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A test for Archimedeanity in bivariate copula models

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ABSTRACT

We propose a new test for the hypothesis that a bivariate copula is an Archimedean copula which can be used as a preliminary step before further dependence modeling. The corresponding test statistic is based on a combination of two measures resulting from the characterization of Archimedean copulas by the property of associativity and by a strict upper bound on the diagonal by the Fréchet–Hoeffding upper bound. We prove weak convergence of this statistic and show that the critical values of the corresponding test can be determined by the multiplier bootstrap method. The test is shown to be consistent against all departures from Archimedeanity. A simulation study is presented which illustrates the finite-sample properties of the new test.

1. Introduction

Let F be a bivariate continuous distribution function with marginal distribution functions F_1 and F_2. By Sklar's Theorem (see Sklar [30]) we can decompose F as follows

$$F(\mathbf{x}) = C\{F_1(x_1), F_2(x_2)\}, \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \tag{1}$$

where C is the unique copula associated to F. By definition, C is a bivariate distribution function on $[0, 1]^2$ whose univariate marginals are standard uniform distributions on $[0, 1]$. Eq. (1) is usually interpreted in the way that the copula C completely characterizes the information about the stochastic dependence contained in F. For an extensive exposition on the theory of copulas, we refer the reader to the monograph Nelsen [22].

In the past decades, various parametric models for copulas have been developed, among which the class of Archimedean copulas forms one the most famous and largest class; see Genest and MacKay [9], Nelsen [22], McNeil and Nešlehová [20] among many others. Many widely used copulas, such as Clayton, Gumbel–Hougaard and Frank copulas are in fact Archimedean copulas. The elements of this class may be characterized by a continuous, strictly decreasing and convex function $\varphi : [0, 1] \to [0, \infty]$ satisfying $\varphi(1) = 0$ such that

$$C(\mathbf{u}) = \varphi^{[-1]}\{\varphi(u_1) + \varphi(u_2)\} \quad \text{for all } \mathbf{u} = (u_1, u_2) \in [0, 1]^2.$$

The function φ is called the *generator* of C and its *pseudo-inverse* $\varphi^{[-1]}(t)$ is defined as the usual inverse $\varphi^{-1}(t)$ for $t \in [0, \varphi(0)]$ and is set to 0 for $t \geq \varphi(0)$. The prominence of the class of Archimedean copulas basically stems from the fact that they are easy to handle and simulate; see Genest et al. [10]. While the estimation of Archimedean copulas has been investigated in Genest and Rivest [13] and recently more thoroughly in Genest et al. [10], the issue of testing for the hypothesis that the copula is an Archimedean only received scant attention in the literature. The present paper fills this gap by developing a consistent test for this hypothesis.

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Our interest in this problem stems from recent work of Genest and Rivest [13], Wang and Wells [33] and Naifar [21] who proposed Archimedean copulas for modeling dependences between bivariate observations (among many others). We also refer to the work of Rivest and Wells [25] who used Archimedean copulas for modeling the dependence in the context of censored data. To the best of our knowledge, the only available test for Archimedeanity hitherto has been discussed in Jaworski [15]. This author proposed a procedure which is based on a characterization of Archimedean copulas similar to the one stated in Theorem 4.1.6 in Nelsen [22] (which dates back to Ling [19]). To be precise recall that a bivariate copula C is called associative if and only if the identity

$$C\{x, C(y, z)\} = C\{C(x, y), z\} \tag{2}$$

holds for all $(x, y, z) \in [0, 1]^3$. Theorem 4.1.6 in Nelsen [22] shows that a bivariate copula C is an Archimedean copula if and only if C is associative and the inequality $C(u, u) < u$ holds for all $u \in (0, 1)$, i.e., on the diagonal C is strictly dominated by the Fréchet–Hoeffding upper bound $M(\mathbf{u}) = \min(u_1, u_2)$. The procedure suggested in Jaworski [15] is in fact to test for associativity in order to check the validity of an Archimedean copula model. The corresponding test statistic is defined as

$$\mathcal{T}_n(x, y) = \sqrt{n}\,[C_n\{x, C_n(y, y)\} - C_n\{C_n(x, y), y\}]\,,$$

where (x, y) is some fixed point in the open cube $(0, 1)^2$ and C_n denotes the empirical copula; see Section 2 for details. The main advantage of this approach is its simplicity, in particular the simple limit distribution of the resulting test statistic, which is in fact normal. However, this simplicity has its price in terms of consistency. The method proposed by Jaworski [15] has at least three mayor drawbacks. First of all, it is clearly not consistent against a large class of alternatives since it only tests for relation (2) with $y = z$. Second, Jaworski [15] uses a pointwise approach in order to test for a global hypothesis as in (2). This means that the test may not reject the hypothesis because (2) is satisfied at the particular point (x, y, y) under investigation, although there may exist many other points where (2) is not satisfied. Third, there exist copulas that are in fact associative but not Archimedean. These problems also have strong implications for the practical applicability of the test, as demonstrated by results in a simulation study in Jaworski [15], where the sample size has to be chosen extremely large in order to get reasonable rejection probabilities.

To the best of our knowledge there exists no test for an Archimedean copula, which is consistent against general alternatives and it is the primary purpose of this paper to develop such a procedure and to investigate its statistical properties. The test can be used as a preliminary step before further dependence modeling. We propose a test statistic which is based on a combination of two measures resulting from the characterization of Archimedean copulas, namely the property of associativity as described in (2) and the strict upper bound on the diagonal $C(u, u) < u$ for all $u \in (0, 1)$. In Section 2, we define a new process which is based on an estimate of the difference of the left- and right-hand side of the defining equation (2) for associativity. We prove weak convergence of this process in the space of all uniformly bounded functions on $[0, 1]^3$. As a consequence, we also obtain weak convergence of a corresponding Cramér–von Mises and a Kolmogorov–Smirnov type statistic. Because the asymptotic distribution depends in a complicated manner on the underlying copula, we propose a multiplier bootstrap procedure to obtain the critical values and show its validity. As a first main result we obtain a test for associativity, which is consistent against all alternatives satisfying weak smoothness assumptions on C. In Section 3, we exploit these findings to develop an asymptotic test for the hypothesis of Archimedeanity. Section 4 is devoted to an investigation of the finite-sample performance of the new test by means of a simulation study; in final Section 5, the results are applied to the classical data set by Cook and Johnson [6]. All proofs are deferred to Appendix.

2. Testing associativity

2.1. The test statistic and its asymptotic behavior

In the following, let $\mathbf{X}_1, \ldots, \mathbf{X}_n$, $\mathbf{X}_i = (X_{i1}, X_{i2})$ denote independent identically distributed bivariate random vectors with continuous distribution function F, marginal distribution functions F_1 and F_2 and copula $C = F(F_1^-, F_2^-)$. Here, F_p^- $(p = 1, 2)$ denotes the generalized inverse function of F_p, i.e., $F_p^-(x) = \inf\{t \in \mathbb{R} : F_p(t) \geq x\}$. In this paragraph, we will introduce a test statistic for the null hypothesis that the underlying copula is associative, i.e., C satisfies condition (2) for all $(x, y, z) \in [0, 1]^3$.

For this purpose, we briefly summarize relevant notations and results on the empirical copula, which is the simplest and most popular nonparametric estimator of the copula. In particular, we define the empirical copula by

$$C_n(\mathbf{u}) = F_n\{F_{n1}^-(u_1), F_{n2}^-(u_2)\},$$

where $F_n(\mathbf{x}) = n^{-1}\sum_{i=1}^n \mathbb{I}(\mathbf{X}_i \leq \mathbf{x})$ and $F_{np}(x_p) = n^{-1}\sum_{i=1}^n \mathbb{I}(X_{ip} \leq x_p)$, $p = 1, 2$ are the joint and marginal empirical distribution functions of the sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$, respectively. It is a well known result that under the assumptions of continuous partial derivatives of C the corresponding empirical copula process

$$\mathbb{C}_n = \sqrt{n}(C_n - C) \tag{3}$$

converges weakly towards a Gaussian limit field $\mathbb{G}_C$ in $\ell^\infty([0, 1]^2)$; see Rüschendorf [26], Gänssler and Stute [8], Fermanian et al. [7], Tsukahara[31] among others. Defining $\dot{C}_p$ as the p-th partial derivative of C $(p = 1, 2)$ the process $\mathbb{G}_C$ can be expressed as

$$\mathbb{G}_C(\mathbf{x}) = \mathbb{B}_C(\mathbf{x}) - \dot{C}_1(\mathbf{x})\mathbb{B}_C(x_1, 1) - \dot{C}_2(\mathbf{x})\mathbb{B}_C(1, x_2) \tag{4}$$

with the copula-Brownian bridge $\mathbb{B}_C$, i.e., $\mathbb{B}_C$ is a centered Gaussian field with $\mathrm{cov}\{\mathbb{B}_C(\mathbf{x}), \mathbb{B}_C(\mathbf{y})\} = C(\mathbf{x} \wedge \mathbf{y}) - C(\mathbf{x})C(\mathbf{y})$, where the minimum of two vectors is defined component-wise. As explained in Segers [29], the assumption of continuity of the partial derivatives of C on the whole unit square does not hold for many (even most) commonly used copula models and as a consequence Segers provides the result that the following nonrestrictive smoothness condition is sufficient in order to obtain weak convergence of the empirical copula process defined in (3).

