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On the Lorenz ordering of order statistics from exponential populations and some applications[☆]

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ABSTRACT

In this paper, the variability of order statistics from heterogeneous random samples is studied. It is shown that, *without any restriction on the parameters*, the variability of order statistics from heterogeneous exponential samples is always larger than that from homogeneous exponential samples in the sense of Lorenz ordering. Finally, some applications to reliability analysis and auction theory are pointed out.

1. Introduction

Order statistics have received considerable attention in the literature since they play an important role in many areas including reliability, data analysis, goodness-of-fit tests, statistical inference, outliers, robustness, and quality control. Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denote the order statistics arising from random variables $X_1, X_2, \ldots, X_n$. In the reliability context, $X_{n-k+1:n}$ denotes the lifetime of a k-out-of-n system. In particular, the parallel and series systems are 1-out-of-n and n-out-of-n systems, respectively. A lot of work on order statistics has been done in the case when the underlying variables are independent and identically distributed (i.i.d.); see [12,7,8] for more details. Studies of order statistics from heterogeneous samples began in early 1970s, motivated by robustness issues. After that, a lot of work has been done on order statistics from single-outlier and multiple-outlier models. Balakrishnan [6] synthesized developments on order statistics arising from independent and non-identically distributed random variables. One may also refer to [15,25] for some reviews on various recent developments.

The variability of order statistics has been studied by a number of authors including David and Groeneveld [11] and Arnold and Villaseñor [5]. Let $X_1, \ldots, X_n$ be independent exponential random variables with hazard rates $\lambda_1, \ldots, \lambda_n$, respectively,

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and $Y_1, \ldots, Y_n$ be i.i.d. exponential random variables with common hazard rate λ. Let $X_{i:n}$ and $Y_{i:n}$, for $i = 1, \ldots, n$, denote the corresponding sets of order statistics. Then, intuitively, $X_{i:n}$ should exhibit more variability than $Y_{i:n}$. This has been partially confirmed in the literature. For example, Sathe [23] proved that if $\lambda = \bar{\lambda}$, where $\bar{\lambda} = \sum_{i=1}^{n} \lambda_i/n$, then

$$\mathrm{Var}(X_{k:n}) \geq \mathrm{Var}(Y_{k:n}).$$

This result has been partially improved by Khaledi and Kochar [13] that if $\lambda = \hat{\lambda} = \left(\prod_{i=1}^{n} \lambda_i\right)^{1/n}$, then

$$\mathrm{Var}(X_{n:n}) \geq \mathrm{Var}(Y_{n:n}). \tag{1.1}$$

More recently, Kochar and Xu [16] further improved this result by showing that (1.1) holds if $\lambda \geq \lambda^*$, where

$$\lambda^* = \sum_{i=1}^{n} \frac{1}{i} \left\{ \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \frac{1}{\sum_{j=1}^{k} \lambda_{i_j}} \right\}^{-1}.$$

It needs to be noted that $\lambda^* \leq \hat{\lambda} \leq \bar{\lambda}$ [16]. It can be seen that all existing results in the literature require certain conditions on the parameters for the comparison of variabilities of order statistics from heterogeneous and homogeneous samples. It turns out that Lorenz ordering is quite suitable for the purpose of this comparison; see, for example, [2,5,18]. For order statistics from the same exponential distribution, Arnold and Nagaraja [4] showed that, for $i \leq j$,

$$(n - i + 1)\mathrm{E}(Y_{i:n}) \leq (m - j + 1)\mathrm{E}(Y_{j:m}) \Longleftrightarrow Y_{j:m} \leq_{\mathrm{Lorenz}} Y_{i:n}, \tag{1.2}$$

where $\leq_{\mathrm{Lorenz}}$ means the Lorenz order defined formally in Section 2. Kochar and Xu [16] showed further that $X_{n:n} \geq_{\mathrm{Lorenz}} Y_{n:n}$ without any restriction on the parameter λ. This result states that the variability of largest order statistic from heterogeneous exponential samples is larger than the one from homogeneous exponential samples in the sense of Lorenz ordering. As a direct consequence of this result, the coefficient of variation (defined in Section 2) $\gamma_{X_{n:n}}$ has the following lower bound:

$$\gamma_{X_{n:n}} \geq \sqrt{\sum_{i=1}^{n} \frac{1}{i^2}} \Bigg/ \sum_{i=1}^{n} \frac{1}{i}.$$

