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Weighted bootstrap for U-statistics

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Abstract

In this paper we investigate the weighted bootstrap for U-statistics and its properties. Under very general choices of random weights and certain regularity conditions, we show that the weighted bootstrap method with U-statistics provides second-order accurate approximations to the distribution of U-statistics. We shall prove this via one-term Edgeworth expansions of weighted U-statistics.

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1. Introduction

Let $X_1, \dots, X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables with common distribution function (d.f.) F. For a symmetric kernel function $h(x, y)$, we define a U-statistic by

$$U_n = \frac{2}{n(n-1)} \sum_{1 \leqslant i < j \leqslant n} h_{ij}, \tag{1}$$

where $h_{ij} \equiv h(X_i, X_j)$ with $E(h(X_1, X_2)) = \theta$. Further, we define

$$g(x) = E(h(X_1, X_2)|X_1 = x) - \theta.$$

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Throughout the paper, it is assumed that $\sigma_g^2 \equiv Var(g(X_1)) > 0$ and let, for each $n \geqslant 2$ and real x,

$$F_n(x) = P\left(\frac{\sqrt{n}(U_n - \theta)}{2\sigma_g} \leqslant x\right).$$

Asymptotic normality of $F_n(x)$ was first established by Hoeffding (1948) [11] under the condition $\sigma_g^2 < \infty$. A more accurate approximation to $F_n(x)$ can be obtained by Edgeworth expansions, which were studied very extensively by various authors in the past two decades, including Callaert et al. [4], Bickel et al. [3], Lai and Wang [15], Maesono [18], Bentkus et al. [2] and Putter and van Zwet [21]. For our purpose in this paper, it is enough to describe an Edgeworth expansion with remainder term of size $o(n^{-1/2})$. Define

$$E_{1n}(x) = \Phi(x) - \frac{\gamma}{6\sqrt{n}}(x^2 - 1)\phi(x), \tag{2}$$

where

$$\gamma = \sigma_g^{-3}\{Eg^3(X_1) + 3E(g(X_1)g(X_2)h(X_1, X_2))\}.$$

Then if $E|h(X_1, X_2)|^3 < \infty$ and $g(X_1)$ is non-lattice, we have

$$\sup_x |F_n(x) - E_{1n}(x)| = o(n^{-1/2}).$$

See Bickel et al. [3] for instance.

In the event that F is unknown, we can estimate $F_n(x)$ by the one-term empirical Edgeworth expansion defined by

$$\hat{E}_{1n}(x) = \Phi(x) - \frac{\hat{\gamma}}{6\sqrt{n}}(x^2 - 1)\phi(x),$$

where $\hat{\gamma}$ is a consistent estimator of γ. Clearly, $\hat{E}_{1n}(x)$ provides a second-order accurate estimate to $F_n(x)$. An alternative way to obtain a second-order accurate estimate to $F_n(x)$ is via Efron's bootstrap in a more direct way. For a description of this, see Helmers [10] or Lai and Wang [15] for instance.

In this paper, we shall investigate yet another method to estimate $F_n(x)$, namely the generalized bootstrap or weighted bootstrap method for U-statistics. As its name implies, the method involves placing random weights to each term $h(X_i, X_j)$ in the original U-statistic U_n. The resulting statistic will be referred to as the weighted bootstrap U-statistic of U_n. The issue of consistency under different weights and kernel functions has been studied by many authors; see Janssen [14], Hušková and Janssen [12,13], Dehling and Mikosch [6] and others. It is the purpose of this paper to study higher-order performance of the weighted bootstrap U-statistics under various choices of random weights.

It is worth mentioning that the weighted bootstrap in the case of means has been well studied so far. See the monograph by Barbe and Bertail [1] and the references therein. In particular, results on the second-order accuracy under various situations in the case of means were given by Weng [24], Haeusler et al. [8], Lo [17] and others.

Finally, we remark that the results of this paper differ from those in Tu [23], in which he constructed a weighted bootstrap U-statistic by using Jackknife pseudo-values.

The paper is organized as follows. In Section 2, we shall show that under rather weak conditions, the weighted bootstrap method for U-statistics provides a second-order accurate estimate to the target distribution function $F_n(x)$. We shall prove this by way of establishing Edgeworth expansions for the weighted (bootstrap) U-statistics, which is treated in Section 3. Proofs of all the main results are given in Section 4. Finally, some technical details will be relegated to Section 5.

2. Main results for weighted bootstrap U-statistics

In this section, we introduce the weighted bootstrap U-statistics and study their properties. The weight function can either be dependent or independent. For ease of exposition, we shall deal with them separately.

2.1. Dependent weights

Let $\{W_{nj}, 1 \leqslant j \leqslant n\}$ be a sequence of random weights independent of the data $\{X_1, \ldots, X_n\}$. Define the weighted bootstrap U-statistic by

$$U_{W_n} = \frac{1}{n^2} \sum_{i \neq j} W_{ni} W_{nj} h_{ij}. \tag{3}$$

There are many ways to choose the weights W_{ni}. For instance, (3) reduces to Efron's bootstrap U-statistics if multinomial weights are chosen, i.e.,

$$(W_{n1}, \ldots, W_{nn}) \sim \textit{Multinomial}\ \{n; n^{-1}, \ldots, n^{-1}\}.$$

Another example is the Bayesian bootstrap U-statistics which can be formed by choosing the weights $\{W_{nj}, 1 \leqslant j \leqslant n\}$ from a Dirichlet distribution, i.e.,

$$(n^{-1} W_{n1}, \ldots, n^{-1} W_{nn}) \sim \textit{Dirichlet}(1, 1, \ldots, 1).$$

For more examples, see Mason and Newton [19], Dehling et al. [5] and Hušková and Janssen [13]. Note that in all these examples, the weights $\{W_{nj}, 1 \leqslant j \leqslant n\}$ are dependent random variables satisfying

$$W_{nj} \geqslant 0, \quad \sum_{j=1}^{n} W_{nj} = n. \tag{4}$$

Here we choose the random weights of the following form:

$$W_{nj} = \frac{Y_j}{\overline{Y}} \quad \text{with } \overline{Y} = \frac{1}{n} \sum_{j=1}^{n} Y_j, \tag{5}$$

where Y_j's are i.i.d. strictly positive random variables (independent of X_j). Clearly, the weights in (5) satisfy (4).

Let $P^*(\cdot)$ denote the conditional probability given $X_1, \dots, X_n$. Let

$$a_{nj} = \frac{1}{n}\sum_{\substack{k=1\\k\neq j}}^{n} h_{jk}, \quad U_{1n} = \frac{2}{n(n+1)}\sum_{i<j} h_{ij} = \frac{n-1}{n+1}U_n,$$

$$\sigma_n^2 = \frac{4}{n}\sum_{j=1}^{n}(a_{nj}-U_n)^2, \quad \sigma_{1n}^2 = \frac{4}{n}\sum_{j=1}^{n}\left(a_{nj} - \frac{n-1}{n}U_{1n}\right)^2. \tag{6}$$

For an arbitrary random variable Z, write

$$\kappa(Z) = 1 - \sup\{|E\exp\{it(Z-EZ)\}| : \ EZ^2/(8E|Z|^3) \leqslant |t| \leqslant 2n^{1/5}\}. \tag{7}$$

We shall elaborate more on the above definitions. The definitions of σ_n^2 and σ_{1n}^2 are two slightly different versions of the variance estimators of σ^2, which are related by $\sigma_{1n}^2 = \sigma_n^2 + 4(n+1)^{-2}U_n^2$. The definition of $\kappa(Z)$ is related to the smoothness of the r.v. Z, see Remark 2.2 below, for instance.

Our first theorem shows that the distribution function of the weighted bootstrap U-statistic, appropriately centered and normalized, can provide a second-order accurate approximation to the distribution function $F_n(x)$ of the standardized U-statistic U_n.

Theorem 2.1. *Suppose that*

(A1) $E|h_{12}|^3 < \infty$,
(A2) $E|Y_1|^3 < \infty, \quad \kappa(Y_1) > 0$,
(A3) $(EY_1)^2 = Var(Y_1), \quad E(Y_1 - EY_1)^3/(Var(Y_1))^{3/2} = 1$,
(A4) $\kappa(g(X_1)) > 0$.