Condition 2.1. *For $p = 1, 2$ the first order partial derivative $\dot{C}_p$ of the copula C with respect to x_p exists and is continuous on the set $V_p = \{\mathbf{u} \in [0, 1]^2 : 0 < u_p < 1\}$.*

Now, in order to test for associativity we consider the process

$$\mathbb{H}_n(x, y, z) = \sqrt{n}\,[C_n\{x, C_n(y, z)\} - C_n\{C_n(x, y), z\}]\,,$$

where $(x, y, z) \in [0, 1]^3$. The asymptotic properties of the process $\{\mathbb{H}_n(x, y, z)\}_{(x,y,z)\in[0,1]^3}$ are summarized in the following theorem. Throughout this paper, $\ell^\infty(T)$ denotes the set of all uniformly bounded functions on T, and the symbol $\rightsquigarrow$ denotes uniform convergence in a metric space (which will be specified in the corresponding statements).

Theorem 2.2. *If the copula C is associative and satisfies Condition 2.1, then $\mathbb{H}_n \rightsquigarrow \mathbb{H}_C$ in $\ell^\infty([0, 1]^3)$, where the limit field $\mathbb{H}_C$ can be expressed as*

$$\mathbb{H}_C(x, y, z) = \mathbb{G}_C\{x, C(y, z)\} - \mathbb{G}_C\{C(x, y), z\} + \dot{C}_2\{x, C(y, z)\}\mathbb{G}_C(y, z) - \dot{C}_1\{C(x, y), z\}\mathbb{G}(x, y).$$

As a consequence of Theorem 2.2 and the Continuous Mapping Theorem (see, e.g., Theorem 1.3.6 in van der Vaart and Wellner [32]), we obtain the weak convergence of a corresponding Cramér–von Mises and Kolmogorov–Smirnov type test statistic, i.e.,

$$\mathbb{T}_{n,\mathrm{L}_2} = \int_{[0,1]^3} \{\mathbb{H}_n(x, y, z)\}^2 \, d(x, y, z) \rightsquigarrow \mathbb{T}_{C,\mathrm{L}_2} = \int_{[0,1]^3} \{\mathbb{H}_C(x, y, z)\}^2 \, d(x, y, z), \tag{5}$$

$$\mathbb{T}_{n,\mathrm{KS}} = \sup_{[0,1]^3} |\mathbb{H}_n(x, y, z)| \rightsquigarrow \mathbb{T}_{C,\mathrm{KS}} = \sup_{[0,1]^3} |\mathbb{H}_C(x, y, z)|\,, \tag{6}$$

which can be used to construct an asymptotic test for the hypothesis of associativity. Given that $\mathbb{T}_{n,\mathrm{M}} \xrightarrow{\mathbb{P}} \infty$ $[\mathrm{M} \in \{\mathrm{L}_2, \mathrm{KS}\}]$ if the copula is not associative, the null hypothesis should be rejected for unlikely large values of $\mathbb{T}_{n,\mathrm{M}}$. This gives rise to the demand for critical values of $\mathbb{T}_{C,\mathrm{M}}$ which can be obtained by multiplier bootstrap methods as described in the subsequent section.

2.2. A multiplier bootstrap approximation

It is the purpose of this section to provide a bootstrap approximation for the distribution of the limiting variables $\mathbb{T}_{C,\mathrm{M}}$ whose variances depend on the unknown copula in a complicated manner. We begin with an approximation of the distribution of the limiting process $\mathbb{H}_C$. For this purpose we rewrite the decomposition of the process $\mathbb{G}_C$ defined in (4) as

$$\begin{aligned}\mathbb{H}_C(x, y, z) = {} & \mathbb{B}_C\{x, C(y, z)\} - \dot{C}_1\{x, C(y, z)\}\mathbb{B}_C(x, 1) - \dot{C}_2\{x, C(y, z)\}\mathbb{B}_C\{1, C(y, z)\} \\ & - \left[\mathbb{B}_C\{C(x, y), z\} - \dot{C}_1\{C(x, y), z\}\mathbb{B}_C\{C(x, y), 1\} - \dot{C}_2\{C(x, y), z\}\mathbb{B}_C(1, z)\right] \\ & + \dot{C}_2\{x, C(y, z)\}\left\{\mathbb{B}_C(y, z) - \dot{C}_1(y, z)\mathbb{B}_C(y, 1) - \dot{C}_2(y, z)\mathbb{B}_C(1, z)\right\} \\ & + \dot{C}_1\{C(x, y), z\}\left\{\mathbb{B}_C(x, y) - \dot{C}_1(x, y)\mathbb{B}_C(x, 1) - \dot{C}_2(x, y)\mathbb{B}_C(1, y)\right\}.\end{aligned} \tag{7}$$

In the following discussion the symbol

$$G_n \underset{\xi}{\overset{\mathbb{P}}{\rightsquigarrow}} G \tag{8}$$

denotes weak convergence in some metric space $\mathbb{D}$ conditionally on the data in probability; see, e.g., Kosorok [17]. More precisely, (8) holds for random variables $G_n = G_n(\mathbf{X}_1, \ldots, \mathbf{X}_n, \xi_1, \ldots, \xi_n)$, $G \in \mathbb{D}$ if and only if

$$\sup_{h\in BL_1(\mathbb{D})} |\mathbb{E}_\xi h(G_n) - \mathbb{E}h(G)| \xrightarrow{\mathbb{P}} 0 \tag{9}$$

(where $\xrightarrow{\mathbb{P}}$ denotes convergence in outer probability) and

$$\mathbb{E}_\xi h(G_n)^* - \mathbb{E}_\xi h(G_n)_* \xrightarrow{\mathbb{P}} 0 \quad \text{for every } h \in BL_1(\mathbb{D}), \tag{10}$$

where

$$BL_1(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{R} \mid \|f\|_\infty \le 1,\ |f(\beta) - f(\gamma)| \le d(\beta, \gamma)\ \forall\, \gamma, \beta \in \mathbb{D}\}$$

denotes the set of all Lipschitz-continuous functions bounded by 1. The subscript ξ in the expectations in (9) and (10) indicates the conditional expectation with respect to the weights $\xi = (\xi_1, \dots, \xi_n)$ given the data and $h(G_n)^*$ and $h(G_n)_*$ denote measurable majorants and minorants with respect to the joint data, including the weights ξ. Note also that condition (9) is motivated by the metrization of weak convergence by the bounded Lipschitz-metric; see, e.g., Theorem 1.12.4 in van der Vaart and Wellner [32].

The process $\mathbb{B}_C$ can be approximated by multiplier bootstrap methods; see Scaillet [27], Rémillard and Scaillet [24], Bücher and Dette[4], Bücher [3], Segers [29]. More precisely, let $\xi_1, \dots, \xi_n$ denote independent identically distributed random variables with mean 0 and variance 1 such that

$$\|\xi_i\|_{2,1} = \int_0^\infty \sqrt{\Pr(|\xi_i| > x)}\, dx < \infty, \tag{11}$$

and consider the process

$$\alpha_n^\xi = \sqrt{n}(C_n^\xi - C_n), \tag{12}$$

where

$$C_n^\xi(\mathbf{x}) = n^{-1} \sum_{i=1}^n \frac{\xi_i}{\bar{\xi}_n} \mathbb{I}\{X_{i1} \le F_{n1}^-(x_1), X_{i2} \le F_{n2}^-(x_2)\}$$

denotes a multiplier bootstrap version of the estimator. The following theorem was shown by Bücher [3]; see Theorem 2.3 in that reference. A new and shorter proof is given in Appendix.

Theorem 2.3. *The process α_n^ξ defined in (12) converges weakly to $\mathbb{B}_C$ in $\ell^\infty([0,1]^2)$ conditionally on the data in probability in the sense of Kosorok [17], i.e., $\alpha_n^\xi \rightsquigarrow_\xi^{\mathbb{P}} \mathbb{B}_C$.*

For the approximation of the partial derivatives in (7) we need some estimator $\widehat{\dot{C}_p}$ for $\dot{C}_p$. In order to prove consistency of the multiplier bootstrap, these estimators need to satisfy the following two conditions.

