\varphi^{-1})$ for some $\varphi \in \Phi$, the *quasi-arithmetic composition of* $f_1, f_2, \ldots, f_n$ *with generating function* $\psi \in \mathfrak{Q}$ *is defined as the functional*

$$\mathcal{Q}_\psi(f_1, \ldots, f_n)(\mathbf{x}) = \psi\left(f_1(\mathbf{x}_1), \ldots f_n(\mathbf{x}_n)\right) \tag{5}$$

for $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)'$, $\mathbf{x}_i \in \mathbb{R}^{d_i}\mathbf{d} = (d_1, \ldots, d_n)'$ *and* $|\mathbf{d}| = d$.

Throughout the paper, we refer to $\psi \in \mathfrak{Q}$ or the corresponding $\varphi \in \Phi$ as the generating functions of $\mathcal{Q}_\psi$. Note that $\mathcal{Q}_\psi(f, \ldots, f) = f$ for any function f and generating function ψ.

Remark 1. A very special case of the quasi-arithmetic compositions is obtained when $\theta_i = 1/n$, $i = 1, \ldots, n$, and it is called Archimedean composition. This setting has been used by Porcu et al. [43], who work with $n = 2$ to build classes of nonstationary spectral densities for spatial data. It is important to observe that, whilst the Archimedean composition of spectral densities defines a permissible structure, the same construction does not hold, in general, for quasi-arithmetic compositions of spectral densities.

Moreover, in this paper quasi-arithmetic functionals are used for the composition of $n \geq 2$ covariance functions. It should be noted that great part of the formalism following subsequently coincides with that in [43], even if the methodology is proposed in a completely different setting and with different purposes.

Ordering relations as well as a minimal element can be found among the set of quasi-arithmetic compositions of $n \in \mathbb{N}$ fixed functions indexed by convex generating functions (a maximal element can also be found only when both fixed functions are upper bounded). For a finite set of weights $\{\theta_i\}$, $i = 1, \ldots, n$ summing up to one, we shall write

$$\mathcal{Q}_{\mathrm{G}}(f_1, \ldots, f_n) = \prod_{i=1}^{n} f_i^{\theta_i},$$

which is the quasi-arithmetic composition associated with $\varphi(t) = \exp(-t)$ that generates a geometric average, with the conventions $\ln 0 = -\infty$ and $\exp(-\infty) = 0$. Also, let

$$\mathcal{Q}_{\mathrm{A}}(f_1, \ldots, f_n) = \sum_{i=1}^{n} \theta_i f_i,$$

for $f_1, \ldots, f_n : \mathbb{R}^d \to [0, M]$), be the quasi-arithmetic composition associated with $\varphi(t) = M(1 - t/M)_+$ that generates the arithmetic average. Finally, we shall call

$$\mathcal{Q}_{\mathrm{H}}(f_1, \ldots, f_n) = \frac{1}{\sum_{i=1}^{n} \frac{\theta_i}{f_i(\mathbf{x}_i)}},$$

the quasi-arithmetic composition associated with $\varphi(t) = 1/t$ and generating the harmonic average, with the conventions $1/\infty = 0$, $1/0 = \infty$ and $0/0 = 0$.

We shall write $g_1 \le g_2$ whenever $g_1(\mathbf{x}_1, \ldots, \mathbf{x}_n) \le g_2(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ for all $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)' \in \mathbb{R}^d$. Finally, recall that a function g is subadditive whenever $g(a + b) \le g(a) + g(b)$ for all a, b in its domain. The following result is the natural extension of that in [43] to the n-variate case and where the involving functions in the compositions are quasi-arithmetic.

Proposition 1. *For any finite system of functions $f_1, \ldots, f_n$ and any arbitrary generating functions $\varphi, \varphi_1, \varphi_2 \in \Phi_{\mathrm{cm}}$, to which $\psi, \psi_1, \psi_2 \in \mathfrak{Q}_{\mathrm{cm}}$ are respectively associated with, we have the following point-wise order relations:*

(i) *If $\varphi_1^{-1} \circ \varphi_2$ is convex, then $\mathcal{Q}_{\psi_1}(f_1, \ldots, f_n) \le \mathcal{Q}_{\psi_2}(f_1, \ldots, f_n)$.*
(ii) *If $\varphi_1^{-1} \circ \varphi_2$ is concave, then $\mathcal{Q}_{\psi_1}(f_1, \ldots, f_n) \ge \mathcal{Q}_{\psi_2}(f_1, \ldots, f_n)$.*
In particular,
(iii) *$\mathcal{Q}_{\psi}(f_1, \ldots, f_n) \le \mathcal{Q}_{\Pi}(f_1, \ldots, f_n) \le \frac{\sum_i \theta_i f_i}{n}$ ($=\mathcal{Q}_{\Sigma}(f_1, \ldots, f_n)$ whenever $f_1, \ldots, f_n$ are bounded).*
(iv) *$\mathcal{Q}_{H}(f_1, \ldots, f_n) \le \mathcal{Q}_{\Pi}(f_1, \ldots, f_n) \le \frac{\sum_i \theta_i f_i}{n}$ ($=\mathcal{Q}_{\Sigma}(f_1, \ldots, f_n)$ whenever $f_1, \ldots, f_n$ are bounded), which is the classical inequality of general means.*

The proof of the result is omitted as it is obtained by the same arguments as that in [43].

Note that this result should extend to the functional case those reported by Nelsen [44]. Surprisingly, this extension needs different requirements on the compositions $\varphi_1^{-1} \circ \varphi_2$, as Nelsen result needs subadditivity of this composition.

2.3. Other useful notions and notation

A real mapping $\gamma : \mathbb{X} \subseteq \mathbb{R}^d \to \mathbb{R}$ such that $\gamma(\mathbf{0}) = 0$, is called an intrinsically stationary variogram [45] if it is conditionally negative definite, that is

$$\sum_{i=1}^{n} \sum_{i=j}^{n} a_i a_j \gamma(\mathbf{x}_i - \mathbf{x}_j) \le 0$$

for all finite collections of real weights a_i summing up to zero and all points $\mathbf{x}_i \in \mathbb{X}$. The restriction to the isotropic case is analogue to that of covariance functions, that is $\gamma(\mathbf{x}) := \tilde{\gamma}(\|\mathbf{x}\|)$, $\mathbf{x} \in \mathbb{X}$ and $\tilde{\gamma} : \mathbb{R}_+ \to \mathbb{R}$ conditionally negative definite.

A completely monotone function φ is a positive function defined on the positive real line and satisfying

$$(-1)^n \varphi^{(n)}(t) \ge 0, \quad t > 0,$$

for all $n \in \mathbb{N}$. Completely monotone functions are characterised in Bernstein's theorem (see [46], p. 439) as the Laplace transforms of positive and bounded measures. By a theorem of Schoenberg [47], a function C is radially symmetric and positive definite on any d-dimensional Euclidean space $\mathbb{R}^d$ if and only if $\mathrm{C}(\mathbf{x}) := \varphi(\|\mathbf{x}\|^2)$, $\mathbf{x} \in \mathbb{R}^d$, with φ completely monotonic on the positive real line.