Then we have

$$\sup_x \left| P^*\left\{\frac{\sqrt{n}(U_{W_n} - U_{1n})}{\sigma_{1n}} \leqslant x\right\} - F_n(x)\right| = o(n^{-1/2}), \quad a.s. \tag{8}$$

The proof of Theorem 2.1 follows easily from Theorem 3.1 in the next section and the next theorem, which gives an Edgeworth expansion of weighted bootstrap U-statistics.

Theorem 2.2. *Suppose that conditions* (A1)–(A3) *in Theorem* 2.1 *hold*, *then we have*

$$\sup_x \left| P^*\left\{\frac{\sqrt{n}(U_{W_n} - U_{1n})}{\sigma_{1n}} \leqslant x\right\} - E_{1n}(x)\right| = o(n^{-1/2}), \quad a.s. \tag{9}$$

where $E_{1n}(x)$ *was given in* (2).

Remark 2.1. The centering value U_{1n} in (8) cannot be replaced by U_n. To see why, note $U_{1n} - U_n = -2U_n/(n+1) = O_p(n^{-1/2})$. Therefore, by replacing U_{1n} by U_n, a bias term of size $O(n^{-1/2})$ will be introduced in the Edgeworth expansion. On the other hand, σ_{1n} can be replaced by σ_n since $\sigma_{1n}^2 = \sigma_n^2 + 4(n+1)^{-2}U_n^2$.

Remark 2.2. If the distribution of Z is non-lattice or the Crámer condition $\lim_{|t|\to\infty}|Ee^{itZ}|<1$ is satisfied, then it can be shown that $\liminf_{n^{1/6}\geqslant EZ^2/(16E|Z|^3)} \kappa(Z)>0$.

Remark 2.3. The weights Y's satisfying conditions (A2) and (A3) in Theorem 2.1 can be easily found. For instance, one can choose Y to follow Gamma distribution function with p.d.f.

$$f_Y(y) = \frac{1}{\Gamma(\alpha)\gamma^\alpha} y^{\alpha-1} e^{-y/\gamma} I_{(0,\infty)}(x),$$

where the two parameters (α, γ) can be determined by the two restrictions in (A3).

2.2. Independent weights

Notice that the above chosen weight function W_{ni} are dependent weights. It is also of interest to consider the case of independent weights. In other words, let ξ_i be i.i.d. random variables independent of X_i. Here, the weights ξ_i's need not be non-negative. Define the weighted bootstrap U-statistics with independent weights ξ's by

$$U_\xi = \frac{1}{n^2} \sum_{i\neq j} \xi_i \xi_j h_{ij}.$$

Note that U_ξ is of random quadratic form. Write

$$\mu_\xi = E\xi_1, \quad \tau_\xi^2 = Var(\xi_1).$$

Dehling and Mikosch [6] discussed the consistency of the distribution function of U_ξ under the conditions $\mu_\xi = 0$ and $\tau_\xi^2 = 1$. The next theorem gives a parallel result to that of Theorem 2.1.

Theorem 2.3. *Assume that*

(B1) $E|h_{12}|^3 < \infty, \quad \kappa(g(X_1))>0,$
(B2) $E|\xi_1|^3 < \infty, \quad \kappa(\xi_1)>0,$
(B3) $(E\xi_1)^2 = Var(\xi_1), \quad E(\xi_1 - E\xi_1)^3/(Var(\xi_1))^{3/2} = 1.$

Then we have

$$\sup_x \left| P^*\left\{ \frac{\sqrt{n}(U_\xi - (n-1)U_n/n)}{\mu_\xi \tau_\xi \sigma_n} \leqslant x \right\} - F_n(x) \right| = o(n^{-1/2}), \quad a.s., \tag{10}$$

where σ_n is defined as in (6).

Variations of the last theorem exist. For instance, define

$$\tilde{U}_\xi = \frac{1}{n^2}\sum_{i\neq j}\left(\xi_i\xi_j + \frac{(\mu_\xi - \tau_\xi)(\xi_i - \mu_\xi)(\xi_j - \mu_\xi)}{\tau_\xi}\right)(h_{ij} - U_n).$$

We have the following theorem.

Theorem 2.4. *Assume that the conditions of Theorem* 2.3 *hold except that* $(E\xi_1)^2 = Var(\xi_1)$ *is now removed. Then we have*

$$\sup_x\left|P^*\left\{\frac{\sqrt{n}\tilde{U}_\xi}{\mu_\xi\tau_\xi\sigma_n}\leqslant x\right\} - F_n(x)\right| = o(n^{-1/2}), \quad a.s. \tag{11}$$

The rates of convergence in (11) can be improved under slightly higher moment conditions.

Theorem 2.5. *Suppose that*

(C1) $E|h_{12}|^4 < \infty, \quad \kappa(g(X_1)) > 0,$

(C2) $E|\xi_1|^4 < \infty, \quad \kappa(\xi_1) > 0,$

(C3) $E(\xi_1 - E\xi_1)^3/(Var(\xi_1))^{3/2} = 1.$

Then we have

$$\sup_x\left|P^*\left\{\frac{\sqrt{n}\tilde{U}_\xi}{\mu_\xi\tau_\xi\sigma_n}\leqslant x\right\} - F_n(x)\right| = O(n^{-2/3}\log n), \quad a.s. \tag{12}$$

3. Edgeworth expansion for weighted U-statistics

To prove the main results of the last section, we need to establish Edgeworth expansions for weighted U-statistics, which could also be of independent interest. Edgeworth expansions for (non-weighted) U-statistics have been studied very extensively in the past two decades. References are Callaert et al. [4], Bickel et al. [3], Lai and Wang [15], Maesono [18], Bentkus et al. [2] and Putter and van Zwet [21].

Let $X_1, \ldots, X_n$ be a sequence of independent and identically distributed (i.i.d.) generic random variables, which could be different from those given in previous sections. Let $\xi(x)$ and $\psi(x, y)$ be real functions in its arguments. For some sequences of real numbers b_{nj}, c_{nj} and d_{nij}, we define

$$T_n = \frac{1}{B_n}\sum_{j=1}^n b_{nj}X_j + \frac{1}{n}\sum_{j=1}^n c_{nj}\xi(X_j) + \frac{1}{n^{3/2}}\sum_{i<j} d_{nij}\psi(X_i, X_j),$$

where $B_n^2 = \sum_{j=1}^n b_{nj}^2$. Many statistics of interest can be put in the form of T_n. In this section, we shall derive an Edgeworth expansion for the distribution of T_n. As an application, we shall refine an Edgeworth expansion obtained by Bickel et al. [3].

Furthermore, the expansion will also be useful in proving the main theorems presented earlier in this paper.

Define

$$E_{2n}(x) = \Phi(x) + L_{1n}(x) + L_{2n}(x),$$

where

$$L_{1n}(x) = \sum_{j=1}^{n} (E\Phi(x - b_{nj}X_j/B_n) - \Phi(x)) - \tfrac{1}{2}\Phi^{(2)}(x),$$

$$L_{2n}(x) = K_{1n}\Phi^{(2)}(x) - K_{2n}\Phi^{(3)}(x)$$

with K_{1n} and K_{2n} given by

$$K_{1n} = \frac{1}{nB_n}\sum_{j=1}^{n} b_{nj}c_{nj}E(X_1\xi(X_1)),$$

$$K_{2n} = \frac{1}{n^{3/2}B_n^2}\sum_{1\leqslant i<j\leqslant n} b_{ni}b_{nj}d_{nij}E(X_1X_2\psi(X_1,X_2)).$$

The next theorem states that $E_{2n}(x)$ provides a second-order accurate approximation to the distribution of T_n under appropriate conditions.

