



Dependence properties and bounds for ruin probabilities in multivariate compound risk models

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Abstract

In risk management, ignoring the dependence among various types of claims often results in over-estimating or under-estimating the ruin probabilities of a portfolio. This paper focuses on three commonly used ruin probabilities in multivariate compound risk models, and using the comparison methods shows how some ruin probabilities increase, whereas the others decrease, as the claim dependence grows. The paper also presents some computable bounds for these ruin probabilities, which can be calculated explicitly for multivariate phase-type distributed claims, and illustrates the performance of these bounds for the multivariate compound Poisson risk models with slightly or highly dependent Marshall–Olkin exponential claim sizes.

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1. Introduction

Consider an insurance or investment portfolio that consists of m sub-portfolios. The claim events occur according to a point process, and each event yields several types of claims, one for each sub-portfolio, that are usually stochastically dependent. Let $N(t)$ denote the number of claim events by time $t > 0$, and $X_{j,n}$ the type j claim size of the n th event, $1 \leq j \leq m$, $n \geq 1$.

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The multivariate claim surplus process of the m sub-portfolios is described by

$$\mathbf{S}(t) = \begin{pmatrix} S_1(t) \\ \vdots \\ S_m(t) \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{N(t)} X_{1,n} - p_1 t \\ \vdots \\ \sum_{n=1}^{N(t)} X_{m,n} - p_m t \end{pmatrix}, \quad t \geq 0, \quad (1.1)$$

where $p_j > 0$ is the premium rate in sub-portfolio j or for type j claim, $j = 1, \dots, m$. We assume throughout that $\{(X_{1,n}, \dots, X_{m,n}), n \geq 1\}$ is a sequence of i.i.d. non-negative random vectors, which is independent of $\{N(t), t \geq 0\}$, but allow $X_{1,n}, \dots, X_{m,n}$ to be dependent. We also assume that $\{N(t), t \geq 0\}$ is a point process and $N(t) < \infty$ almost surely for any fixed $t > 0$.

Let $u_j \geq 0$, $j = 1, \dots, m$, denote the initial capital in sub-portfolio j for such a multivariate compound risk model. A ruin event occurs if the claim surpluses of some sub-portfolios exceed, in certain fashion, their corresponding initial capital reserves. Various ruin probabilities in multivariate risk models are often of fundamental interest in risk management. For example, consider the following three ruin probabilities.

$$\psi_{\text{and}}(u_1, \dots, u_m) = P \left(\bigcap_{j=1}^m \left\{ \sup_{0 \leq t < \infty} (S_j(t)) > u_j \right\} \right), \quad (1.2)$$

$$\begin{aligned} \psi_{\text{or}}(u_1, \dots, u_m) &= P \left(\bigcup_{j=1}^m \left\{ \sup_{0 \leq t < \infty} (S_j(t)) > u_j \right\} \right) \\ &= P \left(\sup_{0 \leq t < \infty} (\max\{S_1(t) - u_1, \dots, S_m(t) - u_m\}) > 0 \right), \end{aligned} \quad (1.3)$$

$$\begin{aligned} \psi_{\text{sim}}(u_1, \dots, u_m) &= P(S_1(t) > u_1, \dots, S_m(t) > u_m \text{ for some } t > 0) \\ &= P \left(\sup_{0 \leq t < \infty} (\min\{S_1(t) - u_1, \dots, S_m(t) - u_m\}) > 0 \right). \end{aligned} \quad (1.4)$$

The ruin probability in (1.2) denotes the probability that ruin occurs, not necessarily at the same time, in all sub-portfolios eventually, whereas the ruin probability in (1.4) denotes the probability that ruin occurs in all sub-portfolios simultaneously or at the same instant in time. The ruin probability in (1.3) represents the probability that ruin occurs in at least one sub-portfolio. The focus of this paper is on these ruin probabilities for the multivariate compound risk models.

In general, these ruin probabilities are intractable. As a matter of fact, even in the univariate case that $m = 1$, it is often difficult to obtain the explicit formula for its ruin probability $\psi(u)$, which can be expressed in terms of (1.2), (1.3), or (1.4) as follows:

$$\psi(u) = \psi_{\text{and}}(u) = \psi_{\text{or}}(u) = \psi_{\text{sim}}(u) = P \left(\sup_{0 \leq t < \infty} S_1(t) > u \right), \quad (1.5)$$

where $u \geq 0$ is the initial capital. A well-known result in the univariate case is due to Asmussen and Rolski [1] who gave an explicit formula of $\psi(u)$ for the compound Poisson risk model when the counting process $N(t)$ is Poisson with rate λ , and the claim size is of phase type in the sense of Neuts [11]. A non-negative random variable X is said to be of phase type with representation

(α, T, d) if X is the time to absorption into the absorbing state 0 in a finite Markov chain with state space $\{0, 1, \dots, d\}$ and initial distribution $(0, \alpha)$, and infinitesimal generator,

$$\begin{bmatrix} 0 & \mathbf{0} \\ -T\mathbf{e} & T \end{bmatrix},$$

where $\mathbf{0}$ is the row vector of zeros of d dimension, and $\mathbf{e}$ is the column vector of 1's, and T is a $d \times d$ non-singular matrix. For a compound Poisson risk model with the relative security loading parameter $\theta = \frac{p_1}{E(X_{1,n})\lambda} - 1 > 0$, if the claim size is of phase type with representation (α, T, d) , then $\psi(u)$ in (1.5) is the tail probability of the stationary waiting time in the $M/PH/1$ queue. Utilizing this fact, Asmussen and Rolski [1] showed that for any $u \geq 0$,

$$\psi(u) = -\frac{\lambda}{p_1} \alpha T^{-1} \exp \left\{ \left(T - \frac{\lambda}{p_1} t_0 \alpha T^{-1} \right) u \right\} \mathbf{e}, \quad (1.6)$$

where $t_0 = -T\mathbf{e}$. The phase-type distributions enjoy many desirable properties [11], and in particular, any distribution on $[0, \infty)$ can be approximated by phase-type distributions. Thus, (1.6) is versatile in applications.

For the multivariate compound Poisson risk models, Sundt [13] studied a recursive approach for the evaluation of the distribution of the multivariate aggregate claim process $\left(\sum_{n=1}^{N(t)} X_{1,n}, \dots, \sum_{n=1}^{N(t)} X_{m,n} \right)$. Chan et al. [5] discussed the ruin probability of the aggregate claim, $\psi_{\text{or}}(u_1, \dots, u_m)$ and $\psi_{\text{sim}}(u_1, \dots, u_m)$ for the case where the claim sizes $X_{1,n}, \dots, X_{m,n}$ are independent for any $n \geq 1$. Cai and Li [3] established the lower bound of $\psi_{\text{and}}(u_1, \dots, u_m)$ for the positively associated claims, and obtained an explicit expression of the ruin probability for the aggregate claim in a multivariate compound Poisson risk model whose claims of various types follow a multivariate phase (MPH)-type distribution. In general, however, the properties and expressions of the multivariate ruin probabilities are largely unknown. In this paper, we investigate the behavior of the ruin probabilities (1.2)–(1.4) with respect to the dependence structure of the claim sizes, and establish the sharp upper and lower bounds of (1.2)–(1.4) whose explicit expressions are intractable even in the simplest cases, such as multivariate compound Poisson risk models with multivariate exponentially distributed claims.

