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Using stochastic prior information in consistent estimation of regression coefficients in replicated measurement error model

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A replicated ultrastructural measurement error regression model is considered where both predictor and response variables are observed with error. Availability of some prior information regarding regression coefficients in the form of stochastic linear restrictions is assumed. Using this prior information, three classes of consistent estimators of regression coefficients are proposed. A two-stage procedure is discussed to obtain feasible version of these Stochastically Restricted estimators. The asymptotic properties of the proposed estimators are studied. No distributional assumption is imposed on any random component of the model. Monte Carlo simulations study is performed to assess the effect of sample size, replicates and non-normality on the estimators. The methods are illustrated using real economic data.

1. Introduction

In real life, there are situations where the data cannot be obtained precisely or observations on some surrogate variables are taken instead of true variable of interest. Thus the data is contaminated by measurement error (ME). For example, variables like air pollutant levels and rainfall etc., cannot be measured accurately. Also in the medical science, the data on biomarkers is taken as surrogate for observing the desired activity. This ME invalidates the results derived through the statistical techniques meant for error-free data. So in order to draw valid conclusions, we require different techniques which take into account the ME. In the past, many researchers have shown interest in situations where ME plays a significant role.

In regression analysis, when predictors are measured with error, the model is called a measurement error regression model. Depending upon the nature of the distribution of true predictors, the ME regression model has two forms. For non-stochastic predictors, the ME regression model is said to be in functional form. In case of independent and identically distributed predictors, the ME regression model takes the structural form [5]. The model is called ultrastructural measurement error (UME) model when the true predictors are independent but not necessarily identically distributed. This was proposed by Dolby [7] as a unified approach to both functional and structural models.

Presence of ME in the data often leads to inconsistent and biased estimators. The literature presents several approaches for finding consistent estimators. One such approach suggests the use of some additional information which is obtained independently from the sample information for example availability of reliability matrix of predictors, variance–covariance matrix of ME and instrumental variables etc. [5,9,10,13]. But such external information is subject to some uncertainties

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or sometimes it is even unavailable [15]. Another approach is to study replicated measurement error (RME) model where replicated observations are taken on variables. For example, Chan and Mak [3] and Isogawa [12] studied the structural form of the RME model under the condition of normally distributed measurement errors. Yam [32] studied the functional form of this model. Ullah et al. [30] studied the relationship between trade balance and exchange rate using the ultrastructural form of the RME model with panel data. For more details, one can refer to Wang et al. [31], Schafer and Purdy [21], Shalabh [22], Shalabh et al. [24] and references cited therein.

In many practical situations, in addition to sample information, some prior information regarding regression coefficients is also available which often may be expressed in the form of stochastic linear restrictions. Stochastic restrictions arise from prior statistical information, usually in the form of previous point or interval estimates of parameters, and take the form of an additional linear model (Ref. Toutenburg [29] and Rao et al. [18]). For example, unbiased pre-estimate of regression coefficient (say $\hat{\beta}$) obtained from earlier studies with smaller sample size or from studies with comparable designs can be expressed as $\hat{\beta} = \beta + \varphi$, where φ is random in nature. Also the prior information that a certain component β_i of vector β may lie in interval (a, b) can be expressed as $(a + b)/2 = \beta_i + \varphi$, where φ may be uniformly distributed over the interval $((a - b)/2, (b - a)/2)$. The methodology of using stochastic prior information provides a framework for attaining new knowledge regarding the phenomenon under study in the light of what is known. The use of stochastic prior information leads to more efficient estimators in terms of variability (Ref. Rao et al. [18]).

Durbin [8] was the first one to use both sample and prior information simultaneously in parameter estimation. Thereafter, Theil and Goldberger [28] and Theil [27] introduced the mixed regression estimator which incorporates stochastic linear restrictions and is more efficient than OLSE. In the without ME case, Shalabh and Toutenburg [25] explored the role of stochastic linear restrictions when there are missing observations. Haupt and Oberhofer [11] discussed the stochastic response restrictions. Jianwen and Yang [14] discussed mixed estimation for a singular linear model. Revan [19] discussed the use of stochastic restrictions with multicollinearity. In the ME regression model, Shalabh et al. [23] provided the consistent estimators that make use of such prior information. Shalabh [22] studied the replicated ultrastructural measurement error (RUME) regression model without incorporating any prior information. For this model, the problem of finding consistent estimators which also use prior stochastic information has not been studied so far. Thus in the present work, we provide the stochastically restricted (SR) consistent estimators for the RUME model by using sample and prior information simultaneously. A two-stage procedure for obtaining feasible version of SR estimators is also discussed. These estimators are found to be more efficient than those suggested by Shalabh [22] in terms of variability.

We consider a RUME multiple regression model under the assumption of stochastic linear restrictions on regression coefficients. The problem of finding estimators that are consistent as well as make use of stochastic linear restrictions is dealt with. Most of the literature assumes the normality of ME, but this assumption often gets violated in practice. Sometimes, the distributional form of ME is also unknown. In the present work, no other assumption except the finiteness of the first four moments of ME is made. The methodology is illustrated using an empirical economic study.

In this paper, Section 2 specifies the RUME multiple regression model and lists various assumptions. In Section 3, we propose the consistent estimators satisfying the stochastic linear restrictions. Section 4 discusses the asymptotic properties of the proposed estimators. Section 5 contains the results from a Monte Carlo simulations study performed to explore the finite sample properties of estimators and the effect of departure from normality. Section 6 deals with the empirical study. Appendix states few definitions, lemmas and provides the derivations of some results.

2. Model specification

Consider the following multiple regression model with p predictor variables

$$\eta_i = \alpha + \sum_{k=1}^{p} \beta_k \xi_{ik}, \tag{2.1}$$

where η_i and ξ_{ik} are ith observations on the dependent and kth predictor respectively for $i = 1, \ldots, n$. β_k's are unknown regression coefficients. We also assume that η_i and ξ_{ik} are unobservable and can be observed through some other variables y_i and x_{ik} with additional measurement error. Further consider that r replicates of y_i and x_{ik} are available for each η_i and ξ_{ik}. Thus for $j = 1, \ldots, r$, we write

$$y_{i:j} = \eta_i + u_{i:j}; \tag{2.2}$$

$$x_{ik:j} = \xi_{ik} + v_{ik:j}, \tag{2.3}$$

where $y_{i:j}$ and $x_{ik:j}$ are the jth replicated observations on y_i and x_{ik} with additional measurement errors $u_{i:j}$ and $v_{ik:j}$ respectively. The model (2.1) does not mention the equation error. Without loss of generality, the possible equation error can be assumed to be submerged with $u_{i:j}$. Thus the model representation remains valid irrespective of the presence of equation error.

To incorporate the ultrastructural property in the model, we consider that ξ_{ik} is a random variable that can be written as

$$\xi_{ik} = m_{ik} + w_{ik}, \tag{2.4}$$

where m_{ik} and w_{ik} are non-stochastic and stochastic components respectively.

Using Eqs. (2.1)–(2.4), the model can be written in the matrix form as

$$Y_{nr\times 1} = \alpha e_{nr} + X_{nr\times p}\beta_{p\times 1} + \left(U_{nr\times 1} - V_{nr\times p}\beta_{p\times 1}\right); \tag{2.5}$$

$$\xi_{n\times p} = M_{n\times p} + W_{n\times p}; \tag{2.6}$$

$$X = (M \otimes e_r) + (W \otimes e_r) + V, \tag{2.7}$$

where '⊗' indicates the Kronecker product of matrices, e_r is a $(r \times 1)$ unit column vector and

$$\begin{aligned}
&X = [X_{1:1} \cdots X_{n:r}]'; && X_{i:j}{}' = \left[x_{i1:j} \cdots x_{ip:j}\right];\\
&V = [V_{1:1} \cdots V_{n:r}]'; && V_{i:j}{}' = \left[v_{i1:j} \cdots v_{ip:j}\right];\\
&\xi = [\xi_1 \cdots \xi_n]'; && \xi_i{}' = \left[\xi_{i1} \cdots \xi_{ip}\right];\\
&M = [M_1 \cdots M_n]'; && M_i{}' = \left[m_{i1} \cdots m_{ip}\right];\\
&W = [W_1 \cdots W_n]'; && W_i{}' = \left[w_{i1} \cdots w_{ip}\right];\\
&Y = [y_{1:1} \cdots y_{n:r}]', && U = [u_{1:1} \cdots u_{n:r}]' \quad \text{and} \quad \beta = \left[\beta_1 \cdots \beta_p\right]'.
\end{aligned}$$

The subscript $i : j$ indicates the row corresponding to the jth replicated observation on the ith subject in the study.