Bernstein functions are positive functions defined on the positive real line, whose first derivative is completely monotonic. Once again, an intimate connection with (negative) definiteness arises, as γ is a radially symmetric and conditionally definite function on any d-dimensional Euclidean space $\mathbb{R}^d$ if and only if $\gamma(\mathbf{x}) := \mathcal{B}(\|\mathbf{x}\|^2)$, with $\mathcal{B}$ a Bernstein function.

Sufficient conditions for positive definiteness are stated in Pólya's criteria [25] in $\mathbb{R}^1$; and in [27,30,48], who extend criteria of the Pólya type to $\mathbb{R}^d$.

3. Theoretical results

In this section we present theoretical results in a general setting, *i.e.* working with arbitrary partitions of d-dimensional spaces as explained previously. In particular, we shall obtain permissibility criteria for quasi-arithmetic averages of covariance functions on $\mathbb{R}^d$. This will be done for (a) a general case in which the respective arguments of the covariance functions used for the quasi-arithmetic average have no restrictions; (b) the restriction to the component-wise isotropic case; and (c) the further restriction to covariances that are even and defined on the real line. Also, we shall show some properties of this construction. In particular, we refer to the associativity of quasi-arithmetic functionals and to the extension of ordering relations in Proposition 1 to the case of compositions of covariance functions. The proof of these new results can be found in the Appendix.

Remark 2. The proposition following subsequently is proven through straightforward arguments. The importance of this result relies on its two corollaries, that establish permissibility criteria for the geometric and harmonic averages of covariance functions, which represents a novel result in the literature.

Proposition 2. (a) ***General case.*** *Let $\varphi \in \Phi_{\mathrm{cm}}$ and $\mathrm{C}_i : \mathbb{R}^{d_i} \to \mathbb{R}$ $(i = 1, \ldots, n)$ be continuous stationary covariance functions such that $\cup_i^n \mathrm{C}_i(\mathbb{R}^{d_i}) \subset D(\varphi^{-1})$ and $\mathbf{d} = (d_1, \ldots, d_n)'$, $|\mathbf{d}| = d$. If the functions $\mathbf{x}_i \mapsto \varphi^{-1} \circ \mathrm{C}_i(\mathbf{x}_i)$, $i = 1, \ldots, n$, are intrinsically stationary variograms on $\mathbb{R}^{d_i}$, then*

$$\mathcal{Q}_\psi\,(\mathrm{C}_1, \ldots, \mathrm{C}_n)\,(\mathbf{x}_1, \ldots, \mathbf{x}_n) \tag{6}$$

is a stationary covariance function on $\mathbb{R}^d$.

(b) ***Component-wise isotropy.*** *Let $\varphi \in \Phi_{\mathrm{cm}}$ and $\mathrm{C}_i : \mathbb{R}^{d_i} \to \mathbb{R}$ $(i = 1, \ldots, n)$ be continuous stationary and* ***isotropic*** *covariance functions such that $\cup_i^n \mathrm{C}_i(\mathbb{R}^{d_i}) \subset D(\varphi^{-1})$ and $\mathbf{d} = (d_1, \ldots, d_n)'$, $|\mathbf{d}| = d$. If the functions $x \mapsto \varphi^{-1} \circ \mathrm{C}_i(x)$ are Bernstein functions on the positive real line, then*

$$\mathcal{Q}_\psi\,(\mathrm{C}_1, \ldots, \mathrm{C}_n)\,(\|\mathbf{x}_1\|, \ldots, \|\mathbf{x}_n\|) \tag{7}$$

is a stationary and fully symmetric covariance function on $\mathbb{R}^d$.

(c) ***Univariate covariances.*** *Let $\varphi \in \Phi_{\mathrm{cm}}$ and $\mathrm{C}_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, \ldots, n)$ be even, continuous stationary covariance functions defined on the real line such that $\cup_i^n \mathrm{C}_i(\mathbb{R}) \subset D(\varphi^{-1})$ and $|\mathbf{d}| = n$. If the functions $x \mapsto \varphi^{-1} \circ \mathrm{C}_i(x)$ are continuous, increasing and concave on the positive real line, then*

$$\mathcal{Q}_\psi\,(\mathrm{C}_1, \ldots, \mathrm{C}_n)\,(|x_1|, \ldots, |x_n|) \tag{8}$$

is a stationary and fully symmetric covariance function on $\mathbb{R}^n$.

It is worth noticing that case (8) represents a covariance permissibility condition that does not depend on the Euclidean metric, as it is function of the Manhattan or city block distance. For a detailed discussion of the limitations of the Euclidean norm-dependent covariances, see [39].

The previous result is of particular importance for its implications of two classes of means, the geometric and harmonic ones. Results are specified subsequently as corollaries.

Corollary 1 (*Geometric Average*). *Let $C_i : \mathbb{R}^{d_i} \to \mathbb{R}$ $(i = 1, \ldots, n)$ be continuous permissible covariance functions. Let θ_i $(i = 1 \ldots, n)$ be nonnegative weights summing up to one. If the functions $x \mapsto -\ln(C_i(x))$, $x > 0$ satisfy any of the relevant conditions described in* (a), (b) *and* (c) *of* Proposition 2 *above, then*

$$\mathcal{Q}_{\mathrm{G}}(C_1, \ldots, C_n) = \prod_{i=1}^{n} C_i^{\theta_i}$$

is a covariance function.

An example of this setting can be found by using the function $x \mapsto (1 + x^\delta)^{-\varepsilon}$, x positive argument, $\delta \in (0, 2]$ and ε positive, also known as generalised Cauchy class [49]. One can verify that the composition of this function with the natural logarithm is continuous, increasing and concave on the positive real line for $\delta \in (0, 1]$. Another function satisfying these requirements is the function $x \mapsto \exp(-x^\delta)$, which is completely monotonic for $\delta \in (0, 1]$. It should be stressed that these permissibility criteria do not apply to compactly supported covariance functions, such as the spherical model [30].

Corollary 2 (*Harmonic Average*). *Let $C_i : \mathbb{R}^{d_i} \to \mathbb{R}$ $(i = 1, \ldots, n)$ be continuous permissible covariance functions. If the functions $x \mapsto C_i(x)^{-1}$, $x > 0$ satisfy any of the relevant conditions described in* (a), (b) *and* (c) *of* Proposition 2, *then for $\theta_i \geq 0$ such that $\sum_i \theta_i = 1$, the*

$$\mathcal{Q}_{\mathrm{H}}(C_1, \ldots, C_n) = \frac{1}{\sum \frac{\theta_i}{C_i}}$$

is a covariance function.

An example of this setting can be found by considering, as previously, the covariance function $x \mapsto (1+x^\delta)^{-\varepsilon}$, x positive argument, $\delta \in (0, 2]$ and restricting ε to belong to the interval $(0, 1]$. It can be readily verified that this function satisfies the requirements of Corollary 2. Another function that satisfies these requirements is the so-called Dagum covariance function [50], having expression $(1 + x^{-\delta})^{-\varepsilon}$, for $\delta \in (0, 2]$ and $\varepsilon \in (0, 1]$. We conjecture that these requirements are also satisfied for the Matérn class (1960), when its smoothing parameter belongs to some specified interval. Certainly, this property is not satisfied when the smoothing parameter is either identically equal to 1/2 or tends to infinity, as these cases respond respectively to the exponential model and the Gaussian one, for which it can be readily verified that Corollary 2 does not apply.