Condition 2.4. (i) *There exists a constant K such that $\|\widehat{\dot{C}_p}\|_\infty \le K$ for all $n \in \mathbb{N}$ and $p = 1, 2$.*
(ii) *For each $p = 1, 2$ and all $\delta \in (0, 1/2)$ one has*

$$\sup_{\mathbf{u}\in[0,1]^2 : u_p \in [\delta, 1-\delta]} \left| \widehat{\dot{C}_p}(\mathbf{u}) - \dot{C}_p(\mathbf{u}) \right| \xrightarrow{\mathbb{P}} 0.$$

Both assumptions can for instance be verified for an estimator based on the differential quotient as considered in Rémillard and Scaillet [24] and more refined in Segers [28] defined by

$$\widehat{\dot{C}_1}(\mathbf{u}) := \begin{cases} \dfrac{C_n(u_1+h, u_2) - C_n(u_1-h, u_2)}{2h} & \text{if } u_1 \in [h, 1-h] \\ \dfrac{C_n(2h, u_2)}{2h} & \text{if } u_1 \in [0, h) \\ \dfrac{u_2 - C_n(1-2h, u_2)}{2h} & \text{if } u_1 \in (1-h, 1] \end{cases} \tag{13}$$

$$\widehat{\dot{C}_2}(\mathbf{u}) := \begin{cases} \dfrac{C_n(u_1, u_2+h) - C_n(u_1, u_2-h)}{2h} & \text{if } u_2 \in [h, 1-h] \\ \dfrac{C_n(u_1, 2h)}{2h} & \text{if } u_2 \in [0, h) \\ \dfrac{u_1 - C_n(u_1, 1-2h)}{2h} & \text{if } u_2 \in (1-h, 1] \end{cases} \tag{14}$$

where $h = h_n \to 0$ such that $\inf_n h_n\sqrt{n} > 0$. Here, part (i) holds without further assumptions while (ii) holds if additionally Condition 2.1 is satisfied; see Segers [28]. For a smooth version of these estimators, see Scaillet [27].

Theorem 2.5. (a) *If Conditions 2.1 and 2.4* (i) + (ii) *hold and if the multipliers ξ_i satisfy (11), then the multiplier process $\mathbb{H}_n^\xi$ defined as*

$$\begin{aligned} \mathbb{H}_n^\xi(x,y,z) = {} & \alpha_n^\xi\{x, C_n(y,z)\} - \widehat{\dot{C}_1}\{x, C_n(y,z)\}\alpha_n^\xi(x_1, 1) - \widehat{\dot{C}_2}\{x, C_n(y,z)\}\alpha_n^\xi\{1, C_n(y,z)\} \\ & - \left[\alpha_n^\xi\{C_n(x,y), z\} - \widehat{\dot{C}_1}\{C_n(x,y), z\}\alpha_n^\xi\{C_n(x,y), 1\} - \widehat{\dot{C}_2}\{C_n(x,y), z\}\alpha_n^\xi(1, z)\right] \\ & + \widehat{\dot{C}_2}\{x, C_n(y,z)\}\left\{\alpha_n^\xi(y,z) - \widehat{\dot{C}_1}(y,z)\alpha_n^\xi(y,1) - \widehat{\dot{C}_2}(y,z)\alpha_n^\xi(1,z)\right\} \\ & + \widehat{\dot{C}_1}\{C_n(x,y), z\}\left\{\alpha_n^\xi(x,y) - \widehat{\dot{C}_1}(x,y)\alpha_n^\xi(x,1) - \widehat{\dot{C}_2}(x,y)\alpha_n^\xi(1,y)\right\} \end{aligned}$$

converges weakly to the process $\mathbb{H}_C$ conditional on the data in probability, i.e., $\mathbb{H}_n^\xi \rightsquigarrow_\xi^{\mathbb{P}} \mathbb{H}_C$.

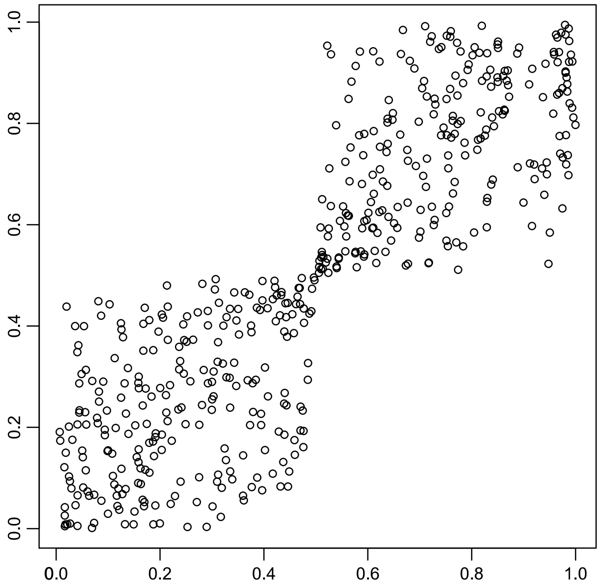

Fig. 1. A random sample of size 500 from the ordinal sum copula of a Gumbel copula C_1 with parameter $\theta_1 = 1.5$ and a Clayton copula C_2 with parameter $\theta_2 = 1$, where $J_1 = [0, 1/2], J_2 = [1/2, 1]$; see (16).

(b) *If Condition 2.4(i) holds,* $\sup_{x,y,z} |\mathbb{H}_n^\xi(x, y, z)| = O_{\mathbb{P}}(1)$.

Note that Theorem 2.5 holds independently of the hypothesis of associativity. As a consequence of the Continuous Mapping Theorem for the bootstrap (see Proposition 10.7 in Kosorok [17]), we can conclude that under the assumptions of (a)

$$\mathbb{T}_{n,\mathrm{L}_2}^\xi = \int_{[0,1]^3} \left\{\mathbb{H}_n^\xi(x, y, z)\right\}^2 d(x, y, z) \underset{\xi}{\overset{\mathbb{P}}{\rightsquigarrow}} \mathbb{T}_{C,\mathrm{L}_2}, \qquad \mathbb{T}_{n,\mathrm{KS}}^\xi = \sup_{[0,1]^3} \left|\mathbb{H}_n^\xi(x, y, z)\right| \underset{\xi}{\overset{\mathbb{P}}{\rightsquigarrow}} \mathbb{T}_{C,\mathrm{KS}} \tag{15}$$

and the latter convergence suggests to use the following approach in order to obtain an asymptotic level-α test for the hypothesis of associativity.

1. Compute the statistic $\mathbb{T}_{n,\mathrm{M}}$ [$\mathrm{M} \in \{\mathrm{L}_2, \mathrm{KS}\}$].
2. Choose the number of bootstrap replications $B \in \mathbb{N}$. For $b = 1, \ldots, B$ simulate independent replications of the random variables $\xi_1, \ldots, \xi_n$ and denote the result form the b-th iteration by $\xi_{1,b}, \ldots, \xi_{n,b}$.
3. For $b = 1, \ldots, B$ compute the statistics $\mathbb{T}_{n,\mathrm{M}}^{(\xi,b)}$ defined in (15) from the data $\mathbf{X}_1, \ldots, \mathbf{X}_n$ and the multipliers $\xi_{1,b}, \ldots, \xi_{n,b}$ and determine the $(1 - \alpha)$-quantile $q_{1-\alpha,\mathrm{M}}^\xi$ of the empirical distribution of the sample $\{\mathbb{T}_{n,\mathrm{M}}^{(\xi,b)}\}_{b=1,\ldots,B}$.
4. Reject the null hypothesis of associativity whenever $\mathbb{T}_{n,\mathrm{M}} > q_{1-\alpha,\mathrm{M}}^\xi$.

Since $\mathbb{T}_{n,\mathrm{M}} \overset{\mathbb{P}}{\to} \infty$ and $\mathbb{T}_{n,\mathrm{M}}^\xi = O_{\mathbb{P}}(1)$ if the copula is not associative, the test is consistent against all alternatives satisfying Condition 2.4(i). In particular, note that we do not need *any* smoothness assumptions apart from continuity of F.

3. Testing Archimedeanity

As stated in the Introduction, a bivariate copula C is Archimedean if and only if C is an associative copula satisfying $C(u, u) < u$ for all $u \in (0, 1)$. Associativity has been dealt with in the preceding section and it remains to handle non-Archimedean copulas which may be associative but satisfy $C(q, q) = q$ for some $q \in (0, 1)$. Due to Theorem 1 in Jaworski [15] or by the results in Section 2.4 of Alsina et al. [1], all those copulas may be expressed as an ordinal sum of Archimedean copulas. According to Section 3.2.2 in Nelsen [22], an ordinal sum copula is defined as following: let $\{J_i\}_{i\in I}$ be a countable partition of non-overlapping closed intervals $J_i = [a_i, b_i]$ whose union is $[0, 1]$. If, moreover, $\{C_i\}_{i\in I}$ is a collection of copulas, then the ordinal sum of $\{C_i\}_{i\in I}$ with respect to $\{J_i\}_{i\in I}$ is the copula C defined by

$$C(\mathbf{u}) = \begin{cases} a_i + (b_i - a_i)C_i\left(\dfrac{u_1 - a_i}{b_i - a_i}, \dfrac{u_2 - a_i}{b_i - a_i}\right) & \text{if } \mathbf{u} \in J_i \times J_i \\ \min\{u_1, u_2\} & \text{otherwise.} \end{cases} \tag{16}$$

Note that $C(b_i, b_i) = b_i$ for all b_i and that ordinal sum copulas put no mass on $[0, 1]^2 \setminus \bigcup_{i\in I} J_i \times J_i$. In Fig. 1, we illustrate the ordinal sum of a Gumbel–Hougaard copula C_1 with parameter $\theta_1 = 1.5$ and a Clayton copula C_2 with parameter $\theta_2 = 1$, where $J_1 = [0, 1/2], J_2 = [1/2, 1]$. Note that Kendall's τ for C is equal to $2/3$, while it equals $1/3$ for both C_1 and C_2.

