JOURNAL OF MULTIVARIATE ANALYSIS **48**, 315–346 (1994)

Holomorphic Processes in Banach Spaces and Banach Algebras

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The main result is that if F is an analytic multifunction and B_t is a complex Brownian motion, then $F(B_t)$ is a subholomorphic process. It has previously been shown that such processes enjoy many interesting sample-path properties. As special cases of the theorem above, we recover

$$f \text{ holomorphic} \Rightarrow f(B_t) \text{ is a local conformal martingale,}$$

$$\varphi \text{ subharmonic} \Rightarrow \varphi(B_t) \text{ is a local submartingale.}$$

We also prove a stochastic form of Radó's theorem, and a holomorphic selection theorem for convex-valued subholomorphic processes of a nature quite different from the usual type of measurable selection theorem.

INTRODUCTION

If F is an analytic multifunction and B_t is a complex Brownian motion, then what sort of stochastic process is $F(B_t)$?

Analytic multifunctions are a set-valued generalization of holomorphic functions. Though originally conceived in the context of several complex variables [13], they have found applications in a number of other areas, including spectral theory [15, 20, 22], uniform algebras [20, 21], interpolation spaces [16, 22, 24], polynomial hulls [3, 23], and complex dynamics [2]. The question above is prompted by the desire to extend the classic theorem of P. Lévy that a holomorphic function f of B_t is itself a time-changed Brownian motion. Lévy's theorem can be proved via the intermediate step that $f(B_t)$ is a local conformal martingale (see [8]), and we attempt to answer our question in these terms. In fact what we prove

Received March 19, 1992; revised June 21, 1993.

AMS 1980 subject classifications: primary 60G48; secondary 60D05.

Key words and phrases: Analytic multifunction, Brownian motion, holomorphic process, conformal martingale, submartingale.

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0047-259X/94 $6.00

is that $F(B_t)$ is a *subholomorphic* process. This class of set-valued processes was introduced in [18], where it was shown that they possess many interesting sample-path properties, among them: a downcrossing inequality, forward and reverse martingale convergence theorems, non-entry into polar sets, local and global maximum principles, an interpolation theorem, and assorted results concerning variation of size. As particular cases of our theorem, we recover:

$$f \text{ holomorphic} \Rightarrow f(B_t) \text{ is a local conformal martingale;}$$

$$\varphi \text{ subharmonic} \Rightarrow \varphi(B_t) \text{ is a local submartingale.}$$

Since there are some important analytic multifunctions defined on infinite-dimensional spaces, for example, the spectrum in a Banach algebra, we develop our theory in the setting of a general Banach space.

This paper is a sequel to [18] and leans heavily on the results therein, but for the convenience of the reader all the main definitions are repeated. Here, in more detail, is a plan of the contents. After establishing some basic notation in Section 0, we extend the theory of holomorphic and subharmonic processes to infinite dimensions in Sections 1 and 2, respectively. The key result that an analytic multifunction of a holomorphic process is subholomorphic is proved in Section 3, and then applied to Brownian motion in Section 4 to give an answer to the question posed above. The paper concludes with two further links between holomorphic, subharmonic and subholomorphic processes: a stochastic form of Radó's theorem in Section 5 and a holomorphic selection theorem in Section 6.

0. Preliminaries

This section summarizes the background material on stochastic processes used in the paper. For the most part we follow the notation and conventions of [4], and this is also a good reference for further details.

Throughout the paper it is assumed that we are given a complete probability space (Ω, Σ, P), together with (except in Section 4) a complete, right continuous filtration $\mathbf{F}$. By the latter is meant a family $\{\mathscr{F}_{0-}\} \cup \{\mathscr{F}_t : 0 \leqslant t \leqslant \infty\}$ of sub-σ-fields of Σ satisfying

(a) $\mathscr{F}_{0-} \subset \mathscr{F}_s \subset \mathscr{F}_t$ $(0 \leqslant s \leqslant t \leqslant \infty)$,

(b) $\mathscr{F}_\infty = \sigma(\bigcup_{s<\infty} \mathscr{F}_s)$,

(c) $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$ $(0 \leqslant t < \infty)$,

(d) $P(\Lambda) = 0 \Rightarrow \Lambda \in \mathscr{F}_{0-}$.

A *process* is a function X defined on $[0, \infty] \times \Omega$ which is $(\mathscr{B}[0, \infty] \otimes \Sigma)$-measurable. It is *adapted* if, for each $t \in [0, \infty]$, the map $\omega \mapsto X(t, \omega)$

is $\mathscr{F}_t$-measurable. It is customary to write $X(t, \omega)$ as $X_t(\omega)$, and the "ω" is usually omitted. Given $\Pi \in \mathscr{B}[0, \infty] \otimes \Sigma$, the *indicator* of Π is the process 1_Π defined by

$$1_\Pi(t, \omega) = \begin{cases} 1 & \text{if } (t, \omega) \in \Pi, \\ 0 & \text{if } (t, \omega) \notin \Pi. \end{cases}$$

If $A \in \Sigma$ then $1_{[0, \infty] \times A}$ is usually written simply as 1_A.

Evanescent sets, namely subsets of $[0, \infty] \times \Lambda$ where $P(\Lambda) = 0$, are always to be treated as negligible. Consequently, we identify *indistinguishable* processes, that is, processes X, Y such that $\{(t, \omega) : X_t(\omega) \neq Y_t(\omega)\}$ is evanescent. Thus a statement like "X is a continuous process" really means

$$P(\{\omega \in \Omega \colon t \mapsto X_t(\omega) \text{ is continuous on } [0, \infty]\}) = 1.$$

Note that a countable union of evanescent sets is again evanescent.

A *random time* is a map $T \colon \Omega \to [0, \infty]$ which is Σ-measurable. We identify random times which are equal almost surely. Let X be a process and let S, T be random times. The *stopped process* X^T is defined by

$$X^T(t, \omega) = X(T(\omega) \wedge t, \omega),$$

and the *delayed process* ${}^S X$ by

$${}^S X(t, \omega) = X(S(\omega) \vee t, \omega).$$

We write $((S, T]\!]$ for the stochastic interval

$$\{(t, \omega) \in [0, \infty] \times \Omega : S(\omega) < t \leqslant T(\omega)\},$$

with similar definitions for $[\![S, T))$, $((S, T))$ and $[\![S, T]\!]$. Notice that unlike [4] we allow points of the form (∞, ω) to be included in $((S, T]\!]$ and $[\![S, T]\!]$. If (T_k) is an increasing sequence of random times, then

$$T_k \uparrow T \text{ means } \bigcup_{k \geqslant 1} ((0, T_k)) = ((0, T)),$$

$$T_k \uparrow\uparrow T \text{ means } \bigcup_{k \geqslant 1} ((0, T_k]\!] = ((0, T)).$$

We shall need a version of Zorn's lemma for random times.

0.1. Theorem [18, Thm. 0.1] *Let Γ be a non-empty set of random times such that, whenever (T_k) is a sequence in Γ and $T_k \uparrow T$, then $T \in \Gamma$. Then Γ has a maximal element, i.e., there exists $S \in \Gamma$ such that $(S' \in \Gamma$ and $S' \geqslant S) \Rightarrow S' = S$.*

A random time T is a *stopping time* if $\{\omega \in \Omega : T(\omega) < t\} \in \mathscr{F}_t$ for all $t \in [0, \infty]$. It is a *predictable time* if there exists an increasing sequence (T_k) of stopping times, satisfying $\{\omega \in \Omega : T_k(\omega) = 0\} \in \mathscr{F}_{0-}$ for each k, such that $T_k \uparrow\uparrow T$. In fact by [4, Chap. IV, Sect. 77] these (T_k) may themselves be chosen to be predictable times.

The *predictable σ-field* on $[0, \infty] \times \Omega$, denoted by $\mathscr{P}$, is the σ-field generated by all sets of the form $\{0\} \times A$ $(A \in \mathscr{F}_{0-})$ and $(t, \infty] \times B$ $(B \in \mathscr{F}_t)$ (together with the evanescent sets). It can be shown that a random time T is a stopping time if and only if $(\!(T, \infty]\!] \in \mathscr{P}$, and that it is a predictable time if and only if $[\![T, \infty]\!] \in \mathscr{P}$ (see [4, Chap. IV]). A process X is called *predictable* if it is $\mathscr{P}$-measurable; examples include all adapted, left continuous processes X such that X_0 is $\mathscr{F}_{0-}$-measurable. Predictable times often arise as the first exit time of a predictable, right continuous process X from an open set U. This is a consequence of the next theorem applied with $\Pi = \{(t, \omega) : X_t(\omega) \notin U\}$.

0.2. THEOREM [4, Chap. IV, Sect. 87(d)]. *Let $\Pi \in \mathscr{P}$, and suppose that Π is closed from the right (i.e., if $(t_k, \omega) \in \Pi$ for all k and $t_k \downarrow t$, then $(t, \omega) \in \Pi$). Let*

$$D_\Pi(\omega) = \inf\{t \geqslant 0 : (t, \omega) \in \Pi\},$$

where $\inf \varnothing = \infty$. *Then D_Π is a predictable time.*

We shall need some notation for set-valued processes. This is taken directly from [18, Sect. 3]. Writing $\kappa(\mathbb{C})$ for the collection of all compact subsets of $\mathbb{C}$ (including $\varnothing$), our processes will be maps K: $[\![0, \infty]\!] \to \kappa(\mathbb{C})$ (defined, as usual, up to evanescent sets) such that, for each open U in $\mathbb{C}$,

$$\{(t, \omega) : K_t(\omega) \subset U\} \in \mathscr{B}[0, \infty] \otimes \Sigma.$$

Given such a K, and given $\Pi \in \mathscr{B}[0, \infty] \otimes \Sigma$, we define the process $K\chi_\Pi$ by

$$K\chi_\Pi(t, \omega) = \begin{cases} K(t, \omega) & \text{if } (t, \omega) \in \Pi, \\ \varnothing & \text{if } (t, \omega) \notin \Pi. \end{cases}$$

When $\Pi = [0, \infty] \times A$ for some $A \in \Sigma$, we frequently abuse notation and write $K\chi_A$ instead of $K\chi_\Pi$. The notations K^T and ${}^SK$ are defined just as for single-valued processes, and we write $K^{(k)} \downarrow K$ to mean that $(K^{(k)})$ is a sequence of processes such that, for all $(t, \omega) \in [\![0, \infty]\!]$,

$$K_t^{(1)}(\omega) \supset K_t^{(2)}(\omega) \cdots \qquad \text{and} \qquad \bigcap_{k \geqslant 1} K_t^{(k)}(\omega) = K_t(\omega).$$

Given a process $K : [\![0, \infty]\!] \to \kappa(\mathbb{C})$, we say that:

(a) K is *bounded* if there exists a bounded subset B of $\mathbb{C}$ such that $K_t(\omega) \subset B$ for all t, ω;

(b) K is *predictable* if, for every open U in $\mathbb{C}$, the set $\{(t, \omega) : K_t(\omega) \subset U\}$ belongs to $\mathscr{P}$;

(c) K is *upper semicontinuous* if, for almost every $\omega \in \Omega$ and every open U in $\mathbb{C}$, the set $\{t : K_t(\omega) \subset U\}$ is open in $[0, \infty]$;

(d) K is *weakly right lower semicontinuous* if, for almost every $\omega \in \Omega$ and every closed C in $\mathbb{C}$, the set $\{t : \partial K_t(\omega) \subset C\}$ is closed from the right.

Measurability problems for such processes are considered in [19], but everything needed in this paper is easily deducible from Theorem 0.2.

A real-valued adapted process X is a *submartingale* (with respect to $\mathbf{F}$) if, whenever $0 \leqslant s \leqslant t \leqslant \infty$, the random variable X_t is integrable and satisfies $E[X_t \mid \mathscr{F}_s] \geqslant X_s$. If both X and $-X$ are submartingales then X is a *martingale*. A complex-valued process is a martingale if both its real and imaginary parts are.

