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Reliability Studies of Bivariate Distributions with Pareto Conditionals

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In this paper we study Arnold's (1987, *Statist. Probab. Lett.* **5**, 263–266) class of bivariate distributions with Pareto conditionals from a reliability point of view. Failure rates and mean residual life function of the marginal distributions and their monotonic properties are studied. The hazard components and their properties are investigated and their relationships with some measures of dependence are established. Finally, the failure rate of the minimum of the two components is examined and its monotonicity is investigated. Some of the results presented here are general and would be useful in studying the dependence structure in other classes of bivariate distributions. © 2001 Academic Press

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1. INTRODUCTION

Arnold (1987) proposed a class of bivariate distributions with Pareto conditionals. This is similar to the class of bivariate distributions with normal conditionals studied by Castillo and Galambos (1987); see also Castillo and Galambos (1989). More specifically, Castillo and Galambos (1987) identified the class of all analytic bivariate densities defined on R^2 for which both of the conditional distributions are normal. In a similar manner, Arnold (1987) derived the class of all bivariate densities $f(x \mid y)$ on R^2 for which $f(x \mid y)$ for every y and $f(y \mid x)$ for every x are Pareto densities. This class of densities includes as special cases the case of independent marginals and the family of bivariate densities introduced by Mardia (1962).

In this paper, we study Arnold's class from a reliability point of view. More specifically, we study the failure rate and the mean residual life function of the marginals and their monotonic properties. We also study the hazard components of the hazard gradient, in the sense of Johnson and

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Kotz (1975), and their monotonic structure. An association measure $\theta(x, y)$, defined by Oakes (1989), is investigated for this class of bivariate distributions, and some relationships between $\theta(x, y)$ and other measures of association defined in the literature are established. Some of the results presented here are general and would be useful in studying the association in other classes of bivariate distributions. Finally, the failure rate of the minimum of the two components is examined and its monotonicity is investigated. Note that, in general, in the nonindependence case the failure rate of the minimum does not necessarily have the same monotonic properties as the failure rates of the individual components; see Nagaraja and Baggs (1996).

2. THE MODEL

To fix the ideas, we define a Pareto density if it has the form

$$f(x, \alpha) = \frac{\alpha}{\sigma}\left(1 + \frac{x}{\sigma}\right)^{-(\alpha+1)}, \qquad x > 0, \quad \alpha, \sigma > 0. \tag{2.1}$$

We shall denote this by $P(\sigma, \alpha)$.

We now consider a two-dimensional continuous random variable (X, Y) with pdf.

$$f_{X,Y}(x, y) = \frac{c}{(\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 xy)^{\alpha+1}}, \qquad x > 0, \quad y > 0. \tag{2.2}$$

The restrictions on the parameters are:

(i) If $\alpha \in (0, 1)$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, and $\lambda_0 \geqslant 0$.

(ii) If $\alpha \in (1, \infty)$, $\lambda_0 > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_3 \geqslant 0$.

(iii) If $\alpha = 1$, $\lambda_i > 0$, $i = 0, 1, 2, 3$.

See Arnold (1987) and Arnold *et al.* (1992, p. 58).

The conditional distributions are

$$X \mid Y = y \sim P(\sigma_1(y), \alpha)$$
$$Y \mid X = x \sim P(\sigma_2(x), \alpha)$$

where $\sigma_1(y) = (\lambda_0 + \lambda_2 y)/(\lambda_1 + \lambda_3 y)$; $\sigma_2(x) = (\lambda_0 + \lambda_1 x)/(\lambda_2 + \lambda_3 x)$.

The marginal densities are given by

$$f_X(x) = \frac{c}{(\lambda_2 + \lambda_3 x)(\lambda_0 + \lambda_1 x)^{\alpha}} \tag{2.3}$$

$$f_Y(y) = \frac{c}{(\lambda_1 + \lambda_3 y)(\lambda_0 + \lambda_2 y)^{\alpha}} \tag{2.4}$$

where c is given by

$$c = \frac{\lambda_0^{\alpha}\lambda_2\phi}{F\left(\alpha, 1; \alpha+1; 1-\frac{1}{\phi}\right) B(1, \alpha)} \tag{2.5}$$

where $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function given by

$$F(\alpha, \beta; \gamma; z) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\, dt,$$

$\operatorname{Re}\gamma > 0$, $\operatorname{Re}\beta > 0$; see Gradshteyn and Ryzhik (196, p. 1040).