Eqs. (2.5)–(2.7) complete the specifications of the RUME multiple regression model. When all rows of M are identical, the rows of X will be independently and identically distributed (iid) with some multivariate distribution. This gives the structural form of the measurement error model. When W is a null matrix, X is fixed but measured with error. This condition specifies a functional measurement error model. In case, both W and V are null matrices, we get the specifications of a classical regression model. Thus, the ultrastructural model combines the three popular regression models in one setup [7].

For a random variable S, using the notations γ_{1S} and γ_{2S} for the Pearson's coefficient of skewness and kurtosis respectively, the following assumptions are made

1. $u_{i:j}$ are iid random variables with mean 0, variance σ_u^2, third moment $\gamma_{1u}\sigma_u^3$ and fourth moment $(\gamma_{2u} + 3)\sigma_u^4$;
2. $v_{ik:j}$ are iid random variables with mean 0, variance σ_v^2, third moment $\gamma_{1v}\sigma_v^3$ and fourth moment $(\gamma_{2v} + 3)\sigma_v^4$;
3. w_{ik} are iid random variables with mean 0, variance σ_w^2, third moment $\gamma_{1w}\sigma_w^3$ and fourth moment $(\gamma_{2w} + 3)\sigma_w^4$;
4. elements of V, W and U are mutually independent;
5. $\lim_{n\to\infty} \frac{1}{n}M'CM = \Sigma_M$ (finite) where $C = I_n - \frac{1}{n}e_n e_n'$;
6. $\lim_{n\to\infty} \frac{1}{n}M'C = \sigma_M$ (finite).

Assumptions 5 and 6 are useful for deriving the asymptotic properties of estimators.

The prior information regarding the regression coefficients is assumed to be available in the form of stochastic linear restrictions given as

$$\theta_{q\times 1} = R_{q\times p}\beta_{p\times 1} + \varphi_{q\times 1}, \tag{2.8}$$

where R and θ are known such that rank $(R) = q \le p$, and φ is a vector of random disturbances with mean zero and known variance–covariance matrix Σ_φ. It is assumed that the random vector φ is independent of U, V and W. This is an essential assumption which ensures the external character of the stochastic prior information. The vector θ may be interpreted as a random variable with expectation $E(\theta) = R\beta$, and hence the stochastic restrictions do not hold exactly but in mean.

3. Estimation of parameters

For the RUME multiple regression model with r replicates, the least squares method provides an estimator of regression coefficient vector β as

$$b_A = \left(X'AX\right)^{-1} X'AY. \tag{3.1}$$

Using the averages of r replicates, the LSE of β is given as

$$b_D = \left(X'DX\right)^{-1} X'DY, \tag{3.2}$$

where $A = I_{nr} - \frac{1}{nr}e_{nr}e_{nr}'$ and $D = \frac{1}{r}\left(I_n \otimes e_r e_r'\right) - \frac{1}{nr}e_{nr}e_{nr}'$ [20,22].

Using (2.5)–(2.7), Assumptions 1–6 and Lemma A.5, it can be easily verified that

$$\operatorname*{plim}_{n\to\infty} b_A = \left(\Sigma_M + \Sigma_W + \Sigma_V\right)^{-1}\left(\Sigma_M + \Sigma_W\right)\beta \quad \text{and} \tag{3.3}$$

$$\operatorname*{plim}_{n\to\infty} b_D = \left(\Sigma_M + \Sigma_W + \frac{1}{r}\Sigma_V\right)^{-1}\left(\Sigma_M + \Sigma_W\right)\beta, \tag{3.4}$$

where $\Sigma_W = \sigma_w^2 I_p$ and $\Sigma_V = \sigma_v^2 I_p$. Eqs. (3.3) and (3.4) indicate that b_A and b_D are inconsistent estimators of β when derived for measurement error ridden data.

Under the assumption of normality of random components in the RUME multiple regression model, Shalabh [22] provided three consistent estimators of β as

$$b_{01} = (r-1)\left[X'(rD-A)X\right]^{-1} X'AY, \tag{3.5}$$

$$b_{02} = (r-1)\left[X'(rD-A)X\right]^{-1} X'DY \quad \text{and} \tag{3.6}$$

$$b_{03} = \left[X'(rD-A)X\right]^{-1} X'(rD-A)Y. \tag{3.7}$$

The estimators b_{01} and b_{02} are obtained by correcting for inconsistency in b_A and b_D. This is done using a consistent estimator of Σ_V, given as

$$\widehat{\Sigma}_V = \frac{1}{n(r-1)} X'(A-D)X. \tag{3.8}$$

The estimator b_{03} is obtained by using the linear combination of b_A and b_D. Using Assumptions 1–6 and Lemma A.5, it can be easily verified that

$$\operatorname*{plim}_{n\to\infty} b_{0s} = \beta; \quad s = 1, 2, 3. \tag{3.9}$$

Remark 3.1. The minimization of the following functions

$$Q_A = (Y - X\beta)' A (Y - X\beta) \quad \text{and}$$
$$Q_D = (Y - X\beta)' D (Y - X\beta),$$

with respect to β yields the estimators which are the same as b_A and b_D respectively. $\square$

3.1. Incorporating stochastic prior information in estimation

The estimators (3.1), (3.2) and (3.5)–(3.7) utilized only sample information. The prior information in the form of stochastic linear restrictions can be incorporated using the methodology of mixed estimation. We first assume that σ_u^2 is known or at least some pre-estimate of σ_u^2 is available. When η and ξ are observable, then following the mixed regression estimation approach, the sample and prior information (2.8), can be utilized simultaneously in the estimation by minimizing

$$Q = (\eta - \xi\beta)'(\eta - \xi\beta) + \sigma_u^2 (\theta - R\beta)' \Sigma_\varphi^{-1} (\theta - R\beta). \tag{3.1.1}$$

The first term captures the information regarding regression coefficients vector in the current sample and the second term contains prior information regarding β. Q cannot be minimized as both η and ξ are unknown. From Remark 3.1, we see that Q_A and Q_D are based on the sample information regarding regression coefficients vector. Hence, we replace $(\eta - \xi\beta)'(\eta - \xi\beta)$ in Q by Q_A and Q_D to get following two forms

$$Q_{AR} = Q_A + \sigma_u^2 (\theta - R\beta)' \Sigma_\varphi^{-1} (\theta - R\beta) \tag{3.1.2}$$

and

$$Q_{DR} = Q_D + \sigma_u^2 (\theta - R\beta)' \Sigma_\varphi^{-1} (\theta - R\beta). \tag{3.1.3}$$

Minimization of Q_{AR} and Q_{DR} with respect to β provide the following estimators

$$b_{AR} = \left(X'AX + \sigma_u^2 R'\Sigma_\varphi^{-1} R\right)^{-1} \left(X'AY + \sigma_u^2 R'\Sigma_\varphi^{-1}\theta\right) \tag{3.1.4}$$

and

$$b_{DR} = \left(X'DX + \sigma_u^2 R'\Sigma_\varphi^{-1} R\right)^{-1} \left(X'DY + \sigma_u^2 R'\Sigma_\varphi^{-1}\theta\right). \tag{3.1.5}$$

Using (2.5)–(2.7) and Lemma A.5, it is observed that these estimators are not consistent since plim $b_{AR} \neq \beta$ and plim $b_{DR} \neq \beta$. In the following subsections, we provide consistent SR estimators of regression coefficients which also incorporate prior information.

3.1.1. Consistent estimation

When there is no measurement error in the data i.e. $\sigma_v^2 = 0$, it can be verified from (3.3) that b_A is consistent. The presence of measurement error in the data results in the inconsistency of this estimator. The estimator b_{01} was obtained by Shalabh [22] by adjusting for the inconsistency in b_A. In the following discussion, we show that the same consistent estimator can also be obtained using the corrected score methodology (refer Buzas and Stefanski [2]). In this methodology, we first make appropriate corrections to the original score function so that the effect of measurement error is eliminated. This corrected score function is then minimized to get the consistent estimator.