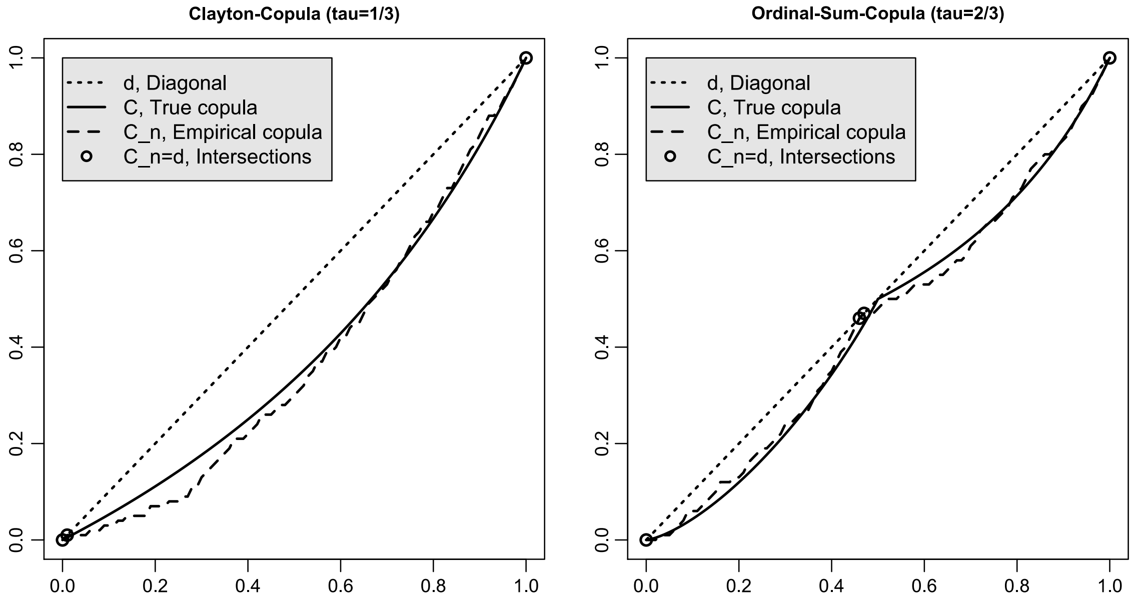

Fig. 2. The solid lines show the diagonal section of a Clayton copula (left) and an ordinal sum copula (right), while the dashed line show one realization of the corresponding empirical copula. The circled points mark the locations where $C_n(i/n, i/n) = i/n$.

In order to check for $C(q, q) = q$ for some $q \in (0, 1)$, we propose the following modification of the statistic $\mathbb{T}_{n,\mathrm{M}}$

$$\mathbb{S}_{n,\mathrm{M}} = \mathbb{T}_{n,\mathrm{M}} + k_n\phi\{A_n(C_n)\},$$

where $\mathbb{T}_{n,\mathrm{M}}$ is defined in (5) and (6), $k_n \sim n^{\alpha}$, $\alpha \in (0, 1/2)$ is some constant chosen by the statistician, ϕ is some increasing function with $\phi(0) = 0$ and

$$A_n(C_n) = \max\left\{\frac{i}{n}\left(1 - \frac{i}{n}\right) : C_n\left(\frac{i}{n}, \frac{i}{n}\right) = \frac{i}{n}\right\}.$$

Intuitively, $A_n(C_n)$ is an estimator for $A(C) = \sup\{u(1-u) : C(u, u) = u\}$ and thus should be "large" for copulas which satisfy $C(q, q) = q$ for some $q \in (0, 1)$. For a decent choice of k_n and ϕ, we refer the reader to Section 4.

In Fig. 2, we illustrate the points i/n at which $C_n(i/n, i/n) = i/n$ for two specific examples, the Clayton copula with $\theta = 1$ and the ordinal sum copula depicted in Fig. 1. The solid and dashed lines correspond to the true copula and the empirical copula calculated for a set of $n = 100$ observations, respectively. For the ordinal sum copula, there always exist some points i/n in a neighborhood of 1/2 such that $C_n(i/n, i/n) = i/n$; see the proof of the following proposition, which is sufficient for the derivation of the asymptotic properties of the statistic $\mathbb{S}_{n,\mathrm{M}}$.

Proposition 3.1. (a) *Suppose C is an Archimedean copula satisfying Condition 2.1 and that the coefficients of tail dependence $\lambda_L = \lim_{u\to 0} C(u, u)/u$ and $\lambda_U = \lim_{u\to 1}\{1 - 2u + C(u, u)\}/\{1 - u\}$ exist and are smaller than 1. Then $A_n(C_n) = o_{\mathbb{P}}(n^{-\alpha})$ for any $\alpha < 1/2$.*

(b) *If there exists a $q \in (0, 1)$ such that $C(q, q) = q$, then $A_n(C_n) \geq q(1-q) + o_{\mathbb{P}}(1)$.*

Remark 3.2. (a) The conditions on the coefficients of tail dependence in part (a) of Proposition 3.1 can be equivalently expressed by conditions on the regular variation of the Archimedean generator of C. For a thorough discussion of these issues the reader is referred to the work of Charpentier and Segers [5] and of Larsson and Nešlehová [18].

(b) Exploiting Theorem G.1 in Genest and Segers [14] and Proposition 4.2 in Segers [29], one can improve the rate of convergence in part (a) of Proposition 3.1 to any $\alpha < 3/4$. It is our conjecture that the term is in fact of order $O_{\mathbb{P}}(1/n)$, but we were not able to derive the asymptotic distribution of $nA_n(C_n)$. Since we do not need a refined rate for our purposes, we omit a deeper discussion and defer these issues to future research.

From now on, suppose that the conditions of Theorem 2.5 and Proposition 3.1 hold. We can conclude that $\mathbb{S}_{n,\mathrm{M}}$ weakly converges to $\mathbb{T}_{C,\mathrm{M}}$ if the copula C is Archimedean and satisfies Condition 2.1, while $\mathbb{S}_{n,\mathrm{M}}$ converges to $+\infty$ in probability if C is non-Archimedean, i.e., if it is either non-associative (by the results of Section 2) or if there exists a $q \in (0, 1)$ such that $C(q, q) = q$ (by Proposition 3.1). The quantiles of $\mathbb{T}_{C,\mathrm{M}}$ can be approximated by the multiplier method described in Section 2.2. Analogously to the discussion at the end of Section 2.2, we can use the multiplier bootstrap to obtain an asymptotic level-α test for the hypothesis of Archimedeanity (for copulas satisfying Condition 2.1), which is consistent against all alternatives under Condition 2.4(i). Its finite-sample properties will be investigated in the following section.

Table 1
Observed rejection probabilities for the multiplier bootstrap-tests for Archimedeanity and for Associtivity (in brackets). The sample size is $n = 200$ or $n = 500$, $B = 200$ Bootstrap-replicates and 1000 simulation runs have been performed. The first four lines are Archimedean copula models, the t-models are not associative and the ordinal sum-models are associative, but not Archimedean.

