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Some Properties of the Homogeneous Multivariate Pareto (IV) Distribution

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In this paper, several properties of the multivariate Pareto (IV) distribution introduced by B. C. Arnold (*Pareto Distributions*, International Cooperative Publ. House, Fairland, MD) are studied. It is shown that in the multivariate Pareto (IV), if all the inequality parameters are the same, then some properties analogous to those of the multivariate Pareto (II) can still be derived. © 1994 Academic Press, Inc.

1. INTRODUCTION

According to a definition of Arnold (1983), an m-dimensional random vector $\mathbf{X} = (X_1, X_2, ..., X_m)$, is said to have a multivariate Pareto (IV) distribution if its joint survival function is of the form

$$F_{\mathbf{X}}(\mathbf{x}) = \left\{1 + \sum_{i=1}^{m} \left(\frac{x_i - \mu_i}{\sigma_i}\right)^{1/\gamma_i}\right\}^{-\alpha}, \qquad \begin{array}{l} x_i \geqslant \mu_i, \\ i = 1, 2, ..., m. \end{array} \tag{1.1}$$

Write

$$\mathbf{X} \sim \mathrm{MP}^{(m)}(\mathrm{IV})(\boldsymbol{\mu}, \boldsymbol{\sigma}, \boldsymbol{\gamma}, \alpha),$$

where

$$\begin{aligned} \boldsymbol{\mu} &= (\mu_1, \mu_2, ..., \mu_m), & \mu_i &\geqslant 0, \\ \boldsymbol{\sigma} &= (\sigma_1, \sigma_2, ..., \sigma_m), & \sigma_i &> 0, \\ \boldsymbol{\gamma} &= (\gamma_1, \gamma_2, ..., \gamma_m), & \gamma_i &> 0, \end{aligned}$$

are respectively the location, scale, and inequality parameter vectors, and $\alpha > 0$ is the shape parameter.

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Two representations of the multivariate Pareto (IV) random vector $\mathbf{X} = (X_1, X_2, ..., X_m)$ are given in Arnold (1983). One is

$$X_i = \mu_i + \sigma_i (W_i/Z)^{\gamma_i}, \qquad i = 1, 2, ..., m, \tag{1.2}$$

where $\{W_i\}_1^m \overset{\text{i.i.d.}}{\sim} \text{Exp}(1)$ and are independent of $Z \sim \text{Gamma}(\alpha, 1)$.
The other representation of $\mathbf{X}$ is that if for each X_i, $i = 1, 2, ..., m$, there exists $Z \sim \text{Gamma}(\alpha, 1)$, and given $Z = z$,

$$\{X_i |_{Z=z}\}_1^m \overset{\text{independent}}{\sim} \text{Weibull}(\sigma_i/z, 1/\gamma_i). \tag{1.3}$$

In (1.1) if all the $\gamma_i = 1$, $i = 1, 2, ..., m$, then $\mathbf{X}$ has the multivariate Pareto (II) or $\text{MP}^{(m)}(\text{II})$ distribution. Arnold (1983) studied some properties of the $\text{MP}^{(m)}(\text{II})$ distribution. However, it is found that if all the inequality parameters γ_i, $i = 1, 2, ..., m$, in $\text{MP}^{(m)}(\text{IV})$ $(\boldsymbol{\mu}, \boldsymbol{\sigma}, \boldsymbol{\gamma}, \alpha)$ are the same, say $\gamma_i = \gamma$, then some properties analogous to the $\text{MP}^{(m)}(\text{II})$ can still be derived in the $\text{MP}^{(m)}(\text{IV})(\boldsymbol{\mu}, \boldsymbol{\sigma}, \gamma, \alpha)$ distribution.
In the univariate case, that is, when $m = 1$ in (1.1),

$$\bar{F}_X(x) = \left\{1 + \left(\frac{x - \mu}{\sigma}\right)^{1/\gamma}\right\}^{-\alpha}, \qquad x \geqslant \mu, \tag{1.4}$$

and write this as $X \sim \text{P(IV)}(\mu, \sigma, \gamma, \alpha)$.