Given a random variable $Y \in L^1(\Sigma; \mathbb{C})$, the process $X_t = E[Y \mid \mathscr{F}_t]$ has a right continuous version which is unique up to indistinguishability (see [4, Chap. VI, Sect. 4]). We always take this version. If, moreover, $Y \in L^2(\mathscr{F}_\infty; \mathbb{C})$, then X is a right continuous martingale which satisfies $\sup_t E|X_t|^2 < \infty$, and conversely, every such martingale has the form $E[Y \mid \mathscr{F}_t]$ for some unique random variable $Y \in L^2(\mathscr{F}_\infty; \mathbb{C})$ (see [4, Chap. VI, Sect. 4, 6]). We denote by $\mathscr{M}$ the set of all such martingales. The vector space $\mathscr{M}$ thus inherits the Hilbert space structure of $L^2(\mathscr{F}_\infty; \mathbb{C})$, namely, the inner product

$$(M, N) := E[M_\infty \bar{N}_\infty] \qquad (M, N \in \mathscr{M}),$$

and the corresponding norm

$$\|M\|_2 := (E|M_\infty|^2)^{1/2} \qquad (M \in \mathscr{M}).$$

Given a subspace $\mathscr{L}$ of $\mathscr{M}$, we write

$$\begin{aligned}
\mathscr{L}^\perp &:= \{M \in \mathscr{M} : (M, L) = 0 \text{ for all } L \in \mathscr{L}\},\\
\bar{\mathscr{L}} &:= \{M \in \mathscr{M} : \bar{M} \in \mathscr{L}\},\\
\mathscr{L}^c &:= \{L \in \mathscr{L} : L \text{ is continuous and } L_0 \in L^2(\mathscr{F}_{0-})\},\\
\mathscr{L}^d &:= \{L \in \mathscr{L} : L \in (\mathscr{L}^c)^\perp\},\\
\mathscr{L}_0 &:= \{L \in \mathscr{L} : L_0 = 0\},\\
\mathscr{L}_{\mathbb{R}} &:= \{L \in \mathscr{L} : L \text{ is real-valued}\}.
\end{aligned}$$

If $\mathscr{L}'$ is another subspace of $\mathscr{M}$ and $\mathscr{L}' \subset \mathscr{L}^{\perp}$, then $\mathscr{L} \oplus \mathscr{L}'$ denotes the orthogonal direct sum.

Finally, we mention briefly how this extends to vector-valued processes. More details can be found in [5]. Let $(E, |\cdot|)$ be a complex Banach space, which we also assume is separable to avoid measurability difficulties. We denote by $\mathscr{M}(E)$ the space of processes $\mathbf{X}$: $[\![0, \infty]\!] \to E$ of the form $\mathbf{X}_t = E[\mathbf{Y} \mid \mathscr{F}_t]$, where $\mathbf{Y} \in L^2(\mathscr{F}_\infty; E)$. Each $\mathbf{X} \in \mathscr{M}(E)$ is a right continuous martingale satisfying $\sup_t E|\mathbf{X}_t|^2 < \infty$, and conversely, if E has the Radon–Nikodým property (in particular, if E is a dual Banach space), then every such martingale belongs to $\mathscr{M}(E)$. The space $\mathscr{M}(E)$ inherits the Banach-space structure of $L^2(\mathscr{F}_\infty; E)$, namely, the norm

$$\|\mathbf{X}\|_2 := (E|\mathbf{X}_\infty|^2)^{1/2} \qquad (\mathbf{X} \in \mathscr{M}(E)).$$

Processes in $\mathscr{M}(E)$ satisfy a form of Doob's inequality.

0.3 Theorem. *If $\mathbf{X} \in \mathscr{M}(E)$ then $|\mathbf{X}|$ is a right continuous submartingale, and*

$$\|\sup_t |\mathbf{X}_t|\|_2 \leqslant 2\, \|\mathbf{X}\|_2.$$

Proof. As E is separable, we can find a sequence (ξ_k) in the unit ball of E^* such that $|\mathbf{x}| = \sup_k |\xi_k(\mathbf{x})|$ for all $\mathbf{x} \in E$. Then for $0 \leqslant s \leqslant t \leqslant \infty$ we have

$$\begin{aligned} E[|\mathbf{X}_t| \mid \mathscr{F}_s] &= E[\sup_k |\xi_k(\mathbf{X}_t)| \mid \mathscr{F}_s] \geqslant \sup_k |\xi_k(E[\mathbf{X}_t \mid \mathscr{F}])| \\ &= \sup_k |\xi_k(\mathbf{X}_s)| = |\mathbf{X}_s|. \end{aligned}$$

Hence $|\mathbf{X}|$ is a submartingale, and it is evidently right continuous. The Doob inequality now follows from [4, Chap. V, Sect. 24]. ∎

1. Vector-Valued Processes and Holomorphic Functions

The idea of a holomorphic stochastic process was developed in [18, Sect. 1], where the interaction with holomorphic functions was also explored. Our aim in this section is to extend these ideas to processes taking values in a Banach space.

We begin by recalling the basic definition in the scalar-valued case.

Definition [18, Defn. 1.2]. A *holomorphic atlas* for $\mathscr{F}$ is a vector subspace $\mathscr{H}$ of $\mathscr{M}^c$ such that:

(a) if $Z \in \mathscr{H}$ then Z^2 is a martingale;

(b) if $Z \equiv Z_0 \in L^2(\mathscr{F}_{0-})$, then $Z \in \mathscr{H}$;

(c) if $M \in \mathscr{M}^c$, then $M = Z + \bar{W}$ for some $Z, W \in \mathscr{H}$.

For the rest of this section $\mathscr{H}$ denotes a fixed holomorphic atlas for $\mathscr{F}$, and $(E, |\cdot|)$ a separable complex Banach space.

1.1. DEFINITION. Let $\mathbf{Z} \in \mathscr{M}(E)$.

(a) $\mathbf{Z}$ is a *simple holomorphic process* if

$$\mathbf{Z}_t = \sum_{j=1}^{n} W_t^{(j)} \mathbf{x}_j,$$

where $W^{(j)} \in \mathscr{H}$ and $\mathbf{x}_j \in E$ $(j = 1, ..., n)$, and $n \geqslant 1$. The space of such processes is denoed by $\mathscr{H}_0(E)$.

(b) $\mathbf{Z}$ is a *holomorphic process* if there are simple holomorphic processes $(\mathbf{Z}^{(k)})$ such that $\|\mathbf{Z} - \mathbf{Z}^{(k)}\|_2 \to 0$. The space of holomorphic processes is denoted by $\mathscr{H}(E)$.

Remarks. (a) The space $\mathscr{H}(E)$ remains unchanged if the norm $|\cdot|$ on E is replaced by an equivalent one.

(b) Clearly all processes in $\mathscr{H}_0(E)$ are continuous. Since the continuous processes in $\mathscr{M}(E)$ form a closed subspace of $\mathscr{M}(E)$ (a consequence of Doob's inequality, Theorem 0.3), it follows that in fact all processes in $\mathscr{H}(E)$ are continuous.

(c) Obviously $\mathbf{Z} \in \mathscr{H}(E)$ implies that $\mathbf{Z}_\infty \in L^2(\Sigma; E)$. Also it follows easily from [18, Thm. 1.4] that $\mathbf{Z} \in \mathscr{H}(E)$ implies $\xi(\mathbf{Z}) \in \mathscr{H}$ for each $\xi \in E^*$. In fact, as we now show, these two properties characterize $\mathscr{H}(E)$, at least for a wide class of spaces E.

Recall that E has the *bounded approximation property* if there exists a constant C such that for each $\varepsilon > 0$ and each compact subset K of E, we can find a finite-rank operator γ on E with

$$|\gamma| \leqslant C \qquad \text{and} \qquad \sup_{\mathbf{x} \in K} |\gamma(\mathbf{x}) - \mathbf{x}| \leqslant \varepsilon.$$

Every Banach space with a Schauder basis has the bounded approximation property, so this includes most of the classical Banach spaces. For more details, see [11].

1.2. THEOREM. *Suppose that E has the bounded approximation property. Let $\mathbf{Z}$: $[\![0, \infty]\!] \to E$ be a process such that $\mathbf{Z}_\infty \in L^2(\Sigma; E)$ and $\xi(\mathbf{Z}) \in \mathscr{H}$ for each $\xi \in E^*$. Then $\mathbf{Z} \in \mathscr{H}(E)$.*

Proof. We first show that $\mathbf{Z} \in \mathcal{M}(E)$. As E is separable, there is a sequence (ξ_k) in E^* which separates points of E. For each k, since $\xi_k(\mathbf{Z}) \in \mathcal{H} \subset \mathcal{M}$, there is a subset Ω_k of Ω with $P(\Omega_k) = 1$ such that

$$E[\xi_k(\mathbf{Z}_\infty) \mid \mathcal{F}_t] = \xi_k(\mathbf{Z}_t) \qquad \text{for all} \quad (t, \omega) \in [0, \infty] \times \Omega_k.$$

As the ξ_k separate points, it follows that

$$E[\mathbf{Z}_\infty \mid \mathcal{F}_t] = \mathbf{Z}_t \qquad \text{for all} \quad (t, \omega) \in [0, \infty] \times \left(\bigcap_k \Omega_k \right).$$

Thus $\mathbf{Z} \in \mathcal{M}(E)$.

Now take $\varepsilon > 0$. Since $\mathbf{Z} \in L^2(\Sigma; E)$, there exists $\delta > 0$ such that

$$P(A) < \delta \Rightarrow \|\mathbf{Z}_\infty 1_A\|_2 < \varepsilon/(C+1),$$

where C is the constant occurring in the definition of the bounded approximation property. As E is separable, we can find a compact subset K such that

$$P(\mathbf{Z}_\infty \in E \setminus K) < \delta.$$

By the bounded approximation property, there is a finite rank operator γ on E with

$$|\gamma| \leqslant C \qquad \text{and} \qquad \sup_{\mathbf{x} \in K} |\gamma(\mathbf{x}) - \mathbf{x}| \leqslant \varepsilon.$$

As $\xi(\mathbf{Z}) \in \mathcal{H}$ for each $\xi \in E^*$, it follows that $\gamma(\mathbf{Z}) \in \mathcal{H}_0(E)$. Also

$$\begin{aligned} \|\gamma(\mathbf{Z}) - \mathbf{Z}\|_2 &\leqslant \|(\gamma(\mathbf{Z}_\infty) - \mathbf{Z}_\infty) 1_{(\mathbf{Z}_\infty \in E \setminus K)}\|_2 + \|(\gamma(\mathbf{Z}_\infty) - \mathbf{Z}_\infty) 1_{(\mathbf{Z}_\infty \in K)}\|_2 \\ &\leqslant (C+1) \|\mathbf{Z}_\infty 1_{(\mathbf{Z}_\infty \in E \setminus K)}\|_2 + \varepsilon \\ &\leqslant 2\varepsilon. \end{aligned}$$

As ε is arbitrary, we deduce that $\mathbf{Z} \in \mathcal{H}(E)$. ∎

The next result lists some basic stability properties of $\mathcal{H}(E)$, mirroring those of $\mathcal{H}$. They are all easy consequences of Definition 1.1 and [18, Thm. 1.4].

1.3. THEOREM. (a) *If $\mathbf{Z} \in \mathcal{H}(E)$ and $A \in \mathcal{F}_{0-}$, then $\mathbf{Z} 1_A \in \mathcal{H}(E)$.*

(b) *If $\mathbf{Z} \in \mathcal{H}(E)$ and T is a stopping time, then $\mathbf{Z}^T \in \mathcal{H}(E)$.*

(c) *If $\mathbf{Z} \in \mathcal{M}(E)$ and $(\mathbf{Z}^{(k)})$ is a sequence in $\mathcal{H}(E)$ such that $\|\mathbf{Z}^{(k)} - \mathbf{Z}\|_2 \to 0$, then $\mathbf{Z} \in \mathcal{H}(E)$.*

(d) *If $\mathbf{Z} \equiv \mathbf{Z}_0 \in L^2(\mathcal{F}_{0-}; E)$, then $\mathbf{Z} \in \mathcal{H}(E)$.*

(e) *If* $\mathbf{Z}^{(m)} \in \mathscr{H}(E_m)$ $(m = 1, \ldots, n)$, *where* $E_1, \ldots, E_n$ *are separable complex Banach spaces, then* $(\mathbf{Z}^{(1)}, \ldots, \mathbf{Z}^{(n)}) \in \mathscr{H}(E_1 \times \cdots \times E_n)$ (*with any of the usual norms on the product*).

The rest of this section explores the interaction between holomorphic processes and holomorphic functions, culminating in Theorem 1.8. We shall be led to consider products of processes, and this makes it more natural to work with H^∞-type spaces than H^2-type ones. Accordingly we make the following definition.

1.4. DEFINITION. $\mathscr{H}^\infty(E) = \{\mathbf{Z} \in \mathscr{H}(E) : \mathbf{Z}_\infty \in L^\infty(\mathscr{F}_\infty; E)\}$.

