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Fourier Transforms of Measures from the Classes $\mathscr{U}_\beta$, $-2<\beta\leqslant -1$

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Subclasses $\mathscr{U}_\beta(E), -2<\beta\leqslant -1$, of the Lévy class L of self-decomposable measures on a Banach space E are examined. They are closed convolution sub-semigroups of the semigroup of infinitely divisible probability distributions on E, defined as limit distributions of some prescribed schemes of summation. Each element of $\mathscr{U}_\beta(E)$ belongs to the domain of normal attraction of a stable measure with exponent $-\beta$. Their Fourier transforms are characterized. The symmetric measures in $\mathscr{U}_\beta(E)$ are shown to decompose uniquely into the convolution product of a symmetric stable measure with exponent $-\beta$ and the probability distribution of a random integral of the form $\int_{(0,1)} t\,dY(t^\beta)$, where Y is a Lévy process with paths in the Skorohod space $D_E[0,\infty)$ and $Y(1)$ has finite $(-\beta)$-moment. Topological and algebraic properties of the random-integral mapping $\mathscr{J}^\beta: \mathscr{L}(Y(1)) \to \mathscr{L}[\int_{(0,1)} t\,dY(t^\beta)]$ are investigated when E is a Hilbert space. As an application of the fact that $\mathscr{J}^\beta$ is a continuous isomorphism, generators for $\mathscr{U}_\beta$ are found as the images of compound Poisson distributions. Finally, the connection between the distributions $\mathscr{U}_\beta$ and thermodynamic limits in the Ising model with zero external field is pointed out.

INTRODUCTION

In a series of papers, random-integral representations were found for many classes of limit distributions in probability theory, including:

(1) self-decomposable (Lévy class L) distributions [11],

(2) s-selfdecomposable distributions [5], and

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(3) classes $\mathscr{U}_\beta$ of distributions similar to the Lévy class L, with $\beta > -1$ [6, 7, 9].

In all of this work, the first aim was to produce a random-integral representation—random integrals with respect to some Lévy process—from which characterizations in terms of Fourier transforms (characteristic functions) easily followed. This approach contrasts with the approach involving Choquet theory on extreme points of compact, convex sets taken earlier by various authors [13–15, 21–23].

Second, the weak continuity of the random-integral mapping was proven. This allowed one to find generators for the given class of limit distributions as images of Poisson-type measures under the random-integral mapping in question (cf. [4, 12]).

The present paper deals with classes $\mathscr{U}_\beta$ for $-2 < \beta \leqslant -1$. It was pointed out in [9] that the random-integral representation cannot be completely achieved for these classes. We will find the Fourier transforms, however, as well as random-integral representations for the symmetric measures in these classes. We also demonstrate the continuity of a natural mapping between the classes $\mathscr{U}_\beta(H)$, where H is a Hilbert space, and a semigroup of infinitely divisible measures on H with finite $(-\beta)$-moments. This leads to generators for the classes $\mathscr{U}_\beta$.

Presenting our results in the context of measures on Banach spaces allows for their application to processes with continuous sample paths, i.e., measures on $C([0, \infty))$. Furthermore, although the classes $\mathscr{U}_\beta$ are defined as limit laws for sequences of independent but not identically distributed random variables, they arise naturally in describing the thermodynamic limits in the Ising model of ferromagnetism. This interesting connection is mentioned briefly in this paper, but we hope to investigate it further and include the case of the quantum lattice model, i.e., self-adjoint-operator-valued random variables.

1. Notation and Definitions

Let E be a real, separable Banach space, E^* its topological dual, and denote the pairing of E and E^* by $\langle \cdot, \cdot \rangle$. By $\mathscr{P} = \mathscr{P}(E)$ and $\mathscr{I}\mathscr{D} = \mathscr{I}\mathscr{D}(E)$ we mean the convolution semigroups of all Borel probability measures on E and those that are infinitely divisible, respectively. Weak convergence in $\mathscr{P}(E)$ is denoted by $\Rightarrow$. We refer the reader to [1, 16, or 20] for the basic facts on the theory of probability measures on Banach spaces.

We shall need the following formula for infinitely divisible measures on E, known as the Lévy–Khinchine formula [1]: A measure $\mu \in \mathscr{P}$ is infinitely divisible if and only if its Fourier transform (characteristic functional) $\hat{\mu}$ satisfies

$$\hat{\mu}(y)=\exp\left[i\langle x_0, y\rangle-\tfrac{1}{2}\langle Ry, y\rangle + \int_{E\setminus\{0\}} [e^{i\langle x,y\rangle}-1-i\langle x, y\rangle 1_B(x)]\, dM(x)\right], \tag{1.1}$$

where $x_0 \in E$, R is a covariance operator, M is a positive measure on E finite outside of every neighborhood of 0, and B denotes the unit ball of E. In the sequel, we shall write $\mu=[x_0, R, M]$ if the Fourier transform of μ can be represented as in (1.1).

It is well known that $\mathscr{I}\mathscr{D}$ is the smallest subsemigroup of $\mathscr{P}$ closed in the topology of weak convergence of measures and containing all Gaussian measures on E (i.e., those for which $M=0$ in (1.1)) and all Poisson measures (i.e., those in whose representation (1.1) $R=0$ and M has one-point support).

For $a>0$, $\rho\in\mathscr{P}$, and G a Borel set in E, set $(T_a\rho)(G)=\rho(a^{-1}G)$. We shall say that a probability measure $\mu\in\mathscr{U}_\beta=\mathscr{U}_\beta(E)$ if there exists a sequence $\nu_n\in\mathscr{I}\mathscr{D}$ such that

$$T_{1/n}(\nu_1 * \nu_2 * \cdots * \nu_n)^{*n^{-\beta}} \Rightarrow \mu \qquad \text{as} \quad n\to\infty. \tag{1.2}$$

Note that the convolution power in (1.2) is well defined because the ν_j are infinitely divisible.

