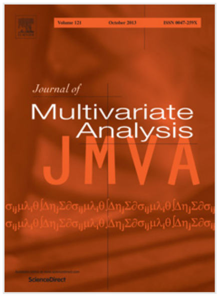
Accepted Manuscript



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PII: S0047-259X(15)00187-6
DOI: http://dx.doi.org/10.1016/j.jmva.2015.08.003
Reference: YJMVA 3977

To appear in: *Journal of Multivariate Analysis*

Received date: 16 December 2014

Construction of Two New General Classes of Bivariate Distributions Based on Stochastic Orders

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Abstract

In this paper, based on the concepts of stochastic orders, we propose two new general classes of bivariate distributions. The usual stochastic order and likelihood ratio order are applied to construct the classes. The joint distributions in each class are derived. It will be seen that the obtained formulas for the joint distributions are very simple and easy to apply. Then, the relationships between the classes are discussed and characterized. We illustrate the practical usefulness of the proposed classes by showing that a number of new families of bivariate distributions can be generated from the classes. Furthermore, to illustrate practical relevance, we apply several developed models to analyze a real bivariate failure time data set.

AMS subject classification: Primary 60E05; Secondary 62H10

Keywords: Usual Stochastic order; Likelihood ratio order; Parametric family; Characterization; Bivariate lack of memory property

1. Introduction

Dependent random quantities can frequently be encountered in practice and they have been modelled by using bivariate distributions. In the literature, various specific parametric models for bivariate distributions have been suggested and studied. See, for instance, Gumbel (1960), Freund (1961), Marshall and Olkin (1967), Downton (1970), Hawkes (1972), Block and Basu (1974), Shaked (1984), Sarkar (1987) and Hayakawa (1994). A nice review on the modelling of multivariate survival models can be found in Hougaard (1987). An excellent encyclopaedic survey of various bivariate distributions can be found in Balakrishnan and Lai (2009).

In practice, the lifetimes of organisms or items are most often stochastically dependent rather than being completely independent. For instance, in reliability application, the failure of one component in a two-component system may considerably shorten the residual lifetime of the remaining component by increasing the load or stress of the remaining one. Several practical examples of similar situations can be found in Section 2 of Lee and Cha (2014). Based on this observation, Lee and Cha (2014) have recently developed a new class of bivariate distributions. More specifically, it was assumed in Lee and Cha (2014) that the residual lifetime is shortened in the sense of failure rate

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order. That is, the failure rate of the remaining component increases after the failure of the other one. In this paper, two new general classes of bivariate distributions will be constructed based on the assumption that the residual lifetime of the remaining component is shortened in the sense of two other types of stochastic orders: usual stochastic order and likelihood ratio order. Thus, this paper provides a general insight and an integrated framework for modelling new classes of bivariate distributions based on the concepts of stochastic orders. It will be seen that the obtained formulas for the joint distributions are very simple and easy to apply (see the joint pdfs in Theorem 1, Theorem 2, and especially those in Corollary 1 of Section 2). Numerous parametric families of bivariate distributions can be generated by just choosing different baseline distributions. In order to illustrate this property, we provide several examples. We will utilize reliability/lifetime modelling tools. However, the applications of the developed models are not necessarily limited to lifetime analysis, but they can generally be applied to the modelling of dependent random quantities in different areas.

The structure of this paper is as follows. In Section 2, we briefly review the general class of bivariate distributions suggested in Lee and Cha (2014). Then, two new general classes of bivariate distributions will be constructed. In Section 3, the characterization of the relationships among those three classes will be made. This characterization would help practical modelling of bivariate distributions and allow more convenient interpretation of the modelling parameters. In Section 4, several specific families of bivariate distributions will be generated for illustrations. In Section 5, the families of bivariate distributions obtained in Section 4 are applied to a real bivariate failure time data set to show the practical relevance of the developed classes. Finally, in Section 6, the results in this paper are briefly summarized.

2. New General Classes of Bivariate Distributions

As given in an example in Section 1, random quantities are commonly positively dependent. Thus, throughout this paper, we will mainly discuss the classes of distributions for modelling positive dependency, unless otherwise specified. However, our discussions can be extended to those for modelling negative dependency without difficulty, which will be formally discussed in subsection 2.4.