In order to complete the picture about quasi-arithmetic covariance functions, it would be desirable to establish at least some of their algebraic properties. The following results show some important features of the theoretical construction obtained from Proposition 2.

Proposition 3 (*Associativity*). *Consider the same arrangements as in Proposition 2 and set $\theta_i = 1/n$ ($i = 1, \ldots, n$). Let $\varphi_1 \in \Phi_{\mathrm{cm}}$ and $\varphi_2 \in \Phi$. If the functions $x \mapsto \varphi^{-1} \circ \varphi_2(x)$, $x \mapsto \varphi_2^{-1} \circ \mathrm{C}_i(x)$ ($i = 1, \ldots, k$) and $x \mapsto \varphi_1^{-1} \circ C_j(x)$ ($j = k+1, \ldots, n$; $t > 0$ and $k < n$) satisfy any of the relevant conditions described in* (a), (b) *and* (c) *of Proposition 2, then*

$$\mathcal{Q}_{\psi_1}\left(\mathcal{Q}_{\psi_2}(\mathrm{C}_1, \ldots, \mathrm{C}_k), \mathrm{C}_{k+1}, \ldots, \mathrm{C}_n\right)(\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n), \tag{9}$$

$$\mathcal{Q}_{\psi_1}\left(\mathcal{Q}_{\psi_2}(\mathrm{C}_1, \ldots, \mathrm{C}_k), \mathrm{C}_{k+1}, \ldots, \mathrm{C}_n\right)(\|\mathbf{x}_1\|, \ldots, \|\mathbf{x}_k\|, \|\mathbf{x}_{k+1}\|, \ldots, \|\mathbf{x}_n\|) \tag{10}$$

and

$$\mathcal{Q}_{\psi_1}\left(\mathcal{Q}_{\psi_2}(\mathrm{C}_1, \ldots, \mathrm{C}_k), \mathrm{C}_{k+1}, \ldots, \mathrm{C}_n\right)(|x_1|, \ldots, |x_k|, |x_{k+1}|, \ldots, |x_n|) \tag{11}$$

and accordingly any coherent permutation of φ_1, φ_2 with C_i ($i = 1, \ldots, n$) are covariance functions.

One may notice that we deviate from the associativity condition defined in [51] and (in a more general form called decomposability) in [5]. Nevertheless, it can be readily verified that the associativity condition of Bemporad is always satisfied for our construction, whereas the strong and weak decomposability of Marichal [5] is certainly satisfied by construction (8), but not, in general, by constructions (6) and (7).

Remark 3. Proposition 1 applies mutatis mutandis to the case of quasi-arithmetic compositions of covariance functions. This fact has some implications on the variance of the stationary random field generated by a quasi-arithmetic operator. This will be discussed in the subsequent sections. Also, it is important to specify that other results, in particular inequalities involving quasi-arithmetic operators, can be readily extended to the case of quasi-arithmetic compositions of covariance functions. This is the case, for instance, of Theorem 1 in [52].

4. Using quasi-arithmetic functionals in the construction of nonseparable space–time covariance functions

4.1. Review of space–time covariance functions

Natural (physical, health, cultural etc.) systems involve various attributes, such as atmospheric pollutant concentrations, precipitation fields, income distributions, and mortality fields. These attributes are characterised by spatial–temporal variability and uncertainty that may be due to epistemic and ontologic factors. In view of the prohibiting costs of spatially dense monitoring networks, one often aims to develop a mathematical model of the natural system in a continuous space–time domain, based on sequential observations at a limited number of monitoring stations. This kind of problem has been a motivation for the development of the *spatio-temporal random field* (S/TRF) theory; see [28–30,41] for a detailed discussion of the ordinary and generalised S/TRF theory and its various applications. In the following, we slightly change notation in order to be consistent with classical nomenclature in the Geostatistical literature [53,54]. Let $\{Z(\mathbf{s}, t), (\mathbf{s}, t) \in \mathbb{R}^d \times \mathbb{R}\}$ be a real-valued S/TRF, where $\mathbf{s}, t$ denote respectively the spatial and temporal positions. Then, the function $\mathrm{C}_{s,t}(\mathbf{s}_1, t_1, \mathbf{s}_2, t_2) = \mathrm{cov}(Z(\mathbf{s}_1, t_1), Z(\mathbf{s}_2, t_2))$, defined on the product space $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$, is the covariance function of the associated Gaussian $S/TRFZ$ if and only if it is permissible, *i.e.* satisfies (1). When referring to the spatial index, the term *homogeneity* instead of weak stationarity is equivalently adopted [28,30]. Thus, under the assumption of spatial homogeneity and temporal stationarity (sometimes, simply called spatio-temporal stationarity in the weak sense), the underlying S/TRF, ergo denoted by H/S, has finite and constant mean and the covariance function, defined on the product space $\mathbb{R}^d \times \mathbb{R}$, is such that $\mathrm{cov}(Z(\mathbf{s}_1, t_1), Z(\mathbf{s}_2, t_2)) = \mathrm{C}_{s,t}(\mathbf{h}, u)$, with $(\mathbf{h}, u) = (\mathbf{s}_i - \mathbf{s}_j, t_i - t_j) \in \mathbb{R}^d \times \mathbb{R}$ denoting the spatio-temporal separation vector, and the $\mathbf{h}$ and u denoting the spatial and the temporal lags, respectively. The special case of isotropy in the spatial component and symmetry in the temporal one is denoted as

$$\mathrm{C}_{s,t}(\mathbf{h}, u) := \widetilde{\mathrm{C}}_{s,t}(\|\mathbf{h}\|, |u|), \tag{12}$$

where the $\|.\|$ denotes the Euclidean norm. Obviously, (12) is fully symmetric.

Another popular assumption concerning the S/TRF model is that of separability, that is [41,83]

$$\mathrm{C}_{s,t}(\mathbf{h}, u) = \mathrm{C}_{s,t}(\mathbf{h}, 0)\mathrm{C}_{s,t}(\mathbf{0}, u). \tag{13}$$

In other words, separability means that the spatio-temporal covariance structure factors into a purely spatial and a purely temporal component, which allows for computationally efficient estimation and inference. Consequently, separable covariance models have been popular even in situations in which they are not physically justifiable. Another interesting aspect is that separable covariances are also fully symmetric, whereas the converse is not necessarily true.

It has been argued in the relevant literature that separable models allow for ease of computation and dimensionality reduction, as the space–time covariance matrix is obtained through the Kronecker product of the marginal spatial and temporal ones. However, separability is an unrealistic assumption for many applications, since it implies a considerable loss of information about important interactions between the spatial and temporal variations. Therefore, various techniques have been introduced for generating different classes of nonseparable spatio-temporal covariance models. Most of these techniques have been developed in the context of applied *stochastic* analysis and include H/S as well as non-H/S covariances (e.g., [55–57,28–30,28,29,38,41,58–60]). Also, covariance models have been developed in the context of spatio-temporal *statistics* (e.g., [61,53,62,63,54,64–67]).

Being the Laplace transform of positive bounded measures, completely monotone functions are particularly appealing for the construction of stationary and nonstationary space–time covariances. In particular, they are intimately connected with the concept of *mixture-based* covariance functions, that has been repeatedly used by several authors. In the stationary case, see [54,68,64,69,65,67]. In the nonstationary case, [70,71,33,43] have made use of this technique.