The space $\mathscr{H}^\infty(E)$ becomes a Banach space when endowed with the norm

$$\|\mathbf{Z}\|_\infty := \|\mathbf{Z}_\infty\|_\infty.$$

However, norm-convergence is too strong for our purposes, and the following lemma is more useful.

1.5. LEMMA. *Let* $\mathbf{Z}^{(k)} \in \mathscr{H}^\infty(E)$ $(k \geqslant 1)$ *and let* $\mathbf{Z}$: $[\![0, \infty]\!] \to E$ *be a right continuous process. Suppose that* $\mathbf{Z}_t^{(k)} \to \mathbf{Z}_t$ *almost surely for each* t, *and that* $\sup_k \|\mathbf{Z}^{(k)}\|_\infty = C < \infty$. *Then* $\mathbf{Z} \in \mathscr{H}^\infty(E)$.

Proof. For each t we have $\|\mathbf{Z}_t^{(k)}\|_\infty \leqslant \|\mathbf{Z}^{(k)}\|_\infty \leqslant C < \infty$. There $\mathbf{Z}_t^{(k)} \to \mathbf{Z}_t$ almost surely and boundedly, and hence in $L^2(\mathscr{F}_t; E)$ by the dominated convergence theorem. It follows that $\mathbf{Z} \in \mathscr{M}(E)$ and that $\|\mathbf{Z}^{(k)} - \mathbf{Z}\|_2 \to 0$. By Theorem 1.3(c) we deduce that $\mathbf{Z} \in \mathscr{H}(E)$, and since $|\mathbf{Z}_\infty| \leqslant \sup_k |\mathbf{Z}_\infty^{(k)}| \leqslant C$ almost surely, we obtain $\mathbf{Z} \in \mathscr{H}^\infty(E)$. ∎

We shall need to know that bounded holomorphic processes are approximable by simple holomorphic processes that are also bounded. The next result fulfils this role, and is actually more general than we need now; its full strength will be used later.

1.6. LEMMA. *Let* $\mathbf{Z} \in \mathscr{H}(E)$ *and let* C *be a closed subset of* E *such that* $\mathbf{Z}_t \in C$ *for all* t, ω. *Then, given* $\varepsilon > 0$, *there exist* $(\mathbf{Z}^{(k)}) \in \mathscr{H}_0(E)$ *such that*

$$\operatorname{dist}(\mathbf{Z}_t^{(k)}, C) \leqslant \varepsilon \qquad \textit{for all } k, t, \omega$$

and

$$\sup_t |\mathbf{Z}_t^{(k)} - \mathbf{Z}_t| \to 0 \textit{ a.s.} \qquad \textit{as} \quad k \to \infty.$$

In particular, if $\mathbf{Z} \in \mathscr{H}^\infty(E)$, *then* (*taking* C *to be the ball of radius* $\|\mathbf{Z}\|_\infty$) *the* $\mathbf{Z}^{(k)}$ *can be chosen so that*

$$\|\mathbf{Z}^{(k)}\|_\infty \leqslant \|\mathbf{Z}\|_\infty + \varepsilon$$

and

$$\|\mathbf{Z}^{(k)}-\mathbf{Z}\|_2\to 0 \qquad as \quad k\to\infty.$$

Proof. We may suppose without loss of generality that $\mathbf{0}\in C$. By definition of $\mathscr{H}(E)$, we can find processes $(\mathbf{W}^{(k)})\in\mathscr{H}_0(E)$ such that $\|\mathbf{W}^{(k)}-\mathbf{Z}\|_2\leqslant 2^{-k}$ for each k. Define random times (T_k) by

$$T_k=\inf\{t\geqslant 0:\operatorname{dist}(\mathbf{W}_t^{(k)}, C)\geqslant\varepsilon\}.$$

By Theorem 0.2 each T_k is a predictable time. For each k set

$$\mathbf{Z}^{(k)}=(\mathbf{W}^{(k)})^{T_k}\,1_{(T_k>0)}.$$

Then $\mathbf{Z}^{(k)}\in\mathscr{H}_0(E)$ and $\operatorname{dist}(\mathbf{Z}_t^{(k)}, C)\leqslant\varepsilon$ for all k, t, ω. Now applying the Chebychev and Doob inequalities yields

$$\begin{aligned}P(T_k<\infty)&\leqslant P(\sup_t|\mathbf{W}_t^{(k)}-\mathbf{Z}_t|\geqslant\varepsilon)\\&\leqslant E[\sup_t|\mathbf{W}_t^{(k)}-\mathbf{Z}_t|^2]/\varepsilon^2\\&\leqslant 4\,\|\mathbf{W}^{(k)}-\mathbf{Z}\|_2^2/\varepsilon^2\\&\leqslant 4.2^{-2k}/\varepsilon^2,\end{aligned}$$

so by the first Borel-Cantelli lemma, $P(T_k=\infty$ for all large enough $k)=1$. Thus $\sup_t|\mathbf{Z}_t^{(k)}-\mathbf{Z}_t|\to 0$ almost surely, provided that $\sup_t|\mathbf{W}_t^{(k)}-\mathbf{Z}_t|\to 0$ almost surely. And indeed, using Doob's inequality once again we have

$$\left\|\sum_1^\infty\sup_t|\mathbf{W}_t^{(k)}-\mathbf{Z}_t|\right\|_2\leqslant 2\sum_1^\infty\|\mathbf{W}^{(k)}-\mathbf{Z}\|_2\leqslant 2\sum_1^\infty 2^{-k}<\infty,$$

so that in fact $\sum_1^\infty\sup_t|\mathbf{W}_t^{(k)}-\mathbf{Z}_t|<\infty$ almost surely. ∎

The proof of the main theorem will make use of the following special case. It may be regarded as a vector-valued generalization of the crucial fact that the product of two processes in $\mathscr{H}^\infty$ again belongs to $\mathscr{H}^\infty$ [18, Coro. 1.10].

1.7. LEMMA. *Let $E_1, ..., E_n$, F be separable complex Banach spaces and let $\beta: E_1\times\cdots\times E_n\to F$ be a continuous n-linear map. If $\mathbf{Z}^{(m)}\in\mathscr{H}^\infty(E_m)$ $(m=1, ..., n)$, then $\beta(\mathbf{Z}^{(1)}, ..., \mathbf{Z}^{(n)})\in\mathscr{H}^\infty(F)$.*

Proof. If the $\mathbf{Z}^{(m)}$ belong to $\mathscr{H}^\infty(E_m)\cap\mathscr{H}_0(E_m)$, then this is an easy consequence of the result about $\mathscr{H}^\infty$ mentioned above [18, Coro. 1.10].

The general case follows by an obvious approximation argument, using Lemmas 1.6 and 1.5. ∎

We now state our main result.

1.8. THEOREM. *Let E and F be separable complex Banach spaces, let U be an open subset of E, and let $f\colon U \to F$ be a bounded holomorphic function. If $\mathbf{Z} \in \mathscr{H}(E)$ and $\mathbf{Z}_t \in U$ for all t, ω, then $f(\mathbf{Z}) \in \mathscr{H}^\infty(F)$.*

Remarks. (a) By holomorphic is meant Fréchet differentiable on U. Thus f has derivatives of all orders on U, the nth derivative at $\mathbf{x}$ being a continuous symmetric n-linear map $D^n f(\mathbf{x})\colon E \times \cdots \times E \to F$. We shall write $D^n f(\mathbf{x}; \mathbf{h})$ for $D^n f(\mathbf{x})(\mathbf{h}, .., \mathbf{h})$. In this notation Cauchy's inequality becomes

$$\|D^n f(\mathbf{x}; \mathbf{h})\| \leqslant n! \sup_U |f| \cdot [|\mathbf{h}|/\mathrm{dist}(\mathbf{x}, \partial U)]^n.$$

For further information about holomorphic functions see, for example, [12].

(b) The finite-dimensional version of Theorem 1.8 was proved in [18, Thm. 1.9]. We have already made use of the special case [18, Coro. 1.10] in the proof of the last lemma.

(c) In the course of the proof of Theorem 1.8, it will be convenient to exploit the following technical device, first introduced in [18, Defn. 1.17]. Given a predictable time S, we define

$${}^S\mathscr{H} = L^2(\mathscr{F}_{S-}) \oplus \{Z \in \mathscr{H} : Z_S = 0\}.$$

It was shown in [18, Thm. 1.18] that ${}^S\mathscr{H}$ is a holomorphic atlas (with respect to a modified filtration).

Proof of Theorem 1.8. Let Γ be the set of predictable times T such that $f(\mathbf{Z}^T) \in \mathscr{H}^\infty(F)$. By Theorem 1.3(d), we certainly have $0 \in \Gamma$. Also from Lemma 1.5 it follows that if $T_k \in \Gamma$ and $T_k \uparrow T$, then $T \in \Gamma$. Thus by Theorem 0.1 Γ has a maximal element, S say. The result will follow once we have proved that $S \equiv \infty$.

Suppose then, for a contradiction, that $S \not\equiv \infty$. Choose $\delta > 0$ such that

$$P(S < \infty \text{ and } \mathrm{dist}(\mathbf{Z}_S, \partial U) > 2\delta) > 0,$$

and then set

$$T = \inf\{t \geqslant S : |\mathbf{Z}_t - \mathbf{Z}_S| \geqslant \delta \text{ or } \mathrm{dist}(\mathbf{Z}_t, \partial U) \leqslant 2\delta\}.$$

By Theorem 0.2 T is a predictable time, and by construction $T \geqslant S$ and $T \not\equiv S$. We claim that $f(\mathbf{Z}^T) \in \mathscr{H}^\infty(F)$. If so, then $T \in \Gamma$, giving the required contradiction to the maximality of S.

It remains to justify the claim. By Taylor's theorem

$$f(\mathbf{Z}^T)-f(\mathbf{Z}^S)=\sum_{n=1}^{\infty}(1/n!)\,D^nf(\mathbf{Z}_S;\mathbf{Z}^T-\mathbf{Z}^S)\,1_{(T>S)},\qquad (*)$$

the convergence being uniform, since by Cauchy's inequality

$$\|(1/n!)\,D^nf(\mathbf{Z}_S;\mathbf{Z}^T-\mathbf{Z}^S)\,1_{(T>S)}\|\leqslant\sup_U|f|\cdot 2^{-n}\qquad\text{for all } n.$$

Now for each n, the random variable $D^nf(\mathbf{Z}_S)\,1_{(T>S)}$ belongs to $L^\infty(\mathscr{F}_{S-},\mathscr{L}^n(E,F))$, where $\mathscr{L}^n(E,F)$ denotes the space of continuous n-linear maps of $E\times\cdots\times E$ to F. Hence in particular

$$D^nf(\mathbf{Z}_S)\,1_{(T>S)}\in({}^S\mathscr{H})^\infty\,(\mathscr{L}^n(E,F)).$$

It is also evident that

$$\mathbf{Z}^T-\mathbf{Z}^S\in({}^S\mathscr{H})^\infty\,(E).$$

Applying Lemma 1.7, with $\beta\colon\mathscr{L}^n(E,F)\times E\times\cdots\times E\to F$ given by

$$\beta(\alpha,\mathbf{x}_1,\ldots,\mathbf{x}_n)=\alpha(\mathbf{x}_1,\ldots,\mathbf{x}_n),$$

and with $\mathscr{H}$ replaced by ${}^S\mathscr{H}$, we conclude that

$$D^nf(\mathbf{Z}_S;\mathbf{Z}^T-\mathbf{Z}^S)\,1_{(T>S)}\in({}^S\mathscr{H})^\infty\,(F).$$

This holds for each $n\geqslant 1$, so from $(*)$ it follows that $f(\mathbf{Z}^T)-f(\mathbf{Z}^S)\in({}^S\mathscr{H})^\infty\,(F)$. Now at $t=0$ we have $f(\mathbf{Z}_0^T)-f(\mathbf{Z}_0^S)=\mathbf{0}$, and so in fact $f(\mathbf{Z}^T)-f(\mathbf{Z}^S)\in\mathscr{H}^\infty(F)$. Finally, since we already know that $f(\mathbf{Z}^S)\in\mathscr{H}^\infty(F)$, it follows that $f(\mathbf{Z}^T)\in\mathscr{H}^\infty(F)$, as claimed. ∎

2. Subharmonic Processes and Plurisubharmonic Functions

Subharmonic processes were introduced in [18, Sect. 2] as a stochastic analogue of subharmonic functions. They possess a wide range of properties, which can be used to analyse holomorphic processes, the key connection being the fact [18, Thm. 2.12] that a plurisubharmonic function of a holomorphic process is a subharmonic process. Our goal in this section is to extend this result to processes with values in a Banach space.