Theorem 3.1. *Suppose that*

(D1) $EX_1 = 0, \quad EX_1^2 = 1, \quad E|X_1|^3 < \infty, \quad \kappa(X_1) > 0,$

(D2) $E\xi(X_1) = 0, \quad E[\psi(X_1,X_2)|X_t] = 0, \quad t = 1,2,$

(D3) *For the sequences b_{nj}, c_{nj} and d_{nij}, there exist absolute constants $l_1, \dots, l_4$, which are independent of n, F, ξ and ψ, such that*

$$\frac{1}{n}\sum_{j=1}^{n} b_{nj}^2 \geqslant l_1 > 0, \quad \frac{1}{n}\sum_{j=1}^{n} |b_{nj}|^3 \leqslant l_2 < \infty, \tag{13}$$

$$\frac{1}{n}\sum_{j=1}^{n} c_{nj}^2 \leqslant l_3 < \infty, \quad \frac{1}{mn}\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{n} d_{nij}^2 \leqslant l_4 < \infty, \quad \textit{for all } m \geqslant 1. \tag{14}$$

Then we have that for any $n \geqslant 2$,

$$\sup_x |P(T_n \leqslant x) - E_{2n}(x)| \leqslant C_1\kappa^{-1}(X_1)(\lambda+\rho+\beta)n^{-2/3}\log n + C_2\beta^2 n^{-1}, \tag{15}$$

where

$$\rho = E\xi^2(X_1), \quad \beta = E|X_1|^3, \quad \textit{and} \quad \lambda = E\psi^2(X_1,X_2).$$

In the remainder of this section, we shall give a more explicit expression of $E_{2n}(x)$. Write

$$II_n = \sup_x \left| L_{1n}(x) + \frac{EX_1^3}{6B_n^3}\sum_{j=1}^{n} b_{nj}^3\Phi^{(3)}(x) \right|.$$

Under condition (13), we shall show that $II_n = o(n^{-1/2})$. In fact, similar to Theorem 2.3 of Hall [9], it follows that

$$II_n \leqslant C_1 B_n^{-3} \sum_{j=1}^{n} |b_{nj}|^3 E(|X_1|^3 I\{|b_{nj} X_1| \geqslant B_n\}) + C_2 B_n^{-4} \sum_{j=1}^{n} b_{nj}^4 E(X_1^4 I\{|b_{nj} X_1| \leqslant B_n\}), \tag{16}$$

where (here and below) $I\{B\}$ denotes the indicator function for a set B. By applying (13), we get that for all $1 \leqslant j \leqslant n$,

$$B_n \geqslant l_1^{1/2} n^{1/2} \quad \text{and} \quad |b_{nj}| \leqslant l_2^{1/3} n^{1/3}.$$

Therefore, there exists a constant $C_0 > 0$ such that

$$E(|X_1|^3 I\{|b_{nj} X_1| \geqslant B_n\}) \leqslant E(|X_1|^3 I\{|X_1| \geqslant C_0 n^{1/6}\}),$$

$$\begin{aligned} &|b_{nj}| E(X_1^4 I\{|b_{nj} X_1| \leqslant B_n\}) \\ &\quad \leqslant B_n^{4/5} E|X_1|^3 + |b_{nj}| E(X_1^4 I\{B_n^{4/5} \leqslant |b_{nj} X_1| \leqslant B_n\}) \\ &\quad \leqslant B_n^{4/5} E|X_1|^3 + B_n E(|X_1|^3 I\{|X_1| \geqslant C_0 n^{1/15}\}). \end{aligned}$$

These estimates, together with (16), imply $II_n = o(n^{-1/2})$. Therefore, combining (16) and Theorem 3.1 yields the following theorem.

Theorem 3.2. *Suppose that*

$$E h_{12}^2 < \infty, \quad E|g(X_1)|^3 < \infty, \quad \kappa(g(X_1)) > 0.$$

Then for all $n \geqslant 2$,

$$\begin{aligned} &\sup_x |F_n(x) - E_{1n}(x)| \\ &\quad \leqslant \frac{C_1}{\kappa(g(X_1))} \left(\frac{E h_{12}^2}{\sigma_g^2} + \frac{E|g(X_1)|^3}{\sigma_g^3} \right) n^{-2/3} \log n + C_2 \left(\frac{E|g(X_1)|^3}{\sigma_g^3} \right)^2 n^{-1} \\ &\qquad + \frac{C_3}{\sigma_g^3} E(|g(X_1)|^3 I\{|g(X_1)| \geqslant \sqrt{n}\sigma_g\}) n^{-1/2} \\ &\qquad + \frac{C_4}{\sigma_g^4} E(|g(X_1)|^4 I\{|g(X_1)| \leqslant \sqrt{n}\sigma_g\}) n^{-1}, \end{aligned} \tag{17}$$

where $E_{1n}(x)$ *was given in* (2).

4. Proofs of main results

In this section, we give proofs of the theorems. Since Theorem 3.1 will be used in the proofs of other theorems, we shall provide its proof first.

From here on, we shall denote by A, $A_0, A_1, \ldots,$ some positive constants independent of n, and by C, $C_0, C_1, \ldots,$ some positive absolute constants independent of n, F and h. All these constants may be different at each occurrence. For ease of presentation, we write $\sum_{i<j}$ *and* $\sum_{i\neq j\neq k}$ for $\sum_{1\leqslant i<j\leqslant n}$ and $\sum_{1\leqslant i\neq j\neq k\leqslant n}$, respectively.

4.1. Proof of Theorem 3.1

We may assume $n\geqslant n_0$, where n_0 is chosen suitably large. Write

$$\begin{aligned}
&\gamma_j(t) = E(e^{itb_{nj}X_j/B_n}),\\
&\varphi_{1n}(t) = \left(1+\sum_{j=1}^{n}[\gamma_j(t)-1]+\frac{1}{2}t^2\right)e^{-t^2/2},\\
&\varphi_{2n}(t) = (itK_{1n}-t^2K_{2n})e^{-t^2/2},\\
&S_n = \frac{1}{B_n}\sum_{j=1}^{n} b_{nj}X_j,\\
&\Delta_{n,m} = \frac{1}{n}\sum_{j=1}^{m} c_{nj}\xi(X_j) + n^{-3/2}\sum_{i=1}^{m-1}\sum_{j=i+1}^{n} d_{nij}\psi(X_i,X_j).
\end{aligned}$$

Simple calculation shows that

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{itx}\,d[\Phi(x)+L_{1n}(x)] = \varphi_{1n}(t),\\
&\int_{-\infty}^{\infty} e^{itx}dL_{2n}(x) = it\varphi_{2n}(t).
\end{aligned}$$

From these and Esseen's smoothing lemma [20], it follows that (noting $E|X_1|^3\geqslant 1$)

$$\begin{aligned}
&\sup_x|P(T_n\leqslant x)-E_{2n}(x)|\\
&\quad\leqslant \int_{|t|\leqslant n^{2/3}}|t|^{-1}|Ee^{itT_n}-\varphi_{1n}(t)-it\varphi_{2n}(t)|\,dt + Cn^{-2/3}\sup_x\left|\frac{dE_{2n}(x)}{dx}\right|\\
&\quad\leqslant \sum_{j=1}^{4} I_{jn} + C_1n^{-2/3}(|K_{1n}|+|K_{2n}|+\beta),
\end{aligned}\tag{18}$$

where

$$\begin{aligned}
&I_{1n} = \int_{|t|\leqslant n^{1/10}}|t|^{-1}|Ee^{itT_n}-Ee^{itS_n}-itE(\Delta_{n,n}e^{itS_n})|\,dt\\
&I_{2n} = \int_{|t|\leqslant n^{1/10}}|t|^{-1}|Ee^{itS_n}-\varphi_{1n}(t)|\,dt,
\end{aligned}$$

$$I_{3n} = \int_{|t| \leqslant n^{1/10}} |E(\Delta_{n,n} e^{itS_n}) - \varphi_{2n}(t)|\, dt,$$
$$I_{4n} = \int_{n^{1/10} \leqslant |t| \leqslant n^{2/3}} |t|^{-1} |Ee^{itT_n}|\, dt.$$

To prove (15), in the following, we obtain bounds for each term in (18).