Also, mixture-based covariances have been developed with less sophisticated instruments than completely monotone functions. This is the case of the so-called product sum model [62] and their extensions [72,73]. These groups of authors build nonseparable space–time covariances through simple application of the basic properties of covariances seen as a convex cone. The mixture-based procedures and the basic properties of covariance functions are properly combined in the present paper.

In the following, a stationary RF with a quasi-arithmetic covariance function will be called *quasi-arithmetic random field* and denoted with the acronym QARF.

4.2. On the representation and smoothness properties of QARF

In this section we focus on the representation of a QARF and discuss the smoothness properties in terms of (mean square) partial differentiability. These properties are intimately related to those of the associated covariance function.

Let $Z_i(s)$ be univariate mutually independent continuous weakly stationary Gaussian random processes defined on the real line ($i = 1, \dots, (d+1)$; $s \in \mathbb{R}$; and $d \in \mathbb{Z}_+$). In particular, let the process Z_{d+1} be continuously indexed by time t. Consider also a $(d+1)$-dimensional nonnegative random vector $\mathbf{R} = (R_1, \dots, R_{d+1})'$ with R_i independent of Z_i. Let the univariate covariances C_{s_i} and the temporal covariance C_t be respectively associated with Z_i, $i = 1, \dots, d$ and Z_{d+1}. In the following we shall assume these covariances to be stationary, even, and of the type $\mathrm{C}_{s_i}(h_i) = \exp(-\nu_i(|h_i|))$, $i = 1, \dots, d$, and $C_t(u) = \exp(-\nu_t(|u|))$, with $\nu_i = \varphi^{-1} \circ \mathrm{C}_i$, $\nu_t = \varphi^{-1} \circ \mathrm{C}_t$, where the $\varphi \in \Phi_{\mathrm{cm}}$ and C_i are positive definite and such that the compositions ν_i are continuous, increasing and concave on the positive real line. Positive definiteness of this construction is guaranteed by direct application of the theorem of [47] and according to a Pólya-type criterion (see [25], Proposition 10.6).

We are interested in inspecting the properties of the following stationary spatio-temporal scale mixture-based random field, defined on $\mathbb{R}^d \times \mathbb{R}$,

$$Z(\mathbf{s}, t) := Z_{d+1}(tR_{d+1}) \prod_{i=1}^{d} Z_i(s_i R_i), \tag{14}$$

with $\mathbf{s} = (s_1, \dots, s_d)' \in \mathbb{R}^d$ and $t \in \mathbb{R}$. It can be easily seen that the covariance structure associated to this random field is nonseparable, as

$$\mathrm{C}_{s,t}(\mathbf{h}, u) = \int_{\mathbb{R}_+^{d+1}} \exp\left(-\sum_{i=1}^{d} \nu_i(|h_i|) r_i - \nu_t(|u|) r_{d+1}\right) \mathrm{d}F(\mathbf{r}), \tag{15}$$

with $\mathbf{h} = (h_1, \dots, h_d)' \in \mathbb{R}^d$, $u \in \mathbb{R}$ and $\mathbf{r} = (r_1, \dots, r_{d+1})' \in \mathbb{R}^{d+1}$, and where F is the $(d+1)$-variate distribution function associated to the random vector $\mathbf{R}$. If F is absolutely continuous with respect to the Lebesgue measure, then previous representation can be reformulated with respect to the $(d+1)$-variate density, say f, that is

$$\mathrm{C}_{s,t}(\mathbf{h}, u) = \int_{\mathbb{R}_+^{d+1}} \exp\left(-\sum_{i=1}^{d} \nu_i(|h_i|) r_i - \nu_t(|u|) r_{d+1}\right) f(\mathbf{r}) \mathrm{d}\mathbf{r}.$$

It can be seen easily that this construction allows for the case of separability if and only if the integrating $(d+1)$-dimensional measure F (or equivalently its associated density f) factorises into the product of $(d+1)$ marginal ones, *i.e.* if the nonnegative random vector $\mathbf{R}$ has mutually independent components.

Now, if we suppose that the measure F is concentrated on the line $r_1 = \dots = r_{d+1} = r$ and that φ^{-1} is such that $\varphi := \mathcal{L}[F]$, *i.e.* the Laplace transform of the positive univariate measure F, then we obtain that the QARF is a special case of (15). For this random field, further inferences may be made about its mean square partial differentiability if, additionally, we assume the function $t \mapsto \exp(-\nu_i(t))$ to be absolutely integrable on the positive real line ($i = 1, \dots, (d+1)$). In this case, one can show (technicalities can be found in the Appendix) that the kth order mean square partial derivative with respect to the ith coordinate exists and is finite whenever the function $\chi(r) := \int_{[0,\infty)} \omega_i^{2k} \hat{c}_i(\omega_i; r) d\omega_i$, with $\hat{c}_i := \mathcal{F}^{-1}[\exp(-r\nu_i)] = \int_{\mathbb{R}} \exp(-i\omega_i h_i - r\nu_i(h_i)) dh_i$ is measurable with respect to F ($i = 1, \ldots, d$).

It should be stressed that Proposition 1 gives some more information about the characteristics of the underlying QARF in terms of variance, as QARF can be ordered with respect to their minimum or maximum variance. Thus, the QARF generated by $\psi := A$ has the largest variance among all the other QARF, for any choice of ψ.

Finally, observe that the trivial quasi-arithmetic composition, obtained by setting $C_i := \varphi \in \Phi_{cm}$ ($i = 1, \ldots, d$) preserves permissibility and results in a model of the type

$$C_s(\mathbf{h}) = \varphi\left(\boldsymbol{\theta}'\mathbf{h}\right)$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d)' \in \mathbb{R}^d$ and $\mathbf{h} = (\mathbf{h}_1, \ldots, \mathbf{h}_d)'$ is the arbitrary partition of the spatial lag vector. Thus, the trivial case results in a composition of a completely monotonic function with an affine function, and can be used for modelling geometric anisotropies.

It should be noted that, recently, there has been some interest in statistical modelling of gradients [74,75]. Large-scale inference for random spatial surfaces over a region using spatial process models has been studied. Under such models, local analysis of the surface (e.g., gradients at given points) has received recent attention. Learning about where the surface exhibits rapid change is called wombling. The concept of rapid change in the spatial surface is central to wombling and is mathematically formalized using spatial gradients. In this context, smooth covariance functions play a role in the statistical modelling. Thus, our QARF constructions could have potential applications in such timely problems.

4.3. *Applications of quasi-arithmeticity in the construction of stationary nonseparable space–time covariances*

The results presented in the previous section can be useful in the construction of space–time covariance functions. For this purpose, some remarks are in order. The functional in Eq. (4) should be adapted in the spatio-temporal case. It does not make sense to consider a weighted average of a spatial covariance with a temporal one. Also, the use of weights forbids, in this case, the construction of a nonseparable model admitting separability as a special case. For this reason, we suggest to reduce the class in Eq. (4) to the case of trivial weights, *e.g.* $\theta_i = 1, \forall i$. Then, it is more appropriate to call the class (4) *Archimedean*, in analogy with the class built in [17]. It should be mentioned that one can easily prove that the restriction to trivial weights does not affect the permissibility of the resulting covariance function, provided that either one of the constraints imposed in cases (a), (b) or (c) of Proposition 2 is fulfilled. By analogy with the construction of copulas, and following [17], the definition of a *generator* for $\varphi \in \Phi_{cm}$ is more meaningful.