In Section 2, we utilize the supermodular comparison method to obtain the dependence comparisons of (1.2)–(1.4). Our results show that both $\psi_{\text{and}}(u_1, \dots, u_m)$ and $\psi_{\text{sim}}(u_1, \dots, u_m)$ increase, whereas $\psi_{\text{or}}(u_1, \dots, u_m)$ decreases, as the dependence among various types of claims grows. This further illustrates the fact that ignoring the dependence among various types of claims often results in over-estimating or under-estimating the portfolio ruin probabilities. In Section 3, we develop the bounds for the ruin probabilities (1.2)–(1.4). In Section 4 we obtain the explicit expressions of these bounds for the multivariate compound Poisson risk models whose claims of various types follow a MPH-type distribution, and also present some illustrative examples. Our bounds incorporate the dependence structure of various types of claims, and as the dependence grows, the upper and lower bounds collapse in the sense that the upper bound becomes smaller and the lower bound becomes larger.

Throughout this paper, the term ‘increasing’ and ‘decreasing’ mean ‘non-decreasing’ and ‘non-increasing’, respectively, and the measurability of sets and functions as well as the existence of expectations are often assumed without explicit mention. Any inequality between two vectors with finite or infinite dimensions means the inequalities component-wise.

2. Dependence properties of multivariate ruin probabilities

To compare the dependence of random vectors, we fix the marginal distributions, and compare their joint distributions in some sense. There are several dependence comparison methods and we here utilize the *supermodular comparison*.

Definition 2.1. Let $\mathbf{X} = (X_1, \dots, X_m)$ and $\mathbf{Y} = (Y_1, \dots, Y_m)$ be two $\mathcal{R}^m$ -valued random vectors.

1. $\mathbf{X}$ is said to be larger than $\mathbf{Y}$ in stochastic order, denoted by $\mathbf{X} \geq_{\text{st}} \mathbf{Y}$, if $Ef(\mathbf{X}) \geq Ef(\mathbf{Y})$ for all increasing functions f .
2. $\mathbf{X}$ is said to be more dependent than $\mathbf{Y}$ in supermodular order, denoted by $\mathbf{X} \geq_{\text{sm}} \mathbf{Y}$, if $Ef(\mathbf{X}) \geq Ef(\mathbf{Y})$ for all supermodular functions f ; that is, functions satisfying that for all $\mathbf{x}, \mathbf{y} \in \mathcal{R}^m$,

$$f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y}) \geq f(\mathbf{x}) + f(\mathbf{y}),$$

where $\mathbf{x} \vee \mathbf{y}$ denotes the vector of component-wise maximums, and $\mathbf{x} \wedge \mathbf{y}$ denotes the vector of component-wise minimums.

The stochastic orders have many useful properties and applications, and are studied in details in [9,12,10], and references therein. The following properties are frequently used in this and the next sections.

Lemma 2.2. Let $\mathbf{X} = (X_1, \dots, X_m)$ and $\mathbf{Y} = (Y_1, \dots, Y_m)$ be two $\mathcal{R}^m$ -valued random vectors such that $\mathbf{X} \geq_{\text{sm}} \mathbf{Y}$.

1. $(f_1(X_1), \dots, f_m(X_m)) \geq_{\text{sm}} (f_1(Y_1), \dots, f_m(Y_m))$ for any functions $f_1, \dots, f_m$ that are all increasing or all decreasing.
2. X_j and Y_j have the same distribution for any $j = 1, \dots, m$, and

$$P(X_1 > x_1, \dots, X_m > x_m) \geq P(Y_1 > x_1, \dots, Y_m > x_m), \quad (2.1)$$

$$P(X_1 \leq x_1, \dots, X_m \leq x_m) \geq P(Y_1 \leq x_1, \dots, Y_m \leq x_m), \quad (2.2)$$

for any $(x_1, \dots, x_m)$.

Thus, if $\mathbf{X} \geq_{\text{sm}} \mathbf{Y}$, then $\text{Cov}(X_i, X_j) \geq \text{Cov}(Y_i, Y_j)$ for any $i \neq j$.

Consider two multivariate compound risk models $\mathcal{M}_1$ and $\mathcal{M}_2$ introduced in Section 1. To compare the effect of the dependence of claim sizes on the ruin probabilities, we suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same claim event arrival process $\{N(t), t \geq 0\}$, same premium rates $p_j, 1 \leq j \leq m$, and same initial reserves $u_j, 1 \leq j \leq m$, but different claim size vectors $\mathbf{X}_n = (X_{1,n}, \dots, X_{m,n})$ and $\mathbf{Y}_n = (Y_{1,n}, \dots, Y_{m,n})$, respectively. Let $\psi_{\text{and}}^X(u_1, \dots, u_m)$ ($\psi_{\text{and}}^Y(u_1, \dots, u_m)$), $\psi_{\text{or}}^X(u_1, \dots, u_m)$ ($\psi_{\text{or}}^Y(u_1, \dots, u_m)$), and $\psi_{\text{sim}}^X(u_1, \dots, u_m)$ ($\psi_{\text{sim}}^Y(u_1, \dots, u_m)$) denote the ruin probabilities of types (1.2)–(1.4), respectively, in model $\mathcal{M}_1$ ($\mathcal{M}_2$).

Theorem 2.3. If $\mathbf{X}_n \geq_{\text{sm}} \mathbf{Y}_n$, then we have, for any non-negative $u_1, \dots, u_m$,

1. $\psi_{\text{and}}^X(u_1, \dots, u_m) \geq \psi_{\text{and}}^Y(u_1, \dots, u_m)$,
2. $\psi_{\text{or}}^X(u_1, \dots, u_m) \leq \psi_{\text{or}}^Y(u_1, \dots, u_m)$, and
3. $\psi_{\text{sim}}^X(u_1, \dots, u_m) \geq \psi_{\text{sim}}^Y(u_1, \dots, u_m)$.

Proof. (1) It suffices to show that given that $N(t) = n(t)$, $t \geq 0$,

$$\psi_{\text{and}}^X(u_1, \dots, u_m) \geq \psi_{\text{and}}^Y(u_1, \dots, u_m). \quad (2.3)$$

Without loss of generality, we assume that $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are independent. For fixed positive integer k , let

$$Z_n = Y_n, \quad n = 1, \dots, k,$$

$$Z_n = X_n, \quad n > k.$$

Let $\psi_k^Z(u_1, \dots, u_m)$ denote the ruin probabilities of type (1.2) in the multivariate compound risk model with the claim event arrival process $N(t)$, premium rates p_j , $1 \leq j \leq m$, initial reserves u_j , $1 \leq j \leq m$, and claim size vectors $\{Z_n, n \geq 1\}$. Also let

$$S_j^X(t) = \sum_{n=1}^{N(t)} X_{j,n} - p_j t, \quad j = 1, \dots, m,$$

$$S_j^Y(t) = \sum_{n=1}^{N(t)} Y_{j,n} - p_j t, \quad j = 1, \dots, m,$$

$$S_j(t) = \sum_{n=1}^{N(t)} Z_{j,n} - p_j t, \quad j = 1, \dots, m.$$

Conditioning on $X_n = \mathbf{x}_n$, $n > k$, $\sup_{0 \leq t < \infty} S_j^X(t)$ is an increasing function of $X_{j,1}, \dots, X_{j,k}$, and $\sup_{0 \leq t < \infty} S_j(t)$ is an increasing function of $Y_{j,1}, \dots, Y_{j,k}$, $1 \leq j \leq m$. Since $X_1, \dots, X_k$ are i.i.d., and $Y_1, \dots, Y_k$ are i.i.d., and $X_n \geq_{\text{sm}} Y_n$, we invoke Lemma 2.2(1) k times, and obtain that conditioning on $X_n = \mathbf{x}_n$, $n > k$,

$$\left(\sup_{0 \leq t < \infty} S_1^X(t), \dots, \sup_{0 \leq t < \infty} S_m^X(t) \right) \geq_{\text{sm}} \left(\sup_{0 \leq t < \infty} S_1(t), \dots, \sup_{0 \leq t < \infty} S_m(t) \right).$$

It follows from unconditioning and (2.1) that for any k ,

$$\psi_{\text{and}}^X(u_1, \dots, u_m) \geq \psi_k^Z(u_1, \dots, u_m).$$

Observe that as $k \rightarrow \infty$, $\psi_k^Z(u_1, \dots, u_m)$ converges to $\psi_{\text{and}}^Y(u_1, \dots, u_m)$ for any $u_1, \dots, u_m$. Thus, we establish (2.3) conditioning on $N(t) = n(t)$, $t \geq 0$.