Another way of looking at the classes $\mathscr{U}_\beta$ involves the averaging of stochastic processes. If $\xi_1, \xi_2, \ldots$ are independent Lévy processes (i.e., processes starting at 0 with independent and stationary increments) such that $\nu_j=\mathscr{L}(\xi_j(1))$, then

$$\mathscr{L}(n^{-1}(\xi_1+\xi_2+\cdots+\xi_n)(n^{-\beta}))=T_{1/n}(\nu_1 * \nu_2 * \cdots * \nu_n)^{*n^{-\beta}}.$$

These classes arose in the study of unimodality for class L distributions [18, 19].

If μ in (1.2) is not degenerate, then $\beta\geqslant -2$ [6]. Moreover, $\mathscr{U}_{-2}$ consists of Gaussian measures only, while for $\beta>-1$, $\mathscr{U}_\beta$ can be characterized in terms of random integrals. Namely,

$$\text{If } \beta>0, \text{ then } \mu\in\mathscr{U}_\beta \text{ if and only if } \mu=\mathscr{L}\left[\int_{(0,1)} t\, dY(t^\beta)\right], \tag{1.3}$$

where Y is any $D_E([0,1])$-valued random variable with independent and stationary increments, and $Y(0)=0$ (i.e., Y is a Lévy process) [6]. The class $\mathscr{U}_0$ coincides with the Lévy class L of self-decomposable measures, and

$$\mu \in \mathscr{U}_0 \text{ if and only if } \mu = \mathscr{L}\left[\int_{(0,\infty)} e^{-t}\, dY(t)\right], \tag{1.4}$$

where Y is a $D_E([0,\infty))$-valued Lévy process such that $\mathbb{E}[\log(1+\|Y(1)\|)] < \infty$ [11]. For negative β we have

$$\text{If } -1<\beta<0, \text{ then } \mu \in \mathscr{U}_\beta \text{ if and only if } \mu = \gamma_\beta * \mathscr{L}\left[\int_{(0,1)} t\, dY(t^\beta)\right], \tag{1.5}$$

where γ_β is a strictly stable measure with exponent $-\beta$ and Y is a $D_E([0,\infty))$-valued Lévy process with $\mathbb{E}[\|Y(1)\|^{-\beta}] < \infty$ [7, 9].

The random integrals appearing above are understood to be defined via formal integration by parts. Note that in (1.4) and (1.5) the integrals involve the paths of Y over *infinite* time intervals, and the moment conditions guarantee the existence of those random integrals. Unfortunately, for $-2<\beta\leqslant -1$ and $Y(t)=tx_0$ a.s., $x_0 \neq 0$, the integral in (1.5) does not exist. Thus we cannot expect to have characterizations like those above for $\mu \in \mathscr{U}_\beta$ with $-2<\beta\leqslant -1$. Furthermore, our general approach here is different from those employed by the first author in his earlier papers. Here we will study measures in $\mathscr{U}_\beta$ by concentrating on how a factorization property that they must possess reflects itself in their Lévy–Khinchine representations. In the past the approach involved first developing the appropriate random integral and then relating this to (1.1) (cf. [6, 7, 9, 11]). The descriptions in terms of the Lévy-Khinchine representations follow easily from the random-integral information.

2. Fourier Transform Characterizations

Our main aim here is to describe Fourier transforms $\hat{\mu}$ of measures $\mu \in \mathscr{U}_\beta$ when $-2<\beta\leqslant -1$. The starting point is the following equivalent definition of the classes $\mathscr{U}_\beta$ (for any $\beta \geqslant -2$):

$$\begin{aligned}&\mu \in \mathscr{U}_\beta \text{ if and only if for all } t>0 \text{ there exists } \mu_t \in \mathscr{I}\mathscr{D} \text{ such}\\ &\text{that } \mu = (T_{e^{-t}}\mu)^{*e^{-\beta t}} * \mu_t\end{aligned} \tag{2.1}$$

[6, Theorem 1.1]. Hence the classes $\mathscr{U}_\beta$ are closed subsemigroups of $\mathscr{I}\mathscr{D}$.

Remark 2.1. From (2.1) it is clear that each $\mathscr{U}_\beta$, $\beta<0$, is a subclass of the class L of self-decomposable measures, the latter being characterized by (2.1) with $\beta = 0$. Furthermore, for $\beta \leqslant -1$, we have from (2.1),

$$|\hat{\mu}(y)| \leqslant |\hat{\mu}(e^{-t}y)|^{e^{-\beta t}} \leqslant |\hat{\mu}(e^{-t}y)|^{e^{t}}, \qquad t \geqslant 0,\ y \in E^*.$$

Thus the symmetrization μ^0 of μ satisfies the conditions for the thermodynamic limit distribution in the Ising model for ferromagnetism in statistical physics [3].

In fact, the factor μ_t in (2.1) is given by the following random integral [7]:

$$\mu_t = \mathscr{L}(Z^\beta(t)), \qquad \text{where} \quad Z^\beta(t) = \int_{[e^{-t},1)} s\, dY(s^\beta),\ t \geqslant 0. \tag{2.2}$$

Here Y is a $D_E([0,\infty))$-valued process with stationary, independent increments, and $Y(0) = 0$ a.s. In terms of Fourier transforms, one can write

$$\hat{\mu}_t(y) = \exp\left[-\beta \int_{[e^{-t},1)} \log(\hat{\mathscr{L}}(Y(1))(-sy)\, s^{\beta-1}\, ds \right] \tag{2.3}$$