Since the inconsistency in b_A is caused by the presence of measurement error of explanatory variables, we first correct Q_A for σ_v^2. It is observed, using (2.6) and (2.7) that

$$\begin{aligned} E\left(Q_A|Y,\xi\right) &= E\left\{(Y-X\beta)'A\left(Y-X\beta\right) \mid Y,\xi\right\} \\ &= \left(Y-[\xi\otimes e_r]\,\beta\right)'A\left(Y-[\xi\otimes e_r]\,\beta\right)+\operatorname{tr}(A)\,\beta'\Sigma_V\beta \quad \text{(using Assumptions 1–4)} \\ &= \left(Y-[\xi\otimes e_r]\,\beta\right)'A\left(Y-[\xi\otimes e_r]\,\beta\right)+(nr-1)\,\beta'\Sigma_V\beta. \end{aligned} \tag{3.1.1.1}$$

Only the second term on the right hand side of (3.1.1.1) contains $\Sigma_V = \sigma_v^2 I_p$. Thus adjusting Q_A for the factor $(nr-1)\,\beta'\Sigma_V\beta$ and replacing the unknown Σ_V by its consistent estimator provided in (3.8), we get the following corrected function for sufficiently large sample size

$$Q_{A;cor} = Q_A - \left(\frac{r}{r-1}\right)\beta'X'\,(A-D)\,X\beta. \tag{3.1.1.2}$$

It is observed that minimizing $Q_{A;cor}$ results in an estimator which is the same as b_{01}. This observation motivated us to use the corrected function $Q_{A;cor}$ in Q as a replacement for $(\eta-\xi\beta)'\,(\eta-\xi\beta)$. Thus we get

$$Q_{AR;cor} = Q_A - \left(\frac{r}{r-1}\right)\beta'X'\,(A-D)\,X\beta + \sigma_u^2\,(\theta-R\beta)'\,\Sigma_\varphi^{-1}\,(\theta-R\beta)\,. \tag{3.1.1.3}$$

Minimization of $Q_{AR;cor}$ gives the following estimator

$$b_{11} = \left[\left(\frac{1}{r-1}\right)X'\,(rD-A)\,X + \sigma_u^2 R'\Sigma_\varphi^{-1}R\right]^{-1}\left(X'AY+\sigma_u^2R'\Sigma_\varphi^{-1}\theta\right). \tag{3.1.1.4}$$

Using (2.5)–(2.7) and Lemma A.5, the above estimator can easily shown to be consistent, i.e. plim $b_{11}=\beta$. From (3.1.1.4), it is observed that after adding the stochastic linear restrictions, we only need to add $\sigma_u^2R'\Sigma_\varphi^{-1}R$ and $\sigma_u^2R'\Sigma_\varphi^{-1}\theta$ to the unrestricted consistent estimator b_{01}. These two terms may be interpreted as the adjustments brought in by the stochastic linear restrictions.

Writing $S_F = X'FX$ for some matrix F and applying Lemma A.1 to the first factor on the right hand side of (3.1.1.4), we get

$$\left[\frac{S_{(rD-A)}}{r-1}+\sigma_u^2R'\Sigma_\varphi^{-1}R\right]^{-1} = \frac{S_{(rD-A)}^{-1}}{(r-1)^{-1}} - \frac{S_{(rD-A)}^{-1}}{(r-1)^{-2}}R'\left[\sigma_u^{-2}\Sigma_\varphi + \frac{RS_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\right]^{-1}RS_{(rD-A)}^{-1}. \tag{3.1.1.5}$$

Inserting the above relation in (3.1.1.4), we get another form of the estimator b_{11} as

$$b_{11} = b_{01} + (r-1)\,S_{(rD-A)}^{-1}R'\left[\sigma_u^{-2}\Sigma_\varphi + (r-1)\,RS_{(rD-A)}^{-1}R'\right]^{-1}(\theta - Rb_{01})\,. \tag{3.1.1.6}$$

In contrast to (3.1.1.4), the modified form of the estimator b_{11} given by (3.1.1.6) no longer requires the matrix Σ_φ to be non-singular. Thus the modified form allows the simultaneous use of exact and stochastic prior information. For Σ_φ a null matrix, (3.1.1.6) provides the estimator using only the exact linear restrictions. When Σ_φ is a full rank matrix, the estimator uses only stochastic information. In case Σ_φ is singular, we get the estimator which uses both exact and stochastic prior information.

Proceeding on similar lines, it is observed that the consistent estimator b_{02} can also be obtained by minimizing the following corrected function

$$Q_{D;cor} = Q_D - \left(\frac{1}{r-1}\right)\beta'X'\,(A-D)\,X\beta. \tag{3.1.1.7}$$

This corrected function could be used in the process of finding another estimator which utilizes stochastic prior information. Replacement of $(\eta-\xi\beta)'\,(\eta-\xi\beta)$ in Q by (3.1.1.7) provides the following function

$$Q_{DR;cor} = Q_D - \left(\frac{1}{r-1}\right)\beta'X'\,(A-D)\,X\beta + \sigma_u^2\,(\theta-R\beta)'\,\Sigma_\varphi^{-1}\,(\theta-R\beta)\,. \tag{3.1.1.8}$$

First minimizing (3.1.1.8) with respect to β and then applying Lemma A.1 to the resultant form of the estimator, we get the following modified estimator

$$b_{12} = b_{02} + (r-1)\,S_{(rD-A)}^{-1}R'\left[\sigma_u^{-2}\Sigma_\varphi + (r-1)\,RS_{(rD-A)}^{-1}R'\right]^{-1}(\theta - Rb_{02})\,. \tag{3.1.1.9}$$

Using (3.9) and Lemma A.5, it is easily observed that this estimator is consistent.

Another restricted estimator can be obtained by using the consistent estimator b_{03}. The estimator b_{03} was obtained by using the linear combination of b_A and b_D. We observe that the same estimator is obtained if we minimize the following function

$$Q_{A,D} = (Y - X\beta)' (rD - A) (Y - X\beta) . \tag{3.1.1.10}$$

This provides the necessary motivation to use (3.1.1.10) in (3.1.1) for obtaining another estimator utilizing the stochastic prior information. Minimizing (3.1.1) after replacing $(\eta - \xi\beta)' (\eta - \xi\beta)$ by $Q_{A,D}$ and then applying Lemma A.1, gives the following estimator

$$b_{13} = b_{03} + S^{-1}_{(rD-A)} R' \left[\sigma_u^{-2} \Sigma_\varphi + R S^{-1}_{(rD-A)} R'\right]^{-1} (\theta - Rb_{03}) . \tag{3.1.1.11}$$

Using (3.9) and Lemma A.5, this estimator can be easily shown to be consistent.

Despite being inconsistent, b_{AR} and b_{DR} incorporate the stochastic linear restrictions. These estimators are used to provide a few more stochastically restricted estimators. Using Lemma A.1, the modified forms of b_{AR} and b_{DR} are obtained as

$$b_{AR} = b_A + S_A^{-1} R' \left[\sigma_u^{-2} \Sigma_\varphi + R S_A^{-1} R'\right]^{-1} (\theta - Rb_A) , \tag{3.1.1.12}$$

and

$$b_{DR} = b_D + S_D^{-1} R' \left[\sigma_u^{-2} \Sigma_\varphi + R S_D^{-1} R'\right]^{-1} (\theta - Rb_D) . \tag{3.1.1.13}$$

The inconsistency of b_{AR} and b_{DR} is caused by the inconsistency of b_A and b_D. For eliminating the inconsistency, we replace b_A and b_D by their consistent counterparts b_{0s} for $s = 1, 2, 3$ and obtain the following estimators

$$b_{2s} = b_{0s} + S_A^{-1} R' \left[\sigma_u^{-2} \Sigma_\varphi + R S_A^{-1} R'\right]^{-1} (\theta - Rb_{0s}) , \tag{3.1.1.14}$$

and

$$b_{3s} = b_{0s} + S_D^{-1} R' \left[\sigma_u^{-2} \Sigma_\varphi + R S_D^{-1} R'\right]^{-1} (\theta - Rb_{0s}) . \tag{3.1.1.15}$$

(3.9) and Lemma A.5 lead to the conclusion that plim $b_{2s} = \beta$ and plim $b_{3s} = \beta$.

Remark 3.1.1. For $s = 1, 2, 3$, we consider the weighted function

$$Q_W = (b_{0s} - \beta)' W (b_{0s} - \beta) + \sigma_u^2 (\theta - R\beta)' \Sigma_\varphi^{-1} (\theta - R\beta) , \tag{3.1.1.16}$$

where W is the weight matrix. For $W = (r-1)^{-1} X' (rD - A) X$, the estimator obtained on minimizing Q_W is the same as b_{11} and b_{12}. When $W = X' (rD - A) X$, we get the estimator b_{13}. Similarly, on taking weight matrices as $X'AX$ and $X'DX$, the respective estimators are the same as b_{2s} and b_{3s}. □

The above observations suggest that the proposed stochastically restricted estimators can be obtained from weighted function Q_W by using some appropriate weight matrices. This motivated us to propose one more consistent estimator of β. On minimizing the unweighted function

$$(b_{0s} - \beta)' (b_{0s} - \beta) + \sigma_u^2 (\theta - R\beta)' \Sigma_\varphi^{-1} (\theta - R\beta) , \tag{3.1.1.17}$$

we get the estimator

$$b_{4s} = b_{0s} + R' \left[\sigma_u^{-2} \Sigma_\varphi + RR'\right]^{-1} (\theta - Rb_{0s}) . \tag{3.1.1.18}$$

This estimator can be easily shown to be consistent by using (3.9) and Lemma A.5.