(2) Using a similar idea as in (1) (using (2.2), instead of (2.1)), we can also show that

$$\begin{aligned} & P \left(\sup_{0 \leq t < \infty} S_1^X(t) \leq u_1, \dots, \sup_{0 \leq t < \infty} S_m^X(t) \leq u_1 \right) \\ & \geq P \left(\sup_{0 \leq t < \infty} S_1^Y(t) \leq u_1, \dots, \sup_{0 \leq t < \infty} S_m^Y(t) \leq u_1 \right). \end{aligned}$$

Therefore,

$$\begin{aligned}\psi_{\text{or}}^X(u_1, \dots, u_m) &= 1 - P\left(\sup_{0 \leq t < \infty} S_1^X(t) \leq u_1, \dots, \sup_{0 \leq t < \infty} S_m^X(t) \leq u_1\right) \\ &\leq 1 - P\left(\sup_{0 \leq t < \infty} S_1^Y(t) \leq u_1, \dots, \sup_{0 \leq t < \infty} S_m^Y(t) \leq u_1\right) \\ &= \psi_{\text{or}}^Y(u_1, \dots, u_m).\end{aligned}$$

(3) Notice that $\psi_{\text{sim}}(u_1, \dots, u_m)$ is the probability that ruin occurs at all the sub-portfolios at the *same time*, and unlike (1.2) and (1.3), is not a separate functional of the claim surplus processes of these sub-portfolios. Thus, in this case, we need some extra work.

Let

$$\bar{S}_j^X(t) = S_j^X(t) - u_j, \quad \bar{S}_j^Y(t) = S_j^Y(t) - u_j, \quad 1 \leq j \leq m.$$

Also let

$$\bar{S}_{(1)}^X(t) = \min \left\{ \bar{S}_1^X(t), \dots, \bar{S}_m^X(t) \right\},$$

$$\bar{S}_{(1)}^Y(t) = \min \left\{ \bar{S}_1^Y(t), \dots, \bar{S}_m^Y(t) \right\}.$$

Since

$$\psi_{\text{sim}}^X(u_1, \dots, u_m) = 1 - P\left(\sup_{0 \leq t < \infty} \bar{S}_{(1)}^X(t) \leq 0\right) = 1 - P(\bar{S}_{(1)}^X(t) \leq 0 \text{ for all } t \geq 0),$$

$$\psi_{\text{sim}}^Y(u_1, \dots, u_m) = 1 - P\left(\sup_{0 \leq t < \infty} \bar{S}_{(1)}^Y(t) \leq 0\right) = 1 - P(\bar{S}_{(1)}^Y(t) \leq 0 \text{ for all } t \geq 0),$$

we need to show that

$$P(\bar{S}_{(1)}^X(t) \leq 0 \text{ for all } t \geq 0) \leq P(\bar{S}_{(1)}^Y(t) \leq 0 \text{ for all } t \geq 0).$$

Since the sample paths of the counting process $\{N(t), t \geq 0\}$ are right-continuous with left-limits, it suffices to show that for any $0 \leq t_1 \leq t_2 \leq \dots \leq t_l < \infty$,

$$P(\bar{S}_{(1)}^X(t_1) \leq 0, \dots, \bar{S}_{(1)}^X(t_l) \leq 0) \leq P(\bar{S}_{(1)}^Y(t_1) \leq 0, \dots, \bar{S}_{(1)}^Y(t_l) \leq 0),$$

which can be rephrased as

$$P\left(\bigcap_{i=1}^l \bigcup_{j=1}^m \{\bar{S}_j^X(t_i) \leq 0\}\right) \leq P\left(\bigcap_{i=1}^l \bigcup_{j=1}^m \{\bar{S}_j^Y(t_i) \leq 0\}\right). \quad (2.4)$$

We first observe that for any real numbers $a_1, \dots, a_l$ and any n , we have,

$$\begin{aligned} P\left(\bigcap_{i=1}^l \bigcup_{j=1}^m \{X_{j,n} \leq a_i\}\right) &= P\left(\bigcap_{i=1}^l \{\min\{X_{1,n}, \dots, X_{m,n}\} \leq a_i\}\right) \\ &= P(\min\{X_{1,n}, \dots, X_{m,n}\} \leq \min\{a_1, \dots, a_l\}) \\ &= 1 - P(X_{1,n} > \min\{a_1, \dots, a_l\}, \dots, X_{m,n} > \min\{a_1, \dots, a_l\}) \\ &\leq 1 - P(Y_{1,n} > \min\{a_1, \dots, a_l\}, \dots, Y_{m,n} > \min\{a_1, \dots, a_l\}) \\ &= P\left(\bigcap_{i=1}^l \bigcup_{j=1}^m \{Y_{j,n} \leq a_i\}\right), \end{aligned}$$

where the inequality follows from (2.1). Thus, for any strictly increasing functions $g_1, \dots, g_l$ and any n , we have

$$\begin{aligned} P\left(\bigcap_{i=1}^l \bigcup_{j=1}^m \{g_i(X_{j,n}) \leq 0\}\right) &= P\left(\bigcap_{i=1}^l \bigcup_{j=1}^m \{X_{j,n} \leq g_i^{-1}(0)\}\right) \\ &\leq P\left(\bigcap_{i=1}^l \bigcup_{j=1}^m \{Y_{j,n} \leq g_i^{-1}(0)\}\right) \\ &= P\left(\bigcap_{i=1}^l \bigcup_{j=1}^m \{g_i(Y_{j,n}) \leq 0\}\right). \end{aligned} \quad (2.5)$$

Notice that (2.5) holds for any $l \geq 1$. Conditioning on $N(t) = n(t)$, $t \geq 0$, $\bar{S}_j^X(t_i)$ is a strictly increasing function of $X_{j,n}$ for $1 \leq n \leq n(t_i)$, $1 \leq i \leq l$. Similarly, $\bar{S}_j^Y(t_i)$ is a strictly increasing function of $Y_{j,n}$, $1 \leq n \leq n(t_i)$, $1 \leq i \leq l$. Since $\mathbf{X}_1, \dots, \mathbf{X}_{n(t_l)}$ are i.i.d., and $\mathbf{Y}_1, \dots, \mathbf{Y}_{n(t_l)}$ are i.i.d., and $\mathbf{X}_n \geq_{\text{sm}} \mathbf{Y}_n$, we invoke (2.5) $n(t_l)$ times, and then obtain (2.4) conditioning on $N(t) = n(t)$, $t \geq 0$. Finally, unconditioning yields (2.4). $\square$

Note that $\psi_{\text{sim}}^X(u_1, \dots, u_m) \leq \psi_{\text{and}}^Y(u_1, \dots, u_m) \leq \psi_{\text{or}}^X(u_1, \dots, u_m)$ for any $u_1, \dots, u_m$. Theorem 2.3 shows that, as the claim size vector becomes more correlated in the sense of supermodular order, both $\psi_{\text{sim}}^X(u_1, \dots, u_m)$ and $\psi_{\text{and}}^Y(u_1, \dots, u_m)$ increase, and $\psi_{\text{or}}^X(u_1, \dots, u_m)$ decreases.