[7, Lemmas 2.1 and 3.1]. Note that (2.3) implies that

$$\frac{d}{ds} [\log(\hat{\mu}_{-\log s}(y)]_{s=1} = -\log \hat{\mathscr{L}}(Y(1))(-y), \qquad y \in E^*,$$

and therefore the process Y in (2.2) is uniquely determined by μ. Furthermore, if $\mathscr{L}(Y(1)) = [a, R, M]$ and $\mu_t = [a_t^{(\beta)}, R_t^{(\beta)}, M_t^{(\beta)}]$, then using (1.1) and (2.3) we obtain:

$$a_t^{(\beta)} = \beta \int_{[e^{-t},1)} \left[a + \int_{1 < \|x\| \leqslant s^{-1}} x\, dM(x) \right] s^\beta\, ds \tag{2.4}$$

$$R_t^{(\beta)} = -\frac{\beta}{2+\beta} (1 - e^{-t(\beta+2)}) R \tag{2.5}$$

$$\begin{aligned} M_t^{(\beta)}(A) &= -\beta \int_{[e^{-t},1)} M(-s^{-1}A)\, s^{\beta-1}\, ds \\ &= -\beta \int_{[e^{-t},1)} \int_E 1_A(-sx)\, s^{\beta-1}\, dM(x)\, ds \end{aligned} \tag{2.6}$$

for all Borel subsets A of $E \backslash \{0\}$. We are now in a position to present the main result of this section.

THEOREM 2.2. *Let* $-2<\beta\leqslant -1$ *and* $\mu=[b,S,N]\in\mathscr{U}_\beta$. *Set* $R=-\beta^{-1}(2+\beta)S$. *Then there exists a unique* $\nu=[a,R,M]$ *with finite* $(-\beta)$-*moment and a stable probability measure* θ_β *with exponent* $(-\beta)$ *such that*

$$\mu=\theta_\beta * [0,S,M_\infty^{(\beta)}], \tag{2.7}$$

where

$$M_\infty^{(\beta)}(A)=-\beta\int_{(0,1)} M(-s^{-1}A)\,s^{\beta-1}\,ds, \qquad A\in\mathscr{B}(E\setminus\{0\}).$$

Conversely, each measure of the form (2.7) *belongs to* $\mathscr{U}_\beta$.

Proof. Using (1.1) we have

$$(T_{e^{-t}}\mu)^{*e^{-\beta t}}=\left[e^{-t(\beta+1)}\left(b+\int_{1<\|x\|\leqslant e^t} x\,dN(x)\right), e^{-t(2+\beta)}S, e^{-t\beta}T_{e^{-t}}N\right]. \tag{2.8}$$

Since μ satisfies (2.1), where μ_t is given by (2.3) and $\mu_t=[a_t^{(\beta)}, R_t^{(\beta)}, M_t^{(\beta)}]$ with $a_t^{(\beta)}$, $R_t^{(\beta)}$, and $M_t^{(\beta)}$ as in (2.4)–(2.6), we have the following decomposition:

$$\begin{aligned}\mu=[b,S,N]=&\left[e^{-t(\beta+1)}\left(b+\int_{1<\|x\|\leqslant e^t} x\,dN(x)\right)\right.\\ &\left.+a_t^{(\beta)}, 0, e^{-t\beta}T_{e^{-t}}N\right]\\ &* [0, e^{-t(2+\beta)}S+R_t^{(\beta)}, 0] * [0,0,M_t^{(\beta)}].\end{aligned} \tag{2.9}$$

(As already noted, (2.3) implies that $\nu=\mathscr{L}(Y(1))=[a,R,M]$ is uniquely determined by μ.) Since the Gaussian part in (2.9) converges to $[0,-\beta(2+\beta)^{-1}R,0]$ as $t\to\infty$, we infer that

$$\begin{aligned}&\left[e^{-t(\beta+1)}\left(b+\int_{1<\|x\|\leqslant e^t} x\,dN(x)\right)\right.\\ &\qquad\left.+a_t^{(\beta)}, 0, e^{-t\beta}T_{e^{-t}}N\right] * [0,0,M_t^{(\beta)}]\end{aligned} \tag{2.10}$$

also converges as $t\to\infty$. Hence both factors in (2.10) belong to shift-compact sets of measures. In fact, by [16, Proposition 5.4.12], the second factors are relatively compact; hence so are the first. For appropriate values of $t'\to\infty$ we conclude that

$$M_{t'}^{(\beta)}(A)\nearrow M_\infty^{(\beta)}(A)=-\beta\int_{(0,1)} M(-s^{-1}A)\,s^{\beta-1}\,ds,$$

and $M_\infty^{(\beta)}$ is a Lévy measure. Thus

$$M_\infty^{(\beta)}(\{\|x\|>1\}) = -\beta \int_{(0,1)} M(\{\|x\|>s^{-1}\})\, s^{\beta-1}\, ds$$

$$= \int_{\|x\|>1} \|x\|^{-\beta}\, dM(x) < \infty,$$

which is equivalent to $\nu = [a, R, M] \in \mathscr{I}\mathscr{D}_\beta = \{\rho \in \mathscr{I}\mathscr{D} \colon \int_E \|x\|^{-\beta}\, d\rho(x) < \infty\}$. Furthermore, because of [16, Theorem 5.6.2], we have

$$[0, 0, M_t^{(\beta)}] \Rightarrow [0, 0, M_\infty^{(\beta)}] \qquad \text{as} \quad t \to \infty. \tag{2.11}$$

(Note that $M_\infty^{(\beta)}(\{\|x\|=1\}) = -\beta \int_{(0,1)} M(\{\|x\|=s^{-1}\})\, s^{\beta-1}\, ds = 0$ because M is a σ-finite measure.) Finally, (2.10) and (2.11) imply that

$$\rho_t = T_{e^{-t}}[b, 0, N]^{*e^{-\beta t}} * \delta(a_t^{(\beta)}), \qquad t>0, \tag{2.12}$$

converges to, say, θ_β as $t \to \infty$. Taking $t = -\beta^{-1} \log m$, $m = 1, 2, \ldots$, we conclude that θ_β is a stable measure with exponent $-\beta$, which proves the decomposition (2.7).