We begin by recalling the basic definition.

DEFINITION [18, Defn. 2.1]. A process Φ: $[\![0, \infty]\!] \to [-\infty, \infty)$ is subharmonic if it is bounded above, predictable, right continuous, and satisfies

$$E[\Phi_t \mid \mathscr{F}_s] \geqslant \Phi_s \text{ a.s.}$$

whenever $0 \leqslant s \leqslant t \leqslant \infty$. The class of such processes is denoted by $\mathscr{S}$.

For the rest of this section, $\mathscr{H}$ denotes a fixed holomorphic atlas and $(E, |\cdot|)$ a separable complex Banach space.

We recall that, given an open subset U of E, a function φ: $U \to [-\infty, \infty)$ is *plurisubharmonic* if it is upper semicontinuous, and if for all $\mathbf{x}, \mathbf{y} \in E$ the map

$$\lambda \mapsto \varphi(\mathbf{x} + \lambda \mathbf{y}) : \{\lambda \in \mathbb{C} : \mathbf{x} + \lambda \mathbf{y} \in U\} \to [-\infty, \infty)$$

is subharmonic. For further details, see [12]. Our main result is the following theorem.

2.1. THEOREM. *Let U be open in E and let φ: $U \to [-\infty, \infty)$ be a plurisubharmonic function which is bounded above on U. If $\mathbf{Z} \in \mathscr{H}(E)$ and $\mathbf{Z}_t \in U$ for all t, ω, then $\varphi(\mathbf{Z}) \in \mathscr{S}$.*

Remark. The finite-dimensional version of this result was proved in [18, Thm. 2.12]. It will be used in the proof below.

Proof. Assume first that φ is also continuous, and that there exists $\delta > 0$ such that $\operatorname{dist}(\mathbf{Z}_t, \partial U) \geqslant \delta$ for all t, ω. Using Lemma 1.6 we can find processes $(\mathbf{Z}^{(k)}) \in \mathscr{H}_0(E)$ such that $\mathbf{Z}_t^{(k)} \in U$ for all k, t, ω and

$$\sup_t |\mathbf{Z}_t^{(k)} - \mathbf{Z}_t| \to 0 \text{ a.s.} \qquad \text{as} \quad k \to \infty.$$

Now $\varphi(\mathbf{Z}^{(k)}) \in \mathscr{S}$ for each k, by the finite-dimensional result mentioned above. As φ is continuous, it follows from [18, Thm. 2.9(i)] that $\varphi(\mathbf{Z}) \in \mathscr{S}$ in this case.

Now let us consider the general case. Without loss of generality, we can suppose that U contains the unit ball of E. Let $k \geqslant 2$ and set

$$U_k = \{\mathbf{x} \in U : \operatorname{dist}(\mathbf{x}, \partial U) > 1/k\},$$
$$T_k = \inf\{t \geqslant 0 : \operatorname{dist}(\mathbf{Z}_t, \partial U) \leqslant 2/k\}.$$

Then T_k is a predictable time by Theorem 0.2, and hence $\mathbf{Z}^{T_k} 1_{(T_k > 0)} \in \mathscr{H}(E)$ by Theorem 1.3. Also $\mathbf{Z}_t^{T_k} 1_{(T_k > 0)} \in U_k$ for all t, ω. For $r \in (0, 1/k)$ define φ_r: $U_k \to [-\infty, \infty)$ by

$$\varphi_r(\mathbf{x}) = \sup_{|\mathbf{h}| < r} \varphi(\mathbf{x} + \mathbf{h}).$$

Clearly φ_r is lower semicontinuous on U_k. Also, using Hadamard's three-circles theorem (see, e.g., [1, p. 166, Thm. 4]) we have that for $0<r<s<1/k$ and $\mathbf{x}\in U_k$

$$\varphi_s(\mathbf{x})\leqslant\left(\frac{\log(1/k)-\log s}{\log(1/k)-\log r}\right)\varphi_r(\mathbf{x})+\left(\frac{\log s-\log r}{\log(1/k)-\log r}\right)\sup_U \varphi,$$

from which it follows that φ_r is upper semicontinuous on U_k, and hence continuous there. It is also easily checked that φ_r is plurisubharmonic. Thus by the first part of the proof $\varphi_r(\mathbf{Z}^{T_k}1_{(T_k>0)})\in\mathscr{S}$. Now $\varphi_r\downarrow\varphi$ on U_k as $r\to 0$, so by [18, Thm. 2.9(ii)], $\varphi(\mathbf{Z}^{T_k}1_{(T_k>0)})\in\mathscr{S}$. Finally, $T_k\uparrow\infty$ as $k\to\infty$, so by [18, Thm. 2.9(i)] again we have $\varphi(\mathbf{Z})\in\mathscr{S}$. ∎

When $\mathbf{Z}\in\mathscr{M}(E)$ is bounded and predictable, then it follows from Theorem 0.3 that $|\mathbf{Z}|\in\mathscr{S}$. If further $\mathbf{Z}$ is holomorphic we can improve on this.

2.2. Corollary. (a) *If* $\mathbf{Z}\in\mathscr{H}^\infty(E)$ *then* $\log|\mathbf{Z}|\in\mathscr{S}$.

(b) *If* $\mathbf{Z}\in\mathscr{H}^\infty(A)$, *where A is a separable Banach algebra, then* $\log\rho(\mathbf{Z})\in\mathscr{S}$, *where ρ denotes the spectral radius. Hence also* $\rho(\mathbf{Z})\in\mathscr{S}$.

Proof. Both $\log|\mathbf{x}|$ and $\log\rho(\mathbf{x})$ are plurisubharmonic functions (the latter by Vesentini's theorem [1, Thm. 1.2.1]), so this is an immediate consequence of Theorem 2.1. The final clause in (b) follows from [18, Prop. 2.8(iv)]. ∎

We shall give another proof of Corollary 2.2 in Section 5. Given the result in (b), it is natural to ask how the whole spectrum Sp($\mathbf{Z}$) behaves. An answer is provided by the next section.

3. Subholomorphic Processes and Analytic Multifunctions

Subholomorphic proceses were introduced and studied in [18, Sect. 3, 4], the idea being that they provided a stochastic analogue of analytic multifunctions. In this section we make a direct link between the two concepts by showing that an analytic multifunction of a holomorphic process is a subholomorphic process.

We begin by recalling the basic definitions.

Definition [18, Defn. 3.1, 3.3, 3.4]. (a) We denote by $\mathscr{K}$ the class of all processes K: $[\![0,\infty]\!]\to\kappa(\mathbb{C})$ which are bounded, predictable, upper semicontinuous and weakly right lower semicontinuous (see Section 0).

(b) Given a subclass $\mathscr{D}$ of $\mathscr{K}$ and a process K: $[\![0, \infty]\!] \to \kappa(\mathbb{C})$, we say K has *local $\mathscr{D}$-selections* if it satisfies the following condition: given any predictable time $R \not\equiv \infty$, there exist processes $(L^{(k)})_{k \geqslant 1} \in \mathscr{D}$, and a predictable time $R' \geqslant R$ with $R' \not\equiv R$, such that that

$$\partial K_R \subset \overline{\bigcup_{k \geqslant 1} L_R^{(k)}} \qquad \text{on} \quad \{R < R'\},$$

and such that for all $k \geqslant 1$

$$L^{(k)} \chi_{[\![R, R']\!]} \subset K \chi_{[\![R, R']\!]} \qquad \text{on} \quad \{R < R'\}.$$

(c) Given a holomorphic atlas $\mathscr{H}$, the class $\mathscr{K}\mathscr{H}$ of *subholomorphic processes* is the smallest subclass $\mathscr{D}$ of $\mathscr{K}$ satisfying:

(i) if S is a predictable time and $Z \in ({}^S\mathscr{H})^\infty$, then $\{Z\}_{\chi[\![S, \infty]\!]} \in \mathscr{D}$;

(ii) if $K \in \mathscr{K}$ and there exists a sequence $(K^{(k)})$ in $\mathscr{D}$ such that $K^{(k)} \downarrow K$, then $K \in \mathscr{D}$;

(iii) if $K \in \mathscr{K}$ and K has local $\mathscr{D}$-selections, then $K \in \mathscr{D}$.

For the rest of this section, $\mathscr{H}$ denotes a fixed holomorphic atlas for $\mathscr{F}$, and $(E, |\cdot|)$ a separable complex Banach space.

An open set P in a complex Banach space is called *pseudoconvex* if the function $\mathbf{x} \mapsto -\log \operatorname{dist}(\mathbf{x}, \partial P)$ is plurisubharmonic on P. It remains pseudoconvex if the norm is replaced by any equivalent one. For further information about pseudoconvexity see, for example, [12].

Let U be an open subset of E. A map F: $U \to \kappa(\mathbb{C})$ is *upper semicontinuous* if, for each open $V \subset \mathbb{C}$, the set $\{\mathbf{x} \in U : F(\mathbf{x}) \subset V\}$ is open in U. It is called an *analytic multifunction* if, in addition, the set

$$\{(\mathbf{x}, z) \in U \times \mathbb{C} : z \notin F(\mathbf{x})\}$$

is pseudoconvex in $E \times \mathbb{C}$ (with any of its usual norms). Simple examples include $F(\mathbf{x}) = \{f(\mathbf{x})\}$, where f: $U \to \mathbb{C}$ is holomorphic, and $F(\mathbf{x}) = \bar{\Delta}(0, e^{\varphi(\mathbf{x})})$, where φ: $U \to [-\infty, \infty)$ is plurisubharmonic. For more information about analytic multifunctions see the references cited in the Introduction, particularly [20, 21].

We first consider what happens in finite dimensions.

3.1. THEOREM. *Let $n \geqslant 1$, let V be open in $\mathbb{C}^n$, and let F: $V \to \kappa(\mathbb{C})$ be a bounded analytic multifunction. Let $K^{(1)}, ..., K^{(n)} \in \mathscr{K}\mathscr{H}$, and suppose that $K_t^{(1)} \times \cdots \times K_t^{(n)} \subset V$ for all t, ω. Then $F(K^{(1)}, ..., K^{(n)}) \in \mathscr{K}\mathscr{H}$, where*

$$F(K_t^{(1)}, ..., K_t^{(n)}) = \bigcup \{F(z_1, ..., z_n) : (z_1, ..., z_n) \in K_t^{(1)} \times \cdots \times K_t^{(n)}\}.$$

Remarks. (a) In fact this result goes beyond what was promised, since it covers analytic multifunctions of subholomorphic processes, not just holomorphic ones. If anything, this makes the proof easier!

(b) By taking as particular F's the two examples cited just before Theorem 3.1, we see that it contains [18, Thm. 3.12, 3.17] as special cases. Its proof also follows similar lines, and so we omit some of the details, highlighting only the main points of difference.

Let U be open in E. A map $F: U \to \mathscr{K}(\mathbb{C})$ is *lower semicontinuous* if, for each closed $C \subset \mathbb{C}$, the set $\{\mathbf{x} \in U : F(\mathbf{x}) \subset C\}$ is closed in U.

3.2. LEMMA. *Let n, V, F be as in Theorem* 3.1, *let $K^{(1)}, ..., K^{(n)} \in \mathscr{K}$, and suppose that $K_t^{(1)} \times \cdots \times K_t^{(n)} \subset V$ for all t, ω. Then $F(K^{(1)}, ..., K^{(n)})$ is bounded, predictable and upper semicontinuous. If further F is lower semicontinuous, then $F(K^{(1)}, ..., K^{(n)}) \in \mathscr{K}$.*

Proof. This is just like the proof of [18, Lemma 3.13], the point to note being that

$$\partial F(K_t^{(1)}, ..., K_t^{(n)}) \subset F(\partial K_t^{(1)}, ..., \partial K_t^{(n)}) \qquad \text{for all } t, \omega,$$

which follows easily from the one-variable result [17, Prop. 2.1]. ∎

Let V be open in $\mathbb{C}^n$. Following [25], we call an analytic multifunction $F: V \to \kappa(\mathbb{C})$ *locally trivial* if V can be covered by open subsets on each of which the graph of F is a union of graphs of holomorphic functions. A simple separability argument then shows that there is a countable open cover (V_j) of V, such that for each j there exists a sequence of holomorphic functions $f_{jk}: V_j \to \mathbb{C}$ for which

$$F(\mathbf{z}) = \overline{\bigcup_k \{f_{jk}(\mathbf{z})\}} \qquad (\mathbf{z} \in V_j).$$

3.3. LEMMA. *Let n, V, F be as in Theorem* 3.1, *and assume further that F is locally trivial. Let $Z^{(1)}, ..., Z^{(n)} \in \mathscr{H}^\infty$, and suppose that $\mathbf{Z}_t := (Z_t^{(1)}, ..., Z_t^{(n)}) \in V$ for all t, ω. Then $F(\mathbf{Z}) \in \mathscr{K}\mathscr{H}$.*

Proof. As F is locally trivial, it is certainly lower semicontinuous, so by Lemma 3.2 we have $F(\mathbf{Z}) \in \mathscr{K}$. Thus it suffices to check that $F(\mathbf{Z})$ has local $\mathscr{K}\mathscr{H}$-selections.