2.1 A Class Based on the Failure Rate Order

In this subsection, the general class of bivariate distributions suggested in Lee and Cha (2014) will be briefly reviewed. In Lee and Cha (2014), the following practical situation was considered for modelling dependency of two random quantities. The system is composed of two components (component 1 and component 2) and the original lifetimes of components 1 and 2, when they start to operate, are described by the corresponding failure rates $\lambda_1(t)$ and $\lambda_2(t)$, pdf's $f_1(t)$ and $f_2(t)$, and survival functions $S_1(t)$ and $S_2(t)$, respectively. These original lifetimes of components 1 and 2 are denoted by X_1^* and X_2^*, respectively, assuming that X_1^* and X_2^* are stochastically independent. It was assumed that the failure of one component shortens the residual lifetime of the remaining component due to the increased stress. In this case, after the change point, $\min\{X_1^*, X_2^*\}$, the residual lifetime distribution of the remaining component changes. We denote the corresponding eventual dependent lifetimes of components 1 and 2 by X_1 and X_2, respectively. In the following discussions, the notations $S(x_1, x_2)$, $f(x_1, x_2)$ will be used to denote the joint survival function and the joint pdf of X_1 and X_2, respectively. Define indicator variables $\Psi_1(t)$ and $\Psi_2(t)$ as follows: $\Psi_1(t)=1$ ($\Psi_2(t)=1$) if component 1 (component 2) is functioning at time t, whereas $\Psi_1(t)=0$ ($\Psi_2(t)=0$) if component 1 (component 2) is at failed state at time t. For notational convenience, let $\tilde{i}=2$ when $i=1$; whereas $\tilde{i}=1$ when $i=2$. For component i, $i=1,2$, it was assumed that, depending on the states of component $\tilde{i}$, the failure rates r_i satisfy:

$$r_i(t \mid \Psi_{\tilde{i}}(s)=1, 0 \le s \le t) = \lambda_i(t), \quad i=1,2, \tag{1}$$

and

$$r_i(t \mid \Psi_{\tilde{i}}(s)=1, 0 \le s < u; \Psi_{\tilde{i}}(s)=0, u \le s \le t) = \alpha_i(u, t-u)\lambda_i(t), \ t \ge u, \ i=1,2, \tag{2}$$

where $\alpha_i(s,w) \ge 1$, for all $s, w \ge 0$, $i=1,2$. Detailed practical interpretations for (1) and (2) are given in Lee and Cha (2014). Then, based on the assumptions (1)-(2), the joint survival function was obtained by

$$S(x_1,x_2) = \int_{x_1}^{x_2} \lambda_1(u)\gamma_2(x_2,u)\, S_1(u)S_2(u)\, du + S_1(x_2)S_2(x_2), \text{ for } 0 < x_1 < x_2, \tag{3}$$

$$S(x_1,x_2) = \int_{x_2}^{x_1} \lambda_2(u)\gamma_1(x_1,u)\, S_1(u)S_2(u)\, du + S_1(x_1)S_2(x_1), \text{ for } 0 < x_2 \le x_1, \tag{4}$$

where $\gamma_i(x_i,u) \equiv \exp\left(-\int_0^{x_i-u} \alpha_i(u,w)\lambda_i(u+w)dw\right)$, $i=1,2$. The corresponding joint pdf was also obtained in Lee and Cha (2014), which has similar functional form provided in Cox (1972).

In order to construct new classes of bivariate distributions in this paper, we need to reinterpret the meaning of the stochastic modeling in (1)-(2) in terms of a stochastic order. For our discussions, we need the following formal definition of the failure rate order (hazard rate order) between two random variables (see Shaked and Shanthikumar (2007), Cha and Mi (2007), Finkelstein and Cha (2013)).

Definition 1. *Let Z_1 and Z_2 be two nonnegative, continuous random variables with respective failure rate functions $r_1(t)$ and $r_2(t)$, such that $r_1(t) \ge r_2(t)$, $t \ge 0$. Then Z_1 is said to be smaller than Z_2 in the failure rate order, denoted by $Z_1 \le_{fr} Z_2$.*