For the converse, first note that for a stable measure θ_β with exponent $-\beta$ we have for all $c>0$,

$$\theta_\beta = T_c \theta_\beta^{*c^\beta} * \delta(b_c) \tag{2.13}$$

for some $b_c \in E$. Second, for $0<c<1$,

$$T_c M_\infty^{(\beta)}(A) = c^{-\beta}(-\beta) \int_{(0,c)} M(-u^{-1}A)\, u^{\beta-1}\, du,$$

and therefore

$$c^\beta T_c M_\infty^{(\beta)}(\cdot) + (-\beta) \int_{[c,1)} M(-u^{-1}\cdot)\, u^{\beta-1}\, du = M_\infty^{(\beta)}(\cdot),$$

which implies that $[0, 0, M_\infty^{(\beta)}]$ satisfies (2.1); i.e., it is an element of $\mathscr{U}_\beta$. Finally, all symmetric Gaussian measures $[0, R, 0]$ with covariance operator R satisfy (2.1) with $\mu_t = [0, (1-e^{-t(2+\beta)})R, 0]$. Since $\mathscr{U}_\beta$ forms a convolution semigroup, it follows from (2.13) and the above that measures of the form (2.7) are in $\mathscr{U}_\beta$. This completes the proof of our theorem.

COROLLARY 2.3. *For $-2<\beta\leqslant -1$, each element of the class $\mathscr{U}_\beta$ is contained in the domain of normal attraction of a stable distribution θ_β with exponent $-\beta$. Moreover, shift vectors x_m for the class $\mathscr{U}_\beta$ are of the form* (2.4) *with $t = -\beta^{-1}\log m$, $m = 1, 2, \ldots$.*

Proof. By definition, λ belongs to the domain of normal attraction of θ_β if there exists a sequence $x_m \in E$ such that

$$T_{m^{1/\beta}}\lambda^{*m} * \delta(x_m) \Rightarrow \theta_\beta \quad \text{as} \quad m \to \infty.$$

If $\mu = [b, S, N] \in \mathscr{U}_\beta$, then by (2.12),

$$T_{m^{1/\beta}}[b, S, N]^{*m} * \delta(a^{(\beta)}_{-(\log m)/\beta}) \Rightarrow \theta_\beta \quad \text{as} \quad m \to \infty,$$

since $T_{m^{1/\beta}}[0, S, 0]^{*m} = [0, m^{1+2/\beta}S, 0] \Rightarrow \delta(0)$, which proves the corollary.

A similar argument shows for $-1 < \beta < 0$, each element of the class $\mathscr{U}_\beta$ is in the domain of normal attraction of a *strictly* stable measure γ_β with exponent $-\beta$ [9].

COROLLARY 2.4. *Let $-2 < \beta \leqslant -1$. If μ is symmetric and belongs to $\mathscr{U}_\beta$, then there is a unique symmetric Lévy process Y with $\mathbb{E}[\|Y(1)\|^{-\beta}] < \infty$ and a symmetric stable measure θ_β with exponent $-\beta$ such that*

$$\mu = \theta_\beta * \mathscr{L}\left(\int_{(0,1)} t\, dY(t^\beta)\right) = \theta_\beta * \mathscr{L}(Z^\beta(+\infty)). \tag{2.14}$$

Conversely, if the integral in (2.14) *exists, then $\mu \in \mathscr{U}_\beta$.*

Proof. For symmetric μ satisfying (2.1) with μ_t given by (2.2) and (2.4)–(2.6), the Lévy process Y is symmetric and $a_t^{(\beta)} = 0$. Moreover, both factors in (2.1) are conditionally compact. In fact both converge: the first to a stable measure θ_β and the second to the integral in (2.14). This follows from the fact that

$$[0, R_\infty^{(\beta)}, M_\infty^{(\beta)}] = \mathscr{L}\left(\int_{(0,1)} t\, dY(t^\beta)\right),$$

where $\mathscr{L}(Y(1)) = [0, R, M]$ with symmetric M. The proof is complete.

Remark 2.5. Since $R_\infty^{(\beta)} = -\beta(2+\beta)^{-1}R$ is a Gaussian covariance operator, the existence of the integral (2.14) is equivalent to the fact that the measure $M_\infty^{(\beta)}$ in (2.7) is a Lévy spectral measure on the Banach space E. As was already pointed out, it is necessary that the corresponding Lévy measure M have finite $(-\beta)$-moment outside every neighborhood of zero. In fact, from a general fact in [8], we have

> If $M_\infty^{(\beta)}$ is a Lévy measure on a Banach space E, then so is M, and $\int_{\|x\|>\varepsilon} \|x\|^{-\beta}\, dM(x) < \infty$ for all $\varepsilon > 0$. (2.15)

This leads to the following problem: What are the Banach spaces E on which the converse of (2.15) holds? That is, can one characterize those spaces E such that:

> If M is a Lévy measure on E with finite $(-\beta)$-moment outside every neighborhood of zero, then $M_\infty^{(\beta)}$ is a Lévy measure. (2.16)

Note that if (2.16) holds, then for all finite measures m concentrated on the unit sphere S of E,

$$\tilde{m}(A) = -\beta \int_S \int_0^1 1_A(tx)\, t^{\beta-1}\, dt\, dm(x) - \beta \int_S \int_1^\infty 1_A(tx)\, t^{\beta-1}\, dt\, dm(x) \tag{2.17}$$

is a Lévy measure (corresponding to a stable distribution on E), since the second measure in (2.17) is always finite. Thus [1, Chap. 3, Theorem 7.9] implies that E is of *stable type* $-\beta$. In fact, in spaces E of stable type $-\beta$, (2.16) is true, provided M is a measure (not necessarily a Lévy measure) that integrates $\|x\|^{-\beta}$ over $E\backslash\{0\}$. Thus condition (2.16) defines a subclass of the class of Banach spaces of stable type $-\beta$ (cf. [17] for more on geometric characterizations of normed spaces in terms of convergence properties of random series).