Hence using b_{0s}; $s = 1, 2, 3$, we provide three classes of four estimators each $\left(b_{fs}; f = 1, 2, 3, 4\right)$, which are consistent as well as utilize prior information in the form of stochastic linear restrictions. These estimators are termed as Stochastically Restricted (SR) Estimators.

3.2. Two-Stage Feasible Stochastically Restricted (TSFSR) estimators

The estimators proposed in the previous subsection are based on the assumption that σ_u^2 is known. But generally, this may not be true and hence we propose to replace σ_u^2 by

$$\hat{\sigma}_u^2 = \frac{1}{n(r-1)} \left[\left(Y - X\hat{\beta}\right)' (A - D) \left(Y - X\hat{\beta}\right) - \hat{\beta}' X' (A - D) X\hat{\beta}\right], \tag{3.2.1}$$

where $\hat{\beta}$ is some good estimator of β. Using Lemma A.5, it can be easily shown that $\hat{\sigma}_u^2$ is consistent provided $\hat{\beta}$ is consistent. The algorithm for obtaining the Two-Stage Feasible Stochastically Restricted (TSFSR) Estimators is as follows

Stage 1: Obtain the unrestricted estimator $\hat{\beta}$ and compute $\hat{\sigma}_u^2$;

Stage 2: Use $\hat{\sigma}_u^2$ in the place of σ_u^2 in the expressions of $b_{fs}; f = 1, 2, 3, 4$ to obtain TSFSR estimators.

We denote these TSFSR estimators as $\hat{b}_{fs}; f = 1, 2, 3, 4$.

A natural choice for $\hat{\beta}$ is among the consistent estimators b_{01}, b_{02} and b_{03}. Under the condition of Normally distributed measurement errors, b_{02} dominates both b_{01} and b_{03} according to mean square error criterion (refer Shalabh [22]). Thus one can use b_{02} in (3.2.1) as a good estimator of β. It is to be noted that, in the present work, we have not imposed any distributional assumption on measurement errors. Hence in case of non-normality, the suitability of b_{01}, b_{02} and b_{03} will be explored in the next section.

There is another important point which needs to be discussed here. For $\hat{\sigma}_u^2$ to be a reasonable estimator of σ_u^2, it must be non-negative. Unfortunately, for certain values of Y and X, $\hat{\sigma}_u^2$ may be negative since it is the difference of two non-negative terms. One may take $\hat{\sigma}_u^2 = 0$ in such a situation but use of this estimate in stage 2 of the algorithm does not provide a better estimator of β than the unrestricted estimators given by (3.5)–(3.7). Thus, for negative $\hat{\sigma}_u^2$, it is better to use some pre-estimate (obtained from earlier studies) of σ_u^2 in SR estimators for utilizing stochastic information.

Remark 3.2.2. The above problem does not arise if the covariance matrix of random component φ in (2.8) is parameterized as $\sigma_u^2 K$ for known matrix K, because in this case, the expression of SR estimators does not involve σ_u^2 (refer Rao et al. [18]). But this parameterization may not be valid for all situations. □

4. Large sample properties of estimators

The derivation of the exact distribution of proposed estimators is difficult. Even if derived, the complexity of expressions may not serve any analytical purpose. Thus, in this section, we explore the large sample properties of the SR and TSFSR estimators. The asymptotic properties of consistent unrestricted estimators b_{01}, b_{02} and b_{03} shall also be explored.

We first define some expressions to be used in deriving the asymptotic distribution of estimators. Using $C = I_n - \frac{1}{n}e_n e_n'$, we write

$$\Sigma_\xi = \frac{1}{n}M'CM + \sigma_W^2 I_p,$$
$$\Sigma_{XA} = \Sigma_\xi + \sigma_v^2 I_p,$$
$$\Sigma_{XD} = \Sigma_\xi + \frac{1}{r}\sigma_v^2 I_p \quad \text{and}$$
$$Z = [C\,(M+W)\otimes e_r] = A\,[(M+W)\otimes e_r] = D\,[(M+W)\otimes e_r]\,.$$

Using Assumption 5, it can be easily seen that

$$\lim_{n\to\infty} \Sigma_\xi = \Sigma_M + \sigma_W^2 I_p = \Sigma \quad \text{(say)},$$
$$\lim_{n\to\infty} \Sigma_{XA} = \Sigma + \sigma_v^2 I_p = \Sigma_A \quad \text{and}$$
$$\lim_{n\to\infty} \Sigma_{XD} = \Sigma + \frac{1}{r}\sigma_v^2 I_p = \Sigma_D.$$

Thus on using (2.5)–(2.7) and definitions of Σ_ξ, Σ_{XA}, Σ_{XD} and Z, we can write

$$\frac{1}{nr}X'AY = \Sigma_\xi\beta + \frac{1}{n^{1/2}}h, \tag{4.1}$$
$$\frac{1}{nr}X'DY = \Sigma_\xi\beta + \frac{1}{n^{1/2}}\left(h + h^*\right), \tag{4.2}$$
$$\frac{1}{nr}X'AX = \Sigma_{XA} + \frac{1}{n^{1/2}}H_1, \tag{4.3}$$
$$\frac{1}{nr}X'DX = \Sigma_{XD} + \frac{1}{n^{1/2}}H_2, \tag{4.4}$$

where

$$h^* = \frac{1}{n^{1/2}r}\left[V'\,(D-A)\,U\right], \tag{4.5}$$
$$h = \frac{1}{n^{1/2}}\left[Q\beta - \frac{1}{r}\left(Z'V\beta - Z'U - V'AU\right)\right], \tag{4.6}$$

$$H_1 = \frac{1}{n^{1/2}}\left[Q + \frac{1}{r}\left(V'AV - nr\sigma_v^2 I_p\right)\right], \tag{4.7}$$

$$H_2 = \frac{1}{n^{1/2}}\left[Q + \frac{1}{r}\left(V'DV - n\sigma_v^2 I_p\right)\right], \tag{4.8}$$

for $Q = \left(M'CW + W'CM\right) + \left(W'CW - n\sigma_w^2 I_p\right) + \frac{1}{r}\left(Z'V + V'Z\right)$.
From Assumptions 1–6 and Lemma A.4, we observe that h^*, h, H_1 and H_2 are of order $O_P(1)$.

Using the above results, we now state the following theorem which gives the relationship between the asymptotic distributions of SR and TSFSR estimators.

Theorem 1. *If $\hat{\sigma}_u^2$ is consistent, the following result holds*

$$n^{\frac{1}{2}}\left(\hat{b}_{fs} - \beta\right) = n^{\frac{1}{2}}\left(b_{fs} - \beta\right) + O_p\left(n^{-1}\right), \tag{4.9}$$

for $f = 1, 2, 3, 4$ and $s = 1, 2, 3$. That is, the SR and TSFSR estimators have same asymptotic distribution. □

The proof of the above theorem is given in Appendix. The result of Theorem 1 has an intuitive appeal. Since $\hat{\sigma}_u^2$ is consistent, hence for large sample size $\hat{b}_{fs}$ and b_{fs} are identical. Hence, we only need to evaluate the asymptotic properties of SR estimators. Eqs. (A.18) and (A.21) from Appendix give

$$n^{\frac{1}{2}}\left(b_{fs} - \beta\right) = \Sigma_\xi^{-1}\left[h - \frac{1}{r-1}H\beta + d_s h^*\right] + O_P\left(n^{-\frac{1}{2}}\right), \tag{4.10}$$

where $H = rH_2 - H_1$. The values $d_1 = 0$, $d_2 = 1$ and $d_3 = \frac{r}{r-1}$ characterize three classes of estimators.

We define the function

$$G(F_1, F_2) = \left[e_{nr}e_p'(F_1 * I_p)\right] * \left[(F_2 * I_{nr})e_{nr}e_p'\right], \tag{4.11}$$

where '$*$' indicates the Hadamard product of matrices [17] and F_1 and F_2 matrices of order $p \times p$ and $nr \times nr$ respectively. The above mentioned results lead to the following theorem which gives the asymptotic distribution of unrestricted as well as restricted estimators.