3. Stochastic bounds

Our bounding strategy is to bound the multivariate ruin probabilities (1.2)–(1.4) by some univariate ruin probabilities, which can be calculated for the phase-type distributed claims.

Consider a multivariate compound risk model (1.1). Let

$$\begin{aligned} X_{(1),n} &= \min\{X_{1,n}, \dots, X_{m,n}\}, & X_{(m),n} &= \max\{X_{1,n}, \dots, X_{m,n}\}, \\ p_{(1)} &= \min\{p_1, \dots, p_m\}, & p_{(m)} &= \max\{p_1, \dots, p_m\}, \\ u_{(1)} &= \min\{u_1, \dots, u_m\}, & u_{(m)} &= \max\{u_1, \dots, u_m\}. \end{aligned}$$

Also let

$$\psi_{\min}(u) = P \left(\sup_{0 \leq t < \infty} \left(\sum_{n=1}^{N(t)} X_{(1),n} - p_{(1)}t \right) > u \right), \quad (3.1)$$

$$\psi_{\max}(u) = P \left(\sup_{0 \leq t < \infty} \left(\sum_{n=1}^{N(t)} X_{(m),n} - p_{(1)}t \right) > u \right). \quad (3.2)$$

Clearly, for any non-negative $(u_1, \dots, u_m)$,

$$\psi_{\min}(u_{(m)}) \leq \psi_{\text{sim}}(u_1, \dots, u_m) \leq \psi_{\text{or}}(u_1, \dots, u_m) \leq \psi_{\max}(u_{(1)}).$$

Consider now the following two ruin probabilities, for any

$$\mathbf{a} \in \mathcal{A} = \left\{ (a_1, \dots, a_m) : a_j \geq 0, 1 \leq j \leq m, \text{ and } \sum_{j=1}^m a_j > 0 \right\},$$

let

$$\psi_{\mathbf{a}}(u) = P \left(\sup_{0 \leq t < \infty} \left\{ \sum_{n=1}^{N(t)} \left(\sum_{j=1}^m a_j X_{j,n} \right) - \left(\sum_{j=1}^m a_j p_j \right) t \right\} > u \right), \quad (3.3)$$

$$\psi_{\text{sum}}(u) = P \left(\sup_{0 \leq t < \infty} \left\{ \sum_{n=1}^{N(t)} \left(\sum_{j=1}^m X_{j,n} \right) - \left(\sum_{j=1}^m p_j \right) t \right\} > u \right). \quad (3.4)$$

Using the notations in (1.1), we observe that

$$\psi_{\mathbf{a}} \left(\sum_{j=1}^m a_j u_j \right) = P \left(\sup_{0 \leq t < \infty} \left\{ \sum_{j=1}^m a_j (S_j(t) - u_j) \right\} > 0 \right). \quad (3.5)$$

We point out that for any $\mathbf{a} \in \mathcal{A}$ if $p_j > 0$, $u_j \geq 0$, and $t p_j > E(N(t)) E(X_{j,1}) > 0$ for all $j = 1, \dots, m$ and $t > 0$, then $\sum_{j=1}^m a_j p_j > 0$, $\sum_{j=1}^m a_j u_j \geq 0$, and $t \sum_{j=1}^m a_j p_j > E(N(t)) \sum_{j=1}^m a_j E(X_{j,1}) > 0$. Hence, there are positive premium rates, non-negative initial capitals, and positive relative security loadings in the ruin probabilities (3.3)–(3.5).

On one hand, for any $(a_1, \dots, a_m) \in \mathcal{A}$ and any $t > 0$, the event $\{S_1(t) > u_1, \dots, S_m(t) > u_m\}$ implies the event $\left\{ \sum_{j=1}^m a_j (S_j(t) - u_j) > 0 \right\}$. Hence,

$$\psi_{\text{sim}}(u_1, \dots, u_m) \leq \psi_{\mathbf{a}} \left(\sum_{j=1}^m a_j u_j \right) \quad (3.6)$$

for any $(a_1, \dots, a_m) \in \mathcal{A}$.

On the other hand, for any $(a_1, \dots, a_m) \in \mathcal{A}$ and any $t > 0$, the event $\left\{ \sum_{j=1}^m a_j (S_j(t) - u_j) > 0 \right\}$ implies that the event $\{S_j(t) - u_j > 0\}$ holds for at least one j . Thus, we also have

$$\psi_{\mathbf{a}} \left(\sum_{j=1}^m a_j u_j \right) \leq \psi_{\text{or}}(u_1, \dots, u_m) \quad (3.7)$$

for any $(a_1, \dots, a_m) \in \mathcal{A}$. We summarize all these results in the following proposition.

Proposition 3.1. Let $\mathcal{A} = \{(a_1, \dots, a_m) : a_j \geq 0, 1 \leq j \leq m, \text{ and } \sum_{j=1}^m a_j > 0\}$.

1. $\psi_{\min}(u_{(m)}) \leq \psi_{\text{sim}}(u_1, \dots, u_m) \leq \inf_{\mathbf{a} \in \mathcal{A}} \psi_{\mathbf{a}}\left(\sum_{j=1}^m a_j u_j\right)$.
2. $\sup_{\mathbf{a} \in \mathcal{A}} \psi_{\mathbf{a}}\left(\sum_{j=1}^m a_j u_j\right) \leq \psi_{\text{or}}(u_1, \dots, u_m) \leq \psi_{\max}(u_{(1)})$.
3. In particular,

$$\psi_{\min}(u_{(m)}) \leq \psi_{\text{sim}}(u_1, \dots, u_m) \leq \psi_{\text{sum}}\left(\sum_{j=1}^m u_j\right) \leq \psi_{\text{or}}(u_1, \dots, u_m) \leq \psi_{\max}(u_{(1)}).$$

The upper bound in (1) and the lower bound in (2) of Proposition 3.1 have been discussed in [5]. The bounds presented in Proposition 3.1 are the ruin probabilities of univariate risk processes, but depend on the dependence structure of the underlying multivariate compound risk process. To see this, consider two multivariate compound risk models $\mathcal{M}_1$ and $\mathcal{M}_2$ introduced in Section 1. Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same claim event arrival process $\{N(t), t \geq 0\}$, same premium rates $p_j, 1 \leq j \leq m$, and same initial reserves $u_j, 1 \leq j \leq m$, but different claim size vectors $\mathbf{X}_n = (X_{1,n}, \dots, X_{m,n})$ and $\mathbf{Y}_n = (Y_{1,n}, \dots, Y_{m,n})$, respectively. Let $\psi_{\min}^X(u)$ ($\psi_{\min}^Y(u)$), $\psi_{\max}^X(u)$ ($\psi_{\max}^Y(u)$), and $\psi_{\text{sum}}^X(u)$ ($\psi_{\text{sum}}^Y(u)$) denote the ruin probabilities of types (3.1), (3.2), and (3.4), respectively, in model $\mathcal{M}_1$ ($\mathcal{M}_2$).