3. MONOTONICITY OF FAILURE RATES

In this section, we shall investigate the monotonicity of various kinds of failure rates. The failure rate of a random variable X having pdf $f(x)$ is defined as

$$r(t) = f(t)/(1-F(t)),$$

where $F(t)$ is the distribution function corresponding to $f(t)$. In order to examine the monotonicity of failure rates, we use the following result due to Glaser (1980), since direct evaluation of failure rates is difficult.

LEMMA 1. *Define* $\eta(t) = -(d/dt)\ln f(t)$. *If* $\eta'(t) > 0$ *for all* t, X *has an increasing failure rate* (*IFR*). *If* $\eta'(t) < 0$ *for all* t, X *has a decreasing failure rate* (*DFR*).

3.1. *Failure Rate and Mean Residual Life Function of the Marginals*

In order to examine the monotonicity of the failure rate of X, we form

$$\eta(t) = -\frac{d}{dt}\ln f_X(t)$$

$$= \frac{\lambda_3}{\lambda_2+\lambda_3 t} + \frac{\alpha\lambda_1}{\lambda_0+\lambda_1 t}$$

$$\therefore$$

$$\eta'(t) = \frac{-\lambda_3^2}{(\lambda_2+\lambda_3 t)^2} - \frac{\alpha\lambda_1^2}{(\lambda_0+\lambda_1 t)^2} < 0.$$

Hence $X \in \mathrm{DFR}$ (class of decreasing failure rate distributions). So marginals $\in \mathrm{DFR}$.

Mean Residual Life Functions. The mean residual life function (MRLF) $\mu(t)$ of a random X is defined as

$$\mu(t) = E(X - t \mid X > t) = \int_t^\infty S(x)\, dx / S(t), \tag{3.1}$$

provided $E(X) < \infty$; where $S(x) = P\ (X > x)$ is the survival function. It is well known that the failure rate, the MRLF, and the survival function are equivalent in the sense that given one of them, the other two can be determined; see Gupta (1981). The failure rate $r(t)$ and the MRLF $\mu(t)$ are connected by the relation

$$r(t) = \frac{1 + \mu'(t)}{\mu(t)}.$$

In order to compute $\mu(t)$, note that in our case it is not feasible to use formula (3.1) as the expression for $S(x)$ is not in compact form. Since

$$E(g(X) \mid X > t) = \int_t^\infty g(x)\, f(x)\, dx / S(t),$$

we find that

$$E(\lambda_2 + \lambda_3 X \mid X > t) = \int_t^\infty (\lambda_2 + \lambda_3 x)\, f(x)\, dx / S(t)$$

$$= \frac{1}{\lambda_1(\alpha - 1)(\lambda_0 + \lambda_1 t)^{\alpha - 1}\, S(t)}, \qquad \text{if} \quad \alpha > 1.$$

Assuming $\alpha > 1$ and $\lambda_3 > 0$,

$$F(X \mid X > t) = \left[\frac{1}{\lambda_1(\alpha - 1)(\lambda_0 + \lambda_1 t)^{\alpha - 1}\, S(t)} - \lambda_2 \right] \lambda_3^{-1}.$$

Hence

$$\mu(t) = E(X - t \mid X > t)$$

$$= \left[\frac{1}{\lambda_1(\alpha - 1)(\lambda_0 + \lambda_1 t)^{\alpha - 1}\, S(t)} - \lambda_2 \right] \lambda_3^{-1} - t. \tag{3.2}$$

This will give

$$E(X) = \lambda_3^{-1} \left[\frac{1}{\lambda_1(\alpha - 1)\, \lambda_0^{\alpha - 1}} - \lambda_2 \right].$$

3.2. *Hazard Components*

Johnson and Kotz (1975) defined the hazard gradient as a vector $(h_1(x, y), h_2(x, y))$ where $h_1(x, y)$ is the hazard rate of the conditional distribution of X given $Y > y$ and $h_2(x, y)$ is the hazard rate of the conditional distribution of Y given $X > x$. It can be verified that

$$h_1(x, y) = -\frac{d}{dx} \ln S(x, y),$$

where $S(x, y) = P(X > x, Y > y)$ is the joint survivor function.