Let R be a predictable time with $R \not\equiv \infty$. Take (V_j) and (f_{jk}) as in the preamble to the Lemma. Then there exists j such that

$$P(R < \infty \text{ and } \mathbf{Z}_R \in V_j) > 0.$$

Using Theorem 0.2, we can find a predictable time $R' \geqslant R$ with $R' \not\equiv R$ such that

$$\{\mathbf{Z}\} \chi_{[\![R, R']\!]} \subset V_j \qquad \text{on} \quad \{R < R'\}.$$

For $m = 1, \ldots, n$, define $L^{(m)}: [\![0, \infty]\!] \to \kappa(\mathbb{C})$ by

$$L^{(m)} = \{\mathbf{Z}^{(m)}\}^{R'} \chi_{[\![R, R']\!]} \chi_{\{R < R'\}}.$$

By [18, Thm. 3.7] each $L^{(m)} \in \mathscr{K}\mathscr{H}$, and so by [18, Thm. 3.12] it follows that $f_{jk}(L^{(1)}, \ldots, L^{(n)}) \in \mathscr{K}\mathscr{H}$ for all k. Also

$$F(\mathbf{Z}) \chi_{[\![R, R']\!]} = \overline{\bigcup_k f_{jk}(L^{(1)}, \ldots, L^{(n)})} \chi_{[\![R, R']\!]} \qquad \text{on} \quad \{R < R'\}.$$

Hence $F(\mathbf{Z})$ does indeed have local $\mathscr{K}\mathscr{H}$-selections. ∎

3.4. Lemma. *Let n, V, F be as in Theorem 3.1, and assume further that F is locally trivial. Let $U_1, \ldots, U_n$ be open subsets of $\mathbb{C}$ such that $U_1 \times \cdots \times U_n \subset V$. Let m be an integer with $0 \leqslant m \leqslant n$, let $K^{(j)} \in \mathscr{K}\mathscr{H}(U_j)$ $(j = 1, \ldots, m)$ and let $Z^{(j)} \in ({}^{S_j}\mathscr{H})^{\infty}(U_j)$ $(j = m+1, \ldots, n)$, where the S_j are predictable times. Then*

$$F(K^{(1)}, \ldots, K^{(m)}, \{Z^{(m+1)}\} \chi_{[\![S_{m+1}, \infty]\!]}, \ldots, \{Z^{(n)}\} \chi_{[\![S_n, \infty]\!]}) \in \mathscr{K}\mathscr{H}.$$

Proof. This is by induction on m, just as in the proof of [18, Lemma 3.14], using Lemma 3.3 as its starting point instead of [18, Thm. 1.9]. ∎

3.5. Lemma. *Let n, V, F and $K^{(1)}, \ldots, K^{(n)}$ be as in Theorem 3.1, and assume further that F is locally trivial. Then $F(K^{(1)}, \ldots, K^{(n)}) \in \mathscr{K}\mathscr{H}$.*

Proof. By Lemma 3.2 we have $F(K^{(1)}, \ldots, K^{(n)}) \in \mathscr{K}$, so it is enough to show that $F(K^{(1)}, \ldots, K^{(n)})$ has local $\mathscr{K}\mathscr{H}$-selections. This is done just as in the proof of [18, Thm. 3.12], using Lemma 3.4 instead of [18, Lemma 3.14]. ∎

Proof of Theorem 3.1. Let $(V_j)_{j \geqslant 0}$ be relatively compact open subsets of V such that $\bar{V}_j \subset V_{j+1}$ for all j and $\bigcup_j V_j = V$. Given $j \geqslant 1$, define a predictable time T_j by

$$T_j = \inf\{t \geqslant 0 : K_t^{(1)} \times \cdots \times K_t^{(n)} \not\subset V_{j-1}\},$$

and for $m = 1, \ldots, n$, define $L^{(m)}: [\![0, \infty]\!] \to \kappa(\mathbb{C})$ by

$$L^{(m)} = (K^{(m)})^{T_j} \chi_{(T_j > 0)}.$$

Then each $L^{(m)} \in \mathscr{K}\mathscr{H}$ by [18, Thm. 3.7], and $L_t^{(1)} \times \cdots \times L_t^{(n)} \subset V_j$ for all t, ω. Now by an approximation theorem of Słodkowski [25], there exist locally trivial analytic multifunctions $F_k: V_j \to \kappa(\mathbb{C})$ such that $F_k(\mathbf{z}) \downarrow F(\mathbf{z})$ for each $\mathbf{z} \in V_j$. From Lemma 3.5 we have $F_k(L^{(1)}, \ldots, L^{(n)}) \in \mathscr{K}\mathscr{H}$ for each k, and so it follows by [18, Coro. 3.20] that $F(L^{(1)}, ..., L^{(n)}) \in \mathscr{K}\mathscr{H}$. In other words

$$F(K^{(1)}, ..., K^{(n)})^{T_j} \chi_{(T_j > 0)} \in \mathscr{K}\mathscr{H}. \qquad (*)$$

This is true for each j. Since $T_j \uparrow \infty$, we deduce that $F(K^{(1)}, ..., K^{(n)})$ is weakly right lower semicontinuous. From Lemma 3.2 we already know that it is bounded, predictable and upper semicontinuous, and hence $F(K^{(1)}, ..., K^{(n)}) \in \mathscr{K}$. Also $(*)$ implies that $F(K^{(1)}, ..., K^{(n)})$ has local $\mathscr{K}\mathscr{H}$-selections, and so finally we obtain $F(K^{(1)}, ..., K^{(n)}) \in \mathscr{K}\mathscr{H}$. ▮

We now extend to infinite dimensions.

3.6. THEOREM. *Let E be a separable complex Banach space, let U be an open subset of E, and let $F: U \to \kappa(\mathbb{C})$ be a bounded analytic multifunction. Let $\mathbf{Z} \in \mathscr{H}(E)$, and suppose that $\mathbf{Z}_t \in U$ for all t, ω. Then $F(\mathbf{Z}) \in \mathscr{K}\mathscr{H}$.*

Remark. By taking as particular F's the two examples cited just before Theorem 3.1, we recover Theorem 1.8 (at least for scalar-valued functions) and Theorem 2.1. Nevertheless, it seems worth including the more elementary proofs given in Sections 1 and 2 because they avoid the difficult theorem in [25].

Proof. We can suppose throughout, without loss of generality, that U contains the unit ball of E.

Assume first that $\mathbf{Z}$ is also bounded, and that there exists $\delta \in (0, 1)$ such that $\operatorname{dist}(\mathbf{Z}_t, \partial U) \geqslant 3\delta$ for all t, ω. Put

$$U' = \{\mathbf{x} \in U : \operatorname{dist}(\mathbf{x}, \partial U) > \delta\},$$

and for $0 < r < \delta$ define $F_r: U' \to \kappa(\mathbb{C})$ by

$$F_r(\mathbf{x}) = \bigcap_{s > r} \overline{\bigcup_{|\mathbf{h}| \leqslant s} F(\mathbf{x} + \mathbf{h})}.$$

Then F_r is an analytic multifunction for each r (this follows from standard properties of pseudoconvex sets—see, e.g., [12, Sect. 37]), and $F_r(\mathbf{x}) \downarrow F(\mathbf{x})$ as $r \to 0$ for each $\mathbf{x} \in U'$. Now by Lemma 1.6 we can find processes $(\mathbf{Z}^{(j)}) \in \mathscr{H}_0(E)$ such that $\mathbf{Z}_t^{(j)} \in U'$ for all j, t, ω and

$$\sup_t |\mathbf{Z}_t^{(j)} - \mathbf{Z}_t| \to 0 \text{ a.s.} \qquad \text{as } j \to \infty.$$

Replacing the $\mathbf{Z}^{(j)}$ by a subsequence, if necessary, we can further ensure that

$$P(\sup_t |\mathbf{Z}_t^{(j)} - \mathbf{Z}_t| \geqslant 2^{-j}) \leqslant 2^{-j} \qquad \text{for all } j.$$

Define predictable times (R_j) and (S_k) by

$$R_j = \inf\{t \geqslant 0 : |\mathbf{Z}_t^{(j)} - \mathbf{Z}_t| \geqslant 2^{-j}\},$$
$$S_k = \inf_{j \geqslant k} R_j.$$

Then $P(S_k < \infty) \leqslant \sum_{j \geqslant k} 2^{-j} = 2.2^{-k}$, so by Borel–Cantelli $S_k \uparrow \infty$ as $k \to \infty$. Fix $k \geqslant 2$. Then for all $j \geqslant k$ we have

$$\|\mathbf{Z}^{(j)S_k} 1_{(S_k > 0)} - \mathbf{Z}^{S_k} 1_{(S_k > 0)}\|_\infty \leqslant 2^{-j},$$

and so

$$F_{1/j}(\mathbf{Z}^{(j)S_k} 1_{(S_k > 0)}) \downarrow F(\mathbf{Z}^{S_k} 1_{(S_k > 0)}) \qquad \text{as} \quad j \to \infty.$$

Moreover, since $\mathbf{Z}^{(j)S_k} 1_{(S_k > 0)} \in \mathscr{H}_0(E) \cap \mathscr{H}^\infty(E)$, it follows by Theorem 3.1 that $F_{1/j}(\mathbf{Z}^{(j)S_k} 1_{(S_k > 0)}) \in \mathscr{K}\mathscr{H}$ for each j. Hence by [18, Coro. 3.20]

$$F(\mathbf{Z}^{S_k} 1_{(S_k > 0)}) \in \mathscr{K}\mathscr{H}.$$

Since $S_k \uparrow \infty$, we deduce, as in the proof of Theorem 3.1, that $F(\mathbf{Z}) \in \mathscr{K}\mathscr{H}$.

Now let us consider the general case. Let $k \geqslant 2$, and set

$$U_k = \{\mathbf{x} \in U : |\mathbf{x}| < k \text{ and } \operatorname{dist}(\mathbf{x}, \partial U) > 1/k\},$$
$$T_k = \inf\{t \geqslant 0 : |\mathbf{Z}_t| \geqslant k/2 \text{ and } \operatorname{dist}(\mathbf{Z}_t, \partial U) \leqslant 2/k\}.$$

Then $\mathbf{Z}_t^{T_k} 1_{(T_k > 0)} \in U_k$ for all t, ω, so by the first part of the proof

$$F(\mathbf{Z}^{T_k} 1_{(T_k > 0)}) \in \mathscr{K}\mathscr{H},$$

and hence, once again, we deduce that $F(\mathbf{Z}) \in \mathscr{K}\mathscr{H}$. ∎

Finally, we can answer the question raised at the end of the previous section.

3.7. COROLLARY. *Let A be a separable complex Banach algebra, and let $\mathbf{Z} \in \mathscr{H}^\infty(A)$. Then* $\operatorname{Sp}(\mathbf{Z}) \in \mathscr{K}\mathscr{H}$, *where* Sp *denotes the spectrum.*

Proof. This follows immediately from the fact that $\operatorname{Sp}: A \to \kappa(\mathbb{C})$ is an analytic multifunction (see, e.g., [15, 20]). ∎

4. BROWNIAN MOTION

We asked in the Introduction what sort of process one gets by taking an analytic multifunction of Brownian motion. In this section we apply the preceding theory to obtain an answer.

For the time being, we assume that (Ω, Σ, P) is a complete probability space, but with no filtration $\mathscr{F}$ or holomorphic atlas $\mathscr{H}$ given as yet.