However, due to the complicated form of distributions of order statistics from independent and non-identical variables, an analogous result for general order statistics has remained as an open problem. In this paper, we solve this problem by showing precisely that

$$X_{k:n} \geq_{\mathrm{Lorenz}} Y_{k:n}$$

for $k = 1, \ldots, n$. This result does finally confirm that the variability in heterogeneous exponential samples is always larger than that in homogeneous exponential samples. Consequently, a sharp lower bound for the coefficient of variation of order statistics from heterogeneous exponential random variables can be readily obtained as

$$\gamma_{X_{k:n}} \geq \sqrt{\sum_{i=1}^{k} \frac{1}{(n-i+1)^2}} \Bigg/ \sum_{i=1}^{k} \frac{1}{n-i+1}.$$

The rest of this paper is organized as follows. Section 2 introduces some pertinent notation and stochastic orders. The Lorenz order between order statistics from heterogeneous and homogeneous random samples is studied in Section 3, wherein the above stated general result is established. Finally, in Section 4, we mention some applications of the established result to reliability analysis and auction theory.

2. Preliminaries

In this section, we recall some stochastic orders that will be used in the sequel. Let the random variables X and Y have distribution functions F and G, survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively.

Definition 2.1. X is said to be smaller than Y in the *star order*, denoted by $X \leq_{\star} Y$ (or $F \leq_{\star} G$), if the function $G^{-1}F(x)$ is star-shaped in the sense that $G^{-1}F(x)/x$ is increasing in x on the support of X.

Definition 2.2. X is said to be smaller than Y in the *convex order*, denoted by $X \leq_{\mathrm{cx}} Y$, if

$$\mathrm{E}[\phi(X)] \leq \mathrm{E}[\phi(Y)]$$

for all convex functions.

Definition 2.3. Let X and Y be non-negative random variables having finite positive expectations. Then, X is said to be smaller than Y in the *Lorenz order*, denoted by $X \leq_{\mathrm{Lorenz}} Y$ (or $F \leq_{\mathrm{Lorenz}} G$), if

$$\frac{X}{\mathrm{E}X} \leq_{\mathrm{cx}} \frac{Y}{\mathrm{E}Y}.$$

The Lorenz order is closely connected to the so-called *Lorenz curve*, which is used in economics to measure the inequality of incomes. The Lorenz curve L_X, corresponding to X, is defined as

$$L_X(p) = \frac{\int_0^p F^{-1}(u)\mathrm{d}u}{\int_0^1 F^{-1}(u)\mathrm{d}u}, \quad p \in [0, 1],$$

where $F^{-1}(u) = \sup\{x : F(x) < u\}$. Then, the Lorenz order defined above can be equivalently defined in terms of the Lorenz curve as

$$X \leq_{\text{Lorenz}} Y \iff L_X(p) \geq L_Y(p), \quad p \in [0, 1].$$

It is well-known that if $\mathsf{E}X = \mathsf{E}Y$, then $X \leq_{\text{cx}} Y \iff X \leq_{\text{Lorenz}} Y$. However, if $\mathsf{E}(X) \neq \mathsf{E}(Y)$, those orders are not equivalent. It has been shown in the literature [18, p. 69] that

$$X \leq_\star Y \Longrightarrow X \leq_{\text{Lorenz}} Y \Longrightarrow \gamma_X \leq \gamma_Y,$$

where $\gamma_X = \sqrt{\mathrm{Var}(X)}/\mathrm{E}(X)$ denotes the coefficient of variation of X.

It needs to be mentioned that the star ordering and Lorenz ordering are both scale invariant. A detailed discussion of these orders can be found in [18].

Definition 2.4. X is said to be smaller than Y in the *usual stochastic order*, denoted by $X \leq_{\text{st}} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for all x.

For a thorough discussion on various stochastic orders, one may refer to [24].

We shall also use the concept of majorization in our discussion. Let $\{x_{(1)}, \ldots, x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, \ldots, x_n)$.

Definition 2.5. The vector $\mathbf{x}$ is said to *majorize* the vector $\mathbf{y}$, denoted by $\mathbf{x} \succeq_{\text{m}} \mathbf{y}$, if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$$

for $j = 1, \ldots, n-1$ and $\sum_{i=1}^n x_{(i)} = \sum_{i=1}^n y_{(i)}$.

For extensive and comprehensive details on the theory of the majorization order and its applications, we refer the reader to [19].

3. Lorenz ordering of order statistics

In this section, we study the variabilities of order statistics from heterogeneous and homogeneous exponential samples. We first present some lemmas, which will be used in the sequel.