Remark 2.6. In view of Theorem 2.2, Corollary 2.4, and [8, Proposition 2], it may be of interest to study the following notion. A Banach space E may be said to be of *Lévy type* λ, where λ is a finite measure on $(0, \infty)$, if:

> For all Lévy measures M on E such that
>
> $$\int_{E\backslash\{0\}} \lambda(\{t\colon t \geq \|x\|^{-1}\})\, dM(x) < \infty,$$
>
> $$M^{\langle\lambda\rangle}(A) = \int_{E\backslash\{0\}} \int_0^\infty 1_A(tx)\, d\lambda(t)\, dM(x), \qquad A \in \mathscr{B}(E\backslash\{0\}), \tag{2.18}$$
>
> is a Lévy measure.

Note that when $d\lambda(t) = -\beta 1_{(0,1)}(t)\, t^{\beta-1}\, dt$, then $M^{\langle\lambda\rangle} = M_\infty^{(\beta)}$. Of course, on a Hilbert space, $M_\infty^{(\beta)}$ is a Lévy measure if and only if M is a Lévy measure and $\int_{\|x\|>1} \|x\|^{-\beta}\, dM(x) < \infty$ [7].

3. Continuity of the Mapping $\mathscr{I}^\beta$, $-2<\beta\leqslant 1$

In this section we will prove the continuity of a mapping $\mathscr{I}^\beta$ which is naturally suggested by Theorem 2.2. In the sequel, we restrict our attention to symmetric measures on a Hilbert space H. This is necessitated by the fact that an analogue of Theorem 3.1 below is not available for general Banach spaces. Let

$$\mathscr{I}\mathscr{D}^s_\beta=\mathscr{I}\mathscr{D}^s_\beta(H)=\left\{\rho\in\mathscr{I}\mathscr{D}(H)\text{: }\rho\text{ is symmetric, }\int_H\|x\|^{-\beta}\,d\rho(x)<\infty\right\}. \tag{3.1}$$

For $\nu\in\mathscr{I}\mathscr{D}^s_\beta$, let Y be a $D_H([0,\infty))$-valued random variable with stationary, independent increments and $Y(0)=0$ such that $\nu=\mathscr{L}(Y(1))$. (In short, Y is a Lévy process.) Let

$$\mathscr{I}^\beta(\nu)=\mathscr{L}\left(\int_{(0,1)}t\,dY(t^\beta)\right), \tag{3.2}$$

and define $\tilde{\mathscr{U}}_\beta=\{\mathscr{I}^\beta(\nu)\colon\nu\in\mathscr{I}\mathscr{D}^s_\beta\}\subset\mathscr{U}_\beta$. Because of Remark 2.6, the integral exists, or equivalently, $M^{(\beta)}_\infty$ in Theorem 2.2 is a Lévy spectral measure. Of course, the class $\tilde{\mathscr{U}}_\beta$ is a proper subclass of $\mathscr{U}_\beta$. The mapping $\mathscr{I}^\beta$ can be expressed in terms of Gaussian covariance operators (positive-definite, trace-class operators on H) and Lévy (symmetric) spectral measures. Namely, if $\nu=[0,R,M]$ and $\mathscr{I}^\beta(\nu)=[0,R^{(\beta)}_\infty,M^{(\beta)}_\infty]$, then

$$R^{(\beta)}_\infty=-\beta(2+\beta)^{-1}R,\qquad M^{(\beta)}_\infty(A)=-\beta\int_{(0,1)}M(-t^{-1}A)\,t^{\beta-1}\,dt. \tag{3.3}$$

(Pass to the limit in (2.5) and (2.6), respectively.) For reference we quote Theorem VI.5.5 in [20]:

Theorem 3.1. *$[b_n,S_n,N_n]\Rightarrow[b,S,N]$ if and only if:*

(a) *$b_n\to b$ in H.*

(b) *For all bounded continuous functions f on $H\backslash\{0\}$ that vanish in some neighborhood of zero,*

$$\lim_{n\to\infty}\int_H f(x)\,dN_n(x)=\int_H f(x)\,dN(x).$$

(c) *For every $\varepsilon>0$, the sequence $T_{n,\varepsilon}$ of S-operators defined by*

$$\langle y,T_{n,\varepsilon}y\rangle=\langle y,S_ny\rangle+\int_{\|x\|\leqslant\varepsilon}\langle y,x\rangle^2\,dN_n(x)$$

is relatively compact, and

$$\lim_{\varepsilon \to 0} \liminf_{n \to \infty} \langle y, T_{n,\varepsilon} y \rangle = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \langle y, T_{n,\varepsilon} y \rangle = \langle y, Sy \rangle.$$

Also recall that a family $\{U_\alpha\}$ of S-operators is relatively compact provided

$$\sup_\alpha \operatorname{Tr} U_\alpha = \sup_\alpha \sum_{n=1}^{\infty} \langle e_n, U_\alpha e_n \rangle < \infty \tag{3.4}$$

and

$$\lim_{m \to \infty} \sup_\alpha \sum_{n=m}^{\infty} \langle e_n, U_\alpha e_n \rangle = 0 \tag{3.5}$$

for some (whence all) orthonormal basis $\{e_n\}$ of H.

Before stating our main result, we need the following lemma.

LEMMA 3.2. *Let E be a separable Banach space, and denote by $C_0(E)$ the space of all bounded, continuous functions on E vanishing on a neighborhood of 0. For measures M_n and M on $E \backslash \{0\}$ such that $\int_{\|x\| > \varepsilon} \|x\|^{-\beta} \, dM_n(x) < \infty$ for all $\varepsilon > 0$, and similarly for M, the following are equivalent:*

(i) $\displaystyle\lim_{n \to \infty} \int_E g \, dM_n = \int_E g \, dM, \; g \in C_0(E),$

$$\textit{and } \lim_{s \to \infty} \sup_n \int_{\|x\| > s} \|x\|^{-\beta} \, dM_n(x) = 0.$$

(ii) $\displaystyle\lim_{n \to \infty} \int_E g(x) \, \|x\|^{-\beta} \, dM_n(x) = \int_E g(x) \, \|x\|^{-\beta} \, dM(x), \; g \in C_0(E).$