We first deal with the terms $|K_{1n}|$ and $|K_{2n}|$ in (18). Note that (13) implies that

$$l_1 n \leqslant B_n^2 \leqslant \sum_{j=1}^{n} (1 + |b_{nj}|^3) \leqslant (1 + l_2)\, n. \tag{19}$$

Also by the Cauchy–Schwartz inequality, we have

$$\sum_{i<j} (|b_{ni}|^k b_{nj}^2) \leqslant \left(\sum_{j=1}^{n} |b_{nj}|^k\right)\left(\sum_{j=1}^{n} b_{nj}^2\right), \quad k = 2, 3. \tag{20}$$

Hence it follows from (19) and (20) that

$$|K_{1n}| \leqslant \frac{1}{4nB_n} \sum_{j=1}^{n} (b_{nj}^2 + c_{nj}^2)(1+\rho) \leqslant C_1 n^{-1/2}(1+\rho),$$
$$|K_{2n}| \leqslant \frac{1}{4n^{3/2}\, B_n^2} \sum_{i<j} (b_{ni}^2 b_{nj}^2 + d_{nij}^2)(1+\lambda) \leqslant C_2 n^{-1/2}(1+\lambda).$$

Next we investigate the terms I_{in} for $1 \leqslant i \leqslant 4$ in (18).

First we estimate I_{1n}. By Taylor's expansion,

$$e^{itT_n} = e^{itS_n} e^{it\Delta_{n,n}} = e^{itS_n}\left(1 + it\Delta_{n,n} + \frac{1}{2}(it)^2 \Delta_{n,n}^2 e^{it\Delta_{n,n}\eta}\right), \quad |\eta| < 1.$$

Using (14), it can be easily shown that

$$E\Delta_{n,m}^2 \leqslant Cn^{-2} m(\lambda + \rho).$$

Thus, we have

$$I_{1n} \leqslant \frac{1}{2} \int_{|t| \leqslant n^{1/10}} |t| E(\Delta_{n,n}^2)\, dt \leqslant Cn^{-2/3}(\lambda + \rho).$$

Secondly we estimate I_{2n}. Using similar arguments to the proof of Lemma 5.2.1 in Hall [9] and also noting that $|b_{nj}| \leqslant l_2^{1/3} n^{1/3}$ for all j, we have

$$I_{2n} \leqslant \frac{C_1}{B_n^4} \sum_{j=1}^{n} b_{nj}^4 + C_2 \left(\frac{1}{B_n^3} \sum_{j=1}^{n} |b_{nj}|^3 E|X_1|^3\right)^2$$
$$\leqslant C_1 n^{-2/3} + C_2 n^{-1} (E|X_1|^3)^2.$$

Thirdly we estimate I_{3n}. For simplicity, we write

$$Z_j = \frac{b_{nj} X_j}{B_n}, \quad \psi_{ij} = d_{nij} \psi(X_i, X_j).$$

Let $R(z) = e^{iz} - 1 - iz$. Using the inequality $|R(z)| \leqslant |z|^{\alpha}$ for all $1 \leqslant \alpha \leqslant 2$ and the assumption $EX_1^2 = 1$, we find by Taylor's expansion that

$$\begin{aligned} &E(\psi_{ij} e^{it(Z_i+Z_j)}) \\ &\quad = E(\psi_{ij}(1 + itZ_i + R(tZ_i))(1 + itZ_j + R(tZ_j))) \\ &\quad = -t^2 E(\psi_{ij} Z_i Z_j) + E\{\psi_{ij}[(it)(Z_i R(tZ_j) + Z_j R(tZ_i)) + R(tZ_i)R(tZ_j)]\} \\ &\quad \equiv -t^2 l_{ij} + \theta_{1ij}(t), \end{aligned}$$

where l_{ij} and $\theta_{1ij}(t)$ satisfy

$$|l_{ij}| = |b_{ni} b_{nj} d_{nij}| B_n^{-2} |E(X_1 X_2 \psi(X_1, X_2))| \leqslant C\lambda^{1/2}(b_{ni}^2 b_{nj}^2 + d_{nij}^2)/n, \tag{21}$$

$$\begin{aligned} |\theta_{1ij}(t)| &\leqslant 2|t|^{5/2} E(|\psi_{ij} Z_i Z_j^{3/2}| + |\psi_{ij} Z_j Z_i^{3/2}|) \\ &\leqslant 2|t|^{5/2} E|X_1 X_2^{3/2} \psi(X_1, X_2)|(|b_{ni}|^{3/2}|b_{nj}| + (|b_{nj}|^{3/2}|b_{ni}|)|d_{nij}|/B_n^{5/2} \\ &\leqslant C|t|^{5/2}(\lambda\beta)^{1/2}(d_{nij}^2 + |b_{ni}|^3 b_{nj}^2 + |b_{nj}|^3 b_{ni}^2)/n^{5/4}. \end{aligned} \tag{22}$$

Therefore, we have

$$\begin{aligned} n^{-3/2} \sum_{i<j} E(\psi_{ij} e^{itS_n}) &= n^{-3/2} \sum_{i<j} E(\psi_{ij} e^{it(Z_i+Z_j)}) E(e^{it(S_n - Z_i - Z_j)}) \\ &= n^{-3/2} \sum_{i<j} (t^2 l_{ij} + \theta_{1ij}(t)) \prod_{k \neq i,j} \gamma_k(t) \\ &= n^{-3/2} \sum_{i<j} \left(t^2 l_{ij}(e^{-t^2/2} + \theta_{2ij}(t)) + \theta_{1ij}(t) \prod_{k \neq i,j} \gamma_k(t) \right) \\ &= n^{-3/2} \sum_{i<j} (t^2 e^{-t^2/2} l_{ij} + \theta_{3ij}(t)) \\ &= -K_{2n} t^2 e^{-t^2/2} + R_{n4}(t), \end{aligned} \tag{23}$$

where, for $|t| \leqslant n^{1/10}$ and suitably large n, we can apply Lemma A.4 in the Appendix to obtain

$$|\theta_{2ij}(t)| = \left| \prod_{k \neq i,j} \gamma_k(t) - e^{-t^2/2} \right| \leqslant C\left(\frac{\beta}{\sqrt{n}} + \frac{1}{n}(b_{ni}^2 + b_{nj}^2) \right)(t^2 + t^4) e^{-t^2/8},$$

$$|\theta_{3ij}(t)| \leqslant 4|\theta_{1ij}(t)| e^{-t^2/8} + |l_{ij}|\, |\theta_{2ij}(t)| t^2,$$

$$\begin{aligned} |R_{n4}(t)| &\leqslant \frac{1}{n^{3/2}} \sum_{i<j} \left(4|\theta_{1ij}(t)| + |l_{ij}| \left(\frac{\beta}{\sqrt{n}} + \frac{1}{n}(b_{ni}^2 + b_{nj}^2) \right)(t^2 + t^6) \right) e^{-t^2/8} \\ &\leqslant C(n^{-3/4}(\lambda\beta)^{1/2} + n^{-1}\lambda^{1/2}\beta)(|t| + |t|^6) e^{-t^2/8} \\ &\leqslant C_1(n^{-3/4}(\lambda + \beta) + \beta^2 n^{-1})(|t| + |t|^6) e^{-t^2/8}. \end{aligned}$$

Similarly, we have that

$$\frac{1}{n}\sum_{j=1}^{n} E(\xi(X_j)e^{itS_n}) = itK_{1n}e^{-t^2/2} + R_{n5}(t), \tag{24}$$

where $R_{n5}(t)$ satisfies $|R_{n5}(t)| \leqslant C(n^{-3/4}(\rho+\beta) + \beta^2 n^{-1})(|t|+|t|^5)e^{-t^2/8}$. It follows from (23) and (24) that

$$I_{3n} \leqslant C_1 n^{-3/4}(\lambda+\rho+\beta) + C_2 n^{-1}\beta^2. \tag{25}$$

Finally we estimate I_{4n}. First define

$$\Omega = \{k : \min(1/2, l_2/l_1^{3/2}) \leqslant \sqrt{n}|b_{nk}|/B_n \leqslant 2l_2/l_1^{3/2}\}. \tag{26}$$

Then from Lemma A.5 in the Appendix, there exists $0 < k_0 < 1$ such that

$$\#\{\Omega\} \geqslant k_0 n, \tag{27}$$

where $\#\{\Omega\}$ denotes the number of elements in Ω.