2. Functions of Standard $\text{MP}^{(m)}(\text{IV})$ Random Vectors

Let $\mathbf{X} = (X_1, X_2, ..., X_m) \sim \text{MP}^{(m)}(\text{IV})(\mathbf{0}, \mathbf{1}, \gamma, \alpha)$. Then according to (1.1), the joint survival is of the form

$$\bar{F}_{\mathbf{X}}(\mathbf{x}) = \left\{1 + \sum_{i=1}^m x_i^{1/\gamma}\right\}^{-\alpha}, \tag{2.1}$$

where $\mathbf{x} = (x_1, x_2, ..., x_m)$, $x_i > 0$, $i = 1, 2, ..., m$.
The corresponding pdf of $\mathbf{X}$ is

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\gamma^m} \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)} \left\{1 + \sum_{i=1}^m x_i^{1/\gamma}\right\}^{-\alpha - m} \left(\prod_{i=1}^m x_i\right)^{1/\gamma - 1}. \tag{2.2}$$

The following two properties deal with the distributions of certain functions of $\mathbf{X}$ which have Feller-Pareto distributions, (denoted by FP distribution). Feller (1971, V.II) defined the FP distribution as

Definition. If $Y \sim \text{Beta}(\lambda_1, \lambda_2)$ and if for μ real, $\sigma > 0$, and $\gamma > 0$, we define $W = \mu + \sigma(Y^{-1} - 1)^{\gamma}$, then W has a Feller–Pareto (FP) distribution and we write $W \sim \mathrm{FP}(\mu, \sigma, \gamma, \lambda_1, \lambda_2)$. The pdf of W is derived from the transformation of Y as

$$f_W(w) = \frac{1}{\beta(\lambda_1, \lambda_2)} \frac{1}{\gamma\sigma} \left(\frac{w-\mu}{\sigma}\right)^{(\lambda_2+1)/\gamma - 2} \left\{1 + \left(\frac{w-\mu}{\sigma}\right)^{1/\gamma}\right\}^{-(\lambda_1+\lambda_2)}, \qquad w \geqslant \mu. \tag{2.3}$$

Arnold (1983, Chap. 3) showed that the FP distribution is a generalization of the P(IV) distribution (and thus of the other Pareto families (I), (II), (III)); i.e., Pareto (IV) distribution can be identified as

$$\mathrm{P(IV)}(\mu, \sigma, \gamma, \alpha) = \mathrm{FP}(\mu, \sigma, \gamma, \alpha, 1).$$

In particular, if we let $\mu = 0$, $\sigma = 1$, $\gamma = 1$ in FP distribution, then $\mathrm{FP}(0, 1, \gamma, \lambda_1, \lambda_2)$ is the so-called scaled-F distribution with pdf

$$f_W(w) = \frac{1}{\beta(\lambda_1, \lambda_2)} w^{\lambda_2 - 1}(1+w)^{-(\lambda_1+\lambda_2)}, \qquad w > 0. \tag{2.4}$$

Now, back to the main properties of this section.

All the proofs of the following properties are obtained by Jacobian transformations and are straightforward. Hence they are stated without proof.

PROPERTY 2.1. *Let* $\mathbf{X} = (X_1, X_2, ..., X_m) \sim \mathrm{MP}^{(m)}(\mathrm{IV})(\mathbf{0}, \mathbf{1}, \gamma, \alpha)$. *If we define* $Y = \sum_{i=1}^m X_i^{1/\gamma}$, *then the distribution of* Y *is a scaled-F distribution, i.e.,*

$$Y \sim \mathrm{FP}(0, 1, 1, \alpha, m).$$

COROLLARY. *Let* $\mathbf{X} = (X_1, X_2, ..., X_m) \sim \mathrm{MP}^{(m)}(\mathrm{IV})(\mathbf{0}, \mathbf{1}, \gamma, \alpha)$, *and define the sum of any* $Y = \sum_{i=1}^l X_i^{1/\gamma}$, $1 \leqslant l \leqslant m$. Then $Y \sim \mathrm{FP}(0, 1, 1, \alpha, l)$.