In our case,

$$\begin{aligned} h_1(x, y) &= -\left[\frac{d}{dx} S(x, y)\right] \Big/ S(x, y) \\ &= -\left[\frac{d}{dx} \int_x^\infty \int_y^\infty \frac{c\, du\, dv}{(\lambda_0 + \lambda_1 u + \lambda_2 v + \lambda_3 uv)^{\alpha+1}}\right] \Big/ S(x, y) \\ &= \left[\int_y^\infty \frac{c\, dv}{(\lambda_0 + \lambda_1 x + \lambda_2 v + \lambda_3 xv)^{\alpha+1}}\right] \Big/ S(x, y) \\ &= \frac{c}{\alpha(\lambda_2 + \lambda_3 x)[\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 xy]^\alpha\, S(x, y)}. \end{aligned} \tag{3.3}$$

Similarly,

$$h_2(x, y) = \frac{c}{\alpha(\lambda_1 + \lambda_3 y)[\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 xy]^\alpha\, S(x, y)}. \tag{3.4}$$

It can be verified that the direct derivative method leads to

$$\frac{\partial}{\partial x} \ln h_1(x, y) = \frac{-\lambda_3}{\lambda_2 + \lambda_3 x} - \frac{\alpha(\lambda_1 + \lambda_3 y)}{\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 xy} + h_1(x, y).$$

From this, we are not able to gain any information about the monotonicity of the hazard component. We proceed as follows.

The corresponding pdf is given by

$$\begin{aligned} f(x \mid Y > y) &= h_1(x, y)\, P(X > x \mid Y > y) \\ &= h_1(x, y)\, S(x, y)/P(Y > y) \\ &= \frac{c}{\alpha(\lambda_2 + \lambda_3 x)[\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 xy]^\alpha\, P(Y > y)}. \end{aligned}$$

We now use Lemma 3.1 to obtain

$$\eta(x) = -\frac{d}{dx} \ln f(x \mid Y > y)$$

$$= \frac{\lambda_3}{\lambda_2 + \lambda_3 x} + \frac{\alpha(\lambda_1 + \lambda_3 y)}{\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 xy}.$$

This gives

$$\eta'(x) = \frac{-\lambda_3^2}{(\lambda_2 + \lambda_3 x)^2} - \frac{\alpha(\lambda_1 + \lambda_3 y)^2}{(\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 xy)^2} < 0.$$

Hence $h_1(x, y)$ is decreasing in x. Similarly, $h_2(x, y)$ is decreasing in y.

3.3. *Failure Rates of the Conditionals*

For the Pareto model $P(\sigma, \alpha)$ defined in Section 2, the failure rate is given by

$$r(x) = \frac{\alpha}{x + \sigma}.$$

Since the conditional distribution of $X \mid Y = y$ is $P(\sigma(y), \alpha)$, the failure rate of the conditional distribution of $X \mid Y = y$ is given by

$$r(x \mid Y = y) = \frac{\alpha}{x + \sigma_1(y)},$$

where $\sigma_1(y) = (\lambda_0 + \lambda_2 y)/(\lambda_1 + \lambda_3 y)$. So $r(x \mid Y = y)$ is a decreasing function of x. The monotonicity of $r(x \mid y)$ as a function of y is also of interest in order to study the association between X and Y.

$$\frac{d}{dy} r(x \mid Y = y) = \frac{-\alpha \sigma_1'(y)}{[x + \sigma_1(y)]^2}$$

$$= \frac{-\alpha(\lambda_1 \lambda_2 - \lambda_0 \lambda_3)}{(\lambda_1 + \lambda_3 y)^2}$$

$$\therefore$$

$$\frac{d}{dy} r(x \mid Y = y) < 0 \qquad \text{if} \quad \lambda_1 \lambda_2 - \lambda_0 \lambda_3 > 0.$$

Since $\operatorname{sign} \rho(X, Y) = \operatorname{sign}(\lambda_1 \lambda_2 - \lambda_0 \lambda_3)$, we conclude that $r(x \mid Y = y)$ is decreasing (increasing) in y according to whether $\rho > (<) 0$.

3.4. *Relation between* $h_1(x, y)$ *and* $r(x \mid Y = y)$

Recall that

$$h_1(x, y) = -\frac{d}{dx} \ln P(X > x \mid Y > y)$$

$$r(x \mid Y = y) = -\frac{d}{dx} \ln P(X > x \mid Y = y).$$

In the context of bivariate survival models induced by frailties, Oakes (1989) studied the association measure

$$\theta(x, y) = \frac{SS_{12}}{S_1 S_2},$$

where $S = S(x, y)$ is the survival function, $S_{12} = \partial^2 S(x, y)/\partial x\, \partial y$, $S_1 = (\partial/\partial x)\, S(x, y)$, and $S_2 = (\partial/\partial y)\, S(x, y)$; see Clayton (1978).

Clayton (1978) presented the above association measure, derived from the Cox model, in a study of the association between the life spans of fathers and their sons.