Without loss of generality, we assume that $l_2/l_1^{3/2} \geqslant 1/2$ and $b_{n1}, \ldots, b_{n,k_0 n} \in \Omega$. For $2 \leqslant m \leqslant k_0 n$, write

$$S_m = \frac{1}{B_n}\sum_{k=1}^{m} b_{nk}X_k, \quad S_m^{i,j} = \frac{1}{B_n}\sum_{k\neq i,j}^{m} b_{nk}X_k.$$

Similar to (17)–(22) of Bickel et al. [3], we find that for any $2 \leqslant m \leqslant k_0 n$,

$$|Ee^{itT_n}| \leqslant |Ee^{itS_{m-1}}| + |t|n^{-1/2}m(\lambda+\rho)^{1/2} \sup_{1\leqslant i\neq j\leqslant n} |Ee^{itS_m^{i,j}}| + Cn^{-2}m(\lambda+\rho)t^2. \tag{28}$$

Since $b_{nk} \in \Omega$ and $l_2/l_1^{3/2} \geqslant 1/2$, i.e. there exists C such that

$$1/2 \leqslant \sqrt{n}|b_{nk}|/B_n \leqslant C < \infty, \tag{29}$$

it is well known that there exists $c_0 > 0$ such that when $|t| \leqslant \frac{1}{4}(E|X_1|^3)^{-1}\sqrt{n}$,

$$|Ee^{itS_m}| \leqslant e^{-c_0 mt^2/n}, \quad |Ee^{itS_m^{i,j}}| \leqslant e^{-c_0(m-2)t^2/n}. \tag{30}$$

Choosing $m = \left[\frac{6n\log n}{c_0 t^2}\right] + 1$, it follows from (28) and (30) that

$$\int_{n^{1/10}\leqslant|t|\leqslant\frac{1}{4}(E|X_1|^3)^{-1}\sqrt{n}} |t|^{-1}|Ee^{itT_n}|dt \leqslant C(1+\lambda+\rho)n^{-3/4}. \tag{31}$$

On the other hand, it follows from $\kappa(X_1) > 0$ and (29) that if $\frac{1}{4}(E|X_1|^3)^{-1}\sqrt{n} \leqslant |t| \leqslant 2n^{7/10}$ and n is suitable large, then

$$\left|Ee^{\frac{itb_{nk}}{B_n}X_k}\right| \leqslant 1 - \kappa(X_1).$$

Therefore, it can be easy shown that when $\frac{1}{4}(E|X_1|^3)^{-1}\sqrt{n} \leqslant |t| \leqslant 2n^{7/10}$ and n is suitable large,

$$|Ee^{itS_m}| \leqslant e^{-m\kappa(X_1)} \quad \text{and} \quad |Ee^{itS_m^{i,j}}| \leqslant e^{-(m-2)\kappa(X_1)}.$$

These inequalities, together with (28) by choosing $m = [4 \log n/\kappa(X_1)] + 2$, imply that

$$\int_{\frac{1}{4}(E|X_1|^3)^{-1}\sqrt{n} \leqslant |t| \leqslant n^{2/3}} |t|^{-1}|Ee^{itT_n}|dt \leqslant C\kappa^{-1}(X_1)(1+\lambda+\rho)n^{-2/3}\log n. \tag{32}$$

From (31) and (32), it follows that

$$I_{4n} \leqslant C\kappa^{-1}(X_1)(1+\lambda+\rho)n^{-2/3}\log n.$$

Substituting the above estimates for I_{in}'s and K_{in}'s into (18), (15) follows immediately and hence we complete the proof of Theorem 3.1.

4.2. Proof of Theorem 2.2

To use Theorem 3.1, put

$$\mu_y = EY_1, \quad \tau_y^2 = Var(Y_1),$$

$$\eta_j = \frac{1}{\tau_y}(Y_j - \mu_y), \quad \overline{\eta} = \frac{1}{n}\sum_{j=1}^n \eta_j,$$

$$b_{nj} = a_{nj} - \frac{n-1}{n}U_{1n}, \quad d_{nij} = h_{ij} - U_{1n}.$$

From the above definitions, it follows that

$$\overline{Y} = \mu_y + \overline{\eta}\tau_y, \quad \frac{1}{n^2}\sum_{j\neq i} d_{nij} = \frac{2}{n}U_{1n}, \quad \frac{1}{n}\sum_{\substack{j=1 \\ j\neq i}}^n d_{nij} = b_{ni},$$

$$Y_iY_j - \mu_y^2 = \mu_y(Y_i - \mu_y) + \mu_y(Y_j - \mu_y) + (Y_i - \mu_y)(Y_j - \mu_y).$$

Therefore, we obtain (recall $\mu_y^2 = \tau_y^2$ and hence $EY_1^2 = 2\mu_y^2$)

$$\begin{aligned}
\varDelta_n(x) &\equiv P^*\left(\frac{\sqrt{n}}{\sigma_{1n}}(U_{W_n} - U_{1n}) \leqslant x\right) \\
&= P^*\left\{\frac{\sqrt{n}}{\sigma_{1n}}\left(\frac{1}{n^2}\sum_{i\neq j} Y_iY_jh_{ij} - \overline{Y}^2U_{1n}\right) \leqslant \overline{Y}^2x\right\} \\
&= P^*\left\{\frac{\sqrt{n}}{\sigma_{1n}}\left(\frac{1}{n^2}\sum_{i\neq j}(Y_iY_j - \mu_y^2)d_{nij} + \frac{1}{n^2}\sum_{j=1}^n (2\mu_y^2 - Y_j^2)U_{1n}\right) \leqslant \overline{Y}^2x\right\} \\
&= P^*\left(\frac{2}{\sqrt{n}\sigma_{1n}}\sum_{j=1}^n \eta_jb_{nj} + \frac{2}{n^{3/2}\sigma_{1n}}\sum_{i<j}\eta_i\eta_jd_{nij} + R_n \leqslant \left(1 + 2\overline{\eta} + \overline{\eta}^2\right)x\right) \\
&= P^*\left(G_n + \frac{1}{n}\sum_{j=1}^n \tilde{\eta}_j(x) \leqslant (1+\overline{\eta}^2)x + R_{n1}\right),
\end{aligned} \tag{33}$$

where

$$q_j = \frac{2\mu_y^2 - Y_j^2}{\mu_y^2}, \quad R_n = \frac{U_{1n}}{n^{3/2}\sigma_{1n}} \sum_{j=1}^{n} q_j,$$

$$G_n = \frac{2}{\sqrt{n}\sigma_{1n}} \sum_{j=1}^{n} \eta_j b_{nj} + \frac{2}{n^{3/2}\sigma_{1n}} \sum_{i<j} \eta_i \eta_j d_{nij}.$$

$$\tilde{\eta}_j(x) = \frac{U_{1n}}{\sqrt{n}\sigma_{1n}} (q_j I\{|q_j| \leqslant \sqrt{n}\} - E\{q_j I\{|q_j| \leqslant \sqrt{n}\}\}) - 2x\eta_j,$$

$$R_{n1} = \frac{U_{1n}}{n^{3/2}\sigma_{1n}} \sum_{j=1}^{n} (E\{q_j I\{|q_j| \geqslant \sqrt{n}\}\} - q_j I\{|q_j| \geqslant \sqrt{n}\}).$$

Note that in (33), we have turned the distribution function of the weighted bootstrap U-statistic $\sqrt{n}(U_{W_n} - U_{1n})/\sigma_{1n}$ into the distribution of some statistic of the form given in Theorem 3.1, which can now be used for our proof below. First we shall prove the following relations:

$$\sup_{x^2 \geqslant 5 \log n} P^*(|G_n| \geqslant |x|) = O(n^{-2/3} \log n) \quad \text{a.s.}, \tag{34}$$

$$\sup_{x^2 \leqslant 10 \log n} \sup_y \left| P^*\left(G_n + \frac{1}{n} \sum_{j=1}^{n} \tilde{\eta}_j(x) \leqslant y \right) - E_{1n}(y) \right| = o(n^{-1/2}), \quad \text{a.s.} \tag{35}$$