PROPERTY 2.2. *Let* $\mathbf{X} = (X_1, X_2, ..., X_m) \sim \mathrm{MP}^{(m)}(\mathrm{IV})(\mathbf{0}, \mathbf{1}, \gamma, \alpha)$. *Define*

$$Y_j = \frac{X_j^{1/\gamma}}{1 + X_1^{1/\gamma} + X_2^{1/\gamma} + \cdots + X_{j-1}^{1/\gamma}}, \tag{2.5}$$

for $j = 1, 2, ..., m$. *Then*

(i) *for each* $j = 1, 2, ..., m$,

$$Y_j = \mathrm{FP}(0, 1, 1, \alpha + j - 1, 1)$$

(ii) $\{Y_1, Y_2, ..., Y_m\}$ *are statistically independent.*

The converse of Property 2.2 is also true, i.e.,

PROPERTY 2.3. *Suppose for each* $j = 1, 2, \dots, m$, $Y_j \overset{\text{independent}}{\sim} \mathrm{FP}(0, 1, 1, \alpha + j - 1, 1)$. *Let* $\gamma > 0$ *be any fixed inequality parameter and define* $X_1 = Y_1^\gamma$,

$$X_j = Y_j^\gamma (1 + Y_{j-1})^\gamma \cdots (1 + Y_1)^\gamma, \qquad j = 2, \dots, m.$$

Then it follows

$$\mathbf{X} = (X_1, X_2, \dots, X_m) \sim \mathrm{MP}^{(m)}(\mathrm{IV})(\mathbf{0}, \mathbf{1}, \gamma, \alpha).$$

PROPERTY 2.4. *Let* $\mathbf{V} = (V_1, V_2, \dots, V_m, V_{m+1})$ *be an* $(m+1)$ *variate Dirichlet*

$$\begin{pmatrix} 1, 1, \dots, 1, \alpha; 1 \\ m\text{'s} \end{pmatrix}.$$

Given $\gamma > 0$, *define* $X_i = (V_i / V_{m+1})^\gamma$ *for* $i = 1, 2, \dots, m$. *Then* $\mathbf{X} = (X_1, X_2, \dots, X_m) \sim \mathrm{MP}^{(m)}(\mathrm{IV})(\mathbf{0}, \mathbf{1}, \gamma, \alpha)$.

Property 2.3 and Property 2.4 can be treated as two different approaches to constructing the standard $\mathrm{MP}^{(m)}(\mathrm{IV})(\mathbf{0}, \mathbf{1}, \gamma, \alpha)$ via independent FP distributions and via a Dirichlet random vector, respectively.

3. DISTRIBUTIONS OF SOME RATIOS AND ORDERED COORDINATES IN THE $\mathrm{MP}^{(m)}(\mathrm{IV})$ RANDOM VECTORS

PROPERTY 3.1. *If* $\mathbf{X} = (X_1, X_2, \dots, X_m) \sim \mathrm{MP}^{(m)}(\mathrm{IV})(\mathbf{0}, \boldsymbol{\sigma}, \gamma, \alpha)$, *then for any* $i, j \in \{1, 2, \dots, m\}$, $i \neq j$, *the ratio* $X_i / X_j \sim \mathrm{P}(\mathrm{IV})(0, \sigma_i/\sigma_j, \gamma, 1)$.

The proof follows easily using Eq. (1.2), $X_i = \sigma_i (W_i / Z)^{\gamma_i}$, $i = 1, 2, \dots, m$, and hence is omitted.

In the following, consider the ordered coordinates of an $\mathrm{MP}^{(m)}(\mathrm{IV})$ random vector.

Let $\mathbf{X} = (X_1, X_2, \dots, X_m) \sim \mathrm{MP}^{(m)}(\mathrm{IV})(\mathbf{0}, \boldsymbol{\sigma}, \gamma, \alpha)$. Although the X_i's, $i = 1, 2, \dots, m$, are dependent, it still makes sense to order them as

$$X_{1:m} \leqslant X_{2:m} \leqslant \cdots \leqslant X_{m-1:m} \leqslant X_{m:m}.$$

PROPERTY 3.2. *If* $\mathbf{X} \sim \mathrm{MP}^{(m)}(\mathrm{IV})(\mathbf{0}, \boldsymbol{\sigma}, \gamma, \alpha)$, *then*

$$X_{1:m} \sim \mathrm{P}(\mathrm{IV}) \left(0, \left(\frac{1}{\sum_{i=1}^m (1/\sigma_i)^{1/\gamma}} \right)^\gamma, \gamma, \alpha \right).$$

The proof is straightforward by (1.1) and (1.4).