Working this way, one can obtain some new families of covariance functions whose analytical expressions are familiar for those interested in probabilistic modelling through copulas. It should be noticed that all the families we propose in this section include separable covariance models as special cases, depending on the parameter values of the generators.

Example 1 (*The Clayton Family*). Consider the completely monotone function

$$\varphi(x) = (1 + x)^{-1/\lambda_1}, \quad x > 0, \tag{16}$$

where λ_1 is a nonnegative parameter with inverse $\varphi^{-1}(y) = y^{-\lambda_1} - 1$. Observe that (16) is the generator of the Clayton family of copulas; refer to [17] for mathematical details about this class.

Also, consider the covariance functions $C_s(\mathbf{h}) = (1+\|\mathbf{h}\|)^{-1/\lambda_2}$ and $C_t(u) = (1+|u|)^{-1/\lambda_3}$, for λ_2, λ_3 positive parameters. It is easy to verify that $\varphi^{-1}(C_i(y)) = (1 + y)^{\lambda_1/\lambda_i}$ ($i = 2, 3$) is a positive function whose first derivative is completely monotone if and only if $\lambda_1 < \lambda_i$. Under this constraint, and applying case (b) of Proposition 2, we find that

$$C_{s,t}(\mathbf{h}, u) = \mathcal{Q}_\psi(C_s, C_t)(\mathbf{h}, u) = \sigma^2 \left[(1 + \|\mathbf{h}\|)^{\rho_1} + (1 + |u|)^{\rho_2} - 1\right]^{-1/\lambda_1} \tag{17}$$

is a valid nonseparable stationary fully symmetric spatio-temporal covariance function, with $\rho_i = \lambda_1/\lambda_i$, $\lambda_i > 0$ ($i = 2, 3$) and σ^2 a nonnegative parameter denoting the variance of the underlying S/T process. It is interesting that both margins (the spatial and the temporal one) belong to the generalised Cauchy class. This is desirable for those interested in the local and global behaviour of the underlying random field.

Another covariance function that preserves Cauchy margins can be obtained as follows. Consider the function $x \mapsto \varphi(x) = x^{-\alpha}$, $t > 0$, that belongs to Φ_{cm} for any positive α, being the Laplace transform of the function $x \mapsto x^{\alpha-1}/\Gamma(\alpha)$, with α positive parameter and Γ the Euler Gamma function. Also, for the spatial margin consider $C_s(\mathbf{h}) = (1 + \|\mathbf{h}\|^\delta)^{-\varepsilon}$ that belongs to Φ_{cm} if and only if $\delta \in (0, 2]$ and ε is strictly positive. Finally, let the temporal margin be of the type $C_t(u) = |u|^{-\rho}$, which is not a stationary covariance function but respects the composition criteria, as $\varphi^{-1} \circ C_t$ is continuous, increasing and concave on the positive real line if and only if $\alpha < \rho$. Then, it is easy to prove that $\mathcal{Q}_\psi(C_S, C_t)(\mathbf{h}, u)$ is a valid space–time function if and only if $\alpha < \varepsilon$ and $\alpha < \rho$, and that this covariance function has margins of the Cauchy type.

Example 2 (*The Gumbel–Hougard Family*). Consider the completely monotone function, the so-called positive stable Laplace transform

$$\varphi(x) = e^{-x^{1/\lambda_1}}, \quad x > 0, \tag{18}$$

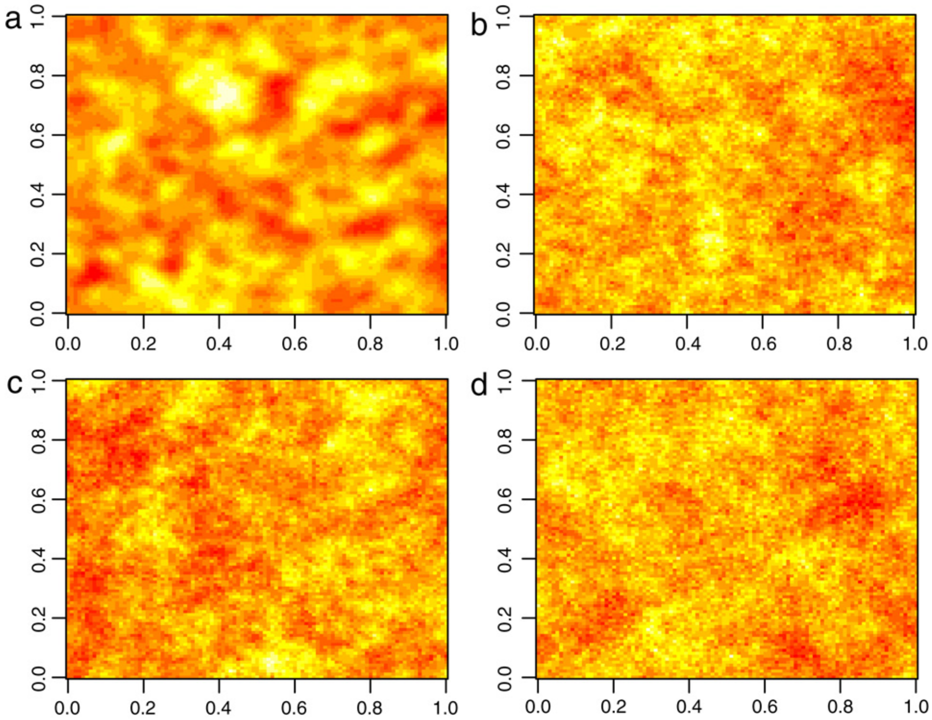

Fig. 1. Realizations of Gaussian RF with a Gumbel–Hougard covariance function: (a) $\lambda_1 = 1.01, \lambda_2 = 1.05, \lambda_3 = 5$; (b) $\lambda_1 = 1.01, \lambda_2 = 2.9, \lambda_3 = 5$; (c) $\lambda_1 = 2.5, \lambda_2 = 2.9, \lambda_3 = 5$; (d) $\lambda_1 = 4.9, \lambda_2 = 4.5, \lambda_3 = 5$.

where $\lambda_1 \geq 1$. Eq. (18) admits the inverse $\varphi^{-1}(y) = (-\ln(y))^{\lambda_1}$. This is the generator of the Gumbel–Hougard family of copulas, whose mathematical construction and details are exhaustively described in [44]. Consider two respectively spatial and temporal covariance functions admitting the same analytical form, *i.e.* $C_s(\mathbf{h}) = \exp(-\|\mathbf{h}\|^{1/\lambda_2})$ and $C_t(u) = \exp(-|u|^{1/\lambda_3})$. Then, it can be easily verified that $\varphi^{-1}(C_s(y)) = y^{\lambda_1/\lambda_2}$, and $\varphi^{-1}(C_t(y)) = y^{\lambda_1/\lambda_3}$ are always positive for $y > 0$ and possess completely monotone first derivatives if and only if $\lambda_1 < \lambda_i, i = 2, 3$. So we get that

$$C_{s,t}(\mathbf{h}, u) = \mathcal{Q}_\psi(C_s, C_t)(\mathbf{h}, u) = \sigma^2 \exp\left(-(\|\mathbf{h}\|^{\rho_1} + |u|^{\rho_2})^{1/\lambda_1}\right), \tag{19}$$

is a permissible nonseparable stationary fully symmetric spatio-temporal covariance function, with $\rho_i = \lambda_1/\lambda_i, i = 2, 3$ and σ^2 as before. Fig. 1 shows realizations of Gaussian RF with this type of covariance under different settings of the parameters.