	L^2-test		KS-test	
	0.1	0.05	0.1	0.05
$n = 200$:				
Clayton($\tau = 1/3$)	0.071 (0.071)	0.038 (0.037)	0.088 (0.088)	0.050 (0.050)
Clayton($\tau = 2/3$)	0.030 (0.016)	0.011 (0.009)	0.124 (0.078)	0.068 (0.036)
Gumbel($\tau = 1/3$)	0.079 (0.079)	0.043 (0.043)	0.082 (0.082)	0.046 (0.045)
Gumbel($\tau = 2/3$)	0.034 (0.032)	0.015 (0.013)	0.108 (0.098)	0.065 (0.057)
$t(\tau = 1/3, df = 1)$	0.953 (0.953)	0.886 (0.884)	0.562 (0.558)	0.380 (0.376)
$t(\tau = 2/3, df = 1)$	0.748 (0.726)	0.592 (0.564)	0.392 (0.355)	0.258 (0.231)
Aneglog($\lambda_U = 0.25$)	0.112 (0.112)	0.061 (0.061)	0.105 (0.105)	0.059 (0.059)
Aneglog($\lambda_U = 0.5$)	0.641 (0.641)	0.536 (0.536)	0.363 (0.356)	0.225 (0.222)
Ordinal$_A$($\tau = 1/3$)	0.996 (0.000)	0.827 (0.000)	1.000 (0.012)	1.000 (0.005)
Ordinal$_A$($\tau = 2/3$)	1.000 (0.004)	1.000 (0.001)	1.000 (0.079)	1.000 (0.041)
Ordinal$_B$($\tau = 1/3$)	1.000 (0.000)	1.000 (0.000)	1.000 (0.045)	1.000 (0.021)
Ordinal$_B$($\tau = 2/3$)	1.000 (0.006)	1.000 (0.001)	1.000 (0.057)	1.000 (0.030)
$n = 500$:				
Clayton($\tau = 1/3$)	0.082 (0.082)	0.051 (0.051)	0.088 (0.088)	0.036 (0.036)
Clayton($\tau = 2/3$)	0.062 (0.059)	0.032 (0.027)	0.090 (0.082)	0.046 (0.039)
Gumbel($\tau = 1/3$)	0.091 (0.091)	0.045 (0.045)	0.105 (0.105)	0.050 (0.050)
Gumbel($\tau = 2/3$)	0.072 (0.070)	0.033 (0.032)	0.124 (0.121)	0.068 (0.066)
$t(\tau = 1/3, df = 1)$	1.000 (1.000)	1.000 (1.000)	0.954 (0.953)	0.871 (0.871)
$t(\tau = 2/3, df = 1)$	0.998 (0.990)	0.998 (0.990)	0.818 (0.811)	0.655 (0.650)
Aneglog($\lambda_U = 0.25$)	0.237 (0.237)	0.173 (0.173)	0.124 (0.124)	0.069 (0.069)
Aneglog($\lambda_U = 0.5$)	0.979 (0.979)	0.947 (0.947)	0.716 (0.716)	0.584 (0.584)
Ordinal$_A$($\tau = 1/3$)	1.000 (0.000)	0.980 (0.000)	1.000 (0.022)	1.000 (0.007)
Ordinal$_A$($\tau = 2/3$)	1.000 (0.021)	1.000 (0.009)	1.000 (0.093)	1.000 (0.038)
Ordinal$_B$($\tau = 1/3$)	1.000 (0.000)	1.000 (0.000)	1.000 (0.005)	1.000 (0.023)
Ordinal$_B$($\tau = 2/3$)	1.000 (0.013)	1.000 (0.004)	1.000 (0.082)	1.000 (0.037)

4. Finite-sample properties

This section is devoted to a simulation study regarding the finite sample performance of the proposed tests for Archimedeanity and Associativity. We consider the following six copula models.

- The Gumbel–Hougaard copula, which is Archimedean.
- The Clayton copula, which is Archimedean.
- The t-copula with fixed degree of freedom $df = 1$, which is non-associative.
- The asymmetric negative logistic model from Joe [16] with fixed parameters $\psi_1 = 2/3$, $\psi_2 = 1$, which is non-associative.
- An ordinal sum model based on the partition $J_1 = [0, 1/2], J_2 = [1/2, 1]$ and the Gumbel–Hougaard (C_1) and Clayton (C_2) copula, denoted by Ordinal$_A$. The model is associative.
- An ordinal sum model based on the partition $J_1 = [0, 1/2], J_2 = [1/2, 1]$ and the two Clayton ($C_1 = C_2$) copulas, denoted by Ordinal$_B$. The model is associative.

The parameters of the models are chosen in such a way that the coefficient of upper tail dependence λ_U is either 1/4 or 1/2 (for the asymmetric negative logistic model) or that Kendalls-τ is either 1/3 or 2/3 (for the remaining five models). For $\tau_{\text{Ordinal}_A} = 1/3$ (or 2/3, resp.) we chose $\tau_{\text{Gumbel}} = 0\ (1/3)$ and $\tau_{\text{Clayton}} = -2/3\ (1/3)$, while $\tau_{\text{Clayton}} = -1/3\ (1/3)$ for $\tau_{\text{Ordinal}_B} = 1/3\ (2/3)$.

We generated 1000 random samples of sample sizes $n = 200$ and $n = 500$ and calculated the empirical probability of rejecting the null hypotheses of Archimedeanity or Associativity for $\mathrm{M} \in \{\mathrm{L}^2, \mathrm{KS}\}$. For each sample of size $n = 200, 500$ we carried out $B = 200$ Bootstrap replications based on the multiplier method, where we chose a $\mathcal{U}(\{0, 2\})$-distribution for the multipliers (i.e., $\Pr(\xi = 0) = \Pr(\xi = 2) = 1/2$, such that $\mu = \tau = 1$) and used $h = n^{-1/4}$ to estimate the partial derivatives. The critical values of the tests are determined by the method described in Section 2. The penalty term $\mathbb{S}_{n,\mathrm{M}} - \mathbb{T}_{n,\mathrm{M}} = k_n \phi\{A_n(C_n)\}$ is chosen in the following data-adaptive way: first of all, we set $\phi(x) = (4x)^2$ in order to give more emphasis to values around the maximal value of $A_n(C_n)$, which equals 1/4. The constant k_n is chosen according to the distribution of the bootstrap approximation: if $q^{\xi}_{0.05,\mathrm{M}}$ denotes the 0.05-quantile of the sample $\{\mathbb{T}^{\xi,b}_{n,\mathrm{M}}\}_{b=1,\dots,B}$ we set $k_n = q^{\xi}_{0.05,\mathrm{M}} n^{1/4}$. The latter choice guarantees that under H_0 the error term is small compared to the distribution of $\mathbb{T}_{C,\mathrm{M}}$.

The results are stated in Table 1. The entries of the table represent the empirical probabilities of rejecting the null hypothesis of Archimedeanity and of Associativity (in brackets) for both the L^2-test (first two columns) and the KS-test (last two columns). We observe that the nominal level of the four tests is accurately approximated for the four Archimedean copulas under investigation. The L^2-test tends to be more conservative than the KS-test. Also note that the values for $\mathbb{S}_{n,\mathrm{M}}$

Table 2
P-values in percent of the multiplier bootstrap tests for Archimedeanity for the 21 possible pairs of variables in the uranium exploration data set, based on $B = 1000$ Bootstrap replicates.

Pair	L^2-test	KS-test
(U, Li)	**12.6**	**47.2**
(U, Co)	0.0	3.4
(U, K)	0.0	0.8
(U, Cs)	0.1	4.9
(U, Sc)	0.0	3.6
(U, Ti)	0.7	**14.4**
(Li, Co)	3.1	**5.6**
(Li, K)	**19.8**	**76.9**
(Li, Cs)	**6.9**	**14.4**
(Li, Sc)	**7.7**	3.7
(Li, Ti)	**12.2**	**63.1**
(Co, K)	0.1	0.4
(Co, Cs)	0.0	0.0
(Co, Sc)	1.3	**70.2**
(Co, Ti)	0.0	0.2
(K, Cs)	0.3	**27.7**
(K, Sc)	0.0	0.1
(K, Ti)	2.0	**6.0**
(Cs, Sc)	0.0	0.0
(Cs, Ti)	**10.4**	**17.9**
(Sc, Ti)	0.0	4.9

and $\mathbb{T}_{n,\mathrm{M}}$ differ only by a very small amount, meaning that the penalty term $k_n\phi\{A_n(C_n)\}$ is of negligible magnitude under the null hypothesis.

The t-copula and the asymmetric negative logistic models are non-associative and the results in Table 1 reveal that these deviations are detected by both tests for Associativity, with better results for the t-copula and for stronger dependence measured by either τ or λ_U. The power properties of the L^2-test outclass the properties of the KS-test for all four non-associative models under investigation, such that the former test seems to be generally preferable.

Regarding the (associative) ordinal sum models both tests for associativity are very conservative. Note that the asymptotic theory developed in Section 2 does not apply for these models since the partial derivatives of the corresponding copulas are not continuous on $\{1/2\} \times [0, 1]$ and $[0, 1] \times \{1/2\}$. Regarding the power properties the KS-test for Archimedeanity performs slightly better for the ordinal sum alternatives.

5. Data application

As a simple illustration of the procedures described in this paper, we apply the new tests for Archimedeanity to the data set of Cook and Johnson [6]. This seven-dimensional data set contains the log-concentration of uranium (U), lithium (Li), cobalt (Co), potassium (K), cesium (Cs), scandium (Sc) and Titanium (Ti) measured in $n = 655$ data samples taken near Grand Junction, Colorado, US. Recently, the pairs of this data set have been tested for a possible modeling by certain selected Archimedean copula models (Ali–Mikhail–Haq, Clayton, Frank, Gumbel–Hougaard) in Genest et al. [11]. Ben Ghorbal et al. [2] and Quessy [23] carried out various tests on the same data set for the hypothesis that the pairs can be modeled by an arbitrary extreme-value copula.