Theorem 2. $n^{\frac{1}{2}}\left(b_{fs} - \beta\right); f = 0, 1, 2, 3, 4; s = 1, 2, 3$ *asymptotically follow a Multivariate Normal distribution, that is*

$$n^{\frac{1}{2}}\left(b_{fs} - \beta\right) \xrightarrow{d} N_p\left(0_{p\times 1}, \Sigma^{-1}\Omega_s\Sigma^{-1}\right) \tag{4.12}$$

where $0_{p\times 1}$ is the mean vector with all elements zero and

$$\Omega_1 = \Theta + \frac{1}{r}\sigma_u^2\sigma_v^2 I_p; \tag{4.13}$$

$$\Omega_2 = \Theta + \frac{1}{r^2}\sigma_u^2\sigma_v^2 I_p; \tag{4.14}$$

$$\Omega_3 = \Theta + \frac{1}{r(r-1)}\sigma_u^2\sigma_v^2 I_p; \tag{4.15}$$

$$\Theta = \frac{1}{r}\left(\sigma_u^2 + \sigma_v^2\beta'\beta\right)\left(\Sigma_M + \sigma_W^2 I_p\right) + \frac{1}{r(r-1)}\sigma_v^4\left(\beta\beta' + \mathrm{tr}\left(\beta'\beta\right)I_p\right);$$

$$\Sigma^{-1} = \lim_{n\to\infty}\Sigma_\xi^{-1}. \quad \square$$

The proof of the above theorem is included in Appendix.

Since mean of the asymptotic distribution of $n^{\frac{1}{2}}\left(b_{fs} - \beta\right); f = 0, 1, 2, 3, 4; s = 1, 2, 3$ is zero, hence all the estimators are asymptotically unbiased. (4.13)–(4.15) indicate that the asymptotic variance–covariance matrix of estimators is not affected by deviation from normality. This suggests that non-normality of the elements of U, V and W does not affect the asymptotic properties of the estimators.

From (4.12), it can be easily observed that for $s = 1, 2, 3$, the asymptotic distribution of $n^{\frac{1}{2}}\left(b_{fs} - \beta\right); f = 1, 2, 3, 4$ is the same as that of $n^{\frac{1}{2}}\left(b_{0s} - \beta\right)$. Hence in each class, the SR estimators have the same asymptotic distribution as that of the unrestricted estimator of that class. This indicates that the effect of using additional information in the form of stochastic linear restrictions vanishes with an increase in the sample size as sufficiently large information regarding the parameter of interest is available from the sample alone.

From Theorem 2, it is observed that the differences $(\Omega_1 - \Omega_2)$, $(\Omega_3 - \Omega_2)$ and $(\Omega_1 - \Omega_3)$ are positive definite. This implies that the estimator b_{f2} dominates b_{f1} and b_{f3} even in the case of non-normality. Also b_{f3} dominate b_{f1} for each $f = 0, 1, 2, 3, 4$.

The small sample properties of the estimators are studied in the next section.

5. Simulations

In this section, the small sample properties of the estimators are assessed using Monte-Carlo simulations. Coding is done in MATLAB. To get an idea about the effect of non-normality on the properties of estimators, we study the following distributions for the measurement error and random components in the model

I. Normal distribution (symmetric and non-kurtic);
II. t-distribution (symmetric but kurtic);
III. Gamma distribution (non symmetric and kurtic).

The effect of kurtosis is studied by comparing the results for Normal and t distributions. Comparison of t and Gamma distributions gives an idea about the effect of skewness. Simulations are performed for various sample sizes and replicates. The different combinations of $\left(\sigma_u^2, \sigma_w^2, \sigma_v^2\right)$ used are (0.5, 0.5, 0.5), (0.5, 0.5, 1.0), (0.5, 1.0, 0.5), (0.5, 1.0, 1.0), (1.0, 0.5, 0.5), (1.0, 0.5, 1.0), (1.0, 1.0, 0.5) and (1.0, 1.0, 1.0). The random numbers are generated from $N(0, 1)$, t_6 and $G(2, 1)$ distributions. These numbers have been suitably scaled to have mean zero and variance specified by different values of $\left(\sigma_u^2, \sigma_w^2, \sigma_v^2\right)$. The vector β is fixed a priori as $\beta = \begin{pmatrix} 2.4 & 1.3 & 1.9 \end{pmatrix}'$. The stochastic restriction imposed is of the form given by (2.8), with $R = \begin{bmatrix} 0.3 & 0.5 & 0.8 \\ -0.45 & 0.57 & 0.33 \end{bmatrix}$. The random term φ is assumed to follow a multivariate normal distribution with mean $\binom{0}{0}$ and variance–covariance matrix $\Sigma_\varphi = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.3 \end{bmatrix}$. The vector θ is computed at each iteration using (2.8).

It is well known that consistent estimators in measurement error models may not have finite expectations [4]. Thus in the simulation study, we use empirical medians instead of empirical expectations. The median square error matrix (MedSEM) and median bias (MedBias) vector are defined as

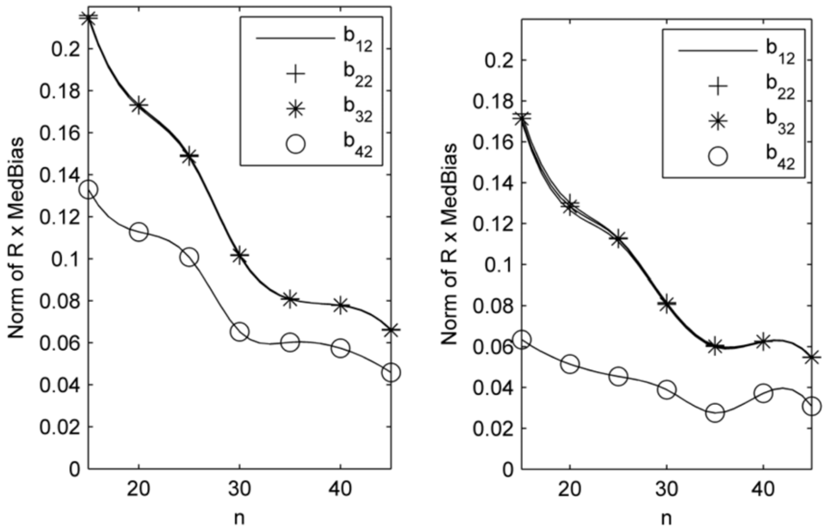
$$\mathrm{MedSEM}\left(b_{fs}\right) = \mathrm{median}\left\{\left(b_{fs} - \beta\right) \times \left(b_{fs} - \beta\right)'\right\}, \quad \text{and}$$
$$\mathrm{MedBias}\left(b_{fs}\right) = \left(\mathrm{median}\left(b_{fs}\right) - \beta\right).$$

Simulations are performed for SR as well as TSFSR estimators. 20,000 iterations are used for each parametric combination and MedSEM and MedBias computed empirically for the unrestricted and SR estimators. For TSFSR estimators, only those iterations are used where $\hat{\sigma}_u^2 > 0$. We denote the trace of MedSEM by TrMedSEM and the norm of MedBias vector by MedAB (median absolute bias). TrMedSEM and MedAB are used for comparison purpose because any change in these reflects the increase/decrease in variances and biases of the estimators. The simulation results can be seen on the web page http://statistics.puchd.ac.in/includes/noticeboard/20120307101646-Tables.pdf in the form of Tables.

From the simulation results given in Tables, it is observed that

- The use of stochastic information provides more efficient estimators since, the MedAB and TrMedSEM for both SR and TSFSR estimators are less as compared to those for unrestricted estimators. The only exception is b_{4s}; $s = 1, 2, 3$. Although it provides the largest reduction in bias as compared to other restricted estimators, it does not provide reduction in variability except for small samples and large σ_v^2.
- MedAB and TrMedSEM tend towards zero as the sample size increases. This validates the theoretical findings that estimators are asymptotically unbiased and consistent.
- For both SR and TSFSR estimators, b_{1s} gives the largest reduction in variability and b_{4s} gives largest reduction in bias followed by b_{1s}.
- MedAB and TrMedSEM for TSFSR estimators are lower than those for SR estimators. This suggests that for $\hat{\sigma}_u^2 > 0$, the TSFSR estimators should be preferred to SR estimators.
- The SR estimators b_{f2}; $f = 1, 2, 3, 4$ dominate the other two classes of estimators in terms of reduction in variability. No clear dominance is observed in terms of reducing the bias.
- For TSFSR estimators, $\hat{b}_{f1}$; $f = 1, 2, 3, 4$ dominate other classes in terms of both variability and bias.
- No clear conclusions can be drawn about the effect of non-normality on the properties of estimators since the differences in MedAB and TrMedSEM of the estimators for Normal, t and Gamma distributions are not very large. This suggests that to some extent, the estimators are robust to the assumption of normality of measurement error and other random components in the model.
- Bias and variance increase as σ_v^2 increases and decrease with increasing σ_w^2.