The following lemma, due to Lefèvre and Utev [17], presents a sufficient condition for the closure property of Lorenz ordering under convolution.

Lemma 3.1. *Let $\{X_i, i = 1, \ldots, n\}$ and $\{Y_i, i = 1, \ldots, n\}$ be two sequences of independent random variables such that*

$$X_i \geq_{\text{Lorenz}} Y_i,$$

and

$$\frac{\mathsf{E}X_i}{\mathsf{E}Y_i} = c, \quad i = 1, \ldots, n$$

for some constant c. Then,

$$\sum_{i=1}^{n} X_i \geq_{\text{Lorenz}} \sum_{i=1}^{n} Y_i.$$

The following lemma discusses the mixture of Lorenz order, which is also of independent interest.

Lemma 3.2. *Let $X_1, \ldots, X_n$ be non-negative independent random variables with respective distribution functions $F_1, \ldots, F_n$. Let Y be another non-negative random variable with distribution function G. If $X_i \geq_{\text{Lorenz}} Y$ for $i = 1, \ldots, n$, then*

$$F = \sum_{i=1}^{n} p_i F_i \geq_{\text{Lorenz}} G,$$

where p_i's are positive weights such that $\sum_{i=1}^n p_i = 1$.

Proof. Assume that $X(\Theta)$ has the mixture distribution F, where Θ is a discrete random variable with $P(\Theta = \theta_i) = p_i$. Then, we may represent X_i as $X(\theta_i)$, which is independent of Θ. We, therefore, need to prove that

$$X(\Theta) \geq_{\text{Lorenz}} Y,$$

i.e.,

$$X(\Theta) \geq_{\text{cx}} \frac{\mathsf{E}[X(\Theta)]}{\mathsf{E}Y} Y.$$

It is enough to prove that

$$\mathsf{E}\left[f\left(X(\Theta)\right)\right] \geq \mathsf{E}\left[f\left(\frac{\mathsf{E}[X(\Theta)]}{\mathsf{E}Y} Y\right)\right]$$

for all convex functions f. Note that

$$\mathsf{E}[X(\Theta)] = \sum_{i=1}^{n} p_i \mathsf{E}[X(\theta_i)] = \sum_{i=1}^{n} p_i \mu_i,$$

where $\mu_i = \mathsf{E}[X(\theta_i)]$. Hence, it is equivalent to proving that

$$\sum_{i=1}^{n} p_i \mathsf{E}\left[f\left(X(\theta_i)\right)\right] \geq \mathsf{E}\left[f\left(\frac{\sum p_i \mu_i}{\mathsf{E}Y} Y\right)\right]. \tag{3.1}$$

Since $X(\theta_i) \geq_{\text{Lorenz}} Y$, it follows that

$$\frac{X(\theta_i)}{\mu_i} \geq_{\text{cx}} \frac{Y}{\mathsf{E}Y},$$

i.e.,

$$\mathsf{E}\left[f\left(X(\theta_i)\right)\right] \geq \mathsf{E}\left[f\left(\frac{\mu_i Y}{\mathsf{E}Y}\right)\right].$$

Therefore, we have

$$\sum_{i=1}^{n} p_i \mathsf{E}\left[f\left(X(\theta_i)\right)\right] \geq \sum_{i=1}^{n} p_i \mathsf{E}\left[f\left(\frac{\mu_i Y}{\mathsf{E}Y}\right)\right]. \tag{3.2}$$

Since f is a convex function, using Jensen's inequality, we have

$$\sum_{i=1}^{n} p_i \mathsf{E}\left[f\left(\frac{\mu_i Y}{\mathsf{E}Y}\right)\right] \geq \mathsf{E}\left[f\left(\frac{\sum_{i=1}^{n} p_i \mu_i}{\mathsf{E}Y} Y\right)\right]. \tag{3.3}$$

Combining inequalities (3.2) and (3.3), inequality (3.1) follows immediately. ■

Remark 1. Xu and Balakrishnan [26] showed a similar result that if the density function g of G is such that $g(e^x)$ is log-concave, and $X_i \geq_\star Y$ for $i = 1, \ldots, n$, then

$$F = \sum_{i=1}^{n} p_i F_i \geq_\star G,$$

where $\geq_\star$ denotes the star ordering. It is known that star ordering implies the Lorenz ordering. Thus, Lemma 3.2 presents a weaker result without any restriction on the density function.