Table 2 shows the corresponding p-values of all 21 possible pairs of variables for the L^2- and the KS-test for Archimedeanity, based on $B = 1000$ repetitions of the multiplier bootstrap procedure. The results for the corresponding tests for associativity are exactly the same. The following observations can be made from Table 2.

(a) The L^2-test rejects 15 of the 21 hypotheses at the 5%-level whereas the KS-test only rejects 11 hypotheses. In general, the p-values for the L^2-test are smaller than the ones for the KS-test with the only exception for the pair (Li, Sc). For three of the six cases in which both tests yield contradictory results the p-values differ by at most four percentage points.

(b) The lower triangular part of Table 3 in Ben Ghorbal et al. [2] shows p-values for a test for the hypotheses that a given pair can be modeled by a Gumbel–Hougaard copula. For seven of the 21 pairs, the test does not reject this hypothesis. The results are in accordance with Table 2 where all of these seven models are also not rejected to be Archimedean at the 5%-level.

(c) In Genest et al. [11], the eight pairs (U, Li), (U, Co), (U,Sc), (Li, Cs), (Li, Ti), (Co, Cs), (Co, Ti) and (Cs, Sc) are tested for four selected Archimedean copulas (with the Gumbel copula being both an Archimedean and an extreme-value copula; see Genest and Rivest [12]) by three different goodness-of-fit tests at the 5%-level. It is their conclusion that only the pair (U, Sc) cannot be modeled by any of the copulas under investigation, which is in accordance with the results from Table 2. Regarding the seven other pairs the results are ambiguous: Table 2 shows that the hypothesis of an Archimedean copula is not rejected for the three pairs (U, Li), (Li, Sc) and (Li, Ti), which reflects the results from Genest et al., whereas the hypothesis of an Archimedean copula for the remaining four pairs is rejected. In Genest et al. [11], only for one of these four pairs, (Co, Ti), all four models are rejected by one of the three tests under consideration, while all other pairs can be modeled by at least one of the models adequately.

(d) A comparison of our results with the tests for extreme-value dependence in Ben Ghorbal et al. [2] and Quessy [23] allows for some further strengthening conclusions. For instance, note that all tests in both papers reveal that the pair (Co, K) can be modeled by an extreme-value copula, while Table 2 shows that the hypothesis of an Archimedean copula must be rejected at a 1%-level. Hence, a Gumbel–Hougaard copula, which is both extreme-value and Archimedean is not suitable for the modeling of this pair. This result is in accordance with the small p-value for that pair in the lower triangular part of Table 3 in Ben Ghorbal et al. [2].

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Appendix A. Proofs

Proof of Theorem 2.2. Consider the functional $\Psi : \mathbb{D}_\Psi \to \ell^\infty([0,1]^3)$ defined for $\alpha \in \mathbb{D}_\Psi = \{F : F \text{ cdf on } [0,1]^2\}$ by

$$\Psi(\alpha)(x,y,z) = \alpha\{x, \alpha(y,z)\} - \alpha\{\alpha(x,y), z\}.$$

If the copula C is associative we can write $\mathbb{H}_n = \sqrt{n}\{\Psi(C_n) - \Psi(C)\}$. Observing that $\mathbb{B}_C \in \mathbb{D}_0$ a.s., where

$$\mathbb{D}_0 = \left\{\gamma \in \ell^\infty([0,1]^2) \mid \gamma \text{ continuous}, \gamma(\mathbf{u}) = 0 \text{ for all } \mathbf{u} \in [0,1]^2 \text{ such that } C(\mathbf{u}) \in \{0,1\}\right\},$$

an application of the Functional Delta Method (see, e.g., Theorem 3.9.4 in van der Vaart and Wellner [32]) and of the following proposition yields the assertion. $\square$

Proposition A.1. *Under Condition 2.1, the mapping Ψ is Hadamard-differentiable at C tangentially to the space $\mathbb{D}_0$ with derivative given by*

$$\Psi_C'(\alpha)(x,y,z) = \alpha\{x, C(y,z)\} - \alpha\{C(x,y), z\} + \dot{C}_2\{x, C(y,z)\}\alpha(y,z) - \dot{C}_1\{C(x,y), z\}\alpha(x,y).$$

Proof. Let $t_n \to 0$ and $\alpha_n \in \ell^\infty([0,1]^2)$ with $\alpha_n \to \alpha \in \mathbb{D}_0$ such that $C + t_n\alpha_n \in \mathbb{D}_\Psi$. Then

$$t_n^{-1}\{\Psi(C + t_n\alpha_n) - \Psi(C)\} = L_{n1} + L_{n2} - L_{n3}$$

where

$$\begin{aligned}
L_{n1}(x,y,z) &= \alpha_n\{x, (C + t_n\alpha_n)(y,z)\} - \alpha_n\{(C + t_n\alpha_n)(x,y), z\} \\
L_{n2}(x,y,z) &= t_n^{-1}\left[C\{x, (C + t_n\alpha_n)(y,z)\} - C\{x, C(y,z)\}\right] \\
L_{n3}(x,y,z) &= t_n^{-1}\left[C\{(C + t_n\alpha_n)(x,y), z\} - C\{C(x,y), z\}\right].
\end{aligned}$$

Exploiting the fact that α_n converges uniformly to a bounded function and that α is uniformly continuous, one can conclude that $L_{n1}(x,y,z) = \alpha\{x, C(y,z)\} - \alpha\{C(x,y), z\} + o(1)$ uniformly in $(x,y,z) \in [0,1]^3$. Regarding the summand L_{n2}, we have to distinguish two cases. First, we consider all those $(x,y,z) \in [0,1]^3$ for which $C(y,z) \in (0,1)$. A Taylor expansion of $C(x,\cdot)$ at $C(y,z)$ yields

$$L_{n2}(x,y,z) = \dot{C}_2\{x, C(y,z)\}\alpha_n(y,z) + r_n(x,y,z),$$

where the error term can be written as

$$r_n(x,y,z) = \left[\dot{C}_2(x, u_n) - \dot{C}_2\{x, C(y,z)\}\right]\alpha_n(y,z)$$

with some intermediate point u_n between $C(y,z)$ and $(C + t_n\alpha_n)(y,z)$. The main term uniformly converges to $\dot{C}_2\{x, C(y,z)\}\alpha(y,z)$ (note that partial derivatives of copulas are uniformly bounded by 1) and it remains to show that $r_n(x,y,z) = o(1)$ uniformly in (x,y,z) with $C(y,z) \in (0,1)$.

To see this, we will show at the end of this proof that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\limsup_{n\to\infty} \sup_{\mathbf{v}\in A_\delta} |\alpha_n(\mathbf{v})| \le \varepsilon, \tag{A.1}$$

where $\mathbf{v} = (y,z)$, $A_\delta = \{\mathbf{v} \in [0,1]^2 \mid C(\mathbf{v}) \in [0,\delta) \cup (1-\delta, 1]\}$. Then, since partial derivatives of copulas are bounded by 1, we can conclude that

$$\limsup_{n\to\infty} \sup_{x\in[0,1],(y,z)\in A_\delta} |r_n(x,y,z)| \le \varepsilon.$$

Due to Condition 2.1, the partial derivative $\dot{C}_2$ is uniformly continuous on the quadrangle $[0, 1] \times [\delta, 1-\delta]$. Thus, since α is uniformly bounded and $u_n \to C(y, z)$, we obtain uniform convergence of $r_n(x, y, z)$ to 0 for all (y, z) such that $C(y, z) \in [\delta, 1-\delta]$, i.e., for $(y, z) \in [0, 1]^2 \setminus A_\delta$. Combining the two facts derived above, it follows that

$$\limsup_{n\to\infty} \sup_{x\in[0,1], C(y,z)\in(0,1)} |r_n(x, y, z)| \le \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this lim sup must be zero. Summarizing, the case $(x, y, z) \in [0, 1]^3$ such that $C(y, z) \in (0, 1)$ is finished.

In the remaining case $C(y, z) \in \{0, 1\}$, i.e. $(y, z) \in A_0$, Lipschitz-continuity of C entails that

$$|L_{n2}(x, y, z)| = t_n^{-1}|C\{x, C(y, z) + t_n\alpha_n(y, z)\} - C\{x, C(y, z)\}| \le \alpha_n(y, z) = \alpha(y, z) + o(1) = o(1)$$

uniformly in (x, y, z) since in this case $\alpha(y, z) = 0$. Finally, the summand L_{n3} may be treated analogously.