Example 3 (*The Power Series Family*). The so-called power series

$$\varphi(x) = 1 - (1 - \exp(-x))^{1/\lambda_1}, \quad x > 0, \tag{20}$$

with $\lambda_1 \geq 1$, admits the inverse $\varphi^{-1}(y) = -\ln(1 - (1 - y)^{\lambda_1})$. Suppose a spatial and a temporal covariance function of the same analytical form as in Example 2. The composition $\varphi^{-1}(C_i(y)) = -\ln(1 - (1 - \exp(-y))^{\lambda_1/\lambda_i}), i = 2, 3$, is always positive for $y > 0$ and admits a completely monotone first derivative if and only if $\lambda_1 < \lambda_i$. So we get that

$$\mathcal{Q}_\psi(C_S, C_T)(\mathbf{h}, u) = 1 - (1 - \exp(-\|\mathbf{h}\|))^{\rho_1} - (1 - \exp(-|u|))^{\rho_2} + (1 - \exp(-\|\mathbf{h}\|))^{\rho_1}(1 - \exp(-|u|))^{\rho_2} \tag{21}$$

is a permissible nonseparable stationary fully symmetric spatio-temporal covariance function, with $\rho_i = \lambda_1/\lambda_i\ (i = 2, 3)$.

Example 4 (*The Semiparametric Frank Family*). Here we show that a stationary covariance function can be obtained, starting from the proposed setting, even if the arguments of the quasi-arithmetic functional are not covariance functions.

The Frank family of copulas [76] is generated by the function $\varphi(x) = \frac{1}{\lambda}\ln\left(1 - (1 - e^{-\lambda})e^{-x}\right)$ with inverse $\varphi^{-1}(y) = -\ln\left((1 - e^{-\lambda y})/(1 - e^{-\lambda})\right)$. Nelsen [44] shows that, for λ positive, φ is the composition of an absolutely monotonic function with a completely monotonic one, *i.e.* a completely monotonic function. As far as the inverse is concerned, it is easy to see that $\varphi^{-1} \circ \gamma$ (γ an intrinsically stationary variogram) is negative definite. Thus, the

$$C_{s,t}(\mathbf{h}, u) = -\frac{1}{\lambda}\ln\left(1 + \frac{(1 - e^{-\lambda\gamma_S(\mathbf{h})})(1 - e^{-\lambda\gamma_T(u)})}{1 - e^{-\lambda}}\right),$$

for γ_S, γ_T intrinsically stationary variograms defined on $\mathbb{R}^d$ and $\mathbb{R}$, respectively, is a stationary space–time covariance function.

4.4. Quasi-arithmeticity and nonstationarity in space

The direct construction of spatial covariances that are nonstationary is anything but a trivial fact. Only few contributions refer to this kind of construction: see, among them, Christakos [28,29], Christakos and Hristopoulos [41] and Kolovos et al. [60]. More recent contributions can be found in [33,71]. It seems that something more could be done concerning the construction of nonstationary covariances, knowing that stationarity is very often an unrealistic assumption for several physical and natural processes.

In this section, we show that both approaches discussed in [33,71], admit a natural extension through the use of quasi-arithmetic functionals. We need to introduce some more notation concerning the restriction of the class Φ_{cm}. For abuse of notation, a completely monotonic function is the Laplace transform of some positive and bounded measure, so that $\varphi \in \Phi_{\mathrm{cm}}$ if and only if $\varphi := \mathcal{L}[F]$.

Theorem 1. *Let Σ be a mapping from $\mathbb{R}^p \times \mathbb{R}^p$ to positive definite $p \times p$ matrices, F a nonnegative measure on $\mathbb{R}_+$, $\varphi_1, \varphi_2 \in \Phi_{\mathrm{cm}}$ and g a nonnegative function such that, for any fixed $\mathbf{s} \in \mathbb{R}^p$, $h_{\mathbf{s}} = \varphi_2^{-1} \circ g(.; \mathbf{s}) \in L^1(F)$. Define $\Sigma(\mathbf{s}_1, \mathbf{s}_2) = 1/2(\Sigma(\mathbf{s}_1) + \Sigma(\mathbf{s}_2))$ and $Q(\mathbf{s}_1, \mathbf{s}_2) = (\mathbf{s}_1 - \mathbf{s}_2)' \Sigma(\mathbf{s}_1, \mathbf{s}_2)^{-1}(\mathbf{s}_1 - \mathbf{s}_2)$. Then,*

$$C_s(\mathbf{s}_1, \mathbf{s}_2) = \frac{|\Sigma(\mathbf{s}_1)|^{1/4} |\Sigma(\mathbf{s}_2)|^{1/4}}{|\Sigma(\mathbf{s}_1, \mathbf{s}_2)|^{1/2}} \int_0^\infty \varphi_1\left(Q(\mathbf{s}_1, \mathbf{s}_2)\tau\right) \mathcal{Q}_{\psi_2}(g_{\mathbf{s}_1}, g_{\mathbf{s}_2})(\tau) \mathrm{d}F(\tau) \tag{22}$$

with $\theta_i = 1$, $i = 1, 2$, is a nonstationary covariance function on $\mathbb{R}^d \times \mathbb{R}^d$.

Some comments are in order. One can see that Stein's [33] result is a special case of (22), under the choice $\varphi_1(t) = \exp(-t)$, t positive, and $\psi_2 := G$. So is the result in [71]. Nevertheless, the form we propose has some drawbacks that need to be noticed explicitly. The first problem is that it is very difficult to obtain a closed form for expression (22), unless one chooses $\psi_2 := G$. Secondly, if one's purpose is to generalise the Matérn covariance function [35] to the nonstationary case, then the problem has already been solved by Stein [33] and his approach is in our opinion the most suitable, as he finds a Matérn-type covariance that allows for a spatially varying smoothness parameter and for local geometric anisotropy. On the one hand, the Matérn covariance possesses some desirable features (i.e., it allows for arbitrary levels of smoothness of the associated random field). On the other hand, there are other covariance functions that are of considerable interest to the statistical and scientific communities. In particular, we refer to the Cauchy class, whose properties (in terms of decoupling of the global and local behaviour of the associated random field) have been discussed in [49]. After several trials, we did not succeed in obtaining a Cauchy-type nonstationary covariance. Several examples of covariances can be derived that belong to the class (22) and can be numerically integrated. Here, we propose some closed form that is obtained by letting $\psi_2 := G$. As a first example, take $\mathrm{d}F(\tau) = \mathrm{d}\tau$, $\varphi_1(\tau) = \tau^{\lambda-1}$, that is completely monotonic for $\lambda \in (0, 1)$, $g(\tau; \alpha_i, \nu_i) = (1 + \alpha(\mathbf{s}_i)\tau)^{-\nu(\mathbf{s}_i)}$, where α and ν are supposed to be strictly positive functions of $\mathbf{s}_i$ $(i = 1, 2)$ and additionally $0 < \alpha(\mathbf{s}_i), \nu(\mathbf{s}_i) < \pi$. One can readily verify that all integrability conditions in Theorem 1 are satisfied. Letting $k = \frac{|\Sigma(\mathbf{s}_1)|^{1/4}|\Sigma(\mathbf{s}_2)|^{1/4}}{|\Sigma(\mathbf{s}_1, \mathbf{s}_2)|^{1/2}} Q(\mathbf{s}_1, \mathbf{s}_2)^{\lambda-1}$ and using [3.259.3] in [82], one obtains the following covariance function