Proof. (i) $\Rightarrow$ (ii) Pick $\varepsilon > 0$, and let g be an element of $C_0(E)$ bounded by $c > 0$. Select $s > 0$ such that $M(\{x : \|x\| = s\}) = 0$ and $\sup_n \int_{\|x\| > s} \|x\|^{-\beta} \, dM_n(x) < \varepsilon / c$. Apply (i) to $f(x) = g(x) \, \|x\|^{-\beta} \, 1_{\{\|x\| \leq s\}}(x)$, which is bounded and whose discontinuity points have M-measure 0 (cf. [2, Theorem 5.2.iii]). Then

$$\lim_{n \to \infty} \int_{\|x\| \leq s} g(x) \, \|x\|^{-\beta} \, dM_n(x) = \int_{\|x\| \leq s} g(x) \, \|x\|^{-\beta} \, dM(x).$$

Since $\sup_n \left| \int_{\|x\| > s} g(x) \, \|x\|^{-\beta} \, dM_n(x) \right| \leq \varepsilon$, (ii) follows.

(ii) $\Rightarrow$ (i) From (ii) we infer that the measures $\|x\|^{-\beta} \, dM_n(x)$, $n > 0$, are tight outside every neighborhood of 0. Hence the second condition of (i) follows. Furthermore, if g is bounded, continuous, and vanishes on a neighborhood of 0, then the same is true for $g(x) \, \|x\|^{\beta}$. Applying (ii) to the latter function gives (i).

THEOREM 3.3. *For $\nu_n, \nu \in \mathscr{I}\mathscr{D}^s_\beta(H)$, the following are equivalent*:

(a) $\nu_n \Rightarrow \nu$ *and* $\lim_{n\to\infty} \int_H \|x\|^{-\beta}\, d\nu_n(x) = \int_H \|x\|^{-\beta}\, d\nu(x)$.

(b) $\mathscr{I}^\beta(\nu_n) \Rightarrow \mathscr{I}^\beta(\nu)$.

Proof. Write $\nu_n = [0, R_n, M_n]$ and $\nu = [0, R, M]$. By [1, Proposition 3.2] (restated in terms of the distributions of the random variables), (a) is equivalent to

$$\nu_n \Rightarrow \nu \qquad \text{and} \qquad \lim_{s\to\infty} \sup_n \int_{\|x\|>s} \|x\|^{-\beta}\, d\nu_n(x) = 0. \tag{3.6}$$

Applying [10, Theorem 2], (3.6) is equivalent to

$$\nu_n \Rightarrow \nu \qquad \text{and} \qquad \lim_{s\to\infty} \sup_n \int_{\|x\|>s} \|x\|^{-\beta}\, dM_n(x) = 0. \tag{3.7}$$

If we now apply Theorem 3.1 and Lemma 2.2, the proof of our theorem is reduced to showing that the conditions

(i) $\lim_{n\to\infty} \int_H g(x)\, \|x\|^{-\beta}\, dM_n(x) = \int_H g(x)\, \|x\|^{-\beta}\, dM(x)$ for all g bounded, continuous, and vanishing on a neighborhood of 0

and

(ii) $\lim_{\varepsilon\to\infty} \liminf_{n\to\infty} \langle y, T_{n,\varepsilon} y\rangle = \lim_{\varepsilon\to\infty} \limsup_{n\to\infty} \langle y, T_{n,\varepsilon} y\rangle = \langle y, Ry\rangle$, where for each $\varepsilon > 0$ the sequence $T_{n,\varepsilon}$ is as in Theorem 3.1 with $S_n = R_n$ and $N_n = M_n$, and is relatively compact

are equivalent to

(i′) $\lim_{n\to\infty} \int f\, d(M_n)^{(\beta)}_\infty = \int f\, dM^{(\beta)}_\infty$ for all f bounded, continuous, and vanishing on a neighborhood of 0

and

(ii′) for each $\varepsilon > 0$, the sequence $(T_{n,\varepsilon})^{(\beta)}_\infty$ of S-operators as in Theorem 3.1, with $S_n = (R_n)^{(\beta)}_\infty$ and $N_n = (M_n)^{(\beta)}_\infty$ as in (3.3), is relatively compact and satisfies

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \langle y, (T_{n,\varepsilon})^{(\beta)}_\infty\, y\rangle$$
$$= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \langle y, (T_{n,\varepsilon})^{(\beta)}_\infty\, y\rangle = \langle y, R^{(\beta)}_\infty y\rangle.$$

[(i), (ii)] $\Rightarrow$ [(i′), (ii′)] For any bounded, continuous function f on H vanishing on a neighborhood of 0, the function

$$\tilde{f}(x) = \int_0^{\|x\|} f(tx/\|x\|)\, t^{\beta-1}\, dt$$

has the same properties. Since

$$\int_{H\setminus\{0\}} f(x)\, d(M_n)_\infty^{(\beta)}(x) = -\beta \int_{H\setminus\{0\}} \int_0^1 f(tx)\, t^{\beta-1}\, dt\, dM_n(x)$$

$$= -\beta \int_{H\setminus\{0\}} \tilde{f}(x)\, \|x\|^{-\beta}\, dM_n(x),$$

we obtain (i′) from (i). Furthermore, by (3.3),

$$\int_{\|x\|\leqslant\varepsilon} \langle y, x\rangle^2\, d(M_n)_\infty^{(\beta)}(x)$$

$$= -\frac{\beta}{2+\beta} \int_{\|x\|\leqslant\varepsilon} \langle y, x\rangle^2\, dM_n(x)$$

$$-\beta \int_0^1 \int_{\varepsilon\leqslant\|x\|\leqslant\varepsilon t^{-1}} \langle y, x\rangle^2\, dM_n(x)\, t^{\beta+1}\, dt$$

$$= -\frac{\beta}{2+\beta} \Bigg[\int_{\|x\|\leqslant\varepsilon} \langle y, x\rangle^2\, dM_n(x)$$

$$+ \varepsilon^{2+\beta} \int_{\|x\|>\varepsilon} \langle y, x\rangle^2\, \|x\|^{-(2+\beta)}\, dM_n(x) \Bigg].$$