Proposition 3.2. If $\mathbf{X}_n \geq_{\text{sm}} \mathbf{Y}_n$, then we have, for any non-negative u ,

1. $\psi_{\min}^X(u) \geq \psi_{\min}^Y(u)$,
2. $\psi_{\max}^X(u) \leq \psi_{\max}^Y(u)$, and
3. $\psi_{\text{sum}}^X(u) \geq \psi_{\text{sum}}^Y(u)$.

Proof. Clearly, $\mathbf{X}_n \geq_{\text{sm}} \mathbf{Y}_n$ implies that

$$X_{(1),n} \geq_{\text{st}} Y_{(1),n}, \quad X_{(m),n} \leq_{\text{st}} Y_{(m),n}.$$

Thus, (1) and (2) follow from the fact that $\psi_{\min}^X(u)$ ($\psi_{\max}^X(u)$) is the increasing function of $X_{(1),n}$ ($X_{(m),n}$), $n \geq 1$. The proof of (3) can be found in [3]. $\square$

The univariate bounds established in Proposition 3.1 hold for any claim size vector $\mathbf{X}_n$. If the claim size vector satisfies some positive dependence property, then the product-type bounds can be also established. We first review the notions of *positive association* and *supermodular dependence*, which can be found, for example, in [14,10].

Definition 3.3. Let $\mathbf{X} = (X_1, \dots, X_m)$ be a real random vector.

1. $\mathbf{X}$ is said to be positively associated if

$$E[f(\mathbf{X})g(\mathbf{X})] \geq Ef(\mathbf{X})Eg(\mathbf{X}) \quad (3.8)$$

for any real increasing functions f, g defined on $\mathcal{R}^m$.

2. $\mathbf{X}$ is said to be supermodular dependent if

$$(X_1, \dots, X_m) \geq_{\text{sm}} (X_1^I, \dots, X_m^I), \quad (3.9)$$

where $X_1^I, \dots, X_m^I$ are independent, and X_j^I and $X_j, 1 \leq j \leq m$, have the same distribution.

Both association and supermodular dependence yield the following lower bounds of product type for the joint distribution and survival functions:

$$P(X_1 \leq x_1, \dots, X_m \leq x_m) \geq \prod_{j=1}^m P(X_j \leq x_j), \quad (3.10)$$

$$P(X_1 > x_1, \dots, X_m > x_m) \geq \prod_{j=1}^m P(X_j > x_j). \quad (3.11)$$

As we will illustrate in Section 4, some random vectors possess both types of positive dependence. However, in general, neither positive association nor supermodular dependence implies one another.

Assuming that the event arrival process $N(t)$ is a Poisson process, Cai and Li [3] established the product-type lower bound for $\psi_{\text{and}}(u_1, \dots, u_m)$, by showing that if the claim size vector $\mathbf{X}_n$ is associated, then

$$\left(\sup_{0 \leq t < \infty} S_1(t), \dots, \sup_{0 \leq t < \infty} S_m(t) \right) \quad (3.12)$$

is also associated. This result also yields the product-type upper bound for $\psi_{\text{or}}(u_1, \dots, u_m)$.

Proposition 3.4. *For the multivariate compound Poisson risk model with a Poisson event arrival process and positively associated claim vector, we have*

$$\prod_{j=1}^m \psi_j(u_j) \leq \psi_{\text{and}}(u_1, \dots, u_m) \leq \psi_{\text{or}}(u_1, \dots, u_m) \leq 1 - \prod_{j=1}^m (1 - \psi_j(u_j))$$

for any non-negative $u_1, \dots, u_m$, where $\psi_j(u_j) = P(\sup_{0 \leq t < \infty} S_j(t) > u_j)$, $1 \leq j \leq m$.

Proof. The first inequality follows from the association property of (3.12) and the inequality (3.11). The third inequality follows from the fact that

$$\psi_{\text{or}}(u_1, \dots, u_m) = 1 - P \left(\sup_{0 \leq t < \infty} S_1(t) \leq u_1, \dots, \sup_{0 \leq t < \infty} S_m(t) \leq u_m \right) \quad (3.13)$$

and the association property of (3.12). $\square$

If the claim size vector possesses the supermodular dependence, the same bounds still hold.

Proposition 3.5. *For the multivariate compound Poisson risk model with a Poisson event arrival process and supermodular dependent claim vector, we have*

$$\prod_{j=1}^m \psi_j(u_j) \leq \psi_{\text{and}}(u_1, \dots, u_m) \leq \psi_{\text{or}}(u_1, \dots, u_m) \leq 1 - \prod_{j=1}^m (1 - \psi_j(u_j))$$

for any non-negative $u_1, \dots, u_m$, where $\psi_j(u_j) = P(\sup_{0 \leq t < \infty} S_j(t) > u_j)$, $1 \leq j \leq m$.

Proof. We only establish the first inequality, and the third inequality follows in the same way as Proposition 3.4 via (3.13). For this, consider (1.1) and (1.2) with a Poisson event arrival process $N(t)$.

For each claim size vector $\mathbf{X}_n = (X_{1,n}, \dots, X_{m,n})$, let $\mathbf{X}_n^I = (X_{1,n}^I, \dots, X_{m,n}^I)$ be the vector in which $X_{1,n}^I, \dots, X_{m,n}^I$ are independent, and $X_{j,n}^I$ and $X_{j,n}$ have the same distribution, $1 \leq j \leq m$. Also let

$$\psi_{\text{and}}^I(u_1, \dots, u_m) = P \left(\bigcap_{j=1}^m \left\{ \sup_{0 \leq t < \infty} (S_j^I(t)) > u_j \right\} \right),$$

where $S_j^I(t) = \sum_{n=1}^{N(t)} X_{j,n}^I - p_j t$, $1 \leq j \leq m$. Since $\mathbf{X}_n$ is supermodular dependent, we have $\mathbf{X}_n \geq_{\text{sm}} \mathbf{X}_n^I$. Thus, from Theorem 2.3, we have

$$\psi_{\text{and}}(u_1, \dots, u_m) \geq \psi_{\text{and}}^I(u_1, \dots, u_m).$$

We need to show that $\psi_{\text{and}}^I(u_1, \dots, u_m) \geq \prod_{j=1}^m \psi_j(u_j)$. Since $N(t)$ is a Poisson process, then

$$N(t) = \max \left\{ n : \sum_{i=1}^n E_i \leq t \right\},$$

where E_i 's are i.i.d. exponential random variables with mean $1/\lambda$. From the Lorentz's inequality (see, for example, [10]), we have, for any $i \geq 1$,

$$\underbrace{(E_i, \dots, E_i)}_m \geq_{\text{sm}} (E_{1,i}, \dots, E_{m,i}), \quad (3.14)$$

where $E_{j,i}$'s are i.i.d. exponential random variables with mean $1/\lambda$. For any $1 \leq j \leq m$, let $N_j(t)$ denote a Poisson process with interevent arrival times $E_{j,i}$, $i \geq 1$. Obviously, Poisson processes $\{N_j(t), t \geq 0\}$, $1 \leq j \leq m$, are independent. Let, for each $1 \leq j \leq m$, $N_j^k(t) = N_j(t)$ given that $E_{j,i} = z_i$, $i \geq k+1$.