Let $\tilde{X}_i$, $i=1,2$, denote the lifetimes of component 1 and component 2 when the two lifetimes are completely independent, i.e., when $\alpha_i(s,w) \equiv 1$, $i=1,2$. It is then clear that the failure rate of the random variable

$$(\tilde{X}_i \mid \Psi_{\tilde{i}}(s)=1, 0 \le s < u, \Psi_{\tilde{i}}(s)=0, s \ge u, \tilde{X}_i > u)$$

is just given by $\lambda_i(t)$, $t \ge u$, $i=1,2$. On the other hand, from (1) and (2), that of the random variable

$$(X_i \mid \Psi_{\tilde{i}}(s)=1, 0 \le s < u, \ \Psi_{\tilde{i}}(s)=0, s \ge u, X_i > u)$$

is given by $\alpha_i(u,t-u)\lambda_i(t)$, $t \ge u$, $i=1,2$. Therefore,

$$\alpha_i(u,t-u)\lambda_i(t) \ge \lambda_i(t), \text{ for all } t \ge u, \ u > 0, \ i=1,2,$$

which implies the following relationship:

$$(X_i \mid \Psi_{\tilde{i}}(s)=1, 0 \le s < u, \ \Psi_{\tilde{i}}(s)=0, s \ge u, X_i > u)$$
$$\le_{fr} (\tilde{X}_i \mid \Psi_{\tilde{i}}(s)=1, 0 \le s < u, \Psi_{\tilde{i}}(s)=0, s \ge u, \tilde{X}_i > u), \text{ for all } u > 0, \ i=1,2. \tag{5}$$

In the following two sections, we develop two new general classes of bivariate distributions by employing different types of stochastic orders between the two random variables in (5).

2.2 A Class Based on the Usual Stochastic Order

The assumptions regarding two lifetimes X_1 and X_2 when both the two components 1 and 2 are in the operating state are the same as before (e.g., (1)). For our discussion, it is necessary to define the following usual stochastic order (see also Shaked and Shanthikumar (2007)):

Definition 2. *Let Z_1 and Z_2 be two nonnegative, continuous random variables with respective cdfs $G_1(t)$ and $G_2(t)$ (survival functions $\overline{G}_1(t)$ and $\overline{G}_2(t)$), such that $G_1(t) \geq G_2(t)$ (or equivalently, $\overline{G}_1(t) \leq \overline{G}_2(t)$), $t \geq 0$. Then Z_1 is said to be smaller than Z_2 in the usual stochastic order, denoted by $Z_1 \leq_{st} Z_2$.*

In accordance with the above motivation, we now assume relationship (5) with $\leq_{fr}$ replaced by $\leq_{st}$. Note that the survival function of the random variable on the right-hand side of (5) is

$$P(\tilde{X}_i > t \mid \Psi_{\tilde{i}}(s) = 1, 0 \leq s < u, \Psi_{\tilde{i}}(s) = 0, s \geq u, \tilde{X}_i > u) = S_i(t) / S_i(u).$$

Therefore, relationship (5) with $\leq_{fr}$ replaced by $\leq_{st}$ is equivalent to

$$P(X_i > t \mid \Psi_{\tilde{i}}(s) = 1, 0 \leq s < u,\ \Psi_{\tilde{i}}(s) = 0, s \geq u, X_i > u) = S_i(u + \rho_i(u, t-u)) / S_i(u), \tag{6}$$

where $\rho_i(s,w)$ is an increasing function of w satisfying $\rho_i(s,w) \geq w$, with $\rho_i(s,0) = 0$, for all $s, w \geq 0$, $i = 1,2$.

In the following theorem, we obtain the joint distribution of X_1 and X_2 under the assumed condition (6).

Theorem 1. *Suppose that $\rho_i(s,w)$ is differentiable with respect to s and w, respectively, $i = 1,2$. Under the assumed condition in (6), the joint survival function $S(x_1, x_2)$ is given by*

$$S(x_1, x_2) = \int_{x_1}^{x_2} f_1(u) S_2(u + \rho_2(u, x_2 - u))\, du + S_1(x_2) S_2(x_2), \quad \text{for } 0 < x_1 < x_2, \tag{7}$$

$$S(x_1, x_2) = \int_{x_2}^{x_1} f_2(u) S_1(u + \rho_1(u, x_1 - u))\, du + S_1(x_1) S_2(x_1), \quad \text{for } 0 < x_2 \leq x_1; \tag{8}$$

and the corresponding joint pdf is given by

$$f(x_1, x_2) = \frac{\partial \rho_2(x_1, x_2 - x_1)}{\partial x_2} f_1(x_1)\ f_2(x_1 + \rho_2(x_1, x_2 - x_1)) \quad , \text{for} \quad 0 < x_1 < x_2,$$

$$f(x_1, x_2) = \frac{\partial \rho_1(x_2, x_1 - x_2)}{\partial x_1} f_2(x_2) f_1(x_2 + \rho_1(x_2, x_1 - x_2)) \quad , \text{for} \quad 0 < x_2 \leq x_1.$$