4.1. DEFINITION. (a) A process B: $[\![0, \infty))\to \mathbb{C}$ is a *complex Brownian motion* if it is continuous, $B_0 = 0$, and for some constant $c \in \mathbb{R}$

$$E[e^{i \operatorname{Re}(\lambda(B_t - B_s))} \mid \mathscr{G}_s] = e^{-c^2 |\lambda|^2 (t-s)/2} \qquad (\lambda \in \mathbb{C}, 0 \leqslant s \leqslant t < \infty),$$

where $\mathscr{G}_s$ denotes the σ-field generated by $\{B_r : 0 \leqslant r \leqslant s\}$ together with the P-null sets. We call B *normalized* if, in addition, $c = 1$. (This represents a slight change of terminology from [18, p. 145].)

(b) A processs $\mathbf{B}$: $[\![0, \infty)) \to E$ is an *E-valued Brownian motion* if $\xi(\mathbf{B})$ is a complex Brownian motion for each $\xi \in E^*$.

4.2. THEOREM. *Let* $\mathbf{B}$ *be an E-valued Brownian motion on* (Ω, Σ, P), *and let* $\mathscr{F}_{0-}$ *be a complete sub-σ-field of* Σ *which is independent of* $\sigma(\mathbf{B}_t : t \in [0, \infty))$. *Define*

$$\mathscr{F}_t = \sigma(\mathscr{F}_{0-}, \{\mathbf{B}_s : s \in [0, t]\}) \qquad (0 \leqslant t < \infty)$$

$$\mathscr{F}_\infty = \sigma\left(\bigcup_{t<\infty} \mathscr{F}_t\right).$$

Then:

(a) $\mathscr{F}$ *is a complete, right continuous filtration on* (Ω, Σ, P) *with respect to which* $\mathbf{B}$ *is a martingale*;

(b) *there is a unique holomorphic atlas* $\mathscr{H}$ *for* $\mathscr{F}$ *such that* $\mathbf{B}^T \in \mathscr{H}(E)$ *for each bounded stopping time T.*

In proving Theorem 4.2, we shall need some information about the structure of E-valued Brownian motion. This is provided by the following result, whose proof is deferred to the end of the section as it is rather technical.

4.3. LEMMA. *Let* $\mathbf{B}$ *be an E-valued Browian motion. Then* $\mathbf{B}$ *is a continuous process with* $\mathbf{B}_t \in L^2(\Sigma; E)$ *for all* $t \in [0, \infty)$. *Moreover, there exist independent normalized complex Brownian motions* $B^{(1)}, B^{(2)}, \ldots$ *and strongly linearly independent vectors* $\mathbf{x}_1, \mathbf{x}_2, \ldots \in E$ (*i.e.,* $\mathbf{x}_k \notin \overline{\operatorname{span}}\{\mathbf{x}_j : j \neq k\}$ *for each k*) *such that for all* $t \in [0, \infty)$

$$E\left[\sup_{s \leqslant t} \left|\mathbf{B}_s - \sum_{j=1}^{n} B_s^{(j)} \mathbf{x}_j\right|^2\right] \to 0 \qquad \textit{as} \quad n \to \infty. \tag{$*$}$$

Remark. The sequences $(B^{(j)})$ and $(\mathbf{x}_j)$ may be finite, in which case we simply have $\mathbf{B} = \sum_1^n B^{(j)} \mathbf{x}_j$ for some n.

Proof of Theorem 4.2. (a) Choose $B^{(j)}$, $\mathbf{x}_j$ satisfying the conclusion of Lemma 4.3. As the $\mathbf{x}_j$ are strongly linearly independent, the Hahn–Banach theorem provides functionals $(\xi_j) \in E^*$ such that $\xi_j(\mathbf{x}_k) = \delta_{jk}$ for all j, k. In particular, $B^{(j)} = \xi_j(\mathbf{B})$ for all j. It follows easily that for all $t \in [0, \infty)$

$$\mathscr{F}_t = \sigma(\mathscr{F}_{0-}, \{B_s^{(j)} : 0 \leqslant s \leqslant t, j = 1, 2, ...\}),$$

so by [18, Lemma 1.11 or 1.13 (whichever is appropriate)], $\mathscr{F}$ is a complete, right continuous filtration on (Ω, Σ, P). By the same result, each $B^{(j)}$ is a martingale with respect to $\mathscr{F}$, and so it follows from $(*)$ that $\mathbf{B}$ is also a martingale.

(b) Let $\mathscr{H}$ be the class of processes of the form

$$Z_t = U + \sum_j \int_0^t H_0^{(j)}\, dB_s^{(j)}, \qquad (**)$$

where $U \in L^2(\mathscr{F}_{0-})$, and the $H^{(j)}$ are $\mathscr{F}$-predictable processes with

$$\sum_j E\left[\int_0^\infty |H_s^{(j)}|^2\, ds\right] < \infty.$$

By [18, Thm. 1.12 or 1.14 (whichever is appropriate)], this is a holomorphic atlas for $\mathscr{F}$. Now given a bounded stopping time T, we have

$$(B^{(j)})^T = \int_0^\infty 1_{[\![0, T]\!]}\, dB^{(j)}$$

and so $(B^{(j)})^T \in \mathscr{H}$ for all j. Hence $\sum_{j=1}^n (B^{(j)})^T \mathbf{x}_j \in \mathscr{H}_0(E)$ for each n. Finally, putting $t = \operatorname{ess\,sup} T$, we have

$$\left\| \mathbf{B}^T - \sum_{j=1}^n (B^{(j)})^T \mathbf{x}_j \right\|_2^2 \leqslant E\left[\sup_{s \leqslant t} \left|\mathbf{B}_s - \sum_{j=1}^n B_s^{(j)} \mathbf{x}_j\right|^2\right] \to 0 \qquad \text{as } n \to \infty,$$

and so $\mathbf{B}^T \in \mathscr{H}(E)$.

To show that $\mathscr{H}$ is unique, suppose that $\mathscr{H}'$ is another holomorphic atlas satisfying (b). Then for each bounded stopping time T we have $(B^{(j)})^T = \xi_j(\mathbf{B}^T) \in \mathscr{H}'$ for all j, and using [18, Thm. 1.6] we deduce that $Z \in \mathscr{H}'$ whenever Z has the form $(**)$. In other words, $\mathscr{H} \subset \mathscr{H}'$. Therefore also $\bar{\mathscr{H}}_0 \subset \bar{\mathscr{H}}'_0$, where, for example,

$$\bar{\mathscr{H}}_0 = \{Z : \bar{Z} \in \mathscr{H}, Z_0 = 0\}$$

(see Section 0), and since

$$\mathcal{M}^c = \mathcal{H} \oplus \bar{\mathcal{H}}_0 = \mathcal{H}' \oplus \bar{\mathcal{H}}'_0$$

(see [18, Thm. 1.4]), it follows that $\mathcal{H} = \mathcal{H}'$. ∎

Remark. Using the strong Markov property of Brownian motion, it is not hard to show that if T is a stopping time with $P(T < \infty) = 1$, then

$$E\,|\mathbf{B}^T|^2 = E(T)E\,|\mathbf{B}_1|^2.$$

Hence if $E(T) < \infty$, then we still have $\mathbf{B}^T \in \mathcal{H}(E)$.

We can now at last prove the promised result about analytic multifunctions and Brownian motion.

4.4. THEOREM. *Let U be an open subset of E containing $\mathbf{0}$, and let $F: U \to \kappa(\mathbb{C})$ be an analytic multifunction. Let $\mathbf{B}$ be an E-valued Brownian motion, and set*

$$T = \inf\{t \geqslant 0 : \mathbf{B}_t \notin U\}.$$

Then there exist $\mathcal{F}$-predictable times (T_k) with $T_k \uparrow\uparrow T$ such that $F(\mathbf{B}^{T_k}) \in \mathcal{K}\mathcal{H}$ for all k, where $\mathcal{F}$ and $\mathcal{H}$ are respectively the filtration and the holomorphic atlas associated to $\mathbf{B}$ by Theorem 4.2.

Proof. By scaling we can suppose, without loss of generality, that $F(\mathbf{0}) \subset \Delta(0, 1)$. For $k \geqslant 1$ define

$$U_k = \{\mathbf{x} \in U : F(\mathbf{x}) \subset \Delta(0, k)\},$$

$$T_k = k \wedge \inf\{t \geqslant 0 : \operatorname{dist}(\mathbf{B}_t, \partial U_k) \leqslant 1/k\}.$$

Then the T_k are $\mathcal{F}$-predictable times such that $T_k \uparrow\uparrow T$. Also each T_k is bounded, so $\mathbf{B}^{T_k} \in \mathcal{H}(E)$ by Theorem 4.2. Moreover $\mathbf{B}_t^{T_k} \in U_k$ for all t, ω, a set upon which F is bounded. Hence by Theore 3.6 $F(\mathbf{B}^{T_k}) \in \mathcal{K}\mathcal{H}$. ∎

Appendix: *Proof of Lemma* 4.3

The proof proceeds via two auxiliary lemmas.

We shall call a complex-valued random variable *complex Gaussian* if its real and imaginary parts are independent normal random variables with the same variance. It is easily checked that two complex Gaussian random variables G_1, G_2 are independent if and only if $E[G_1 \bar{G}_2] = 0$.

4.5. LEMMA. *Let $\mathbf{G}$: $(\Omega, \Sigma, P) \to E$ be a random variable such that $\xi(\mathbf{G})$*

is complex Gaussian for each $\xi \in E^*$. *Then* $\mathbf{G} \in L^2(\Sigma; E)$, *and there exist* $\mathbf{x}_1, \mathbf{x}_2, \ldots \in E$ *and* $\xi_1, \xi_2, \ldots \in E^*$ *such that*:

$$\text{(a)}\quad E[\xi_j(\mathbf{G})\overline{\xi_k(\mathbf{G})}] = 2\delta_{jk};$$

$$\text{(b)}\quad \left\| \mathbf{G} - \sum_{j=1}^{n} \xi_j(\mathbf{G})\mathbf{x}_j \right\|_2 \to 0 \qquad as \quad n \to \infty.$$

Remarks. (a) The sequences $(\mathbf{x}_j)$ and (ξ_j) may be finite, in which case we just have $\mathbf{G} = \sum_1^n \xi_j(\mathbf{G})\mathbf{x}_j$ for some n. All that follows will be written as if the sequences were infinite, but the interpretation for the finite case will be obvious.

(b) This lemma appears to folklore (see, e.g., [14]), but a proof is included for the sake of completeness.

Proof. By a theorem of Fernique [7], there exists $\varepsilon > 0$ such that $E[e^{\varepsilon |\mathbf{G}|^2}] < \infty$, so certainly $\mathbf{G} \in L^2(\Sigma, E)$.

As E is separable, there is a sequence (η_k) in E^* that separates points of E. The sequence $(\eta_k(\mathbf{G}))$ is contained in $L^2(\Sigma)$, so its closed linear span in $L^2(\Sigma)$ is a separable Hilbert space, H say. Using the Gram–Schmidt algorithm, we can find a sequence $(\xi_j) \in E^*$ such that $(\xi_j(\mathbf{G})/\sqrt{2})$ is orthonormal basis for H. In particular this ensures that (a) holds. For each j set $\mathbf{x}_j = E[\overline{\xi_j(\mathbf{G})}\,\mathbf{G}/2]$: we shall show that (b) holds.

As the $\xi_j(\mathbf{G})$ are orthogonal complex Gaussians, they are independent. Thus if we set $\mathscr{G}_n = \sigma(\xi_1(\mathbf{G}), \ldots, \xi_n(\mathbf{G}))$, then

$$E[\xi_j(\mathbf{G}) \mid \mathscr{G}_n] = \begin{cases} \xi_j(\mathbf{G}), & j \leqslant n, \\ 0, & j > n. \end{cases}$$

Also, since $(\xi_j(\mathbf{G})/\sqrt{2})$ is an orthonormal basis for H, for each η_k we have

$$\eta_k(\mathbf{G}) = \sum_{j=1}^{\infty} E[\eta_k(\mathbf{G})\,\overline{\xi_j(\mathbf{G})}/\sqrt{2}]\,\xi_j(\mathbf{G})/\sqrt{2},$$

with convergence in $L^2(\Sigma)$, and therefore

$$E[\eta_k(\mathbf{G}) \mid \mathscr{G}_n] = \sum_{j=1}^{n} E[\eta_k(\mathbf{G})\overline{\xi_j(\mathbf{G})}/2]\,\xi_j(\mathbf{G}) = \sum_{j=1}^{n} \eta_k(\mathbf{x}_j)\xi_j(\mathbf{G}).$$

As the η_k separate points of E, we conclude that for each n

$$E[\mathbf{G} \mid \mathscr{G}_n] = \sum_{j=1}^{n} \xi_j(\mathbf{G})\mathbf{x}_j.$$

Thus (b) follows from the L^2-martingale convergence theorem. ∎

4.6. LEMMA. *Let* $\mathbf{B}$ *be an E-valued Brownian motion on* (Ω, Σ, P), *and suppose that* $\xi_1, \xi_2, \ldots \in E^*$ *satisfy* $E[\xi_j(\mathbf{B}_1)\overline{\xi_j(\mathbf{B}_1)}] = 2\delta_{jk}$. *Then* $\xi_1(\mathbf{B}), \xi_2(\mathbf{B}), \ldots$ *are independent normalized complex Brownian motions on* (Ω, Σ, P).