We also tried to explore the extent to which the stochastic restrictions are satisfied by proposed estimators. Since the estimators under study may not have finite expectations, thus we use the MedBias vector to explore whether stochastic restrictions are satisfied at least in the central part of the distribution of estimators. The norm of vector ($R \times$ MedBias) is plotted in Fig. 1. It is clear that TSFSR estimators satisfy stochastic restrictions more closely as compared to SR estimators. This further strengthens the preference of TSFSR estimators over SR estimators.



(a) SR estimators.

(b) TSFSR estimators.

Fig. 1. Norm of $R \times$ MedBias vs sample size when $\left(\sigma_u^2, \sigma_w^2, \sigma_v^2\right) = (0.5, 0.5, 1.0)$ and $r = 2$.

6. Empirical study

The trade of a country with other countries is an essential activity for the economic development. The wealth of the country increases when export revenue is high. On the other hand, higher import expenditure puts extra pressure on the resources and thus affects the growth of that country. This Trade balance (TB, the difference of export and import) may be related with other economic variables, viz. exchange rate and gross domestic product (GDP). Increase in the exchange rate reduces the cost of the goods produced by the country. This causes an increase in export whereas imported goods get costlier. Similarly, higher GDP increases the volume of export. At the same time, the purchasing power of the people increases. This increases the demand and in turn increases the import expenditure. In the past, many researchers tried to relate the TB to these economic variables. Breda et al. [1] and Shirvani and Wilbratte [26] explored the effect of exchange rate on TB using time series. The relationship of GDP and exchange rate with TB was explored by Chiu et al. [6] using panel data. Ullah et al. [30] used the replicated ultrastructural measurement error model for exploring the effect of exchange rate on TB from a cross-sectional point of view.

For the purpose of illustrating the estimators proposed in Section 3, we explore the effect of exchange rate averaged over whole year (x_1) and GDP (x_2) on TB (y) from a cross-sectional point of view under the RUME model setup defined in Section 3. The data used is for 40 countries and the observations for two different periods (years 1992 and 2002) are taken as replicated observations. The data is obtained from the Penn-World Table and the International Monetary Fund Database. The variables x_1, x_2 and y are expected to be contaminated with measurement error and thus satisfy Eqs. (2.2) and (2.3). The effect of different time periods on the cross-sectional relation is captured by taking replications over time. Thus, for $i = 1, \ldots, 40$ and $j = 1, 2$, the linear relationship takes the following form

$$y_{i:j} = \beta_1 x_{i1:j} + \beta_2 x_{i2:j} + u_{i:j}.$$

Ullah et al. [30] provided the consistent estimate of regression coefficient of exchange rate on TB (estimate $= 9.4$, SE $= 4.5$) using the data for the years 1977 and 1987. Our model setup is similar to their setup except that one additional variable x_2 is included and more recent data is used (years 1992 and 2002). Thus, we use the results reported by them, in the form of stochastic linear restriction (2.8) by taking $R = \begin{pmatrix} 1 & 0 \end{pmatrix}, \theta = 9.4$ and $\text{var}\,(\varphi) = 4.5 \times 4.5 = 20.25$. Unknown σ_u^2 is estimated using (3.2.1) by taking $\hat{\beta} = b_{02}$ and it is found that $\hat{\sigma}_u^2 = 7.5725$. The results are reported for TSFSR estimators in Table 1. The bootstrap method is used to estimate the standard error of estimates.

Table 1 provides the values of the estimates and corresponding SE in parentheses. The estimators of class 1 have the least SE as compared to the other two classes and estimator b_{11} dominates in this class in terms of variability. The findings are consistent with the simulation results for TSFSR estimators. It can be easily observed that, by using the stochastic information in estimation, SE is reduced. Although this reduction is not very large, but this could be due to the fact that var (φ) is very large and hence the stochastic information used here is highly variable. It is also observed that the effect of this additional information is negligible on the estimates of other variables.

Table 1
Estimate of regression coefficients and SE.

	b_{01}	b_{02}	b_{03}
Exchange rate	1.8877 (0.7906)	2.3727 (0.8547)	2.8577 (1.1986)
GDP	−0.0097 (0.0100)	−0.0129 (0.0115)	−0.0161 (0.0135)
	b_{11}	b_{12}	b_{13}
Exchange rate	1.8858 (0.7769)	2.3709 (0.8446)	2.8561 (1.1769)
GDP	−0.0097 (0.0100)	−0.0129 (0.0117)	−0.0161 (0.0135)
	b_{21}	b_{22}	b_{23}
Exchange rate	1.8878 (0.7916)	2.3728 (0.8528)	2.8578 (1.1858)
GDP	−0.0097 (0.0100)	−0.0129 (0.0115)	−0.0161 (0.0135)
	b_{31}	b_{32}	b_{33}
Exchange rate	1.8879 (0.7924)	2.3729 (0.8518)	2.8579 (1.1748)
GDP	−0.0097 (0.0100)	−0.0129 (0.0115)	−0.0161 (0.0135)
	b_{41}	b_{42}	b_{43}
Exchange rate	3.9323 (1.2033)	4.2853 (1.1767)	4.6383 (1.1613)
GDP	−0.0097 (0.0100)	−0.0129 (0.0115)	−0.0161 (0.0135)

7. Conclusions

A replicated ultrastructural measurement error (RUME) multiple regression model is considered where replicated observations are available on both study and predictor variables. Some prior information regarding regression coefficients is assumed to be available in the form of stochastic linear restrictions. Three classes of consistent stochastically restricted (SR) estimators are proposed. When σ_u^2 is unknown, the SR estimators cannot be used. To overcome this problem, a two-stage procedure of obtaining restricted estimators, known as TSFSR estimators, is described. No distributional assumption is imposed on any random component in the model. The asymptotic properties of unrestricted and restricted consistent estimators are reported. It is observed that asymptotically, the estimators follow a Multivariate Normal distribution and are unbiased. Monte Carlo simulations are performed to explore the small sample properties of estimators. It is observed that inclusion of prior information improves the estimators in terms of both bias and variability. The effect of stochastic information vanishes with increasing sample size. In small samples, the TSFSR estimators dominate SR estimators in terms of both bias and variability. To some extent, the proposed estimators are robust to the assumption of normality. The utility of the proposed estimators is illustrated using a real economic data set on trade balance, exchange rate and GDP.

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Appendix

Lemma A.1. *If $A : p \times p, B : p \times n, C : n \times n$, and $D : n \times p$, then*

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B\left(C^{-1} + DA^{-1}B\right)^{-1} DA^{-1}. \quad \square$$

The above lemma is taken from Rao et al. [18]

Lemma A.2. *Let $C = \left(c_{ij}\right)$ be a $(m \times m)$ matrix and let $\|C\|_1 = \max_{1\le i\le m} \sum_{j=1}^{m} \left|c_{ij}\right|$ and $\|C\|_2 = \max_{1\le j\le m} \sum_{i=1}^{m} \left|c_{ij}\right|$ be the maximum column sum and maximum row sum matrix norms respectively. If $\|C\|_1 < 1$ and/or $\|C\|_2 < 1$, then $(I_m - C)$ is invertible and*
$(I_m - C)^{-1} = \sum_{i=0}^{\infty} C^i$, *where $C^o = I_m$.* $\square$

For the proof, one can refer to Rao and Rao [17].

Lemma A.3. *Let $V_n = \sum_{j=1}^{n} U_{jn}X_j$ where $X_1, \dots, X_n$ are $(p \times 1)$ independent and identically distributed random vectors with $E\left(X_j\right) = 0$, and $U_{1n}, \dots, U_{nn}$ are $(q \times p)$ non-stochastic matrices. Suppose that $\lim_{n\to\infty} \operatorname{cov}(V_n) = \Lambda$; where $\left|\Lambda_{ij}\right| < \infty$, for each i, j and Λ is positive definite. If there exists a function $\omega(n)$ such that $\lim_{n\to\infty} \omega(n) = \infty$, and the elements of $\omega(n)U_{jn}$ are bounded, then $V_n \xrightarrow{d} N_q(0, \Lambda)$ as $n \to \infty$.* $\square$

The above result, known as the Central Limit Theorem, is due to Malinvaud [16].