To complete the proof it remains to show (A.1). Exploiting uniform convergence of α_n, uniform continuity of α and the fact that $\alpha(\mathbf{v}) = 0$ for all $\mathbf{v} \in A_0 = \{\mathbf{v} | C(\mathbf{v}) \in \{0, 1\}\}$, we can conclude that there exists a $\kappa > 0$ such that $|\alpha_n(\mathbf{v})| \le \varepsilon$ for all $\mathbf{v} \in A_0^\kappa = \{\mathbf{v} | \exists\, \mathbf{u} \in A_0 \text{ s.t. } \|\mathbf{u}-\mathbf{v}\| \le \kappa\}$ and sufficiently large n. For $v_1 \in [\kappa, 1]$ let $\delta(v_1) = \sup\{C(v_1, z) | (v_1, z) \in A_0^\kappa\}$ (which equals $C\{v_1, z(v_1)\}$ for some $z(v_1)$ such that $(v_1, z(v_1)) \in \overline{\partial A_0^\kappa \cap (0, 1)^2}$ since for any fixed v_1 the function $u \mapsto C(v_1, u)$ is increasing) and set $\delta = \inf_{v_1\in[\kappa,1]} \delta(v_1)$, which is strictly positive due to compactness of $\overline{\partial A_0^\kappa \cap (0, 1)^2}$ and continuity of C. We will now show that this choice of δ yields (A.1). Now, if $C(\mathbf{v}) \le \delta$, we have either $v_1 < \kappa$ (in which case $\mathbf{v} \in A_0^\kappa$ since $C(0, v_2) = 0$) or $v_1 \ge \kappa$. In the latter case, $C(\mathbf{v}) \le \delta(v_1)$ and monotonicity of C imply $\mathbf{v} \in A_0^\kappa$. This proves (A.1) and completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Without loss of generality, we may assume that $\mathbf{X_i} \sim C$. We only perform the proof of condition (9); the asymptotic measurability in (10) follows along similar lines. By Theorem 2.6 in Kosorok [17], we have $\beta_n^\xi = \sqrt{n}(F_n^\xi - F_n) \rightsquigarrow_\xi^{\mathbb{P}} \mathbb{B}_C$. Observing that $\alpha_n^\xi = \beta_n^\xi(F_n^-)$, where $F_n^- = (F_{n1}^-, F_{n2}^-)$, and due to the estimation

$$\begin{aligned}\sup_{h\in BL_1(\mathbb{D})} \left|\mathbb{E}_\xi h(\alpha_n^\xi) - \mathbb{E}h(\mathbb{B}_C)\right| &\le \sup_{h\in BL_1(\mathbb{D})} \left|\mathbb{E}_\xi h\{\beta_n^\xi(F_n^-)\} - \mathbb{E}h(\beta_n^\xi)\right| + \sup_{h\in BL_1(\mathbb{D})} \left|\mathbb{E}_\xi h(\beta_n^\xi) - \mathbb{E}h(\mathbb{B}_C)\right| \\ &\le \mathbb{E}_\xi \left(\|\beta_n^\xi(F_n^-) - \beta_n^\xi\|_\infty \wedge 2\right)^* + o_{\mathbb{P}}(1)\end{aligned}$$

we have to show that $\mathbb{E}_\xi \delta_n^* = o_{\mathbb{P}}(1)$, where $\delta_n = \|\beta_n^\xi(F_n^-) - \beta_n^\xi\|_\infty \wedge 2$. The asterisk in the latter estimation denotes a measurable majorant with respect to the joint data. By Theorem 10.1 in Kosorok [17], we have $\beta_n^\xi \rightsquigarrow \mathbb{B}_C$ unconditionally, and hence Theorem 1.5.7 and its addendum in van der Vaart and Wellner [32] yield $\delta_n^* = o_{\mathbb{P}}(1)$. By boundedness of δ_n^* we also obtain L_1-convergence and the assertion follows by Fubini's Theorem; see Lemma 1.2.7 in van der Vaart and Wellner [32]. □

Proof of Theorem 2.5. (a) Define the process $\tilde{\mathbb{H}}_n^\xi$ by substituting the estimators $\widehat{\dot{C}}_1$, $\widehat{\dot{C}}_2$ and C_n in the definition of $\mathbb{H}_n^\xi$ by the true but unknown objects $\dot{C}_1$, $\dot{C}_2$ and C. By Lemma B.1 in Appendix B it suffices to show that

$$\|\mathbb{H}_n^\xi - \tilde{\mathbb{H}}_n^\xi\|_\infty = \sup_{(x,y,z)\in[0,1]^3} |\mathbb{H}_n^\xi(x, y, z) - \tilde{\mathbb{H}}_n^\xi(x, y, z)| \xrightarrow{\mathbb{P}} 0.$$

Using the triangle inequality we have to estimate the following 12 summands

$$\begin{aligned}\|\mathbb{H}_n^\xi - \tilde{\mathbb{H}}_n^\xi\|_\infty \le\ & \|\alpha_n^\xi\{x, C_n(y, z)\} - \alpha_n^\xi\{x, C(y, z)\}\|_\infty + \|\widehat{\dot{C}}_1\{x, C_n(y, z)\}\alpha_n^\xi(x, 1) - \dot{C}_1\{x, C(y, z)\}\alpha_n^\xi(x, 1)\|_\infty \\ &+ \|\widehat{\dot{C}}_2\{x, C_n(y, z)\}\alpha_n^\xi(1, C_n(y, z)) - \dot{C}_2\{x, C(y, z)\}\alpha_n^\xi\{1, C(y, z)\}\|_\infty \\ &+ \|\alpha_n^\xi\{C_n(x, y), z\} - \alpha_n^\xi\{C(x, y), z\}\|_\infty \\ &+ \|\widehat{\dot{C}}_1\{C_n(x, y), z\}\alpha_n^\xi\{C_n(x, y), 1\} - \dot{C}_1\{C(x, y), z\}\alpha_n^\xi\{C(x, y), 1\}\|_\infty \\ &+ \|\widehat{\dot{C}}_2\{C_n(x, y), z\}\alpha_n^\xi(1, z) - \dot{C}_2\{C(x, y), z\}\alpha_n^\xi(1, z)\|_\infty \\ &+ \|\widehat{\dot{C}}_2\{x, C_n(y, z)\}\alpha_n^\xi(y, z) - \dot{C}_2\{x, C(y, z)\}\alpha_n^\xi(y, z)\|_\infty \\ &+ \|\widehat{\dot{C}}_2\{x, C_n(y, z)\}\widehat{\dot{C}}_1(y, z)\alpha_n^\xi(y, 1) - \dot{C}_2\{x, C(y, z)\}\dot{C}_1(y, z)\alpha_n^\xi(y, 1)\|_\infty \\ &+ \|\widehat{\dot{C}}_2\{x, C_n(y, z)\}\widehat{\dot{C}}_2(y, z)\alpha_n^\xi(1, z) - \dot{C}_2\{x, C(y, z)\}\dot{C}_2(y, z)\alpha_n^\xi(1, z)\|_\infty \\ &+ \|\widehat{\dot{C}}_1\{C_n(x, y), z\}\alpha_n^\xi(x, y) - \dot{C}_1\{C(x, y), z\}\alpha_n^\xi(x, y)\|_\infty \\ &+ \|\widehat{\dot{C}}_1\{C_n(x, y), z\}\widehat{\dot{C}}_1(x, y)\alpha_n^\xi(x, 1) - \dot{C}_1\{C(x, y), z\}\dot{C}_1(x, y)\alpha_n^\xi(x, 1)\|_\infty \\ &+ \|\widehat{\dot{C}}_1\{C_n(x, y), z\}\widehat{\dot{C}}_2(x, y)\alpha_n^\xi(1, y) - \dot{C}_1\{C(x, y), z\}\dot{C}_2(x, y)\alpha_n^\xi(1, y)\|_\infty,\end{aligned}$$

of which one of the hardest cases will be considered exemplarily in the following, namely the third summand

$$\sup_{(x,y,z)\in[0,1]^3}\left|\widehat{\dot{C}}_2\{x, C_n(y,z)\}\alpha_n^\xi\{1, C_n(y,z)\} - \dot{C}_2\{x, C(y,z)\}\alpha_n^\xi\{1, C(y,z)\}\right|.$$

The treatment of the other summands is similar and is omitted for the sake of brevity. We estimate

$$\begin{aligned}
&\left|\widehat{\dot{C}}_2\{x, C_n(y,z)\}\alpha_n^\xi\{1, C_n(y,z)\} - \dot{C}_2\{x, C(y,z)\}\alpha_n^\xi\{1, C(y,z)\}\right| \\
&\quad\le \left|\widehat{\dot{C}}_2\{x, C_n(y,z)\} - \dot{C}_2\{x, C_n(y,z)\}\right| \times \left|\alpha_n^\xi\{1, C_n(y,z)\}\right| \\
&\qquad + \left|\dot{C}_2\{x, C_n(y,z)\} - \dot{C}_2\{x, C(y,z)\}\right| \times \left|\alpha_n^\xi\{1, C_n(y,z)\}\right| \left|\dot{C}_2\{x, C(y,z)\}\right| \times \left|\alpha_n^\xi\{1, C_n(y,z)\} - \alpha_n^\xi\{1, C(y,z)\}\right| \\
&\quad =: A_1(x,y,z) + A_2(x,y,z) + A_3(x,y,z)
\end{aligned}$$

and consider each term separately. For arbitrary $\varepsilon > 0$ and $\delta \in (0, 1/2)$ we estimate

$$\Pr\{\sup A_1(x,y,z) > \varepsilon\} \le \Pr\left\{\sup_{C_n(y,z)\in[\delta,1-\delta]} A_1(x,y,z) > \varepsilon\right\} + \Pr\left\{\sup_{C_n(y,z)\notin[\delta,1-\delta]} A_1(x,y,z) > \varepsilon\right\}$$

where we suppressed the index $(x, y, z) \in [0, 1]^3$ at the suprema. The first probability can be made arbitrary small by the assumptions on $\widehat{\dot{C}}_2$ and by the asymptotic tightness of the process α_n^ξ, which follows from Theorem 2.3. For the second summand use uniform boundedness of $\widehat{\dot{C}}_2$ and the fact that the (unconditional) limit process $\mathbb{B}_C(1,\cdot)$ of $\alpha_n^\xi(1,\cdot)$ is a standard Brownian bridge having continuous trajectories which vanish at 0 and 1. By decreasing δ the probability can be made arbitrary small; see Segers [29] for a rigorous treatment of this argument.