Denote by E^* the conditional expectation given the data $X_1, X_2, \ldots, X_n$. In terms of Lemmas A.1–A.2, $E\eta_1^2 = 1$ and $n\sigma_{1n}^2 = 4\sum_{j=1}^{n} b_{nj}^2$, it follows from Theorem 3.1 that for any fixed $x \in R$,

$$\begin{aligned}
&\sup_y \left| P^*\left(G_n + \frac{1}{n} \sum_{j=1}^{n} \tilde{\eta}_j(x) \leqslant y \right) - E_{2n}^*(y) \right| \\
&\quad \leqslant C\kappa^{-1}(\eta_1)(1 + E^*\eta_1^{*2}(x) + E|\eta_1|^3) n^{-2/3} \log n + C_1 (E|\eta_1|^3)^2 n^{-1} \\
&\quad \leqslant A(1 + x^2) n^{-2/3} \log n, \quad \text{a.s.},
\end{aligned} \tag{36}$$

where $E_{2n}^*(y) = \Phi(y) + L_{1n}(y) + L_{2n}(y)$,

$$L_{1n}(y) = \sum_{j=1}^{n} E^*\left\{ \Phi\left(y - \frac{2\eta_j b_{nj}}{\sqrt{n}\sigma_{1n}} \right) - \Phi(y) \right\} - \frac{1}{2}\Phi^{(2)}(y),$$

$$L_{2n}(y) = \frac{2}{n^{3/2}\sigma_{1n}} \sum_{j=1}^{n} b_{nj} E^*(\eta_1 \eta_1^*(x)) \Phi^{(2)}(y) - \frac{8}{n^{5/2}\sigma_{1n}^3} \sum_{i<j} b_{ni} b_{nj} d_{nij} \Phi^{(3)}(y),$$

and we also used the following estimate:

$$E^*\eta_1^{*2}(x) \leqslant 4x^2 E\eta_1^2 + 8\sigma_{1n}^{-2} U_{1n}^2 \leqslant A(1 + x^2), \quad \text{a.s.}$$

Recalling (16) and using (A.3) and (A.4) in Lemma A.2, we have that

$$\begin{aligned}&\sup_y\left|L_{1n}(y)+\frac{Eg^3(X_1)}{6\sqrt{n}\sigma_g^3}\Phi^{(3)}(y)\right|\\&\quad\leqslant An^{-1/2}\left|\frac{8}{\sigma_{1n}^3 n}\sum_{j=1}^n b_{nj}^3-\frac{Eg(X_1)^3}{\sigma_g^3}\right|+o(n^{-1/2})=o(n^{-1/2}),\quad \text{a.s.}\end{aligned}\tag{37}$$

Noting $\sum_{j=1}^n b_{nj}=2U_{1n}$, $E^*|\eta_1\eta_1^*(x)|\leqslant(E^*\eta_1^{*2}(x))^{1/2}\leqslant A(1+|x|)$ and using Lemmas A.1 and (A.5) in Lemma A.2, we have that

$$\begin{aligned}&\sup_{x^2\leqslant 10\log n}\ \sup_y\left|L_{2n}(y)+\frac{Eg(X_1)g(X_2)h_{12}}{2\sqrt{n}\sigma_g^3}\Phi^{(3)}(y)\right|\\&\quad\leqslant A_1\sup_{x^2\leqslant 10\log n}\frac{E^*|\eta_1\eta_1^*(x)|}{n^{3/2}\sigma_{1n}}\left|\sum_{j=1}^n b_{nj}\right|+\frac{A_2}{n^{1/2}}\left|\frac{8\sum_{i<j}b_{ni}b_{nj}d_{nij}}{\sigma_{1n}^3n^2}-\frac{Eg(X_1)g(X_2)h_{12}}{2\sigma_g^3}\right|\\&\quad=o(n^{-1/2}),\quad\text{a.s.}\end{aligned}\tag{38}$$

It follows from (36)–(38) that

$$\begin{aligned}&\sup\nolimits_{x^2\leqslant 10\log n}\ \sup_y\left|P^*\left(G_n+\frac{1}{n}\sum_{j=1}^n\tilde\eta_j(x)\leqslant y\right)-E_{1n}(y)\right|\\&\quad\leqslant\sup_{x^2\leqslant 10\log n}\ \sup_y|E_{2n}^*(y)-E_{1n}(y)|+O(n^{-2/3}\log^2 n)\\&\quad\leqslant\sup_y\left|L_{1n}(y)+\frac{Eg^3(X_1)}{6\sqrt{n}\sigma_g^3}\Phi^{(3)}(y)\right|\\&\qquad+\sup_{x^2\leqslant 10\log n}\ \sup_y\left|L_{2n}(y)+\frac{Eg(X_1)g(X_2)h_{12}}{2\sqrt{n}\sigma_g^3}\Phi^{(3)}(y)\right|+o(n^{-1/2})\\&\quad=o(n^{-1/2}),\quad\text{a.s.,}\end{aligned}$$

which implies (35).

Similarly to the proof of (35), we can obtain that (note that G_n does not include x)

$$\sup_{x^2\geqslant 5\log n}|P^*(G_n\geqslant|x|)-(1-E_{1n}(|x|))|=O(n^{-2/3}\log n),\quad\text{a.s.,}$$

$$\sup_{x^2\geqslant 5\log n}|P^*(G_n\leqslant-|x|)-E_{1n}(-|x|)|=O(n^{-2/3}\log n),\quad\text{a.s.}$$

These, together with $\sup_{x^2\geqslant 5\log n}|1-E_{1n}(|x|)|\leqslant An^{-2}$, imply (34).

Next we finish the proof of Theorem 2.2. Recall $EY_1^2=2\mu_y^2$ and $E|Y_1|^3<\infty$, by Markov's inequality, we get

$$P(\overline{\eta}^2\geqslant A_0n^{-3/5})\leqslant A_1n^{9/10}E|\overline{\eta}|^3\leqslant A_2n^{-3/5}\tag{39}$$

and

$$\varepsilon_n \equiv E(|q_1|^{3/2} I\{|q_1| \geqslant \sqrt{n}\}) \to 0. \tag{40}$$

It follows from (40) that

$$\begin{aligned} P^*(|R_{n1}| \geqslant \varepsilon_n^{1/2} n^{-1/2}) &\leqslant n^{3/4}/\varepsilon_n^{3/4} E^*|R_{n1}|^{3/2} \\ &\leqslant A_1 n^{-1/2} \varepsilon_n^{1/4} \left|\frac{U_{1n}}{\sigma_{1n}}\right|^{3/2} = o(n^{-1/2}), \quad \text{a.s.} \end{aligned} \tag{41}$$

In terms of (39)–(41), we get

$$\begin{aligned} \Delta_n(x) &\leqslant P^*\left(G_n + \frac{1}{n}\sum_{j=1}^n \tilde{\eta}_j(x) \leqslant (1+n^{-3/5})x + \varepsilon_n^{1/2} n^{-1/2}\right) \\ &\quad + P^*(\overline{\eta}^2 \geqslant n^{-3/5}) + P^*(|R_{n1}| \geqslant \varepsilon_n^{1/2} n^{-1/2}) \\ &\leqslant P^*\left(G_n + \frac{1}{n}\sum_{j=1}^n \eta_j^*(x) \leqslant (1+n^{-3/5})x + \varepsilon_n^{1/2} n^{-1/2}\right) + o(n^{-1/2}), \quad \text{a.s.} \end{aligned}$$

Similarly, we have

$$\Delta_n(x) \geqslant P^*\left(G_n + \frac{1}{n}\sum_{j=1}^n \tilde{\eta}_j(x) \leqslant (1-n^{-3/5})x - \varepsilon_n^{1/2} n^{-1/2}\right) - o(n^{-1/2}), \quad \text{a.s.}$$