Lemma A.4. (i) $n^{-\frac{1}{2}}\left[W'C\otimes e_r'\right]U = n^{-\frac{1}{2}}\left[W'C\otimes e_r'\right]V = O_p(1)$;
(ii) $n^{-\frac{1}{2}}\left[M'C\otimes e_r'\right]U = n^{-\frac{1}{2}}\left[M'C\otimes e_r'\right]V = n^{-\frac{1}{2}}M'CW = O_p(1)$;
(iii) $n^{-\frac{1}{2}}V'AU = n^{-\frac{1}{2}}V'DU = O_p(1)$;
(iv) $n^{-\frac{1}{2}}\left[W'CW - n\sigma_W^2 I_p\right] = n^{-\frac{1}{2}}\left[V'AV - nr\sigma_V^2 I_p\right] = n^{-\frac{1}{2}}\left[V'DV - n\sigma_V^2 I_p\right] = O_p(1)$;
(v) $n^{-\frac{1}{2}}\left[U'AU - nr\sigma_u^2\right] = n^{-\frac{1}{2}}\left[U'DU - n\sigma_u^2\right] = O_p(1)$.

The proof can be obtained using the definitions of order in probability and Assumptions 1–6. □

Lemma A.5. *As $n\to\infty$, we have*
(i) $\operatorname{plim} n^{-1}\left[W'C\otimes e_r'\right]U = \operatorname{plim} n^{-1}\left[W'C\otimes e_r'\right]V = 0$;
(ii) $\operatorname{plim} n^{-1}\left[M'C\otimes e_r'\right]U = \operatorname{plim} n^{-1}\left[M'C\otimes e_r'\right]V = \operatorname{plim} n^{-1}M'CW = 0$;
(iii) $\operatorname{plim} n^{-1}V'AU = \operatorname{plim} n^{-1}V'DU = 0$;
(iv) $\operatorname{plim} n^{-1}W'CW = \sigma_W^2 I_p$; $\operatorname{plim} n^{-1}V'AV = r\sigma_V^2 I_p$; $\operatorname{plim} n^{-1}V'DV = \sigma_V^2 I_p$;
(v) $\operatorname{plim} n^{-1}U'AU = r\sigma_u^2$; $\operatorname{plim} n^{-1}U'DU = \sigma_u^2$;
(vi) $\operatorname{plim} n^{-1}X'AX = \Sigma_M + \sigma_W^2 I_p + \sigma_V^2 I_p$; $\operatorname{plim} n^{-1}X'AY = \left(\Sigma_M + \sigma_W^2 I_p\right)\beta$;
(vii) $\operatorname{plim} n^{-1}X'DX = \Sigma_M + \sigma_W^2 I_p + \frac{1}{r}\sigma_V^2 I_p$; $\operatorname{plim} n^{-1}X'DY = \left(\Sigma_M + \sigma_W^2 I_p\right)\beta$.

The proof follows using Lemma A.4 and Assumptions 1–6. □

Now we derive few results which will be useful for proving the theorems. Using (4.3) and Lemma A.2, we observe that

$$\begin{aligned}\left[\frac{1}{nr}X'AX\right]^{-1} &= \left(\Sigma_{XA} + \frac{1}{n^{1/2}}H_1\right)^{-1}\\ &= \left(I_p + \frac{1}{n^{1/2}}\Sigma_{XA}^{-1}H_1\right)^{-1}\Sigma_{XA}^{-1}\\ &= \left(I_p - \frac{1}{n^{1/2}}\Sigma_{XA}^{-1}H_1\right)\Sigma_{XA}^{-1} + O_P\left(n^{-1}\right).\end{aligned} \tag{A.1}$$

Eq. (4.4) and Lemma A.2 lead to the expression

$$\left[\frac{1}{nr}X'DX\right]^{-1} = \left(I_p - \frac{1}{n^{1/2}}\Sigma_{XD}^{-1}H_2\right)\Sigma_{XD}^{-1} + O_P\left(n^{-1}\right). \tag{A.2}$$

Similarly, Eqs. (4.3) and (4.4) and Lemma A.2 give

$$\begin{aligned}\left[\frac{1}{nr}X'\left(rD-A\right)X\right]^{-1} &= \left[(r-1)\,\Sigma_\xi + \frac{1}{n^{1/2}}H\right]^{-1}\\ &= \frac{1}{(r-1)}\left[I_p - \frac{1}{n^{1/2}(r-1)}\Sigma_\xi^{-1}H\right]\Sigma_\xi^{-1} + O_P\left(n^{-1}\right),\end{aligned} \tag{A.3}$$

where $H = rH_2 - H_1$.

Now, consider

$$\begin{aligned}\frac{1}{nr}\left[\sigma_u^{-2}\Sigma_\varphi + \frac{RS_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\right]^{-1} &= \left[I + \frac{\sigma_u^2\Sigma_\varphi^{-1}}{nr}\frac{RnrS_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\right]\left[\frac{\sigma_u^2\Sigma_\varphi^{-1}}{nr}\right]\\ &= \left\{I + \frac{\sigma_u^2\Sigma_\varphi^{-1}}{nr}R\left[\left(I_p - \frac{\Sigma_\xi^{-1}H}{n^{1/2}(r-1)}\right)\Sigma_\xi^{-1} + O_P\left(n^{-1}\right)\right]R'\right\}\left[\frac{\sigma_u^2\Sigma_\varphi^{-1}}{nr}\right]\\ &= O_P\left(n^{-1}\right).\end{aligned} \tag{A.4}$$

Thus, using (A.3) and (A.4), we have

$$S_{(rD-A)}^{-1}R'\left[\sigma_u^{-2}\Sigma_\varphi + \frac{RS_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\right]^{-1} = O_P\left(n^{-1}\right). \tag{A.5}$$

Proceeding on similar lines, using (A.1)–(A.4) and Lemma 2, we observe that

$$S_{(rD-A)}^{-1}R'\left[\sigma_u^{-2}\Sigma_\varphi + RS_{(rD-A)}^{-1}R'\right]^{-1} = O_P\left(n^{-1}\right), \tag{A.6}$$

$$S_A^{-1}R'\left[\sigma_u^{-2}\Sigma_\varphi + RS_A^{-1}R'\right]^{-1} = O_P\left(n^{-1}\right), \tag{A.7}$$

and

$$S_D^{-1}R'\left[\sigma_u^{-2}\Sigma_\varphi + RS_D^{-1}R'\right]^{-1} = O_P\left(n^{-1}\right). \tag{A.8}$$

Now, we have enough details to provide the proof of Theorem 1.

Proof of Theorem 1. Using (A.3),

$$\begin{aligned}\left[\frac{1}{nr}X'\left(rD-A\right)X\right]^{-1} &= \frac{1}{(r-1)}\Sigma_\xi^{-1} + O_P\left(n^{-\frac{1}{2}}\right);\\ &= \frac{1}{(r-1)}\left[\frac{1}{n}M'CM + \sigma_W^2 I_p\right]^{-1} + O_P\left(n^{-\frac{1}{2}}\right).\end{aligned} \tag{A.9}$$

Using (A.9) and (4.2) along with (3.6), we get

$$\begin{aligned} b_{02} &= \left\{\left[\frac{1}{n}M'CM + \sigma_W^2 I_p\right]^{-1} + O_P\left(n^{-\frac{1}{2}}\right)\right\}\left\{\left[\frac{1}{n}M'CM + \sigma_W^2 I_p\right]\beta + O_P\left(n^{-\frac{1}{2}}\right)\right\}\\ &= \beta + O_P\left(n^{-\frac{1}{2}}\right).\end{aligned} \tag{A.10}$$

Using (A.10) in the expression of $\hat{\sigma}_u^2$ given by (3.2.1) and then applying Lemma A.4, it is observed that

$$\hat{\sigma}_u^2 = \sigma_u^2 + O_P\left(n^{-\frac{1}{2}}\right). \tag{A.11}$$

It can be easily verified that the above result is true even when b_{01} or b_{03} is used instead of b_{02}. After inserting (A.11) in $\hat{b}_{11}$, we write

$$\begin{aligned}\hat{b}_{11} &= b_{01} + \frac{S_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\left[\sigma_u^{-2}\Sigma_\varphi + O_P\left(n^{-\frac{1}{2}}\right) + \frac{RS_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\right]^{-1}(\theta - Rb_{01})\\ &= b_{01} + \frac{S_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\left[I_p + \left(\sigma_u^{-2}\Sigma_\varphi + \frac{RS_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\right)^{-1} O_P\left(n^{-\frac{1}{2}}\right)\right]^{-1}\left(\sigma_u^{-2}\Sigma_\varphi + \frac{RS_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\right)^{-1}\\ &\quad\times(\theta - Rb_{01}).\end{aligned} \tag{A.12}$$

Using (A.4), we get

$$\hat{b}_{11} = b_{01} + \frac{S_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\left[I_p + O_P(1)\,O_P\left(n^{-\frac{1}{2}}\right)\right]^{-1}\left(\sigma_u^{-2}\Sigma_\varphi + \frac{RS_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\right)^{-1}(\theta - Rb_{01}). \tag{A.13}$$