Proof. The $\xi_j(\mathbf{B})$ are complex Brownian motions, because $\mathbf{B}$ is an E-valued one, and are normalized since $E\,|\xi_j(\mathbf{B}_1)|^2 = 2$ for all j. The main point is prove is independence. For this, it suffices to show that, given $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_p$, the random variables

$$\{\xi_j(\mathbf{B}_{t_l}) - \xi_j(\mathbf{B}_{t_{l-1}}) : j \geqslant 1, l = 1, \ldots, p\}$$

are independent. Since they are complex Gaussian, it is enough to prove that they are mutually orthogonal, and as the $\xi_j(\mathbf{B})$ have independent increments anyway, this amounts to showing that, given j, k, l, m with $j \neq k$ and $1 \leqslant l, m \leqslant p$, we have

$$E[\xi_j(\mathbf{B}_{t_l} - \mathbf{B}_{t_{l-1}})\,\overline{\xi_k(\mathbf{B}_{t_m} - \mathbf{B}_{t_{m-1}})}] = 0. \tag{†}$$

Case 1: $l \neq m$. In this case, for each $\xi \in E^*$

$$E[\xi(\mathbf{B}_{t_l} - \mathbf{B}_{t_{l-1}})\overline{\xi(\mathbf{B}_{t_m} - \mathbf{B}_{t_{m-1}})}] = 0$$

because complex Brownian motion has independent increments. In particular, this holds when $\xi = \xi_j \pm i\xi_k$, so (†) follows from the polarization identity.

Case 2: $l = m$. For each $\xi \in E^*$

$$E\,|\xi(\mathbf{B}_{t_l}) - \xi(\mathbf{B}_{t_{l-1}})|^2 = (t_l - t_{l-1}) \cdot E\,|\xi(\mathbf{B}_1)|^2,$$

so by the polarization identity again

$$E[\xi_j(\mathbf{B}_{t_l} - \mathbf{B}_{t_{l-1}})\overline{\xi_k(\mathbf{B}_{t_l} - \mathbf{B}_{t_{l-1}})}] = (t_l - t_{l-1}) \cdot E[\xi_j(\mathbf{B}_1)\overline{\xi_k(\mathbf{B}_1)}] = 0.$$

Thus (†) holds in this case too. ∎

Proof of Lemma 4.3. Set $\mathbf{G} = \mathbf{B}_1$, and choose $\mathbf{x}_1, \mathbf{x}_2, \ldots \in E$ and $\xi_1, \xi_2, \ldots \in E^*$ as in Lemma 4.5. By Lemma 4.6, if $B^{(j)} = \xi_j(\mathbf{B})$ $(j = 1, 2, \ldots)$, then $B^{(1)}, B^{(2)}, \ldots$ are independent normalized complex Brownian motions on (Ω, Σ, P). In particular, for each $t \in [0, \infty)$ and each $n \geqslant m \geqslant 1$, the $(n - m + 1)$-tuples

$$(B_t^{(m)}, \ldots, B_t^{(n)}) \qquad \text{and} \qquad (\sqrt{t}\, B_1^{(m)}, \ldots, \sqrt{t}\, B_1^{(n)})$$

have the same distribution, and using this together with Theorem 0.3 yields

$$E\left[\sup_{s\leqslant t}\left|\sum_{j=m}^{n} B_s^{(j)}\mathbf{x}_j\right|^2\right]\leqslant 4E\left|\sum_{j=m}^{n} B_t^{(j)}\mathbf{x}_j\right|^2$$

$$=4t\cdot E\left|\sum_{j=m}^{n} B_1^{(j)}\mathbf{x}_j\right|^2$$

$$=4t\cdot E\left|\sum_{j=m}^{n} \xi_j(\mathbf{G})\mathbf{x}_j\right|^2.$$

Now the partial sums $\sum_1^n \xi_j(\mathbf{G})\mathbf{x}_j$ are Cauchy in $L^2(\Sigma; E)$, because by Lemma 4.5 they converge to $\mathbf{G}$. By the inequality above and completeness, it follows that there exists a process $\mathbf{X}$: $[\![0, \infty))\to E$ such that for each $t\in[0, \infty)$

$$E\left[\sup_{s\leqslant t}\left|\mathbf{X}_s-\sum_{j=1}^{n} B_s^{(j)}\mathbf{x}_j\right|^2\right]\to 0 \qquad \text{as} \quad n\to\infty.$$

In particular, $\mathbf{X}$ is continuous, and $\mathbf{X}_t\in L^2(\Sigma; E)$ for all $t\in[0, \infty)$. To finish the proof, we show that $\mathbf{B}=\mathbf{X}$.

For each $\xi=E^*$ and each $t\in[0, \infty)$

$$E\,|\xi(\mathbf{B}_t)-\xi(\mathbf{X}_t)|^2=\lim_{n\to\infty} E\left|\left(\xi-\sum_{j=1}^{n}\xi(\mathbf{x}_j)\xi_j\right)(\mathbf{B}_t)\right|^2$$

$$=\lim_{n\to\infty} t\cdot E\left|\left(\xi-\sum_{j=1}^{n}\xi(\mathbf{x}_j)\xi_j\right)(\mathbf{B}_1)\right|^2$$

$$=\lim_{n\to\infty} t\cdot E\left|\xi\left(\mathbf{G}-\sum_{j=1}^{n}\xi_j(\mathbf{G})\mathbf{x}_j\right)\right|^2$$

$$=0.$$

Hence $\xi(\mathbf{B}_t)=\xi(\mathbf{X}_t)$ almost surely for each $\xi\in E^*$ and each $t\in[0, \infty)$. As $\xi(\mathbf{B})$ and $\xi(\mathbf{X})$ are both continuous processes, it follows that $\xi(\mathbf{B})=\xi(\mathbf{X})$ for each $\xi\in E^*$. Finally, as countably many ξ separate points of E, we deduce that $\mathbf{B}=\mathbf{X}$. ∎

5. Radó's Theorem

Radó's theorem for subharmonic functions states that, given a non-negative function φ in the plane, $\log\varphi(z)$ is subharmonic if and only if $|e^{\alpha z}|\,\varphi(z)$ is subharmonic for each $\alpha\in\mathbb{C}$ (see [1, p. 169, Thm. 9]). A weaker version, which suffices for many purposes, is that $\log\varphi(z)$ is subharmonic if and only if $e^{h(z)}\varphi(z)$ is subharmonic for each harmonic function h. In this section we prove a stochastic analogue of this theorem, and then give two applications.

We first prove a lemma, which may be of independent interest. Recall from Section 0 that $\mathcal{M}^c$ denotes the subspace of continuous, predictable processes in $\mathcal{M}$, and that $\mathcal{M}^d$ is its orthogonal complement, the purely discontinuous processes.

5.1. Lemma. *Let $\Phi \in \mathcal{S}$ be non-negative, let $N \in \mathcal{M}^d_{\mathbb{R}}$ be bounded, and let $\psi: \mathbb{R} \to [0, \infty)$ be a convex function. Then $\psi(N)\Phi$ is a submartingale.*

Proof. There exist C^∞ convex functions $\psi^{(k)}: \mathbb{R} \to [0, \infty)$ such that $\psi^{(k)} \downarrow \psi$. If $\psi^{(k)}(N)\Phi$ is a submartingale for each k, then by the monotone convergence theorem so is $\psi(N)\Phi$. Therefore we can suppose, without loss of generality, that ψ is C^∞. Also, by [18, Thm. 2.2] there exist bounded, continuous, predictable submartingales $(\Phi^{(k)})$ such that $\Phi^{(k)} \downarrow \Phi$. Thus we can similarly assume that Φ is bounded and continuous.

We now proceed to make a further reduction. By [4, Chap. VII, Sect. 8] Φ may be decomposed as $\Phi = M + A$, where $M: [\![0, \infty]\!] \to \mathbb{R}$ is a right continuous martingale, and $A: [\![0, \infty]\!] \to \mathbb{R}$ is a predictable, increasing process with $A_0 = 0$. As both Φ and A are predictable, so is M, which implies that M is continuous (see [6, Part 2, Chap. IV, Sect. 23(a)]), and hence A is also continuous. For $k \geq 1$ define predictable times (T_k) by

$$T_k = \inf\{t \geq 0 : |M_t| \geq k\}.$$

These $T_k \uparrow \infty$ as $k \to \infty$, so if $\psi(N)\, \Phi^{T_k} 1_{(T_k > 0)}$ is a submartingale for each k, then by the dominated convergence theorem so is $\psi(N)\Phi$. Moreover we have

$$\Phi^{T_k} 1_{(T_k > 0)} = M^{T_k} 1_{(T_k > 0)} + A^{T_k} 1_{(T_k > 0)}.$$

Thus we are reduced to the case $\Phi = M + A$, where $M \in \mathcal{M}^c_{\mathbb{R}}$ and is bounded, and where A is bounded, continous, predictable and increasing, with $A_0 = 0$.

Assume now that ψ and Φ have the special form described above. Applying Ito's formula [4, Chap. VIII, Sect. 27] with $F(x, y) = \psi(x)\, y$, and noting that $M \in \mathcal{M}^c_{\mathbb{R}}$ and $N \in \mathcal{M}^d_{\mathbb{R}}$ are orthogonal, we get

$$\psi(N_t)\Phi_t = \left[\psi(N_0)\Phi_0 + \int_{(0,t]} \psi(N_{s-})\, dM_s + \int_{(0,t]} \psi'(N_{s-})\, \Phi_s\, dN_s \right|$$
$$+ \left[\int_0^t \psi(N_{s-})\, dA_s + \sum_{0 \leq s < t} \{\psi(N_s) - \psi(N_{s-}) - \psi'(N_{s-})(N_s - N_{s-})\}\, \Phi_s \right].$$

The first bracket is a martingale. Also, as ψ is convex,

$$\{\psi(y)-\psi(x)-\psi'(x)(y-x)\}\geqslant 0 \qquad \text{for all} \quad x, y\in\mathbb{R},$$

and hence the second bracket is an adapted increasing process. Therefore $\psi(N)\Phi$ is submartingale, as desired. ∎

We can now prove the promised version of Radó's theorem.

5.2. THEOREM. *Let* Φ: $[\![0, \infty]\!] \to [0, \infty)$ *be a process. Then* $\log\Phi\in\mathcal{S}$ *if and only if* $e^M\Phi\in\mathcal{S}$ *for each bounded* $M\in\mathcal{M}^c_{\mathbb{R}}$.

Proof. Suppose that $\log\Phi\in\mathcal{S}$. If $M\in\mathcal{M}^c_{\mathbb{R}}$ is bounded, then $M\in\mathcal{S}$, so that $(M+\log\Phi)\in\mathcal{S}$, and hence $e^M\Phi\in\mathcal{S}$ by [18, Prop. 2.8(iv)] (with $\psi(x)=e^x$).

Conversely, suppose that $e^M\Phi\in\mathcal{S}$ for each bounded $M\in\mathcal{M}^c_{\mathbb{R}}$. Taking $M=0$, we see in particular that $\Phi\in\mathcal{S}$, so Φ is bounded above, predictable and right continuous, and hence so also is $\log\Phi$. We can suppose too, without loss of generality, that $\log\Phi$ is bounded below (otherwise replace Φ by $(\Phi+1/k)$ and then let $k\to\infty$, using [18, Thm. 2.9(ii)]).