Conditioning on $\mathbf{X}_n^I = (x_{1,n}, \dots, x_{m,n})$, $n \geq 1$, and $E_i = z_i$, $i \geq k+1$, $\sup_{0 \leq t < \infty} S_j(t)$, $1 \leq j \leq m$, is a decreasing function of $E_1, \dots, E_k$. Because of (3.14), we invoke Lemma 2.2(1) k times, and obtain that

$$\psi_{\text{and}}^I(u_1, \dots, u_m) \geq P \left(\bigcap_{j=1}^m \left\{ \sup_{0 \leq t < \infty} \left(\sum_{n=1}^{N_j^k(t)} x_{j,n} - p_j t \right) > u_j \right\} \right).$$

As $k \rightarrow \infty$, we obtain that

$$\psi_{\text{and}}^I(u_1, \dots, u_m) \geq P \left(\bigcap_{j=1}^m \left\{ \sup_{0 \leq t < \infty} \left(\sum_{n=1}^{N_j(t)} x_{j,n} - p_j t \right) > u_j \right\} \right).$$

Unconditioning on $\mathbf{X}_n^I$, $n \geq 1$, we have

$$\begin{aligned} \psi_{\text{and}}^I(u_1, \dots, u_m) &\geq P \left(\bigcap_{j=1}^m \left\{ \sup_{0 \leq t < \infty} \left(\sum_{n=1}^{N_j(t)} X_{j,n}^I - p_j t \right) > u_j \right\} \right) \\ &= \prod_{j=1}^m P \left(\left\{ \sup_{0 \leq t < \infty} \left(\sum_{n=1}^{N_j(t)} X_{j,n}^I - p_j t \right) > u_j \right\} \right) = \prod_{j=1}^m \psi_j(u_j). \end{aligned}$$

Hence $\psi_{\text{and}}(u_1, \dots, u_m) \geq \prod_{j=1}^m \psi_j(u_j)$. $\square$

4. Multivariate compound Poisson risk models with MPH-type distributed claims

In this section, we first show that the bounds derived in Section 3 can be calculated explicitly when the claim size vector has the MPH-type distribution. We then illustrate our results using the multivariate Marshall–Olkin distribution,

Let $\{X(t), t \geq 0\}$ be a right-continuous, continuous-time Markov chain on a finite state space $\mathcal{E}$ with generator Q , in which Δ is the only absorbing state while all the other states are transient. Let $\mathcal{E}_i$, $i = 1, \dots, m$, be m non-empty stochastically closed subsets of $\mathcal{E}$ such that $\bigcap_{i=1}^m \mathcal{E}_i = \{\Delta\}$. (A subset of the state space is said to be stochastically closed if once the process $\{X(t), t \geq 0\}$ enters it, $\{X(t), t \geq 0\}$ never leaves.) Thus, $\mathcal{E} = (\bigcup_{i=1}^m \mathcal{E}_i) \cup \mathcal{E}_0$ for some subset $\mathcal{E}_0 \subset \mathcal{E}$ with $\mathcal{E}_0 \cap \mathcal{E}_j = \emptyset$ for $1 \leq j \leq m$. Furthermore, the states in $\mathcal{E}$ are enumerated in such a way that Δ is the first element of $\mathcal{E}$. Thus, the generator of the chain has the form

$$Q = \begin{bmatrix} 0 & \mathbf{0} \\ -A\mathbf{e} & A \end{bmatrix}, \quad (4.1)$$

where $\mathbf{0} = (0, \dots, 0)$ is the d -dimensional row vector of zeros, $\mathbf{e} = (1, \dots, 1)^T$ is the d -dimensional column vector of 1's, sub-generator A is a $d \times d$ non-singular matrix, and $d = |\mathcal{E}| - 1$. Let $\boldsymbol{\beta} = (0, \boldsymbol{\alpha})$ be an initial probability vector on $\mathcal{E}$ such that $\boldsymbol{\beta}(\Delta) = 0$.

We define

$$X_i = \inf\{t \geq 0 : X(t) \in \mathcal{E}_i\}, \quad i = 1, \dots, m. \quad (4.2)$$

As in [3], for simplicity, we shall assume that $P(X_1 > 0, \dots, X_m > 0) = 1$, which means that the underlying Markov chain $\{X(t), t \geq 0\}$ starts within $\mathcal{E}_0$ almost surely. The joint distribution of $(X_1, \dots, X_m)$ is called a *MPH-type* distribution with representation $(\boldsymbol{\alpha}, A, \mathcal{E}, \mathcal{E}_1, \dots, \mathcal{E}_m)$, and $(X_1, \dots, X_m)$ is called a phase type random vector.

When $m = 1$, the distribution of (4.2) reduces to the univariate PH distribution introduced in [11] (see Section 1). Examples of MPH distributions include, among many others, the well-known Marshall–Olkin distribution [8]. The MPH distributions, their properties, and some related applications in reliability theory were discussed in [2]. As in the univariate case, those MPH distributions (and their densities, Laplace transforms and moments) can be written in a closed form. The set of m -dimensional MPH distributions is dense in the set of all distributions on $[0, \infty)^m$. It is also shown in [2,6] that MPH distributions are closed under marginalization, finite mixture, convolution, and the formation of coherent systems. The following lemma, taken from [4], presents the phase type representations of some closure properties.

Lemma 4.1. Let $(X_1, \dots, X_m)$ be of phase type with representation $(\alpha, A, \mathcal{E}, \mathcal{E}_1, \dots, \mathcal{E}_m)$, where $A = (a_{i,j})$. For any $S \subseteq \mathcal{E} - \{\Delta\}$, let A_S denote the sub-matrix of A consisting of all the transition rates from S to S , and α_S is the sub-vector of α consisting all the probability entries on S . Then

1. X_j is of phase type with representation $\left(\frac{\alpha_{\mathcal{E}-\mathcal{E}_j}}{\alpha_{\mathcal{E}-\mathcal{E}_j}\mathbf{e}}, A_{\mathcal{E}-\mathcal{E}_j}, |\mathcal{E} - \mathcal{E}_j|\right)$.
2. $X_{(1)} = \min\{X_1, \dots, X_m\}$ is of phase type with representation $\left(\frac{\alpha_{\mathcal{E}_0}}{\alpha_{\mathcal{E}_0}\mathbf{e}}, A_{\mathcal{E}_0}, |\mathcal{E}_0|\right)$.
3. $X_{(n)} = \max\{X_1, \dots, X_m\}$ is of phase type with representation $(\alpha, A, |\mathcal{E}| - 1)$.
4. $\sum_{i=1}^n X_i$ has a phase type distribution with representation $(\alpha, T, |\mathcal{E}| - 1)$, where $T = (t_{i,j})$ is given by

$$t_{i,j} = \frac{a_{i,j}}{k(i)}, \quad (4.3)$$

and $k(i) = \text{number of indexes in } \{j : i \notin \mathcal{E}_j, 1 \leq j \leq m\}$.

With the help from Lemma 4.1 and (1.6), we obtain the explicit expressions of all the bounds in Propositions 3.1 and 3.5 as follows.

Proposition 4.2. Consider the multivariate compound Poisson risk model (1.1) with a Poisson event arrival process of rate λ , and phase type distributed claim size vectors with representation $(\alpha, A, \mathcal{E}, \mathcal{E}_1, \dots, \mathcal{E}_m)$, where $A = (a_{i,j})$.