Applying Lemma A.2 to (A.13), it is observed that

$$\hat{b}_{11} = b_{01} + \frac{S_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\left[I_p + O_P\left(n^{-\frac{1}{2}}\right)\right]\left(\sigma_u^{-2}\Sigma_\varphi + \frac{RS_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\right)^{-1}(\theta - Rb_{01}). \tag{A.14}$$

Using (A.5) and (A.10) in (A.14), we get

$$\begin{aligned}\hat{b}_{11} &= b_{01} + \left\{\frac{S_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\left(\sigma_u^{-2}\Sigma_\varphi + \frac{RS_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\right)^{-1} + O_P\left(n^{-\frac{3}{2}}\right)\right\}(\theta - Rb_{01})\\ &= b_{01} + \left\{\frac{S_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\left(\sigma_u^{-2}\Sigma_\varphi + \frac{RS_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\right)^{-1} + O_P\left(n^{-\frac{3}{2}}\right)\right\}\left(\theta - R\left[\beta + O_P\left(n^{-\frac{1}{2}}\right)\right]\right)\\ &= b_{01} + \frac{S_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\left(\sigma_u^{-2}\Sigma_\varphi + \frac{RS_{(rD-A)}^{-1}R'}{(r-1)^{-1}}\right)^{-1}(\theta - Rb_{01}) + O_P\left(n^{-\frac{3}{2}}\right).\end{aligned} \tag{A.15}$$

Thus we get the desired result

$$n^{\frac{1}{2}}\left(\hat{b}_{11} - \beta\right) = n^{\frac{1}{2}}\left(b_{11} - \beta\right) + O_P\left(n^{-1}\right). \tag{A.16}$$

For other values of f and s, the result can be proved similarly. □

In the following discussion, we obtain the expression for SR estimators in the form of order in probability. Using (3.5)–(3.7), (4.1), (4.2) and (A.3), we can write

$$b_{0s} = \left\{\left[I_p - \frac{1}{n^{1/2}(r-1)}\Sigma_\xi^{-1}H\right]\Sigma_\xi^{-1} + O_P\left(n^{-1}\right)\right\}\left\{\Sigma_\xi\beta + \frac{1}{n^{1/2}}h + \frac{d_s}{n^{1/2}}h^*\right\}, \tag{A.17}$$

where for $s = 1, 2, 3$, we have $d_1 = 0$, $d_2 = 1$ and $d_3 = \frac{r}{r-1}$.
Solving (A.17), we get

$$n^{\frac{1}{2}}(b_{0s} - \beta) = \Sigma_\xi^{-1}\left[h - \frac{1}{r-1}H\beta + d_s h^*\right] + O_P\left(n^{-\frac{1}{2}}\right). \tag{A.18}$$

(A.10) gives

$$\begin{aligned}(\theta - Rb_{01}) &= \left(\theta - R\left[\beta + O_P\left(n^{-1}\right)\right]\right)\\ &= (\theta - R\beta) + O_P\left(n^{-1}\right).\end{aligned} \tag{A.19}$$

Using (A.5), (A.18) and (A.19) in (3.1.1.6), we get

$$b_{11} - \beta = b_{01} - \beta + O_P\left(n^{-1}\right),$$

that is

$$n^{\frac{1}{2}}(b_{11} - \beta) = n^{\frac{1}{2}}(b_{01} - \beta) + O_P\left(n^{-\frac{1}{2}}\right). \tag{A.20}$$

Proceeding on similar lines, for $s = 1, 2, 3$ and $f = 1, 2, 3, 4$, we observe that

$$n^{\frac{1}{2}}\left(b_{fs} - \beta\right) = n^{\frac{1}{2}}(b_{0s} - \beta) + O_P\left(n^{-\frac{1}{2}}\right). \tag{A.21}$$

Let $M_i^{C'}$ and $W_i^{C'}$ be the ith rows of CM and CW respectively. $V_{i:j}^{A'}$, $V_{i:j}^{D'}$, $V_{i:j}'$ and $U_{i:j}$ are the $(i:j)$th row of AV, DV, V and $(i:j)$th element of U respectively, where $(i:j)$ indicates the row corresponding to jth replicate of ith subject for $i = 1, \ldots, n$ and $j = 1, \ldots, r$. Using these notations, we write $V_i^A = \left[V_{i:1}^A, \ldots, V_{i:r}^A\right]$, $V_i^D = \left[V_{i:1}^D, \ldots, V_{i:r}^D\right]$, $V_i = [V_{i:1}, \ldots, V_{i:r}]$ and $U_i = [U_{i:1}, \ldots, U_{i:r}]$. These notations are helpful in the proof of Theorem 2 which is given below.

Proof of Theorem 2. From (A.18) and (A.21), it is obvious that for $f = 0, 1, \ldots, 4$, the asymptotic distribution of $n^{\frac{1}{2}}\left(b_{fs} - \beta\right)$ is the same as that of $\left[h - \frac{1}{r-1}H\beta + d_s h^*\right]$. We can write

$$\begin{aligned}&\left[h - \frac{1}{r-1}H\beta + d_s h^*\right]\\ &= \frac{1}{n^{1/2}r}\left\{\left[C(M+W)\otimes e_r\right]'(U - V\beta) + V'AU + d_s V'(D-A)U - \frac{1}{r-1}V'(rD-A)V\beta\right\}\\ &= \frac{1}{n^{1/2}r}\left\{\left[\left(\left[M_1^C\cdots M_n^C\right] + \left[W_1^C\cdots W_n^C\right]\right)\otimes e_r'\right]\ \left([U_1\cdots U_n]' - [V_1\cdots V_n]'\beta\right)\right.\\ &\quad + \left((1-d_s)\left[V_1^A\cdots V_n^A\right] + d_s\left[V_1^D\cdots V_n^D\right]\right)[U_1\cdots U_n]'\\ &\quad \left. + \left(\frac{1}{r-1}\left[V_1^A\cdots V_n^A\right] - \frac{r}{r-1}\left[V_1^D\cdots V_n^D\right]\right)[V_1\cdots V_n]'\beta\right\}\\ &= \frac{1}{n^{1/2}r}\sum_i^n\left\{\left[\left(M_i^C + W_i^C\right)\otimes e_r'\right]\left(U_i' - V_i'\beta\right) + \left((1-d_s)V_i^A + d_s V_i^D\right)U_i'\right.\\ &\quad \left. + \left(\frac{1}{r-1}V_i^A - \frac{r}{r-1}V_i^D\right)V_i'\beta\right\}\\ &= \sum_i^n C_i D_i\end{aligned}$$

where for $i = 1, \ldots, n$

$$C_i = \frac{1}{n^{1/2}r}\left[M_i^C\otimes e_r', I_p, -\beta'\otimes\left(M_i^C\otimes e_r'\right), -\beta'\otimes I_p, (1-d_s)I_p, d_s I_p, \frac{-1}{r-1}\beta'\otimes I_p\right] \quad \text{and}$$

$$D_i' = \left[U_i, U_i\left(W_i^{C'}\otimes e_r\right), \operatorname{vec}'\left(V_i'\right), \operatorname{vec}'\left(\left[W_i^C\otimes e_r'\right]V_i'\right), U_i V_i^{A'}, U_i V_i^{D'}, \operatorname{vec}'\left(\left[rV_i^D - V_i^A\right]V_i'\right)\right]$$

are matrices of constants and iid random vectors, respectively. Assumptions 1–6 imply that $\lim_{n\to\infty} E\,(D_i) = 0$ and $n^{1/2}C_i$ is bounded for fixed r. Thus using Lemma A.3, for $s = 1, 2, 3$

$$\left[h - \frac{1}{r-1}H\beta + d_s h^*\right] \xrightarrow{d} N_p\left(0_{p\times 1}, \Omega_s\right), \tag{A.22}$$

where

$$\Omega_s = \lim_{n\to\infty} E\left\{\left[h - \frac{1}{r-1}H\beta + d_s h^*\right]\left[h - \frac{1}{r-1}H\beta + d_s h^*\right]'\right\}.$$

Using Assumptions 1–6 and on evaluating the expectations, we get the expressions for $\Omega_s; s = 1, 2, 3$ as given in Eqs. (4.13)–(4.15).

Thus from (A.18), (A.21) and (A.22), we have for $f = 0, 1, 2, 3, 4$

$$n^{\frac{1}{2}}\left(b_{fs} - \beta\right) \xrightarrow{d} N_p\left(0_{p\times 1}, \Sigma^{-1}\Omega_s\Sigma^{-1}\right),$$

where $\Sigma^{-1} = \lim_{n\to\infty} \Sigma_\xi^{-1}$. $\square$

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