Proof. Let $0 < x_1 < x_2$. The conditional joint survival function given that component 1 has failed first at u, for $u < x_1$ or $x_2 \leq u$, is given by

$$P(X_1 > x_1, X_2 > x_2 \mid X_1^* < X_2^*, X_1^* = u) = \begin{cases} 0, & u < x_1, \\ 1, & x_2 \leq u, \end{cases}$$

whereas, for $x_1 \leq u < x_2$, from the assumptions stated above,

$$\begin{aligned} P(X_1 > x_1, X_2 > x_2 \mid X_1^* < X_2^*, X_1^* = u) &= P(X_2 > x_2 \mid X_1^* < X_2^*, X_1^* = u) \\ &= S_2(u + \rho_2(u, x_2 - u)) / S_2(u),\ x_1 \leq u < x_2. \end{aligned}$$

Then, by applying similar procedures as those described in the proof of Theorem 1 in Lee and Cha (2014), we now have

$$S(x_1,x_2)=\int_{x_1}^{x_2} S_2(u+\rho_2(u,x_2-u))\lambda_1(u)S_1(u)\,du + S_1(x_2)S_2(x_2),\ \text{for } 0<x_1<x_2.$$

The case for $0<x_2\le x_1$ can be proved symmetrically. The joint pdf $f(x_1,x_2)$ can be obtained by differentiation.

■

It is notable that the obtained formulas for the joint pdf in Theorem 1 are very simple and easy to apply (see also specific parametric distributions in Section 4). By letting $x_1\equiv 0$ or $x_2\equiv 0$ in $S(x_1,x_2)$, the marginal distributions can be easily obtained.

Especially when the parameter function is specified as $\rho_i(s,w)=p_i w$, $p_i>1$, $i=1,2$, which does not depend on s, the class has the following much simpler and more useful parametric form.

Corollary 1. *Suppose that* $\rho_i(s,w)=p_i w$, $i=1,2$. Then the joint pdf is given by

$$f(x_1,x_2)=p_2 f_1(x_1)\, f_2(x_1+p_2(x_2-x_1))\ ,\textit{for}\ \ 0<x_1<x_2,$$
$$f(x_1,x_2)=p_1 f_2(x_2) f_1(x_2+p_1(x_1-x_2))\ ,\textit{for}\ \ 0<x_2\le x_1.$$

■

From Corollary 1, it can be seen that numerous parametric families of bivariate distributions can be generated by just choosing different baseline distributions $f_1(x_1)$ and $f_2(x_2)$.

Remark 1.

From the failure mechanism of the model, the joint pdf in Corollary 1 can also be interpreted as that of (X_1,X_2), *where*

$$X_1=Y_1\,I(Y_1<Y_2)+\left\{\frac{Y_1}{p_1}+\left(1-\frac{1}{p_1}\right)Y_2\right\}I(Y_1\ge Y_2),$$
$$X_2=\left\{\frac{Y_2}{p_2}+\left(1-\frac{1}{p_2}\right)Y_1\right\}I(Y_1<Y_2)+Y_2\,I(Y_1\ge Y_2),$$

and (Y_1,Y_2) *are independent with densities* f_1, f_2, *respectively.*

2.3 A Class Based on the Likelihood Ratio Order

In this subsection, another, more restrictive, class will be constructed based on the likelihood ratio order of lifetimes. The definition of likelihood ratio order is as follows (see also Shaked and Shanthikumar (2007)):

Definition 3. *Let* Z_1 *and* Z_2 *be two nonnegative, continuous random variables with respective pdfs* $g_1(t)$ *and* $g_2(t)$, *such that* $\frac{g_1(t)}{g_2(t)}$ *is decreasing in* $t\ge 0$. *Then* Z_1 *is said to be smaller than* Z_2 *in the likelihood ratio order, denoted by* $Z_1\le_{lr} Z_2$.