It remains to show that $\log\Phi$ is a submartingale. Fix $r, s\in[0,\infty]$ with $r\leqslant s$, and define a process X: $[\![0,\infty]\!]\to\mathbb{R}$ by

$$X_t=-E[\log\Phi_s\,|\,\mathcal{F}_t] \qquad (t\in[0,\infty]).$$

This is a bounded, right continuous martingale, so we can decompose it as $X=M+N$, where $M\in\mathcal{M}^c_{\mathbb{R}}$ and $N\in\mathcal{M}^d_{\mathbb{R}}$. For $k\geqslant 1$ define predictable times (T_k) by

$$T_k=\inf\{t\geqslant 0: |M|_t\geqslant k\}.$$

Then $M^{T_k}1_{(T_k>0)}\in\mathcal{M}^c_{\mathbb{R}}$ and is bounded, so by hypothesis $\exp(M^{T_k}1_{(T_k>0)})\Phi\in\mathcal{S}$. Also $N^{T_k}1_{(T_k>0)}\in\mathcal{M}^d_{\mathbb{R}}$ and is bounded, so applying Lemma 5.1 (with $\psi(x)=e^x$) we deduce that

$$\exp(N^{T_k}1_{(T_k>0)})\exp(M^{T_k}1_{(T_k>0)})\Phi$$

is a submartingale, i.e., that $\exp(X^{T_k}1_{(T_k>0)})\Phi$ is a submartingale. Since $T_k\uparrow\infty$ as $k\to\infty$, it follows using the dominated convergence theorem that $e^X\Phi$ is a submargingale. In particular,

$$e^{X_r}\Phi_r\leqslant E[e^{X_s}\Phi_s\,|\,\mathcal{F}_r]=E[1\,|\,\mathcal{F}_r]=1,$$

and so

$$\log\Phi_r\leqslant -X_r=E[\log\Phi_s\,|\,\mathcal{F}_r],$$

showing that $\log\Phi$ is indeed a submartingale. ∎

As an immediate application of Theorem 5.2 we have the following result, which was previously obtained in [18, p. 174] under the extra assumption that $\mathscr{F}$ supports a holomorphic atlas. The new proof shows that this assumption is unnecessary.

5.3. COROLLARY. *If* $\log \Phi$, $\log \Psi \in \mathscr{S}$, *then* $\log(\Phi + \Psi) \in \mathscr{S}$.

We can also use Theorem 5.2 to give an easy proof of an earlier result in this paper.

Second Proof of Corollary 2.2. (a) Given a bounded $M \in \mathscr{M}^c_{\mathbb{R}}$, the definition of holomorphic atlas yields a $W \in \mathscr{H}$ with $\operatorname{Re} W = M$. From Theorem 1.8 we have $e^W \mathbf{Z} \in \mathscr{H}^\infty(E)$, and so by Theorem 0.3 $|e^W \mathbf{Z}| \in \mathscr{S}$, or in other words $e^M |\mathbf{Z}| \in \mathscr{S}$. Hence by Theorem 5.2 we conclude that $\log |\mathbf{Z}| \in \mathscr{S}$.

(b) The spectral radius formula implies that $\log \rho(\mathbf{Z})$ is the limit of the decreasing sequence of processes $2^{-k} \log |\mathbf{Z}^{2^k}|$. By Theorem 1.8 (or indeed Lemma 1.7) we know that $\mathbf{Z}^{2^k} \in \mathscr{H}^\infty(A)$ for all k, and so from part (a) we have $\log |\mathbf{Z}^{2^k}| \in \mathscr{S}$. The result now follows from [18, Thm. 2.9(ii)]. ∎

6. A HOLOMORPHIC SELECTION THEOREM

Holomorphic selection theorems for analytic multifunctions provide a means of establishing the existence of holomorphic functions with prescribed boundary values, a tool which has proved useful in complex analysis, interpolation speces and control theory (see, e.g., [3, 9, 23, 24]). In this section we prove a stochastic analogue of this for holomorphic processes.

As always, $\mathscr{H}$ denotes a fixed holomorphic atlas for $\mathscr{F}$.

6.1. THEOREM. *Assume that* $\mathscr{M} = \mathscr{M}^c$. *Let* $K \in \mathscr{K}\mathscr{H}$, *and suppose that* K *is convex-valued. Then, given* $U \in L^\infty(\mathscr{F}_0)$ *with* $U \in K_0$ *a.s., there exists* $Z \in \mathscr{H}^\infty$ *such that*

$$Z_0 = U \text{ a.s.} \qquad \text{and} \qquad Z_\infty \in K_\infty \text{ a.s.}$$

Remarks. (a) The assumption that $\mathscr{M} = \mathscr{M}^c$ is true, for example, when $\mathscr{F}$ is the filtration of Theorem 4.2 (see [18, Thm. 1.11, 1.13]) and, more generally, whenever $\mathscr{F}$ is generated by a continuous Hunt process. However, Theorem 6.1 may well still be valid even if $\mathscr{M} \neq \mathscr{M}^c$: possibly some result like Lemma 5.1 is needed to take care of $\mathscr{M}^d$.

(b) Clearly, a necessary condition for U to exist is that $K_0 \neq \emptyset$ a.s. By a simple measurable selection theorem (see, e.g., [10, Thm. 1.0]), this is also sufficient. Even if this does not hold, a result can still be salvaged by setting

$$S = \inf\{t \geq 0 : K_t \neq \emptyset\},$$

and working with ${}^S K \chi_{[\![S,\infty]\!]} \chi_{(S<\infty)}$, using [18, Thm. 3.7, 3.8].

(c) By analogy with analytic multifunctions, Theorem 6.1 may still hold if "convex" is replaced by "polynomially convex and connected" (see [9]). In any case, given any $K \in \mathscr{K}\mathscr{H}$, the theorem can be applied to its convex hull conv (K), because $\operatorname{conv}(K) \in \mathscr{K}\mathscr{H}$ ([18, Coro. 3.16]).

Proof. Replacing K by $K - \{U\}$ (the *vector* difference, which belongs to $\mathscr{K}\mathscr{H}$ by [18, Coro. 3.15]), we can suppose, without loss of generality, that $U = 0$ a.s. Let $\mathscr{X}$ be the vector space $L^1(\mathscr{F}_\infty) \times L^1(\mathscr{F}_\infty)$, and define a seminorm $p: \mathscr{X} \to \mathbb{R}$ by

$$p(X^{(1)}, X^{(2)}) = E[\sup_{z \in K_\infty} |X^{(1)} + zX^{(2)}|].$$

Let $\mathscr{X}_0$ be the subspace of $\mathscr{X}$ given by

$$\mathscr{X}_0 = \{(W_\infty^{(1)}, W_\infty^{(2)}) : W^{(1)}, W^{(2)} \in \mathscr{H}^\infty\},$$

and define a linear functional $\xi: \mathscr{X}_0 \to \mathbb{C}$ by

$$\xi(W_\infty^{(1)}, W_\infty^{(2)}) = EW_\infty^{(1)}.$$

Then we claim that

$$|\xi(W_\infty^{(1)}, W_\infty^{(2)})| \leq p(W_\infty^{(1)}, W_\infty^{(2)}) \qquad \text{for all} \quad (W_\infty^{(1)}, W_\infty^{(2)}) \in \mathscr{X}_0.$$

To see this, take $W^{(1)}, W^{(2)} \in \mathscr{H}^\infty$ and define Φ: $[\![0,\infty]\!] \to [0,\infty)$ by

$$\Phi_t = \sup\{|w_1 + w_2 z| : w_1 \in \{W_t^{(1)}\}, w_2 \in \{W_t^{(2)}\}, z \in K_t\}.$$

Then $\Phi \in \mathscr{S}$ by [18, Thm. 3.17], and in particular $E\Phi_0 \leq E\Phi_\infty$. Hence

$$\begin{aligned}|\xi(W_\infty^{(1)}, W_\infty^{(2)})| &= |EW_\infty^{(1)}| = |EW_0^{(1)}| \leq E\,|W_0^{(1)}| \\ &\leq E\Phi_0 \leq E\Phi_\infty = p(W_\infty^{(1)}, W_\infty^{(2)}),\end{aligned}$$

as claimed. The Hahn–Banach theorem now allows us to extend to a linear functional $\xi: \mathscr{X} \to \mathbb{C}$ such that

$$|\xi(X^{(1)}, X^{(2)})| \leq p(X^{(1)}, X^{(2)}) \qquad \text{for all} \quad (X^{(1)}, X^{(2)}) \in \mathscr{X}. \tag{$*$}$$

We next seek to determine the form of this extended ξ. Consider the linear functional $X \mapsto \xi(X, 0): L^1(\mathscr{F}_\infty) \to \mathbb{C}$. From $(*)$ we have

$$|\xi(X, 0)| \leq p(X, 0) = \|X\|_1 \qquad \text{for all} \quad X \in L^1(\mathscr{F}_\infty),$$

so there exists $Y^{(1)} \in L^{\infty}(\mathscr{F}_{\infty})$ with $\|Y^{(1)}\|_{\infty} \leqslant 1$ such that

$$\xi(X, 0) = E[XY^{(1)}] \qquad \text{for all} \quad X \in L^1(\mathscr{F}_{\infty}).$$

Moreover, $E[Y^{(1)}] = \xi(1, 0) = E[1] = 1$, so in fact $Y^{(1)} = 1$ a.s. Similarly, there exists $Y^{(2)} \in L^{\infty}(\mathscr{F}_{\infty})$ such that

$$\xi(0, X) = E[XY^{(2)}] \qquad \text{for all} \quad X \in L^1(\mathscr{F}_{\infty}),$$

though in this case we can only say that $\|Y^{(2)}\|_{\infty} \leqslant \|\sup_{z \in K_{\infty}} |z|\,\|_{\infty}$. Hence

$$\xi(X^{(1)}, X^{(2)}) = E[X^{(1)} + X^{(2)}Y^{(2)}] \qquad \text{for all} \quad (X^{(1)}, X^{(2)}) \in \mathscr{X}.$$

Now define $Z \in \mathscr{M}$ by

$$Z_t = E[Y^{(2)} \mid \mathscr{F}_t] \qquad (t \in [0, \infty]).$$

We shall show that this satisfies the conclusions of the theorem. By assumption $\mathscr{M} = \mathscr{M}^c$, so $Z \in \mathscr{M}^c$. Moreover, given $W \in \mathscr{H}^{\infty}$,

$$E[W_{\infty} Z_{\infty}] = E[W_{\infty} Y^{(2)}] = \xi(0, W_{\infty}) = E[0] = 0,$$

so $Z \in (\overline{\mathscr{H}^{\infty}})^{\perp} = \bar{\mathscr{H}}^{\perp} = \mathscr{H}_0$ (see the end of the proof of Theorem 4.2). Thus $Z \in \mathscr{H}^{\infty}$, and $Z_0 = 0 = U$ a.s. It remains to prove that $Z_{\infty} \in K_{\infty}$ a.s. As K_{∞} is convex-valued, it is enough to show that for each pair $\alpha, \beta \in \mathbb{Q} + i\mathbb{Q}$

$$\|(\alpha + \beta Z_{\infty})\, 1_{\Omega_{\alpha,\beta}}\|_{\infty} \leqslant 1,$$

where

$$\Omega_{\alpha,\beta} = \{\omega \in \Omega : \sup_{z \in K_{\infty}(\omega)} |\alpha + \beta z| \leqslant 1\}.$$

For this to hold, it is sufficient that

$$|E[(\alpha + \beta Z_{\infty})\, 1_{\Omega_{\alpha,\beta}} X]| \leqslant \|X\|_1 \qquad \text{for all} \quad X \in L^1(\mathscr{F}_{\infty}),$$

and this we now verify. Fix $\alpha, \beta \in \mathbb{Q} + i\mathbb{Q}$, and take $X \in L^1(\mathscr{F}_{\infty})$. Then

$$\begin{aligned} |E[(\alpha + \beta Z_{\infty})\, 1_{\Omega_{\alpha,\beta}} X]| &= |\xi(\alpha 1_{\Omega_{\alpha,\beta}} X, \beta 1_{\Omega_{\alpha,\beta}} X)| \\ &\leqslant p(\alpha 1_{\Omega_{\alpha,\beta}} X, \beta 1_{\Omega_{\alpha,\beta}} X) \\ &= E[\sup_{z \in K_{\infty}} |(\alpha + \beta z)\, 1_{\Omega_{\alpha,\beta}} X|] \\ &\leqslant E\,|X|, \end{aligned}$$

as desired. This completes the proof. ∎

ACKNOWLEDGMENT

The author is grateful to the referee for reading this paper carefully, and for several helpful comments and suggestions.

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