The following lemma gives a sufficient condition for comparing convolutions of random variables in terms of Lorenz ordering.

Lemma 3.3. *Let $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_n\}$ be two sets of i.i.d. random variables and $X_1 \overset{\text{st}}{=} Y_1$, where $\overset{\text{st}}{=}$ means equality in law. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ be two sets of positive constants. If*

$$\frac{\beta_{(n)}}{\alpha_{(n)}} \geq \frac{\beta_{(n-1)}}{\alpha_{(n-1)}} \geq \cdots \geq \frac{\beta_{(1)}}{\alpha_{(1)}}, \tag{3.4}$$

then

$$\sum_{i=1}^{n} \beta_i Y_i \geq_{\text{Lorenz}} \sum_{i=1}^{n} \alpha_i X_i,$$

where $\alpha_{(i)}$ and $\beta_{(i)}$ are the ith smallest elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively.

Proof. Since the Lorenz ordering is scale invariant, the required result is equivalent to proving that

$$\frac{\sum_{i=1}^{n} \beta_i Y_i}{\sum_{i=1}^{n} \beta_i} \geq_{\text{Lorenz}} \frac{\sum_{i=1}^{n} \alpha_i X_i}{\sum_{i=1}^{n} \alpha_i}.$$

Since both sides have the same means, it is enough to prove that

$$\sum_{i=1}^{n} \frac{\beta_i}{\sum_{i=1}^{n} \beta_i} Y_i \geq_{\text{cx}} \sum_{i=1}^{n} \frac{\alpha_i}{\sum_{i=1}^{n} \alpha_i} X_i. \tag{3.5}$$

From the result in B.1.b of [19, p. 188], (3.4) implies that

$$\frac{\boldsymbol{\beta}}{\sum_{i=1}^{n} \beta_i} \geq_{\text{m}} \frac{\boldsymbol{\alpha}}{\sum_{i=1}^{n} \alpha_i}.$$

Therefore, the required result in (3.5) follows immediately from Theorem 3.A.35 of Shaked and Shanthikumar [24, p. 129]. ■

Remark 2. It is easy to see that the i.i.d. assumption in Lemma 3.3 can be relaxed to exchangeability by using Theorem 3.A.35 of Shaked and Shanthikumar [24, p. 129]

Păltănea [22] proved that order statistics can be represented as a convolution of two independent random variables as stated in the following lemma.

Lemma 3.4. *Let $S = \{X_1, \ldots, X_n\}$ be a set of independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$. For $i = 1, \ldots, n$, let $S^{[i]} = S \setminus \{X_i\}$, and $X_{k:n}$ and $X_{k:n-1}^{[i]}$ denote the kth order statistics from S and $S^{[i]}$ with distribution functions $F_{k:n}$ and $F_{k:n-1}^{[i]}$, respectively. Then, the $(k+1)$th order statistic from S is the sum of two independent random variables $X_{1:n}$ and Z_k with distribution function*

$$\sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} F_{k:n-1}^{[i]},$$

i.e.,

$$F_{k+1:n} = F_{1:n} * \left(\sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} F_{k:n-1}^{[i]} \right),$$

where "$*$" denotes the convolution, and $\Lambda = \sum_{i=1}^{n} \lambda_i$.

Now, we are ready to establish our main result presented in the following theorem.

Theorem 3.5. *Let $X_1, \ldots, X_n$ be independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$. Let $Y_1, \ldots, Y_n$ be i.i.d. exponential random variables with common hazard rate λ. Then,*

$$X_{k:n} \geq_{\text{Lorenz}} Y_{k:n}. \tag{3.6}$$

for all $k = 1, \ldots, n$.

Proof. The proof is based on mathematical induction and proceeds as follows involving four steps.

Step 1: for $k = 1$ and all $n \geq 1$, the assertion holds since both $X_{1:n}$ and $Y_{1:n}$ are exponential random variables.