It can be easily seen that

$$\theta(x, y) = \frac{r(x \mid Y = y)}{h_1(x, y)}.$$

The numerator is the hazard rate for sons at time x given that their fathers died at y. The denominator is the hazard rate for sons at time x given that their fathers live past y. Also,

$$r(x \mid Y = y) = -S_{12} S_2 \qquad \text{and} \qquad h_1(x, y) = -S_1/S.$$

It can now be verified that

$$\frac{\partial}{\partial y} h_1(x, y) = \frac{S_2}{S} [-h_1(x, y) + r(x \mid Y = y)]$$

$$= \frac{S_2}{S} h_1(x, y)(\theta - 1),$$

suppressing the argument of θ.

Since $S_2 < 0$, $\theta > 1$ is equivalent to $(\partial/\partial y)\, h_1(x, y) < 0$.

We now present the following definition due to Harris (1970).

DEFINITION. The random vector (X, Y), or its distribution function, is said to be right corner set increasing (RCSI) if $P(X > x,\ Y > y \mid X > x', Y > y')$ is increasing in x' and y' for all x and y.

It has been shown by Shaked (1977) that (X, Y) being RCSI is equivalent to $h_1(x, y)$ decreasing in y for all y. Therefore, the following are equivalent:

(i) $\theta > 1$,

(ii) $(\partial/\partial y)\, h_1(x, y) < 0$.

(iii) (X, Y) is RCSI.

Since $r(x \mid Y = y)$ being decreasing in y implies that $h_1(x, y)$ is decreasing in y (see Shaked, 1977), we can state the following in the context of our model:

$$\rho > \lambda_1\lambda_2 - \lambda_2\lambda_3 > 0 \Leftrightarrow \frac{d}{du} r(x \mid Y = y) < 0$$

$$\Rightarrow \frac{d}{dy} h_1(x, y) < 0$$

$$\Leftrightarrow (X, Y) \text{ is RCSI.}$$

3.5. *Derivation of* $\theta(x, y)$

Using the definitions given in the previous section, it can be verified that

$$\frac{\partial^2}{\partial x\, \partial y} \ln S(x, y) = \frac{S_1 S_2}{S^2}(\theta - 1)$$

$$= h_1(x, y)\, h_2(x, y)(\theta - 1)$$

or that

$$\frac{-\partial}{\partial y} h_1(x, y) = h_1(x, y)\, h_2(x, y)(\theta - 1).$$

This gives

$$\theta = 1 - \frac{(\partial/\partial y)\, h_1(x, y)}{h_1(x, y)\, h_2(x, y)}.$$

In a similar manner,

$$\theta = 1 - \frac{(\partial/\partial x)\, h_2(x, y)}{h_1(x, y)\, h_2(x, y)}.$$

Remark. It can be proved that X and Y are independent if and only if $\theta = 1$. For our model,

$$h_1(x, y) = \frac{c}{\alpha(\lambda_2 + \lambda_3 x)(\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 xy)^{\alpha} S(x, y)};$$

see Eq. (3.4).

Taking the logarithmic derivative of the above, it can be seen that

$$\theta = \frac{\alpha(\lambda_2 + \lambda_3 x)}{(\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 xy)\, h_2(x, y)}.$$

In a similar manner, we can show that

$$\theta = \frac{\alpha(\lambda_1 + \lambda_3 y)}{(\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 xy)\, h_1(x, y)}.$$

From the above two expressions for θ, we see that

$$\frac{h_1(x, y)}{h_2(x, y)} = \frac{\lambda_1 + \lambda_3 y}{\lambda_2 + \lambda_3 x}.$$

4. FAILURE RATE OF $T = \text{Min}(X, Y)$

Distribution of T

$$P(T > t) = \int_t^{\infty} \int_t^{\infty} f(x, y)\, dx\, dy$$

$\therefore$ The pdf of T is given by

$$\begin{aligned} f_T(t) &= \frac{-d}{dt} \int_t^{\infty} \int_t^{\infty} f(x, y)\, dx\, dy \\ &= \int_t^{\infty} f(x, t)\, dx + \int_t^{\infty} f(t, y)\, dy \\ &= \int_t^{\infty} f_X(x \mid t)\, f_Y(t)\, dx + \int_t^{\infty} f_Y(y \mid t)\, f_X(t)\, dy \\ &= f_Y(t)\, P(X > t \mid Y = t) + f_X(t)\, P(Y > t \mid X = t). \end{aligned}$$

This gives us a general formula for expressing the pdf of T in terms of the survival functions of the conditionals.