As for the higher ordered coordinates, $X_{i:m}$, in $\mathbf{X}$, $i=1, 2, ..., m$, if one assumes scale homogeneity, i.e., $\sigma_i = 1$, $i=1, 2, ..., m$, then the distribution of $X_{i:m}$ is obtainable.

PROPERTY 3.3. *Let* $\mathbf{X}=(X_1, X_2, ..., X_m) \sim \mathrm{MP}^{(m)}(\mathrm{IV})(\mathbf{0}, \mathbf{1}, \gamma, \alpha)$, *then for any* $i=1, 2, ..., m$, *the survival function of the ith ordered coordinate* $X_{i:m}$ *is for* $u>0$

$$\bar{F}_{X_{i:m}}(u) = (-1)^{i-1} \sum_{l=1}^{i} \left\{\prod \left(\frac{m-h+1}{h-l}\right)\right\} \{1+(m-l+1)\, u^{1/\gamma}\}^{-\alpha},$$
$$h \in \{1, 2, ..., i\} - \{l\}.$$

Proof. Refer to Eq. (1.2), where each $X_i = (W_i/Z)^{\gamma}$ and where $\{W_i\}_1^m$ and Z are as defined in Section 1. The technique used to prove this property is to express $X_{i:m}^{1/\gamma}$ as a linear function of the spacings

$$X_{i:m}^{1/\gamma} = \sum_{j=1}^{i} (X_{j:m}^{1/\gamma} - X_{j-1:m}^{1/\gamma}) = \sum_{j=1}^{i} \left(\frac{W_{j:m}}{Z} - \frac{W_{j-1:m}}{Z}\right), \tag{3.1}$$

where $W_{i:m}$ is the ith order statistic of the $\{W_i\}_1^m \overset{\text{i.i.d.}}{\sim} \mathrm{Exp}(1)$ with $W_{0:m}=0$. Given $Z=z$, the scaled random variables $\{W_i/z\}_1^m \overset{\text{i.i.d.}}{\sim} \mathrm{Exp}(1/z)$. It follows that $X_{i:m}^{1/\gamma}$ is distributed as a sum of i's independent $\mathrm{Exp}(1/(m-i+1)\, z)$ variables. Here two consequences of the well-known results regarding Exponential order statistics (Sukhatme, 1937) and the sum of independent Exponential variables (Feller, 1971, V.II, p. 40) are referred. Applying these two results, then the conditional pdf of $X_{i:m}^{1/\gamma}$ given $Z=z$ takes the form

$$f_{X_{i:m}^{1/\gamma} \mid z}(x) = \left\{\begin{array}{ll} \dfrac{m!}{(m-i)!} \displaystyle\sum_{l=1}^{i} [(-1)^{i-l}(l-1)!\,(i-l)!]^{-1}\, z e^{-(m-l+1)zx}, & \text{if} \quad x>0 \\ 0 & \text{o.w.} \end{array}\right\}. \tag{3.2}$$

The conditional survival function of $X_{i:m}$ given $Z=z$ is for any $u>0$,

$$\begin{aligned} P(X_{i:m}>u \mid Z=z) &= P(X_{i:m}^{1/\gamma} > u^{1/\gamma} \mid Z=z) \\ &= \int_{u^{1/\gamma}}^{\infty} f_{X_{i:m}^{1/\gamma} \mid z}(x)\, dx \\ &= (-1)^{i-1} \sum_{l=1}^{i} \left\{\prod \left(\frac{m-h+1}{h-l}\right)\right\} e^{-(m-l+1)zu^{1/\gamma}}, \\ & \qquad h \in \{1, 2, ..., i\} - \{l\} \qquad (\text{i.e., } h \neq l). \end{aligned} \tag{3.3}$$

Since $Z \sim \Gamma(\alpha, 1)$, the survival function of $X_{i:m}$ is for $u > 0$

$$\bar{F}_{X_{i:m}}(u) = \int_0^\infty P(X_{i:m} > u \mid Z = z) f_Z(z)\, dz$$

$$= (-1)^{i-1} \sum_{l=1}^{i} \left\{ \prod \left(\frac{m-h+1}{h-l} \right) \right\} \{ m-l+1)\, u^{1/\gamma} + 1\}^{-\alpha},$$

$$h \in \{1, 2, ..., i\} - \{l\}. \tag{3.4}$$

The following property deals with the joint distribution of any ordered coordinates $X_{k_1:m}$ and $X_{k_2:m}$ in $\mathbf{X}$ $(1 \leqslant k_1 < k_2 \leqslant m)$ is terms of their ratio, $X_{k_2:m}/X_{k_1:m}$.