Step 2: assume that (3.6) holds for some positive integer k and all $n \geq k$. Then, it is enough to prove that (3.6) also holds true for $k+1$ and $n \geq k+1$. Let $F_{k:n}^{(a)}$ be the distribution function of the kth order statistic from an i.i.d. exponential sample of size n with common hazard rate $a > 0$. The induction assumption implies

$$F_{k:n-1}^{[i]} \geq_{\text{Lorenz}} F_{k:n-1}^{(\gamma)}$$

for $n \geq k+1$ and $\gamma > 0, i = 1, \ldots, n$. From Lemma 3.2, it follows that

$$\sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} F_{k:n-1}^{[i]} \geq_{\text{Lorenz}} F_{k:n-1}^{(\gamma)}, \tag{3.7}$$

where $\sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} F_{k:n-1}^{[i]}$ is the distribution function of the random variable Z_k defined in Lemma 3.4, and $F_{k:n-1}^{(\gamma)}$ is the distribution function of random variable U_γ, which is independent of $X_{1:n}$. Hence, (3.7) can be written as

$$Z_k \geq_{\text{Lorenz}} U_\gamma.$$

Since there is no restriction on the parameter γ, we may choose a proper γ, say γ_0, such that

$$\mathsf{E}Z_k = \mathsf{E}U_{\gamma_0}. \tag{3.8}$$

Then, by using Lemmas 3.1 and 3.4, we have

$$X_{1:n} + Z_k \geq_{\text{Lorenz}} X_{1:n} + U_{\gamma_0};$$

that is,

$$X_{k+1:n} \geq_{\text{Lorenz}} X_{1:n} + U_{\gamma_0}.$$

In the following step, we will prove

$$X_{1:n} + U_{\gamma_0} \geq_{\text{Lorenz}} Y_{k+1:n}. \tag{3.9}$$

Then, the required result follows from the transitive property of Lorenz ordering.

Step 3: since U_{γ_0} is the kth order statistic from an exponential sample of size $n-1$, it can be rewritten as

$$U_{\gamma_0} = \frac{V_1}{(n-1)\gamma_0} + \cdots + \frac{V_k}{(n-k)\gamma_0}$$

where $V_1, \ldots, V_k$ are i.i.d. standard exponential random variables; see, for example, [3]. Similarly, $Y_{k+1:n}$ can be represented as

$$Y_{k+1:n} = \frac{W_1}{n\lambda} + \cdots + \frac{W_{k+1}}{(n-k)\lambda},$$

where W_i's are i.i.d. standard exponential random variables. Thus, it suffices to prove that

$$X_{1:n} + \sum_{i=1}^{k} \frac{V_i}{(n-i)\gamma_0} \geq_{\text{Lorenz}} \sum_{i=1}^{k+1} \frac{W_i}{(n-i+1)\lambda}.$$

Note that $X_{1:n}$ can be represented as V_{k+1}/Λ, where V_{k+1} is a standard exponential random variable, which is independent of $V_1, \ldots, V_k$, and $\Lambda = \sum_{i=1}^{k} \lambda_i$, and so we need to prove that

$$\sum_{i=1}^{k} \frac{V_i}{(n-i)\gamma_0} + \frac{V_{k+1}}{\Lambda} \geq_{\text{Lorenz}} \sum_{i=1}^{k+1} \frac{W_i}{(n-i+1)\lambda}. \tag{3.10}$$

If $\Lambda \geq n\gamma_0$, then

$$\frac{1}{\Lambda} \leq \frac{1}{(n-1)\gamma_0} \leq \cdots \leq \frac{1}{(n-k)\gamma_0},$$

and hence

$$\frac{\frac{1}{(n-k)\gamma_0}}{\frac{1}{(n-k)\lambda}} \geq \frac{\frac{1}{(n-k-1)\gamma_0}}{\frac{1}{(n-k-1)\lambda}} \geq \cdots \geq \frac{\frac{1}{(n-1)\gamma_0}}{\frac{1}{(n-1)\lambda}} \geq \frac{\frac{1}{\Lambda}}{\frac{1}{n\lambda}},$$

where the last inequality follows from the assumption that $\Lambda \geq n\gamma_0$. According to Lemma 3.3, (3.10) follows immediately. Hence, all that is left is to verify that $\Lambda \geq n\gamma_0$.

Step 4: we now prove $\Lambda \geq n\gamma_0$ by contradiction, by first assuming that $\gamma_0 > \Lambda/n = \bar{\lambda}$. It is known from Corollary 2.1 of Kochar and Rojo [14] that if $\gamma_0 > \bar{\lambda}$, then

$$Z_k \geq_{\text{st}} U_{\gamma_0}.$$

However, from (3.8), we have $\mathsf{E}Z_k = \mathsf{E}U_{\gamma_0}$. Hence, it follows from Theorem 1.A.8 of Shaked and Shanthikumar [24] that

$$Z_k \stackrel{\text{st}}{=} U_{\gamma_0}.$$

We shall now invalidate this result.