1. $\psi_j(u_j) = -\frac{\lambda}{p_j} \frac{\alpha_{\mathcal{E}-\mathcal{E}_j}}{\alpha_{\mathcal{E}-\mathcal{E}_j}\mathbf{e}} A_{\mathcal{E}-\mathcal{E}_j}^{-1} \exp \left\{ \left(A_{\mathcal{E}-\mathcal{E}_j} - \frac{\lambda}{p_j} \mathbf{t}_0 \frac{\alpha_{\mathcal{E}-\mathcal{E}_j}}{\alpha_{\mathcal{E}-\mathcal{E}_j}\mathbf{e}} A_{\mathcal{E}-\mathcal{E}_j}^{-1} \right) u_j \right\} \mathbf{e}$, where $\mathbf{t}_0 = -A_{\mathcal{E}-\mathcal{E}_j}\mathbf{e}$.
2. $\psi_{\min}(u_{(m)}) = -\frac{\lambda}{p_{(m)}} \frac{\alpha_{\mathcal{E}_0}}{\alpha_{\mathcal{E}_0}\mathbf{e}} A_{\mathcal{E}_0}^{-1} \exp \left\{ \left(A_{\mathcal{E}_0} - \frac{\lambda}{p_{(m)}} \mathbf{t}_0 \frac{\alpha_{\mathcal{E}_0}}{\alpha_{\mathcal{E}_0}\mathbf{e}} A_{\mathcal{E}_0}^{-1} \right) u_{(m)} \right\} \mathbf{e}$, where $\mathbf{t}_0 = -A_{\mathcal{E}_0}\mathbf{e}$.
3. $\psi_{\max}(u_{(1)}) = -\frac{\lambda}{p_{(1)}} \alpha A^{-1} \exp \left\{ \left(A - \frac{\lambda}{p_{(1)}} \mathbf{t}_0 \alpha A^{-1} \right) u_{(1)} \right\} \mathbf{e}$, where $\mathbf{t}_0 = -A\mathbf{e}$.
4. $\psi_{\text{sum}}(\sum_{j=1}^m u_j) = -\frac{\lambda}{\sum_{j=1}^m p_j} \alpha T^{-1} \exp \left\{ \left(T - \frac{\lambda}{\sum_{j=1}^m p_j} \mathbf{t}_0 \alpha T^{-1} \right) \left(\sum_{j=1}^m u_j \right) \right\} \mathbf{e}$, where $\mathbf{t}_0 = -T\mathbf{e}$, and T is defined as in (4.3).

We now illustrate our results using the multivariate Marshall–Olkin distribution, and also show some interesting effects of different parameters on the bounds.

Let $\{E_S, S \subseteq \{1, \dots, m\}\}$ be a sequence of independent, exponentially distributed random variables, with E_S having mean $1/\lambda_S$. Let

$$X_j = \min\{E_S : S \ni j\}, \quad j = 1, \dots, m. \quad (4.4)$$

The joint distribution of $(X_1, \dots, X_m)$ is called the Marshall–Olkin distribution with parameters $\{\lambda_S, S \subseteq \{1, \dots, m\}\}$ [8].

It follows from (4.4) that any Marshall–Olkin distribution is positively associated and super-modular dependent. Thus, from Proposition 3.5, we have

$$\left(\prod_{j=1}^m \frac{1}{1 + \theta_j} \right) \exp \left(- \sum_{j=1}^m \left\{ \frac{\theta_j}{1 + \theta_j} \left(\sum_{S: S \ni j} \lambda_S \right) u_j \right\} \right) \leq \psi_{\text{and}}(u_1, \dots, u_m),$$

$$\psi_{\text{or}}(u_1, \dots, u_m) \leq 1 - \prod_{j=1}^m \left[1 - \frac{1}{1 + \theta_j} \exp \left(- \left\{ \frac{\theta_j}{1 + \theta_j} \left(\sum_{S: S \ni j} \lambda_S \right) u_j \right\} \right) \right],$$

for any non-negative $u_1, \dots, u_m$, where the relative security loading $\theta_j = (\sum_{S: S \ni j} \lambda_S) p_j / \lambda - 1$, $1 \leq j \leq m$.

It is known that the Marshall–Olkin distribution is a MPH-type distribution [2] and its phase type representation can be found in [3]. Thus, we can calculate the bounds in Proposition 4.2 for the Marshall–Olkin distribution. To illustrate the results, we consider the bivariate Marshall–Olkin distribution in the following example.

Example 4.3. Assume that the claim vector $(X_{1,n}, X_{2,n})$ has the bivariate Marshall–Olkin distribution, or equivalently, $X_{1,n} = \min\{E_{12,n}, E_{1,n}\}$ and $X_{2,n} = \min\{E_{12,n}, E_{2,n}\}$, where $\{E_{12,n}, E_{1,n}, E_{2,n}, n = 1, 2, \dots\}$ is a family of independent exponential random variables and $E_{12,n}, E_{1,n}$ and $E_{2,n}$ have means $1/\lambda_{12}, 1/\lambda_1$ and $1/\lambda_2$, respectively, for all $n = 1, 2, \dots$. Furthermore, assume that $\lambda = 1.6$ and $p_1 = p_2 = 3$. Let ρ be the correlation coefficient between $X_{1,n}$ and $X_{2,n}$. Then, it is not hard to find that

$$\rho = \frac{\lambda_{12}}{\lambda_1 + \lambda_2 + \lambda_{12}}. \quad (4.5)$$

To apply Proposition 4.2, we need the phase type representation for the bivariate Marshall–Olkin distribution. The representation has been given in [3] with the state space $\mathcal{E} = \{12, 2, 1, \emptyset\}$; stochastically closed subsets $\mathcal{E}_j = \{12, j\}$, $j = 1, 2$; initial probability vector $(0, 0, 0, 1)$; and the sub-generator

$$A = \begin{bmatrix} -\lambda_{12} - \lambda_1 & 0 & 0 \\ 0 & -\lambda_{12} - \lambda_2 & 0 \\ \lambda_2 & \lambda_1 & -\Lambda + \lambda_{\emptyset} \end{bmatrix}, \quad (4.6)$$

where 12 is the absorbing state and $\Lambda = \lambda_{12} + \lambda_2 + \lambda_1 + \lambda_{\emptyset}$.

Note that none of ρ and the matrix A in (4.6) involves $\lambda_{\emptyset}$. We introduce $\lambda_{\emptyset}$ in the model because we want to change the model parameters in a systematic fashion according to supermodular order, so that the effect of claim dependence on the ruin probabilities can be investigated.

To study the effect of dependence on the bounds, we calculate $\psi_{\min}(u_{(m)})$, $\psi_{\max}(u_{(1)})$, $\psi_{\text{sum}}(u_1 + u_2)$ and the product-type bounds in Proposition 3.5, respectively, under several different sets of model parameters.

We consider the following three cases of the claim vector $(X_{1,n}, X_{2,n})$. The correlation coefficients in the three cases are increasing, which indicates the increasing (linear) dependence of the claim vector in the three cases. In fact, it follows from [7] that the claim size vector in Case 1 is less dependent than that in Case 2, which, in turn, is less dependent than that in Case 3, all in supermodular order. The bounds for ψ_{sim} in Table 1 are based on Proposition 3.1 and the bounds for ψ_{or} in Tables 2 and 3 are based on Propositions 3.1 and 3.5, respectively. The analytic forms of these bounds in the three cases and the numerical values in Tables 1–3 were easily produced by Mathematica by using the formulas given in Proposition 4.2. The first columns of the tables lists several values of u_1 and u_2 , and the second and third columns list values of the lower and upper bounds, respectively, in the following three cases.