Denoting by $U_{M_n,\varepsilon}$ the S-operator given by the second integral in the bracket, we have

$$\langle y, (T_{n,\varepsilon})_\infty^{(\beta)} y\rangle = -\frac{\beta}{2+\beta} [\langle y, T_{n,\varepsilon}\, y\rangle + \langle y, U_{M_n,\varepsilon}\, y\rangle]. \tag{3.8}$$

Since

$$\langle y, U_{M_n,\varepsilon}\, y\rangle = \int_H 1_{\{\|x\|>\varepsilon\}}(x)\, \varepsilon^{2+\beta} \langle y, x/\|x\|\rangle^2\, \|x\|^{-\beta}\, dM_n(x),$$

(i) and [2, Theorem 5.2.iii] imply $\lim_{n\to\infty} \langle y, U_{M_n,\varepsilon}\, y\rangle = \langle y, U_{M,\varepsilon}\, y\rangle$, $y \in H$, provided $M(\{x\colon \|x\| = \varepsilon\}) = 0$. Furthermore, since M integrates $\langle y, \cdot\rangle^2$ over the unit ball of H, the Lebesgue dominated convergence theorem gives

$$\lim_{\varepsilon\to 0} \int_{\varepsilon<\|x\|\leqslant 1} \varepsilon^{2+\beta} \langle y, x\rangle^2\, \|x\|^{-(2+\beta)}\, dM(x) = 0. \tag{3.9}$$

Finally, we have $\lim_{\varepsilon \to 0} \lim_{n \to \infty} \langle y, U_{M,\varepsilon} y \rangle = 0$. Thus (ii′) follows from (ii) and (3.8), completing the proof that (a) $\Rightarrow$ (b).

[(i′), (ii′)] $\Rightarrow$ [(i), (ii)] From (i′) we have that the sequence $(M_n)_\infty^{(\beta)}$, $n > 0$, is conditionally compact outside every neighborhood of zero. Thus for each $\eta > 0$ and $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq H \setminus \{0\}$ such that for all $n > 0$,

$$\begin{aligned}
\varepsilon &\geqslant (M_n)_\infty^{(\beta)} (K_\varepsilon^c \cap \{x: \|x\| > \eta\}) \\
&\geqslant -\beta \int_{1/2}^{1} M_n(t^{-1}(K_\varepsilon^c \cap \{x: \|x\| > \eta\}))\, t^{\beta-1}\, dt \\
&\geqslant M_n(([1, 2] \cdot K_\varepsilon)^c \cap \{x: \|x\| > 2\eta\})(-\beta) \int_{1/2}^{1} t^{\beta-1}\, dt.
\end{aligned}$$

Since $[1, 2] \cdot K_\varepsilon$ is compact, the sequence $\{M_n\}$ is conditionally compact outside every neighborhood of zero. Suppose M_0 is a limit point of $\{M_n\}$ with corresponding subsequence $\{M_{n'}\}$. By (3.3) for any $r > 0$,

$$\begin{aligned}
(M_n)_\infty^{\beta} &(\{x: \|x\| > r\}) \\
&= -\beta \int_r^\infty \left[\int_{\|x\| > t} \|x\|^{-\beta}\, dM_n(x) \right] t^{\beta-1}\, dt \\
&= r^\beta \int_{\|x\| > r} \|x\|^{-\beta}\, dM_n(x) - M_n(\{x: \|x\| > r\}), \qquad (3.10)
\end{aligned}$$

and similarly for M. If $M_0(\{x: \|x\| = r\}) = M_\infty^{(\beta)}(\{x: \|x\| = r\}) = 0$, then

$$\lim_{n \to \infty} (M_n)_\infty^{(\beta)}(\{x: \|x\| > r\}) = M_\infty^{(\beta)}(\{x: \|x\| > r\})$$

and

$$\lim_{n' \to \infty} M_{n'}(\{x: \|x\| > r\}) = M_0(\{x: \|x\| > r\}),$$

so

$$\lim_{n' \to \infty} \int_{\|x\| > r} \|x\|^{-\beta}\, dM_{n'}(x) = h(r)$$

exists. Set

$$h_n(t) = \int_{\|x\| > t} \|x\|^{-\beta}\, dM_n(x) \qquad \text{and} \qquad g(t) = \int_{\|x\| > t} \|x\|^{-\beta}\, dM(x).$$

Since each h_n is nonincreasing, by (i′), (3.10), and the dominated convergence theorem,

$$\int_r^\infty g(t)\, t^{\beta-1}\, dt = \lim_{n'\to\infty} \int_r^\infty \left[\int_{\|x\|>t} \|x\|^{-\beta}\, dM_{n'}(x)\right] t^{\beta-1}\, dt$$
$$= \int_r^\infty h(t)\, t^{\beta-1}\, dt$$

for all but countably many r. Thus the monotonicity of g and h imply $g = h$; i.e., for all $r > 0$,

$$\lim_{n'\to\infty} \int_{\|x\|>r} \|x\|^{-\beta}\, dM_{n'}(x) = \int_{\|x\|>r} \|x\|^{-\beta}\, dM(x)$$
$$= \int_{\|x\|>r} \|x\|^{-\beta}\, dM_0(x).$$

Now [2, Theorem 5.4] implies that the conditions of Lemma 3.2.i obtain with M_0 in place of M; so applying Lemma 3.2 gives (i), with M replaced by M_0. Because of (3.8) and the monotonicity in ε of all the expressions therein, we also obtain (ii) for the sequence $M_{n'}$ and some Gaussian covariance operator R_0. Applying the implication (a) $\Rightarrow$ (b) to $\nu_{n'} \Rightarrow \nu_0 = [0, R_0, M_0]$, we obtain $\mathcal{I}^\beta(\nu_{n'}) \Rightarrow \mathcal{I}^\beta(\nu) = [0, (R_0)_\infty^{(\beta)}, (M_0)_\infty^{(\beta)}]$. Since the mapping $\mathcal{I}^\beta$ is one-to-one, we see that $R_0 = R$ and $M_0 = M$, yielding (i) and (ii). Thus (b) $\Rightarrow$ (a), and the proof is complete.