If $Z_k \overset{\text{st}}{=} U_{\gamma_0}$, then

$$\sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} F_{k:n-1}^{[i]}(t) = F_{k:n-1}^{(\gamma_0)}(t) \quad \forall t \geq 0. \tag{3.11}$$

Let us denote $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and

$$s_k(\boldsymbol{\lambda}) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad k = 1, \ldots, n.$$

Then, we have (see, for example, [9])

$$F_{k:n-1}^{[i]}(t) = s_k^{[i]}(\boldsymbol{\lambda}) t^k + o(t^k), \quad t \to 0,$$

where

$$s_k^{[i]}(\boldsymbol{\lambda}) = s_k(\boldsymbol{\lambda} \setminus \{\lambda_i\}).$$

Similarly, we have (see [3])

$$F_{k:n-1}^{(\gamma_0)}(t) = \binom{n-1}{k} \gamma_0^k t^k + o(t^k), \quad t \to 0.$$

From (3.11) and the assumption that $\gamma_0 > \bar{\lambda}$, it then follows that

$$\sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} s_k^{[i]}(\boldsymbol{\lambda}) = \binom{n-1}{k} \gamma_0^k \geq \binom{n-1}{k} \bar{\lambda}^k,$$

i.e.,

$$\binom{n-1}{k}^{-1} \sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} s_k^{[i]}(\boldsymbol{\lambda}) \geq \bar{\lambda}^k. \tag{3.12}$$

It can be seen that (see [22])

$$\binom{n-1}{k}^{-1} \sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} s_k^{[i]}(\boldsymbol{\lambda}) = (m_{k+1})^{k+1} \bar{\lambda}^{-1},$$

where

$$m_{k+1} = \left(\binom{n}{k+1}^{-1} s_{k+1}(\boldsymbol{\lambda}) \right)^{\frac{1}{k+1}}, \quad j = 1, \ldots, n.$$

Thus, (3.12) implies that

$$m_{k+1} \geq \bar{\lambda},$$

which obviously contradicts Maclaurin's inequality [21]. Hence, we should have $\gamma_0 \leq \bar{\lambda}$.

Upon combining Steps 1–4, the proof gets completed. ■

In terms of Lorenz curves, Theorem 3.5 states that the Lorenz curves of order statistics from i.i.d. exponential samples are always upper bounds for those from independent heterogeneous exponential variables. In particular, it should be pointed out that the Lorenz curves of order statistics from i.i.d. exponential samples are independent of the hazard rate λ. This point can be easily argued by noting that the star order is preserved under the formation of order statistics from i.i.d. samples (see Theorem 4.B.15 of [24]). In Fig. 1, we have plotted the Lorenz curves of $X_{2:3}$ from exponential random variables X_1, X_2 and X_3 with parameters $(0.05, 0.5, 2)$ and $(0.05, 2, 6)$, respectively, and the Lorenz curve of $Y_{2:3}$ from i.i.d. exponential random variables Y_1, Y_2 and Y_3 with common parameter λ.

Using Theorem 3.5 and (1.2), we readily obtain the following result.

Corollary 3.6. *Let $X_1, \ldots, X_n$ be independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$. Let $Y_1, \ldots, Y_m$ be i.i.d. exponential random variables with common hazard rate λ. Then, for $i \leq j$,*

$$X_{i:n} \geq_{\text{Lorenz}} Y_{j:m}$$

if

$$(n-i+1) \sum_{k=1}^{i} \frac{1}{n-k+1} \leq (m-j+1) \sum_{k=1}^{j} \frac{1}{m-k+1}.$$

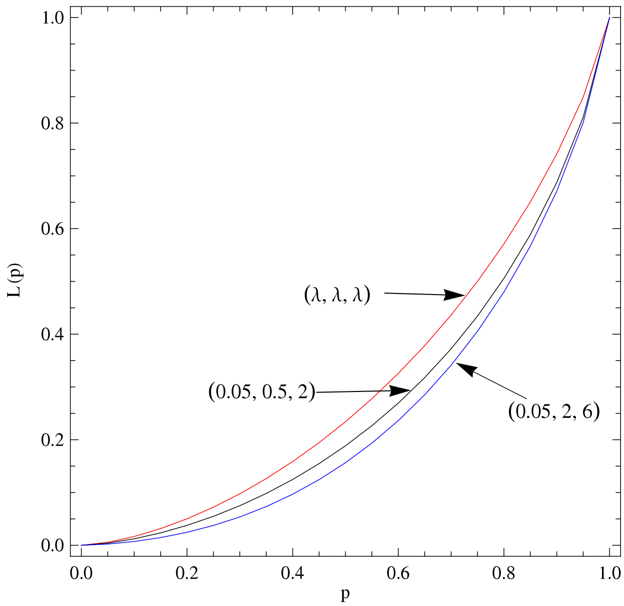

Fig. 1. The upper bound for the Lorenz curves of the second order statistic from three heterogeneous independent exponential variables.