Table 1
Effects of dependence on the bounds for $\psi_{\text{sim}}(u_1, u_2)$

u_1	u_2	$\psi_{\min}(u_{(2)})$			$\psi_{\text{sum}}(u_1 + u_2)$		
		Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
0.5	0.5	0.09409	0.11161	0.31012	0.31640	0.31903	0.33493
1.0	1.0	0.03851	0.04998	0.22000	0.20578	0.21043	0.24295
1.5	1.5	0.01576	0.02238	0.15607	0.13161	0.13659	0.17592
2.0	2.0	0.00645	0.01002	0.11072	0.08373	0.08819	0.12729
2.5	2.5	0.00264	0.00449	0.07854	0.05318	0.05684	0.09207
3.0	3.0	0.00108	0.00201	0.05572	0.03376	0.03661	0.06658

Table 2
Effects of dependence on the bounds for $\psi_{\text{or}}(u_1, u_2)$

u_1	u_2	$\psi_{\text{sum}}(u_1 + u_2)$			$\psi_{\max}(u_{(1)})$		
		Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
0.5	0.5	0.31640	0.31903	0.33493	0.60200	0.57960	0.36230
1.0	1.0	0.20578	0.21043	0.24295	0.51725	0.49368	0.27192
1.5	1.5	0.13161	0.13659	0.17592	0.44229	0.41834	0.20400
2.0	2.0	0.08373	0.08819	0.12729	0.37761	0.35387	0.15301
2.5	2.5	0.05318	0.05684	0.09207	0.32224	0.29914	0.11474
3.0	3.0	0.03376	0.03661	0.06658	0.27495	0.25283	0.08603

Table 3
Effects of dependence on the bounds for $\psi_{\text{or}}(u_1, u_2)$

u_1	u_2	$\prod_{j=1}^2 \psi_j(u_j)$	$1 - \prod_{j=1}^2 (1 - \psi_j(u_j))$
		Cases 1–3	Cases 1–3
0.5	0.5	0.11297	0.55931
1.0	1.0	0.06037	0.43111
1.5	1.5	0.03226	0.32705
2.0	2.0	0.01724	0.24546
2.5	2.5	0.00921	0.18285
3.0	3.0	0.00492	0.13550

Case 1: Independent claim vector: Let $\lambda_{12} = 0$, $\lambda_1 = 1.15$, $\lambda_2 = 1.17$, $\lambda_{\emptyset} = 0$. Then the claim vector $(X_{1,n}, X_{2,n})$ is independent with $\rho = 0$ and

$$\begin{aligned}\psi_{\min}(u) &= 0.229885 e^{-1.7866u}, \\ \psi_{\text{sum}}(2u) &= 0.516615 e^{-0.909286u} - 0.0568111 e^{-3.19738u}, \\ \psi_{\max}(u) &= 0.712874 e^{-0.31756u} + 0.0000250601 e^{-1.16014u} - 0.0231758 e^{-2.62897u}, \\ \psi_1(u) &= 0.463768 e^{-0.616667u}, \quad \psi_2(u) = 0.455840 e^{-0.636667u}.\end{aligned}$$

Note that even the claim sizes are independent, but the claim surplus processes are still positively dependent.

Case 2: Slightly dependent claim vector: Let $\lambda_{12} = 0.18$, $\lambda_1 = 0.97$, $\lambda_2 = 0.99$, $\lambda_\emptyset = 0.18$. Then the claim vector $(X_{1,n}, X_{2,n})$ is slightly dependent with $\rho = 0.0841$ and

$$\begin{aligned}\psi_{\min}(u) &= 0.249221 e^{-1.60667u}, \\ \psi_{\text{sum}}(2u) &= 0.513675 e^{-0.880331u} - 0.0000160242 e^{-2.31993u} - 0.0538543 e^{-3.0464u}, \\ \psi_{\max}(u) &= 0.694215 e^{-0.336674u} + 0.0000266203 e^{-1.16014u} - 0.0238541 e^{-2.42985u}, \\ \psi_1(u) &= 0.463768 e^{-0.616667u}, \quad \psi_2(u) = 0.455840 e^{-0.636667u}.\end{aligned}$$

Case 3: Highly dependent vector: Let $\lambda_{12} = 1.1$, $\lambda_1 = 0.05$, $\lambda_2 = 0.07$, $\lambda_\emptyset = 1.1$. Then the claim vector $(X_{1,n}, X_{2,n})$ is highly dependent with $\rho = 0.9016$ and

$$\begin{aligned}\psi_{\min}(u) &= 0.437158 e^{-0.686667u}, \\ \psi_{\text{sum}}(2u) &= 0.465657 e^{-0.6483u} - 0.00111286 e^{-2.31406u} - 0.0047396 e^{-2.36431u}, \\ \psi_{\max}(u) &= 0.485566 e^{-0.576616u} + 0.000041294 e^{-1.16093u} - 0.00315671 e^{-1.26912u}, \\ \psi_1(u) &= 0.463768 e^{-0.616667u}, \quad \psi_2(u) = 0.455840 e^{-0.636667u}.\end{aligned}$$

In all the three cases, the distributions of $X_{1,n}$ and $X_{2,n}$ are the same. Indeed, $X_{1,n}$ and $X_{2,n}$ have exponential distributions with means $1/(\lambda_1 + \lambda_{12}) = \frac{1}{1.15}$ and $1/(\lambda_2 + \lambda_{12}) = \frac{1}{1.17}$, respectively.

The product-type bounds in Proposition 3.5 are the functions of the ruin probabilities $\psi_1(u)$ and $\psi_2(u)$, which do not depend on the dependence structure of the claim vector $(X_{1,n}, X_{2,n})$. Since these bounds are obtained from *independent claim surplus processes*, the bounds in Proposition 3.5 should out-perform (under-perform) those in Proposition 3.1 when the claim vector $(X_{1,n}, X_{2,n})$ is slightly (highly) dependent. Indeed, the tables show that the bounds in Proposition 3.5 are better than those in Proposition 3.1 in Cases 1 and 2 for independent or slightly dependent claim vectors while the bounds in Proposition 3.1 are better than those in Proposition 3.5 in Case 3 for highly dependent claim vectors. Note, however, that the bounds in Proposition 3.5 are not sharp for *independent claim vectors*.

Tables 1 and 2 also show that, serving as lower and upper bounds for $\psi_{\text{sim}}(u_1, u_2)$, the lower bound $\psi_{\min}(u_{(2)})$ and the upper bound $\psi_{\text{sum}}(u_1 + u_2)$ are tighter in Case 3 than in Cases 1 and 2. Similarly, serving as lower and upper bounds for $\psi_{\text{or}}(u_1, u_2)$, the lower bound $\psi_{\text{sum}}(u_1 + u_2)$ and the upper bound $\psi_{\max}(u_{(1)})$ are tighter in Case 3 than in Cases 1 and 2. Furthermore, Table 2 shows that the difference between the upper and lower bounds or $\psi_{\max}(u_{(1)}) - \psi_{\text{sum}}(u_1 + u_2)$ decreases with increasing strength of dependence from 's 1 to 3. Indeed, in the extremal case or the comonotone case where $X_{1,n} = X_{2,n}$ (that is the case of $\lambda_1 = \lambda_2 = 0$), we have $\psi_{\min}(u_{(2)}) = \psi_{\text{sim}}(u_1, u_2) = \psi_{\text{sum}}(u_1 + u_2) = \psi_{\text{and}}(u_1, u_2) = \psi_{\text{or}}(u_1, u_2) = \psi_{\max}(u_{(1)})$ for $u_1 = u_2$ and $p_1 = p_2$. In this sense, the bounds in Proposition 3.1(3) are attainable for the comonotone case. This further indicates that the bounds in Proposition 3.1 are better for highly dependent claim vectors.

In addition, as proved in Proposition 3.2, the tables display that $\psi_{\min}(u)$ ($\psi_{\text{sum}}(u)$) and $\psi_{\max}(u)$ have opposite monotonicity properties when dependence among the claim sizes increases.

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