From these relations and (35), it follows that

$$\begin{aligned} &\sup_{x^2 \leqslant 10 \log n} |\Delta_n(x) - E_{1n}(x)| \\ &\leqslant \sup_{x^2 \leqslant 10 \log n} \sup_y \left| P^*\left(G_n + \frac{1}{n}\sum_{j=1}^n \tilde{\eta}_j(x) \leqslant y\right) - E_{1n}(y)\right| \\ &\quad + \sup_{x^2 \leqslant 10 \log n} |E_{1n}((1+C_{1n})x + C_{2n}) - E_{1n}(x)| + o(n^{-1/2}) \\ &= o(n^{-1/2}), \quad \text{a.s.,} \end{aligned} \tag{42}$$

where we assume $|C_{1n}| = n^{-3/5}, |C_{2n}| = \varepsilon_n^{1/2} n^{-1/2}$ and use the following elementary estimate: when $|p_n| + |q_n| \to 0$

$$\sup_x |\Phi^{(k)}[(1+p_n)x + q_n] - \Phi^{(k)}(x)| \leqslant A(|p_n| + |q_n|), \quad \text{for } k = 0, 1, 2, \ldots.$$

On the other hand, by applying the second last equality in (33), (34) and (39),

$$\begin{aligned} &\sup_{x^2 \geqslant 10 \log n} |\Delta_n(x) - E_{1n}(x)| \\ &\leqslant \sup_{x^2 \geqslant 10 \log n} P^*(G_n \geqslant (1 + 2\overline{\eta} + \overline{\eta}^2)x + |R_n|) + \sup_{x^2 \geqslant 10 \log n} |1 - E_{1n}(x)| \end{aligned}$$

$$\leqslant \sup_{x^2 \geqslant 10 \log n} P^*(G_n \geqslant x/2) + P^*(|2\overline{\eta} + \overline{\eta}^2| \geqslant 1/4) \\ + P^*(|R_n| \geqslant 1/4) + O(n^{-2}) \\ = o(n^{-1/2}), \quad \text{a.s.,} \tag{43}$$

where we use the estimate $P^*(|R_n| \geqslant 1/4) \leqslant An^{-5/4}E|q_1|^{3/2}$, a.s. which can be proved as in (41).

Note that (9) follows from (42) and (43), we finish the proof of Theorem 2.2.

4.3. Proof of Theorems 2.3–2.5

We first prove Theorem 2.5. Put $\zeta_j = (\xi_j - \mu_\xi)/\tau_\xi$. As in (33), we obtain

$$P^*\left\{\frac{\sqrt{n}}{\mu_\xi \tau_\xi \sigma_n} \tilde{U}_\xi \leqslant x\right\} = P^*\left\{\frac{2}{\sqrt{n}\sigma_n} \sum_{j=1}^n \zeta_j (a_{nj} - U_n) \right. \\ \left. + \frac{2}{n^{3/2}\sigma_n} \sum_{i=1}^n \zeta_i U_n + \frac{1}{n^{3/2}\sigma_n} \sum_{i \neq j} \zeta_i \zeta_j (h_{ij} - U_n) \leqslant x \right\}.$$

So Theorem 2.5 follows easily from Lemmas A.2 and A.3 in the Appendix and Theorems 3.1 and 3.2.

The proofs of Theorems 2.2 and 2.4 can be shown similarly and hence omitted here.

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Appendix. Some technical lemmas

In this section, we give some lemmas which are complement to the main results.

Lemma A.1. *Let $f(x_1, \ldots, x_m)$ be a real-valued function symmetric in its arguments. Define a U-statistic by*

$$U_n(f) = \binom{n}{m}^{-1} \sum_{1 \leqslant i_1 < \cdots < i_m \leqslant n} f(X_{i_1}, \ldots, X_{i_m}), \quad n \geqslant m.$$

Assume that $E|f(X_1, \dots, X_m)|^p < \infty$. *Then*

$$n^{-m} U_n(|f|) \to 0 \quad a.s. \quad \text{for } 0 < p < 1,$$
$$n^{1-1/p} |U_n(f) - Ef(X_1, \dots, X_m)| \to 0 \quad a.s. \quad \text{for } 1 \leqslant p < 2,$$
$$\frac{\sqrt{n}}{\log n} |U_n(f) - Ef(X_1, \dots, X_m)| \to 0 \quad a.s. \quad \text{for } p = 2.$$

Proof. For $0 < p < 2$, the results are from Gine and Zinn [7]. For $p = 2$, see Lee [16] or Serfling [22].

The notations used in Lemmas A.2 and A.3 below are the same as those in Theorem 2.2 and its proof.

Lemma A.2. *Assume that* $E|h_{12}|^3 < \infty$. *Then,*

$$\frac{1}{mn} \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{n} d_{nij}^2 \leqslant C < \infty \quad \text{for any } m, n \geqslant 1, \tag{A.1}$$

$$\frac{1}{n} \sum_{j=1}^{n} |b_{nj}|^3 \leqslant C_1 < \infty, \tag{A.2}$$

$$\frac{1}{n} \sum_{j=1}^{n} b_{nj}^3 \to Eg(X_1)^3 \quad a.s., \tag{A.3}$$

$$\frac{1}{n} \sum_{j=1}^{n} b_{nj}^2 \to Eg(X_1)^2 \quad a.s., \tag{A.4}$$

$$\frac{1}{n^2} \sum_{i<j} b_{ni} b_{nj} d_{nij} \to \frac{1}{2} E(g(X_1) g(X_2) h_{12}) \quad a.s. \tag{A.5}$$

Proof. We shall only prove (A.3) below. The proofs for others are similar and hence will be omitted. By the definition of b_{nj}, it follows that

$$\begin{aligned} b_{nj} &= \frac{1}{n} \sum_{\substack{k=1 \\ k \neq j}}^{n} (h_{jk} - Eh_{12}) - \frac{n-1}{n} (U_{1n} - Eh_{12}) \\ &= \frac{1}{n} \sum_{\substack{k=1 \\ k \neq j}}^{n} (h_{jk} - Eh_{12}) + \left(-\frac{n-1}{n^2(n+1)} \sum_{i<j} (h_{ij} - Eh_{12}) + \frac{(n-2)^2}{n^2(n+1)} Eh_{12} \right) \\ &= S_j + Q_n, \quad say. \end{aligned} \tag{A.6}$$

Write $h^*_{jk} = h_{jk} - Eh_{12}$. Simple calculations show that

$$\frac{1}{n}\sum_{j=1}^{n} S_j^3 = \frac{1}{n^4}\sum_{j\neq k} h^{*3}_{jk} + \frac{2}{n^4}\sum_{j\neq k\neq l} h^{*2}_{jk}h^*_{jl} + \frac{1}{n^4}\sum_{j\neq k\neq l\neq m} h^*_{jk}h^*_{jl}h^*_{jm}$$
$$= Z_{1n} + Z_{2n} + Z_{3n}, \quad say. \tag{A.7}$$

It follows from $E|h_{12}|^3 < \infty$ that

$$E|h^{*2}_{12}h^*_{13}| \leqslant E|h^*_{12}|^3 < \infty, \quad E|h^*_{12}h^*_{13}h^*_{14}| \leqslant E|h^*_{12}|^3 < \infty.$$

Using Lemma A.1, we obtain that

$$Q_n \to 0, \quad Z_{1n} \to 0, \quad Z_{2n} \to 0, \quad \text{a.s.},$$

$$Z_{3n} = \frac{4!}{n^4}\sum_{j<k<l<m} h^*_{jk}h^*_{jl}h^*_{jm} \to Eh^*_{12}h^*_{13}h^*_{14} = Eg(X_1)^3, \quad \text{a.s.}$$

Therefore,

$$\frac{1}{n}\sum_{j=1}^{n} S_j^3 \to Eg(X_1)^3, \quad \text{a.s.}$$

Similarly, we get

$$\frac{1}{n}\sum_{j=1}^{n} S_j^2 \to Eh^*_{12}h^*_{13} = Eg(X_1)^2, \quad \text{a.s.},$$

$$\frac{1}{n}\sum_{j=1}^{n} S_j = \frac{n-1}{n}U_n \to Eh_{12}, \quad \text{a.s.}$$

In view of the above estimates, we have that

$$\frac{1}{n}\sum_{j=1}^{n} b^3_{nj} = \frac{1}{n}\sum_{j=1}^{n} (S_j^3 + 3S_j^2 Q_n + 3S_j Q_n^2 + Q_n^3) \to Eg(X_1)^3, \quad \text{a.s.}$$

Thus (A.3) is proved.