PROPERTY 3.4. *Let* $\mathbf{X} = (X_1, X_2, ..., X_m) \sim \mathrm{MP}^{(m)}(\mathrm{IV})(\mathbf{0}, \mathbf{1}, \gamma, \alpha)$, *given any pair* (k_1, k_2), $(1 \leqslant k_1 < k_2 \leqslant m)$, *the survival function of* $X_{k_2:m}/X_{k_1:m}$ *is for any* $u > 0$

$$P\left(\frac{X_{k_2:m}}{X_{k_1:m}} > u \right)$$

$$= \frac{m!}{(m-k_1)!} \left(\sum_{l=1}^{k_2-k_1} \left\{ \prod_{h \in \{1, 2, ..., k_2-k_1\} - \{l\}} \left(\frac{m-k_1-h+1}{h-l} \right) \right\} \right)$$

$$\cdot \left(\sum_{j=1}^{k_1} (-1)^{k_2-j} \{ (j-1)!\, (k_1-1)!\, (k_1+l-j) \}^{-1} \right.$$

$$\left. \cdot \left\{ 1 + \left(\frac{m-k_1-l+1}{k_1+l-j} \right) u^{1/\gamma} \right\}^{-1} \right).$$

Proof. Referring to Eq. (1.2), each X_i, $i = 1, 2, ..., m$, can be expressed as $X_i = (W_i/Z)^\gamma$, so $X_{k_2:m}/X_{k_1:m} = (X_{k_2:m}/Z)^\gamma/(X_{k_1:m}/Z)^\gamma = (W_{k_2:m}/W_{k_1:m})^\gamma$, hence

$$\left(\frac{X_{k_2:m}}{X_{k_1:m}} \right)^{1/\gamma} = 1 + \frac{\sum_{j=k_1+1}^{k_2} (W_{j:m} - W_{j-1:m})}{W_{k_1:m}},$$

where $W_{k_1:m}$ and $(W_{j:m} - W_{j-1:m})$ are the k_1th order statistics and the spacings of the i.i.d. Exp(1) random variables.

The survival function of $X_{k_2:m}/X_{k_1:m}$ is for $u > 1$

$$P\left(\left(\frac{X_{k_2:m}}{X_{k_1:m}} \right) > u \right) = P\left(\frac{W_{k_2:m}}{W_{k_1:m}} > u^{1/\gamma} \right)$$

$$= \int_0^\infty P\left(\frac{W_{k_2:m}}{W_{k_1:m}} > u^{1/\gamma} \mid W_{k_1:m} = x \right) f_{W_{k_1:m}}(x)\, dx. \tag{3.5}$$

A theorem proved by David (1970, p. 18) is applied, given $W_{k_1:m}=x$,

$$W_{k_2:m} \mid_{W_{k_1:m}=x} = W_{(k_2-k_1):(m-k_1)} = \sum_{j=1}^{k_2-k_1} \{W_{j:(m-k_1)} - W_{(j-1):(m-k_1)}\}.$$

Here, a result of Sukhatme (1937) is applied, and the spacings are independent $\operatorname{Exp}(1/((m-k_1)-j+1))$ random variables truncated from the left at x.

The pdf of $W_{k_2:m} \mid_{W_{k_1:m}=x}$ is calculated by applying a result of Feller (1971) and (3.2). For any $v \geqslant x$,

$$f_{W_{k_2:m} \mid W_{k_1:m}=x}(v \mid x)$$

$$= \begin{cases} \dfrac{(m-k_1)!}{(m-k_1-(k_2-k_1))!} \displaystyle\sum_{l=1}^{k_2-k_1} \{(-1)^{k_2-k_1-l}(l-1)!\,(k_2-k_1-l)!\}^{-1} & \\ \quad \cdot e^{-(m-k_1-l+1)(v-x)}, & \text{if } v \geqslant x \\ 0 & \text{o.w.} \end{cases}.$$