As a direct consequence of Theorem 3.5, we also obtain the following result for general sample spacings.

Corollary 3.7. *Let $X_1, \ldots, X_n$ be independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$. Let $Y_1, \ldots, Y_n$ be i.i.d. exponential random variables with common hazard rate λ. Then,*

$$X_{k:n} - X_{1:n} \geq_{\text{Lorenz}} Y_{k:n} - Y_{1:n}$$

for all $k = 1, \ldots, n$.

Proof. From Lemma 3.4, it is enough to prove that

$$\sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} F_{k:n-1}^{[i]} \geq_{\text{Lorenz}} F_{k:n-1}^{(\lambda)}.$$

From Theorem 3.5, we have

$$F_{k:n-1}^{[i]} \geq_{\text{Lorenz}} F_{k:n-1}^{(\lambda)}$$

and so the required result follows immediately from Lemma 3.2.

4. Some applications

In this section, we present two applications of the results established in the last section.

4.1. Reliability analysis

The lifetime of a k-out-of-n system can be represented as $X_{n-k+1:n}$. Reliability engineers are not only interested in controlling the mean life of a system, but also its variability. One important issue is in designing a system with less variability, subject to given mean life constraints. This naturally leads to Lorenz order comparisons [5]. In this case, Theorem 3.5 leads to the following insight with regard to k-out-of-n systems.

Proposition 4.1. *The k-out-of-n system with homogeneous exponential components has a smaller variability than any k-out-of-n system with heterogeneous exponential components.*

The following result provides a *sharp lower bound* for the variability of the lifetime of an $(n-k+1)$-out-of-n system.

Proposition 4.2. *Let $X_1, \ldots, X_n$ be the lifetimes of independent exponential components. Then,*

$$\gamma_{X_{k:n}} \geq \sqrt{\sum_{i=1}^{k} \frac{1}{(n-i+1)^2}} \Bigg/ \sum_{i=1}^{k} \frac{1}{n-i+1}.$$

Proposition 4.2 can be used to develop a simple test for the heterogeneity of exponential samples, for example. The null hypothesis is that the system is composed of the same type of components. Then, we may formulate the hypothesis testing problem as

$$\mathrm{H}_0 : \gamma = \sqrt{\sum_{i=1}^{k} \frac{1}{(n-i+1)^2}} \Big/ \sum_{i=1}^{k} \frac{1}{n-i+1}$$

versus

$$\mathrm{H}_a : \gamma > \sqrt{\sum_{i=1}^{k} \frac{1}{(n-i+1)^2}} \Big/ \sum_{i=1}^{k} \frac{1}{n-i+1},$$

where γ is the population coefficient of variation of the $(n-k+1)$-out-of-n system. Note that γ can be estimated from the sample as

$$\hat{\gamma} = \frac{S}{\bar{X}},$$

where S is the sample standard deviation and $\bar{X}$ is the sample mean of the available system lifetime data. The distribution of $\hat{\gamma}$ based on large/small samples has been extensively discussed in the literature; see, for example, [1] and the references contained therein. The null and non-null properties of this test procedure can be studied in more detail, which we hope to do in a future work.

4.2. Auction theory

It is well-known that asymmetries (heterogeneities) among bidders are widespread in auction markets. One important question is how the differences in bidders' distributions of valuations affect their behavior, and in turn their profits. Unfortunately, the effect of asymmetries on the auctioneer's revenue has not been well understood. Here, we briefly explain how the asymmetries affect the revenue in the second price auction. The second price auction is an auction mechanism that the bidder who submitted the highest bid is awarded the object being sold and pays a price equal to the second highest amount bid.

Maskin and Riley [20] showed that the revenue ranking between the second price auction depends generally on the kind of asymmetries among bidders. Cantillon [10] proved that, under some suitable conditions, the expected revenue from heterogeneous auction is larger than that from homogeneous auctions in the second price auction. It is well-known that, in the second price auction, bidding one's own valuation is a dominant strategy.