Corollary 3.4. *Let* $\nu_n = [0, R_n, M_n]$, $\nu = [0, R, M] \in \mathcal{ID}_\beta^s(H)$. *The following are equivalent*:

(a) $\nu_n \Rightarrow \nu$ *and* $\lim_{r\to\infty} \sup_n \int_{\|x\|>r} \|x\|^{-\beta}\, dM_n(x) = 0$.

(b) $\mathcal{I}^\beta(\nu_n) \Rightarrow \mathcal{I}^\beta(\nu)$.

Proof. This result follows from the fact that (a) of Theorem 3.3 is equivalent to (3.6), which is in turn equivalent to (a) of our corollary, as mentioned in the beginning of the proof of Theorem 3.3.

The class $\mathcal{ID}(E)$ of all infinitely divisible measures on E can be described as the smallest closed subsemigroup of $\mathcal{P}(E)$ containing all symmetric Gaussian measures and all shifted Poisson measures of the form $[x, 0, \lambda\delta(y)]$ for $x, y \in E$ and $\lambda > 0$. Using the homeomorphism $\mathcal{I}^\beta$ between $\mathcal{ID}_\beta^s(H)$ and the symmetric measures in $\tilde{\mathcal{U}}_\beta(H)$, we shall describe a set of generators of $\tilde{\mathcal{U}}_\beta(H)$.

COROLLARY 3.5. *For* $-2<\beta\leqslant -1$, *the class* $\tilde{\mathscr{U}}^s_\beta(H)$ *of all symmetric measures in* $\tilde{\mathscr{U}}_\beta(H)$ *is the smallest closed subsemigroup of* $\mathscr{I}\mathscr{D}(H)$ *containing the set* $\mathscr{G}_\beta$ *of all symmetric Gaussian measures and all compound Poisson measures* $[0, 0, \lambda M_{\alpha,z}]$, *where*

$$M_{\alpha,z}(A)=\int_0^\alpha 1_A(tz)\, t^{\beta-1}\, dt, \qquad A\in\mathscr{B}(H\backslash\{0\}) \tag{3.11}$$

for $\alpha, \lambda>0$ *and* $\|z\|=1$.

Proof. Since $M_{\alpha,z}$ integrates $\|x\|^2\wedge 1$, it is a Lévy measure on H. Note that for $y\neq 0$ and $A=[U, I]=\{x: x/\|x\|\in U,\ \|x\|\in I\}$,

$$\delta(y)^{(\beta)}_\infty(A)= -\beta\, \|y\|^{-\beta}\int_0^{\|y\|} 1_A(sy/\|y\|)\, s^{\beta-1}\, ds,$$

i.e., $[0, 0, \delta(y)^{(\beta)}_\infty]\in\mathscr{G}_\beta$.

For each $n>0$, let $A_{nj}\subseteq H\backslash\{0\}$ be pairwise disjoint Borel sets with $\operatorname{diam}(A_{nj})<n^{-1}$, $0\notin\bar{A}_{nj}$, $j=1, 2, ...,$ and $\bigcup_j A_{nj}=H\backslash\{0\}$, and choose $x_{nj}\in A_{nj}$ for all n and j. For $v=[0, 0, M]\in\mathscr{I}\mathscr{D}^s_\beta(H)$, set $a_{nj}=M(A_{nj})$, $n, j=1, 2, ...$ Then setting $M_n=\sum_{j=1}^\infty a_{nj}\,\delta(x_{nj})$, it is well known that $[0, 0, M_n]\Rightarrow v$ outside every neighborhood of 0 (cf. [1, p. 18, Ex. 13]). Now, for large r, letting B_r denote the closed ball of radius r about 0,

$$\begin{aligned}
&\int_{\|x\|>r}\|x\|^{-\beta}\, dM_n(x)\\
&\quad\leqslant \sum_{\{j:\, A_{nj}\cap B_r^c\neq\varnothing\}}\int_{A_{nj}}\|x_{nj}\|^{-\beta}\, dM\\
&\quad\leqslant \sum_{\{j:\, A_{nj}\cap B_r^c\neq\varnothing\}} 2^{-\beta}\left[\int_{A_{nj}}\|x_{nj}-x\|^{-\beta}\, dM+\int_{A_{nj}}\|x\|^{-\beta}\, dM\right]\\
&\quad\leqslant 2^{-\beta}\, M(B^c_{r-1})+2^{-\beta}\int_{\|x\|>r-1}\|x\|^{-\beta}\, dM.
\end{aligned}$$

By Corollary 3.4, $\mathscr{I}^\beta([0, 0, M_n])\Rightarrow\mathscr{I}^\beta(v)$.

Since Gaussian measures are invariant under $\mathscr{I}^\beta$, up to multiplication by the constant $-\beta/(2+\beta)$, and $\mathscr{I}^\beta$ is a homomorphism with respect to convolution, the proof of our corollary is complete.

Remarks 3.6. (i) It might be worthwhile to note that the generators $\mathscr{G}_\beta$ are of the same form for all $\beta>-2$ (cf. [6, 8, 10]). Moreover, for

$\beta > -1$, the measures $M_{\alpha,z}$ integrate $\|x\| \wedge 1$, so they are Lévy measures on *any* Banach space [1].

(ii) Since the thermodynamic limit distributions in the Ising model are closely related to the classes $\mathcal{U}_\beta$ with $-2 < \beta \leqslant -1$ (cf. Remark 2.1), one would like to interpret the set of "generators" $\mathcal{G}_\beta$ in the context of statistical physics. This and similar questions will be the subject of a future work.

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