Since $\dot{C}_2$ is uniformly continuous if the second coordinate is bounded away from zero and one the second summand $A_2(x, y, z)$ can be treated similarly. Regarding $A_3(x, y, z)$ note that Theorem 2.3 yields asymptotic uniform equicontinuity of α_n^ξ. Together with the fact that $\sup_{(y,z)\in[0,1]^2} |C_n(y,z) - C(y,z)| \xrightarrow{\mathbb{P}} 0$ this yields

$$\sup_{(y,z)\in[0,1]^2}\left|\alpha_n^\xi\{1, C_n(y,z)\} - \alpha_n^\xi\{1, C(y,z)\}\right| \xrightarrow{\mathbb{P}} 0.$$

By boundedness of $\dot{C}_2$ this yields the assertion $\sup_{(x,y,z)\in[0,1]^3} A_3(x,y,z) \xrightarrow{\mathbb{P}} 0$.

(b) Without loss of generality, we may assume that $\mathbf{X_i} \sim C$. By Theorem 10.1 in Kosorok [17], we have $\beta_n^\xi = \sqrt{n}(F_n^\xi - F_n) \rightsquigarrow \mathbb{B}_C$. Observing that $\alpha_n^\xi = \beta_n^\xi(F_n^-)$, where $F_n^- = (F_{n1}^-, F_{n2}^-)$, Theorem 1.5.7 and its addendum in van der Vaart and Wellner [32] yield $\|\alpha_n^\xi - \beta_n^\xi\|_\infty = o_{\mathbb{P}}(1)$. Since β_n^ξ converges weakly, this yields $\|\alpha_n^\xi\|_\infty = O_{\mathbb{P}}(1)$. Combining this result with Condition 2.4(i) yields the assertion. □

Proof of Proposition 3.1. We start with the proof of a). First choose $\delta > 0$ and $\lambda < 1$ such that

$$\frac{C(u,u)}{u} \vee \frac{1 - 2u + C(u,u)}{1-u} \le \lambda$$

for all $u \in [0, \delta] \cup [1-\delta, 1]$ and use the decomposition

$$A_n(C_n) = A_n(C_n, [0,\delta)) + A_n(C_n, [\delta, 1-\delta]) + A_n(C_n, (1-\delta, 1]), \tag{A.2}$$

where

$$A_n(C_n, B) = \max\{i/n\,(1 - i/n) : C_n\,(i/n, i/n) = i/n \text{ and } i/n \in B\}$$

for some set $B \subset [0, 1]$ (with the convention that $\max\emptyset = 0$). Consider each term separately and define $M_n = \sup_{u\in[0,1]} |C_n(u,u) - C(u,u)|$, which is of order $O_{\mathbb{P}}(n^{-1/2})$ under Condition 2.1. Now let $i/n \in (0, \delta)$ be such that $C_n(i/n, i/n) = i/n$. Due to the estimate

$$i/n(1-\lambda) \le i/n\left\{1 - \frac{C(i/n, i/n)}{i/n}\right\} = i/n - C(i/n, i/n) = C_n(i/n, i/n) - C(i/n, i/n) \le M_n$$

we have $i/n(1 - i/n) \le i/n \le M_n/(1-\lambda)$ and we can conclude that

$$A_n(C_n, [0,\delta)) \le \frac{M_n}{1-\lambda} = O_{\mathbb{P}}(n^{-1/2}). \tag{A.3}$$

A similar calculation shows that for $i/n \in (1-\delta, 1]$ with $C_n(i/n, i/n) = i/n$ we have $(1 - i/n)(1-\lambda) \le M_n$ which in turn implies

$$A_n(C_n, (1-\delta, 1]) \le \frac{M_n}{1-\lambda} = O_{\mathbb{P}}(n^{-1/2}). \tag{A.4}$$

It remains to estimate the second summand $A_n(C_n, [\delta, 1-\delta])$ of decomposition (A.2). For continuity reasons we can choose a $\kappa > 0$ such that $u - C(u, u) \geq \kappa$ for all $u \in [\delta, 1-\delta]$. If there was a $q \in [\delta, 1-\delta]$ such that $C_n(q, q) = q$, it would follow that $M_n \geq C_n(q, q) - C(q, q) \geq \kappa$ and therefore we have for any $\varepsilon > 0$

$$\Pr\{n^\alpha A_n(C_n, [\delta, 1-\delta]\} > \varepsilon) \leq \Pr\{\exists\, q \in [\delta, 1-\delta] : C_n(q, q) = q\} \leq \Pr(M_n \geq \kappa) \xrightarrow{\mathbb{P}} 0.$$

A combination of (A.3) and (A.4) with this result proves part (a) of the proposition.

For the proof of part (b) let $n_1 = \#\{1 \leq i \leq n : (F_1(X_{i1}), F_2(X_{i2})) \in [0, q]^2\}$. Since $C(q, q) = q$ implies that the mass of C is concentrated on $(0, q)^2 \cup (q, 1)^2$ we have $(F_1(X_{i1}), F_2(X_{i2})) \in [0, q]^2$ if and only if $X_{i1} \leq X_{n_1:n,1}$ and $X_{i2} \leq X_{n_1:n,2}$, where $X_{j:n,p} = F_{np}^-(j/n)$ denotes the j-th order statistic of $X_{1p}, \ldots, X_{np}$ (for $p = 1, 2$). This yields $C_n(n_1/n, n_1/n) = n_1/n$, which entails the assertion by

$$A_n(C_n) \geq \frac{n_1}{n}\left(1 - \frac{n_1}{n}\right) \xrightarrow{\mathbb{P}} q(1-q) > 0. \quad \square$$

Appendix B. An auxiliary result

Lemma B.1. *Let $Y_n = Y_n(\mathbf{X}_1, \ldots, \mathbf{X}_n, \xi_1, \ldots, \xi_n)$ and $Z_n = Z_n(\mathbf{X}_1, \ldots, \mathbf{X}_n, \xi_1, \ldots, \xi_n)$ be two (bootstrap) statistics in a metric space $(\mathbb{D}, d)$, depending on the data $\mathbf{X}_1, \ldots, \mathbf{X}_n$ and on some multipliers $\xi_1, \ldots, \xi_n$. If $Y_n \rightsquigarrow_\xi^{\mathbb{P}} Y$ in $\mathbb{D}$, where Y is tight, and $d(Y_n, Z_n) \xrightarrow{\mathbb{P}} 0$, then also $Z_n \rightsquigarrow_\xi^{\mathbb{P}} Y$ in $\mathbb{D}$.*

Proof. We only prove (9) of the definition of the $\rightsquigarrow_\xi^{\mathbb{P}}$-convergence, the assertion about the asymptotic measurability in (10) follows along similar lines. Observing the estimate

$$\sup_{h \in BL_1(\mathbb{D})} \left|\mathbb{E}_\xi h(Z_n) - \mathbb{E}h(Y)\right| \leq \mathbb{E}_\xi\left[d(Y_n, Z_n)^* \wedge 2\right] + \sup_{h \in BL_1(\mathbb{D})} \left|\mathbb{E}_\xi h(Y_n) - \mathbb{E}h(Y)\right|$$

it suffices to show that $\mathbb{E}_\xi\,[d(Y_n, Z_n)^* \wedge 2]$ converges to 0 in outer probability. Now the random variable $d(Y_n, Z_n)^* \wedge 2$ is uniformly integrable and converges in probability by assumption, hence it also converges in L^1. We finally use Markov's inequality to obtain $\mathbb{E}_\xi[d(Y_n, Z_n)^* \wedge 2] \xrightarrow{\mathbb{P}} 0$, which proves the assertion. $\square$

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