Let $X_1, \ldots, X_n$ be n bidders' valuations. Then, the revenue of the auctioneer in this setting is equal to $X_{n-1:n}$; see, for example [20,10]. In this case, Theorem 3.5 leads to the following insight into the effect of asymmetries.

Proposition 4.3. *Let $X_1, \ldots, X_n$ be independent exponential random variables. Then, for the second price auction, the bidders' payoffs from the asymmetric auction always possesses larger variability.*

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References

[1] H. Albrecher, S.A. Ladoucette, J.L. Teugels, Asymptotics of the sample coefficient of variation and the sample dispersion, J. Statist. Plann. Inference 140 (2010) 358–368.
[2] B.C. Arnold, Majorization and the Lorenz Order: A Brief Introduction, in: Lecture Notes in Statistics, vol. 43, Springer-Verlag, New York, 1987.
[3] B.C. Arnold, N. Balakrishnan, H.N. Nagaraja, A First Course in Order Statistics, Wiley, New York, 1992.
[4] B.C. Arnold, H.N. Nagaraja, Lorenz ordering of exponential order statistics, Statist. Probab. Lett. 11 (1991) 485–490.
[5] B.C. Arnold, J.A. Villasenor, Lorenz ordering of order statistics, in: IMS Lecture Notes, vol. 19, Hayward, California, 1991, pp. 38–47.
[6] N. Balakrishnan, Permanents, order statistics, outliers, and robustness, Rev. Mat. Complut. 20 (2007) 7–107.
[7] N. Balakrishnan, C.R. Rao, Handbook of Statistics 16 – Order Statistics: Theory and Methods, North-Holland, Amsterdam, 1998.
[8] N. Balakrishnan, C.R. Rao, Handbook of Statistics 17 – Order Statistics: Applications, North-Holland, Amsterdam, 1998.
[9] J.L. Bon, E. Păltănea, Comparison of order statistics in a random sequence to the same statistics with i.i.d. variables, ESAIM: Probab. Stat. 10 (2006) 1–10.
[10] E. Cantillon, The effect of bidders' asymmetries on expected revenue in auctions, Games Econom. Behav. 62 (2008) 1–25.
[11] H.A. David, R.A. Groeneveld, Measures of local variation in a distribution: expected length of spacings and variances of order statistics, Biometrika 69 (1982) 227–232.
[12] H.A. David, H.N. Nagaraja, Order Statistics, third ed., Wiley, Hoboken, New Jersey, 2003.

[13] B. Khaledi, S.C. Kochar, Some new results on stochastic comparisons of parallel systems, J. Appl. Probab. 37 (2000) 1123–1128.
[14] S.C. Kochar, J. Rojo, Some new results on stochastic comparisons of spacings from heterogeneous exponential distributions, J. Multivariate Anal. 57 (1996) 69–83.
[15] S.C. Kochar, M. Xu, Stochastic comparisons of parallel systems when components have proportional hazard rates, Probab. Engrg. Inform. Sci. 21 (2007) 597–609.
[16] S.C. Kochar, M. Xu, Comparisons of parallel systems according to the convex transform order, J. Appl. Probab. 46 (2009) 342–352.
[17] C. Lefèvre, S. Utev, Comparison of individual risk models, Insurance Math. Econom. 28 (2001) 21–30.
[18] A.W. Marshall, I. Olkin, Life Distributions, Springer, New York, 2007.
[19] A.W. Marshall, I. Olkin, B.C. Arnold, Inequalities: Theory of Majorization and its Applications, second ed., Springer, New York, 2010.
[20] E. Maskin, J.G. Riley, Asymmetric auctions, Rev. Econ. Stud. 67 (2000) 413–438.
[21] D.S. Mitrinović, Analytic Inequalities, Springer-Verlag, Berlin, 1970.
[22] E. Păltănea, Bounds for mixtures of order statistics from exponentials and applications, J. Multivariate Anal. 102 (2011) 896–907.
[23] Y.S. Sathe, On the correlation coefficient between the first and the r-th smallest order statistics based on n independent exponential random variables, Comm. Statist. Theory Methods 17 (1988) 3295–3299.
[24] M. Shaked, J.G. Shanthikumar, Stochastic Orders and Their Applications, Springer, New York, 2007.
[25] M. Xu, Stochastic orders in heterogeneous samples with applications, Ph.D. Thesis, Portland State University, Portland, Oregon, 2010.
[26] M. Xu, N. Balakrishnan, On the sample ranges from heterogeneous exponential variables, J. Multivariate Anal. 109 (2012) 1–9.