In our case $f_X(t)$ and $f_Y(t)$ are given by (2.3) and (2.4). Also,

$$P(X > t \mid Y = t) = (1 + t/\sigma_1(t))^{-\alpha}$$

$$P(Y > t \mid X = t) = (1 + t/\sigma_2(t))^{-\alpha},$$

where $\sigma_1(t)$ and $\sigma_2(t)$ are given before. These give

$$f_T(t) = \frac{c}{(\lambda_0 + (\lambda_1 + \lambda_2)\, t + \lambda_3 t^2)^{\alpha}} \left[\frac{1}{\lambda_1 + \lambda_3 t} + \frac{1}{\lambda_2 + \lambda_3 t} \right]. \tag{4.1}$$

Failure Rate of T

The failure rate of T is given by

$$r_T(t) = f_T(t)/S(t, t).$$

Since it is a complicated function of t, the monotonicity of the failure rate by the direct derivative method is not feasable. We shall, therefore, proceed as follows.

It can be verified that

$$r_T(t) = h_1(t, t) + h_2(t, t),$$

where $h_1(x, y)$ and $h_2(x, y)$ are the hazard components defined earlier. We will now interpret the meaning of $h_1(t, t)$ and $h_2(t, t)$ in the competing risk setup; see Gupta (1979) and Elandt-Johnson and Johnson (1980).

Let us consider a population in which two causes of death, C_1 and C_2 are operating and each cause has its own potential failure time. For example, a person can have heart disease and kidney disease and both of them are competing for a person's life. The methods of analyzing survival data in such a population are called competing risk analysis. We also have a fundamental assumption that each death takes place due to a single cause.

Let X and Y denote the hypothetical (potential) times the person is due to die. X and Y are not observable. Instead, $T = \text{Min}(X, Y)$ and the cause of death are observable.

Define

$$I = \begin{cases} 1 & \text{if death takes place due to } C_1 \\ 2 & \text{if death takes place due to } C_2. \end{cases}$$

Let π_i denote the expected proportion of death from C_i, $i=1, 2$. Then $\pi_1+\pi_2=1$ and the crude probability of eventually dying from C_i at an age greater than T is

$$F_i(t)=P(T>t \text{ and } I=i)=\int_t^\infty \lambda_i(x)\, S_T(x)\, dx,$$

where we are writing $h_i(t, t)$ as $\lambda_i(t)$ for convenience.

Note that $\lambda_i(t)$ is the cause-specific hazard due to cause i, $i=1, 2$; see Elandt-Johnson and Johnson (1980). $\lambda_i(t)$ is not a hazard rate in the usual sense; i.e., it is not necessary that there exists a random variable whose hazard rate is $\lambda_i(t)$. Thus, we express $\lambda_i(t)=\pi_i\lambda_i^*(t)$, where $\lambda_i^*(t)$ is the hazard rate at time t conditional of failing from cause C_i and is given by $\lambda_i^*(t)=-(\partial/\partial t)\ln[1-F_i^*(t)]$. $F_i(t)$ and $F_i^*(t)$ are connected by the relation $F_i^*(t)=F_i(t)/\pi_i$; see Gaynor *et al.* (1993) for details. Note that $\lambda_i^*(t)$ is a proper hazard rate and $\lambda_i^*(t)=F_i^{*\prime}(t)/(1-F_i^*(t))$.

In order to show that $\lambda_i(t)$ is decreasing, it is enough to show that $\lambda_i^*(t)$ is decreasing.

The corresponding pdf is

$$f_i^*(t)=\frac{1}{\pi_i}\lambda_i(t)\, S_T(t).$$

In our case, using (3.4), we have

$$f_1^*(t)=\frac{c}{\alpha(\lambda_2+\lambda_3 t)[\lambda_0+(\lambda_1+\lambda_2)\, t+\lambda_3 t^2]^\alpha}.$$

This gives

$$\eta_1^*(t)=\frac{-d}{dt}\ln f_1^*(t)$$

$$=\frac{\lambda_3}{\lambda_2\lambda_3 t}+\frac{\alpha(\lambda_1+\lambda_2+2\lambda_3 t)}{\lambda_0+(\lambda_1+\lambda_2)\, t+\lambda_3 t^2}.$$

It can now be verified that $\eta_1^{*\prime}(t)<0$. So $\lambda_1(t)$ is decreasing. Similarly, $\lambda_2(t)$ is decreasing. Hence $h(t, t)$ is decreasing and T has DFR.

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