Then in (3.5), the conditional survival function is

$$P(W_{k_2:m} > u^{1/\gamma}x \mid W_{k_1:m}=x)$$

$$= \int_{u^{1/\gamma}x}^{\infty} f_{W_{k_2:m} \mid W_{k_1:m}=x}(v \mid x)\, dv$$

$$= (-1)^{k_2-k_1} \sum_{l=1}^{k_2-k_1} \left\{\prod \left(\frac{m-k_1-h+1}{h-l}\right)\right\} e^{-(m-k_1-l+1)(u^{1/\gamma}-1)x},$$

$$h \in \{1, 2, \ldots, k_2-k_1\} - \{l\} \qquad \text{for} \quad u > 1$$

and hence (3.5) becomes

$$(-1)^{k_2-k_1} \sum_{l=1}^{k_2-k_1} \left\{ \prod_{h \in \{1, 2, \ldots, k_2-k_1\} - \{l\}} \left(\frac{m-k_1-h+1}{h-l}\right)\right\}$$

$$\cdot \int_0^{\infty} e^{-(m-k_1-l+1)(u^{1/\gamma}-1)x} f_{W_{k_1:m}}(x)\, dx,$$

where $f_{W_{k_1:m}}(x)$ is the pdf of the Exponential order statistic $W_{k_1:m}$. Then the survival function of $X_{k_2:m}/X_{k_1:m}$ is

$$P\left(\frac{X_{k_2:m}}{X_{k_1:m}}>u\right)$$
$$=\frac{m!}{(m-k_1)!}\left(\sum_{l=1}^{k_2-k_1}\left\{\prod_{h\in\{1,2,\ldots,k_2-k_1\}-\{l\}}\left(\frac{m-k_1-h+1}{h-l}\right)\right\}\right)$$
$$\cdot\left(\sum_{j=1}^{k_1}(-1)^{k_2-j}\{(j-1)!\,(k_1-1)!\,(k_1+l-j)\}^{-1}\right.$$
$$\left.\cdot\left\{1+\left(\frac{m-k_1-l+1}{k_1+l-j}\right)u^{1/\gamma}\right\}^{-1}\right). \tag{3.6}$$

The following property concerns the relation between the standard $MP^{(m)}(IV)(\mathbf{0}, \mathbf{1}, \gamma, \alpha)$ distribution and its corresponding scaled spacings.

PROPERTY 3.5. *Let* $\mathbf{X}=(X_1, X_2, \ldots, X_m) \sim MP^{(m)}(IV)(\mathbf{0}, \mathbf{1}, \gamma, \alpha)$. *Define* $S_i=(m-i+1)(X_{i:m}^{1/\gamma}-X_{i-1:m}^{1/\gamma})$ *as its associated scaled spacings. Then the random vector* $\mathbf{S}=(S_1, S_2, \ldots, S_m) \sim MP^{(m)}(II)(\mathbf{0}, \mathbf{1}, \alpha)$.

Proof. Following the same notation as in the proof of Property 3.3, we have

$$X_{i:m}^{1/\gamma}-X_{i-1:m}^{1/\gamma}\,|_{Z=z} \stackrel{d}{=} \frac{W_{i:m}}{z}-\frac{W_{i-1:m}}{z} \overset{\text{independent}}{\sim} \text{Exp}\left(\frac{1}{(m-i+1)\,z}\right),$$

for each $i=1, 2, \ldots, m$.

Hence, given $Z=z$, $S_i|_{Z=z} \overset{\text{independent}}{\sim} \text{Exp}(1/z)$, then the joint conditional survival function of $(S_1, S_2, \ldots, S_m \mid Z=z)$ is for $x_i>0$, $i=1, 2, \ldots, m$,

$$P(S_1>x_1, S_2>x_2, \ldots, S_m>x_m \mid Z=z)=e^{-z\sum_{i=1}^m x_i}.$$

Since $Z \sim \Gamma(\alpha, 1)$, then unconditionally, the joint survival function of $\mathbf{S}$ is

$$\bar{F}_{\mathbf{S}}(\mathbf{x})=\int_0^\infty e^{-z\sum_{i=1}^m x_i}\frac{1}{\Gamma(\alpha)}z^{\alpha-1}e^{-z}\,dz=\left(1+\sum_{i=1}^m x_i\right)^{-\alpha} \sim MP^{(m)}(II)(\mathbf{0}, \mathbf{1}, \alpha).$$

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