Lemma A.3. *Assume that* $E|h_{12}|^4 < \infty$. *Then*

$$\frac{1}{n}\sum_{j=1}^{n} (a_{nj} - U_n)^3 - Eg(X_1)^3 = o(n^{-1/4}) \quad a.s., \tag{A.8}$$

$$\frac{1}{n}\sum_{j=1}^{n} (a_{nj} - U_n)^2 - Eg(X_1)^2 = o(n^{-1/4}) \quad a.s., \tag{A.9}$$

$$\frac{1}{n^2}\sum_{i<j}(a_{ni}-U_n)(a_{nj}-U_n)(h_{ij}-U_n) - \frac{1}{2}E(g(X_1)g(X_2)h_{12}) = o(n^{-1/4}) \quad a.s. \tag{A.10}$$

Proof. We only prove (A.10). The others are similar and will be omitted. Again writing $h^*_{jk} = h_{jk} - Eh_{12}$. As in Lemma A.2, we have that

$$a_{nj} - U_n = \frac{1}{n}\sum_{\substack{k=1\\k\neq j}}^{n} h^*_{jk} - \frac{1}{n^2}\sum_{j\neq k} h^*_{jk} = S_j - Q^*_n, \quad say$$

$$h_{ij} - U_n = h^*_{ij} - \frac{n}{n-1}Q^*_n.$$

Therefore, it follows that

$$\frac{1}{n^2}\sum_{i<j}(a_{ni}-U_n)(a_{nj}-U_n)(h_{ij}-U_n) = \frac{1}{n^2}\sum_{i<j} S_iS_jh^*_{ij} - \frac{Q^*_n}{n^2}\sum_{i<j}\left((S_i+S_j+Q^*_n)h^*_{ij} + \frac{n}{n-1}(S_i-Q^*_n)(S_j-Q^*_n)\right).$$

Noting that from Lemma A.1, we get

$$Q^*_n = o(n^{-1/2}\log n), \quad \text{a.s.}$$

So in order to prove (A.10), it remains to show

$$\frac{1}{n^2}\sum_{i<j} S_iS_jh^*_{ij} - \frac{1}{2}E(g(X_1)g(X_2)h_{12}) = o(n^{-1/4}) \quad \text{a.s.}, \tag{A.11}$$

$$\frac{1}{n^2}\sum_{i<j}\left((S_i+S_j+Q^*_n)h^*_{ij} + \frac{n}{n-1}(S_i-Q^*_n)(S_j-Q^*_n)\right) = O(1) \quad \text{a.s.} \tag{A.12}$$

As in (A.7), we have that

$$\frac{1}{n^2}\sum_{i<j} S_iS_jh^*_{ij} = \frac{1}{2n^4}\sum_{i\neq j\neq k\neq m} h^*_{ik}h^*_{jm}h^*_{ij} + \frac{1}{n^4}\sum_{i\neq j\neq k}(h^{*2}_{ij}h^*_{ik} + h^*_{ik}h^*_{jk}h^*_{ij}) + \frac{1}{n^4}\sum_{i\neq j} h^{*3}_{ij}.$$

Since $E|h_{12}|^4 < \infty$, we have that

$$E(|h^*_{13}h^*_{24}h^*_{12}|^{4/3}) \leqslant E|h_{12}|^4 < \infty.$$

Now noting $E(h^*_{13}h^*_{24}h^*_{12}) = E(g(X_1)g(X_2)h_{12})$, (A.11) follows by applying Lemma A.1.

Similarly, we can prove (A.12). We finish the proof of Lemma A.3.

Lemma A.4. *Assume that $EX_1 = 0$, $EX_1^2 = 1$ and $E|X_1|^3 < \infty$. Let $\gamma_k(t) = Ee^{itb_{nk}X_k/B_n}$, where b_{nk} satisfies* (13) *and $B_n^2 = \sum_{j=1}^n b^2_{nj}$. Then for any $i\neq j$, there exists $\eta > 0$ such* *that for* $|t| \leqslant \eta n^{1/6}$ *and suitably large* n,

$$\left| \prod_{k \neq i,j} \gamma_k(t) - e^{-t^2/2} \right| \leqslant A\left(\frac{1}{\sqrt{n}} E|X_1|^3 + \frac{1}{n}(b_{ni}^2 + b_{nj}^2) \right)(t^2 + t^4)e^{-t^2/8}. \tag{A.13}$$

Proof. From (13), $|b_{nj}| \leqslant l_2^{1/3} n^{1/3}$ and $B_n \geqslant nl_1$, it follows that for all k,

$$|\gamma_k(t) - 1| \leqslant \frac{t^2 b_{nk}^2}{2B_n^2} \leqslant \frac{t^2 b_{nk}^2}{nl_1} \leqslant Ct^2 n^{-1/3}.$$

Hence, there exists $\eta > 0$ such that when $|t| \leqslant \eta n^{1/6}$, for all k, $|\gamma_k(t)| \geqslant 1/2$. Now by applying the classical results (e.g., see [20] p. 109), we get for $|t| \leqslant \eta n^{1/6}$ and suitably large n,

$$\left| \prod_{k \neq i,j} \gamma_k(t) \right| \leqslant \left| \prod_{k=1}^{n} \gamma_k(t) \right| \Big/ (|\gamma_i(t)|\,|\gamma_j(t)|) \leqslant 4e^{-t^2/8}$$

and

$$\begin{aligned} \left| \prod_{k \neq i,j} \gamma_k(t) - e^{-t^2/2} \right| &\leqslant \left| \prod_{k \neq i,j} \gamma_k(t) - \prod_{k=1}^{n} \gamma_k(t) \right| + \left| \prod_{k=1}^{n} \gamma_k(t) - e^{-t^2/2} \right| \\ &\leqslant \left| \prod_{k \neq i,j} \gamma_k(t) \right| |\gamma_i(t)\gamma_j(t) - 1| + Cn^{-1/2} E|X_1|^3 |t|^3 e^{-t^2/8} \\ &\leqslant C_1 \left(\frac{1}{\sqrt{n}} E|X_1|^3 + \frac{1}{n}(b_{ni}^2 + b_{nj}^2) \right)(t^2 + t^4)e^{-t^2/8}. \end{aligned}$$

The proof of Lemma A.4 is completed.

Lemma A.5. *If* (13) *holds, then there exists* $0 < k_0 < 1$ *such that* $\#\{\Omega\} \geqslant k_0\, n$, *where* ω *is given in* (26) *and* $\#\{A\}$ *denotes the number of elements in* A.

Proof. Let $\Omega_1 = \{k : \sqrt{n}|b_{nk}|/B_n \geqslant \min(1/2, l_2/l_1^{3/2})\}$, $\Omega_2 = \{k : \sqrt{n}|b_{nk}|/B_n \leqslant 2l_2/l_1^{3/2}\}$. Clearly, $\Omega = \Omega_1 \cap \Omega_2$. In view of (13), it follows that

$$\begin{aligned} 1 &= \left(\sum_{j \in \Omega_2} + \sum_{j \notin \Omega_2} \right) \left(\frac{b_{nj}}{B_n} \right)^2 \\ &\leqslant \sum_{j \in \Omega_2} \left(\frac{b_{nj}}{B_n} \right)^2 + \frac{\sqrt{n} l_1^{3/2}}{2l_2} \sum_{j \notin \Omega_2} \left(\frac{|b_{nj}|}{B_n} \right)^3 \\ &\leqslant \sum_{j \in \Omega} \left(\frac{b_{nj}}{B_n} \right)^2 + \sum_{j \notin \Omega_1} \left(\frac{b_{nj}}{B_n} \right)^2 + \frac{1}{2} \\ &\leqslant \frac{4l_2^2 \#\{\Omega\}}{nl_1^3} + \frac{3}{4}, \end{aligned}$$

which implies $\#\{\Omega\} \geqslant l_1^3/(16 l_2^2) n$. So the lemma is proved by taking $k_0 = l_1^3/(16 l_2^2)$. Also from (13) and applying Cauchy–Schwartz inequality, it is easy to see that $k_0 \leqslant 1/16$.

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