$$C_s(\mathbf{s}_1, \mathbf{s}_2) = k\alpha(\mathbf{s}_1)^{-\lambda} \mathrm{B}(\lambda, \nu(\mathbf{s}_1) + \nu(\mathbf{s}_2) - \lambda)\, {}_2F_1\left(\nu(\mathbf{s}_2), \lambda; \nu(\mathbf{s}_1) + \nu(\mathbf{s}_2); 1 - \frac{\alpha(\mathbf{s}_2)}{\alpha(\mathbf{s}_1)}\right),$$

where $\mathrm{B}(., .)$ is the Beta function, and ${}_2F_1(., ., ., .)$ is the Gauss hypergeometric function.

Another example is obtained by considering $F(\mathrm{d}\tau) = \exp(-\tau)\mathrm{d}\tau$, $\varphi_1(\tau) = \tau^{\nu-1}$, with $\nu \in (0, 1)$, $g(\tau; \mathbf{s}_i) = \exp(-\frac{\alpha(\mathbf{s}_i)}{2}\tau)$, with $\alpha(\mathbf{s}_i)$ strictly positive $(i = 1, 2)$ and using [3.478.4] of [82], we find that

$$C_s(\mathbf{s}_1, \mathbf{s}_2) = 2k \left(\frac{\alpha(\mathbf{s}_1) + \alpha(\mathbf{s}_2)}{2}\right)^{-\nu/2} \mathcal{K}_\nu\left(2\left(\frac{\alpha(\mathbf{s}_1) + \alpha(\mathbf{s}_2)}{2}\right)^{1/2}\right)$$

is a nonstationary spatial covariance that allows for local geometric anisotropy, but has a fixed smoothing parameter.

It should be stressed that numerical integration under the setting (22) could outperform previously proposed models if the objective is to find a different type of interaction between the local parameters characterising the integrating function g. In all the examples proposed by Paciorek and Schervish [71] and Stein [33] the varying-smoothing, spatially adaptive parameter is obtained as the semisum of a parameter acting on the location $\mathbf{s}_1$ with another one depending on the location $\mathbf{s}_2$. This is a serious limitation of the method, as only one type of interaction can be achieved. Starting from a different setting, Pintore and Holmes [77] obtain the same type of spatially adaptive smoothing parameters. Thus, quasi-arithmetic functionals could be of help, at least through numerical integration. Finding a closed form for a functional different than $\mathcal{Q}_G$ remains an open problem to which we did not find any solution for the meantime.

5. Conclusions and discussion

In this work, novel results are presented concerning permissible spatial–temporal covariance functions in terms of the theory of quasi-arithmetic means, and valuable insight is gained about their space–time structure. The theory of quasi-arithmetic means is used, together with the relevant permissibility criteria, to derive new classes of nonseparable space–time covariances and to investigate their properties in considerable detail.

There are several possible avenues for research based on the results of the present paper. From the spatial and spatio-temporal statistics perspectives, the QARF representation considered in this paper seems very promising. E.g., the QARF may provide a naive separability testing procedure, as follows. Consider the set $\Psi_{\rm cm}$ of all possible generators of the QARF class and let Θ be the set of parameter vectors indexing the generators, *i.e.* $\Theta : \{\boldsymbol{\theta} \in \mathbb{R}^p : \varphi_{\boldsymbol{\theta}} \in \Phi_{\rm cm}\}$. Thus, testing for separability of the covariance function associated to the QARF is equivalent to testing for the null hypothesis $\varphi_0 := \varphi_0(x; \boldsymbol{\theta}_0) = \exp(-\boldsymbol{\theta}_0 x)$, $x > 0$. This is a topic worth of further investigation.

Another interesting topic is that of generator estimation. A tempting choice would be to consider the approach proposed by Genest and Rivest [78], and that in [79] who find a serial version of Kendall's tau and a relationship between this concordance index and the generating function φ. In this case, one would extend the result of these authors at least to the lattice $\mathbb{Z}^2$.

Future research effort would also focus on a deeper study of the properties of quasi-arithmetic covariances along several directions. For instance, it would be interesting to find limit properties of the covariance functions specified through this construction. Also, exploring the usefulness of our constructions to cross-covariance modelling would provide additional interest to multivariate spatial and spatio-temporal modelling. Finally, questions such as adequacy of QARF constructions to model large spatial data sets versus the use of tapered covariance functions would be a natural and interesting way to explore.

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Appendix

A.1. Proof of Proposition 2

Recall that, for Bernstein's theorem [46, p. 439], $\varphi \in \Phi_{\rm cm}$ if and only if

$$\varphi(t) = \int_0^\infty e^{-rt} dF(r), \tag{23}$$

with F a positive and bounded measure. Now, for the proof of (a), notice that, if $\varphi \in \Phi_{\rm cm}$, Eq. (6) can be written as

$$\int_0^\infty \exp\left(-r\sum_{i=1}^n \theta_i \varphi^{-1}\left(\mathrm{C}_i(\mathbf{h}_i)\right)\right) dF(r).$$

Now, observe that if, for every i, $\varphi^{-1} \circ \mathrm{C}_i$ is a variogram, then so is the sum $\sum_{i=1}^n \theta_i \varphi^{-1} \circ \mathrm{C}_i$. Then, by Schoenberg theorem (1939), we have that for every positive r, the integrand in the formula above is a covariance function. So is the positive scale mixture of covariances. This completes the proof.

For (b), it is sufficient to notice that, by Schoenberg theorem [47], the mapping $\gamma : \mathbb{R}^d \to \mathbb{R}_+$ is the variogram associated to an intrinsically stationary and isotropic random field if and only if $\gamma(\mathbf{h}) = \psi(\|\mathbf{h}\|^2)$, for ψ a Bernstein function. The rest of the proof comes from the same arguments of point (a).

For (c), notice that, being continuous, increasing and concave on $[0, \infty)$, each of the functions $\varphi^{-1}(\mathrm{C}_i)$, is negative definite on $\mathbb{R}$, according to a Pólya-type criterion (see [25], Proposition 10.6). So is their sum. Since $\varphi \in \Phi_{\rm cm}$, by Schoenberg's theorem (cf. [25]), we get once again the result.

A.2. Proof of Proposition 3

We shall only prove the result in (9), as (10) and (11) follow the same argument. Recall that here we impose $\theta_i = 1/n \forall i$. Call $\mathbf{h}^{(1)} = (\|\mathbf{h}_1\|, \ldots, \|\mathbf{h}_k\|)'$ and $\mathbf{h}^{(2)} = (\|\mathbf{h}_{k+1}\|, \ldots, \|\mathbf{h}_n\|)'$, $k < n$. Also, let $f_1 = 1/n \sum_{j=k+1}^n \varphi_1^{-1} \circ \mathrm{C}_j$, $f_2 = \varphi_1^{-1} \circ \varphi_2$ and $f_3 = 1/n \sum_{i=1}^k \varphi_2^{-1} \circ \mathrm{C}_i$. Thus, Eq. (9) can be written, using Bernstein's theorem, as

$$\int_0^\infty e^{-rf_1(\mathbf{h}^{(2)}) - rf_2 \circ f_3(\mathbf{h}^{(1)})} dF_1(r)$$

where F_1 is the distribution associated to its Laplace transform φ_1. Thus, the proof follows straight by arguments of the previous proofs.

A.3. Proof of Theorem 1

The result is a direct consequence of the work of Stein [33]. First, observe that $\mathrm{C}(\mathbf{s}_1, \mathbf{s}_1) = \int_0^\infty g(\tau; \mathbf{s}_1)\mathrm{d}F(\mathbf{s}_1) < \infty$. Now, we need to show that $\sum_{i=1}^n a_i a_j C(\mathbf{s}_i, \mathbf{s}_j) \geq 0$, for every finite system of arbitrary real constants a_i, $i = 1, \dots, n$. Write $K_i^{\tau,r,r'}$ for the normal density centered at $\mathbf{s}_i$ and with covariance matrix $(\tau r r')^{-1}\Sigma_{\mathbf{s}_i}$. Also, by abuse of notation, $\varphi_1 := \mathcal{L}[F_1]$ and $\varphi_2 := \mathcal{L}[F_2]$. Then, by a convolution argument in [80, p. 27], by Bernstein's theorem, and using Fubini's theorem, we get

$$\begin{aligned}
&\sum_{i,j=1}^n a_i a_j C(\mathbf{s}_i, \mathbf{s}_j)\\
&= \pi^{p/2} \sum_{i,j=1}^n a_i a_j |\Sigma_{\mathbf{s}_i}|^{\frac{1}{4}} |\Sigma_{\mathbf{s}_j}|^{\frac{1}{4}} \int_0^\infty \int_0^\infty \int_0^\infty \left(\int_{\mathbb{R}^p} K_i^{\tau,r,r'}(\mathbf{u}) K_j^{\tau,r,r'}(\mathbf{u})\mathrm{d}\mathbf{u} \right) (\tau r r')^{-p/2}\\
&\quad \times \mathrm{e}^{-r'(1/2\varphi_2^{-1}\circ g(\tau;\mathbf{s}_i)+1/2\varphi_2^{-1}\circ g(\tau;\mathbf{s}_j))} \mathrm{d}F(\tau)\mathrm{d}F_1(r)\mathrm{d}F_2(r')\\
&= \pi^{p/2} \int_0^\infty \int_0^\infty \int_0^\infty \int_{\mathbb{R}^p} \left(a_i |\Sigma_{\mathbf{s}_i}|^{\frac{1}{4}} K_i^{\tau,r,r'}(\mathbf{u}) \mathrm{e}^{-r' 1/2\varphi_2^{-1}\circ g(\tau;\mathbf{s}_i)} \right)^2 \mathrm{d}\mathbf{u} (\tau r r')^{-p/2} \mathrm{d}F(\tau)\mathrm{d}F_1(r)\mathrm{d}F_2(r') \geq 0.
\end{aligned} \tag{24}$$

Thus, the proof is completed.

A.4. M.S. differentiability of the QARF

Recall that the existence of the kth order ith mean square partial derivative of Z is related to the existence of the $2k$th order ith mean square partial derivative of the covariance function. This is in turn related to the spectral moments through the following formula [81, p. 31]:

$$(-1)^k \frac{\delta^{2k}\mathrm{C}(\mathbf{h})}{\delta h_i^{2k}}\bigg|_{\mathbf{h}=0} = \int_{\mathbb{R}^d} \omega_i^{2k} \mathrm{d}\hat{\mathrm{C}}(\boldsymbol{\omega}) < \infty.$$

Under the setting in Section 4.2, we ensure the existence of the spectral density $\hat{\mathrm{c}}$, thus

$$\begin{aligned}
\int_{\mathbb{R}^d} \omega_i^{2k} \hat{\mathrm{c}}(\boldsymbol{\omega})\mathrm{d}\boldsymbol{\omega} &\propto \int_{\mathbb{R}^d} \omega_i^{2k} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\boldsymbol{\omega}'\mathbf{h}} \mathrm{C}(\mathbf{h})\mathrm{d}\mathbf{h}\mathrm{d}\boldsymbol{\omega}\\
&= \int_{\mathbb{R}^d} \omega_i^{2k} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\boldsymbol{\omega}'\mathbf{h}} \int_0^\infty \mathrm{e}^{-r\sum_i \theta_i \nu_i(|h_i|)} \mathrm{d}F(r)\mathrm{d}\mathbf{h}\mathrm{d}\boldsymbol{\omega}\\
&= \int_{\mathbb{R}^d} \int_0^\infty \omega_i^{2k} \left(\int_{\mathbb{R}^{d-1}} \mathrm{e}^{-\mathrm{i}\tilde{\boldsymbol{\omega}}'\tilde{\mathbf{h}}} \mathrm{e}^{-r\sum_{j\neq i}\theta_j\nu_j(|h_j|)} \mathrm{d}\tilde{\mathbf{h}} \right) \left(\int_{\mathbb{R}} \mathrm{e}^{-r\theta_i\nu_i(|h_i|)} \mathrm{d}h_i \right) \mathrm{d}F(r)\mathrm{d}\boldsymbol{\omega}
\end{aligned}$$

where $\tilde{\mathbf{h}} = (h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_d)' \in \mathbb{R}^{d-1}$ and where we repeatedly make use of Fubini's theorem for standard integrability criteria. Now, observe that, for the assumption of absolute integrability of $\exp(-r\theta_i\nu_i(x))$ for any positive r, the last equality can be written as

$$\int_0^\infty \int_{\mathbb{R}^d} \omega_i^{2k} \hat{\mathrm{c}}(\tilde{\boldsymbol{\omega}}; r)\hat{\mathrm{c}}(\omega_i; r)\mathrm{d}\boldsymbol{\omega}\mathrm{d}F(r)$$

where $\hat{\mathrm{c}}(\tilde{\boldsymbol{\omega}}; r) := \mathcal{F}^{-1}[\mathrm{C}_r](\tilde{\boldsymbol{\omega}})$, with $\mathrm{C}_r(\tilde{\mathbf{h}}) = \exp(-r\sum_{j\neq i}\theta_j\nu_j(|h_j|))$ and where $\hat{\mathrm{c}}(\omega_i; r) = \mathcal{F}^{-1}[\mathrm{C}_r](\omega_i)$, with $\mathrm{C}_r(h_i) = \exp(-r\theta_i\nu_i(|h_i|))$. Then, by noticing that $\int_{\mathbb{R}^{d-1}} \hat{\mathrm{c}}(\tilde{\boldsymbol{\omega}}; r)\mathrm{d}\tilde{\boldsymbol{\omega}} = \mathrm{C}_r(\tilde{\mathbf{0}}) < \infty$, one obtains that the integral above is finite if and only if the function $r \mapsto \int_{\mathbb{R}} \omega_i^{2k}\hat{\mathrm{c}}(\omega_i; r)\mathrm{d}\omega_i$ is F-measurable. Thus, the proof is completed.

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