In this regard, we are now employing relationship (5) with $\le_{fr}$ replaced by $\le_{lr}$. The assumptions regarding two lifetimes X_1 and X_2 when both the two components 1 and 2 are in the operating state are the same as before (e.g., (1)). Then, under the independence of X_1 and X_2, the conditional pdf of $\widetilde{X}_i(\widetilde{i}:u) \equiv (\widetilde{X}_i \mid \Psi_{\widetilde{i}}(s)=1, 0 \le s < u, \Psi_{\widetilde{i}}(s)=0, s \ge u, \widetilde{X}_i > u)$ is given by

$$f_{\widetilde{X}_i(\widetilde{i}:u)}(t) = \frac{f_i(t)}{S_i(u)}, \quad t \ge u, \quad i = 1,2 .$$

On the other hand, we will assume that the conditional pdf of $X_i(\widetilde{i}:u) \equiv (X_i \mid \Psi_{\widetilde{i}}(s)=1, 0 \le s < u, \Psi_{\widetilde{i}}(s)=0, s \ge u, X_i > u)$ is given by

$$f_{X_i(\widetilde{i}:u)}(t) = \frac{\beta_i(u, t-u) f_i(t)}{S_i(u)}, \quad t \ge u, \quad i = 1,2, \tag{9}$$

where $\beta_i(s,w)$ is a decreasing function of w for all $s > 0$, $i = 1,2$. Then this assumption implies that the ratio $f_{X_i(\widetilde{i}:u)}(t) / f_{\widetilde{X}_i(\widetilde{i}:u)}(t)$ decreases in $t \ge u$, which implies relationship (5) with $\le_{fr}$ replaced by $\le_{lr}$. Note that the function $\beta_i(s,w)$ in (9) should be carefully taken so that the conditional pdf in (9) should be a legitimate probability density function. That is, it should be satisfied that

$$\int_u^\infty \beta_i(u, t-u) f_i(t) dt = \int_0^\infty \beta_i(u,t) f_i(u+t) dt = S_i(u) .$$

In order to find an appropriate $\beta_i(s,w)$, one can start with a simple decreasing function of w. This technical issue will be discussed in detail in Section 4.

In the following theorem, we obtain the joint distribution of X_1 and X_2 under the assumed condition (9).

Theorem 2. *Under the assumed condition in (9), the joint survival function $S(x_1, x_2)$ is given by*

$$S(x_1,x_2) = \int_{x_1}^{x_2} \int_{x_2-u}^{\infty} \beta_2(u,w) f_2(u+w) dw f_1(u)\, du + S_1(x_2) S_2(x_2), \quad \text{for } 0 < x_1 < x_2, \tag{10}$$

$$S(x_1,x_2) = \int_{x_2}^{x_1} \int_{x_1-u}^{\infty} \beta_1(u,w) f_1(u+w) dw f_2(u)\, du + S_1(x_1) S_2(x_1), \quad \text{for } 0 < x_2 \le x_1; \tag{11}$$

and the corresponding joint pdf is given by

$$f(x_1,x_2) = f_1(x_1) \beta_2(x_1, x_2 - x_1) f_2(x_2), \text{ for } 0 < x_1 < x_2,$$
$$f(x_1,x_2) = f_2(x_2) \beta_1(x_2, x_1 - x_2) f_1(x_1), \text{ for } 0 < x_2 \le x_1.$$

Proof. For $0 < x_1 < x_2$, it is now clear that

$$P(X_1 > x_1, X_2 > x_2 \mid X_1^* < X_2^*, X_1^* = u) = \begin{cases} 0, & u < x_1, \\ \int_{x_2-u}^{\infty} \beta_2(u,w) f_2(u+w) dw / S_2(u), & x_1 \le u < x_2, \\ 1, & x_2 \le u, \end{cases}$$

which enables us to obtain the desired results by applying similar procedure as those described in the proof of Theorem 1.

■

2.4 Classes for Modeling Negative Dependency

As mentioned before, our previous discussions can be readily extended for modelling the classes of bivariate distributions having negative dependency. We discuss this issue in this subsection. Observe that, in the previous discussions, the key idea for modelling positive dependency is that the failure of one component should shorten the residual lifetime of the remaining component. This idea was mathematically modelled by the fact that $(X_i \mid \Psi_{\tilde{i}}(s)=1, 0\le s<u,\ \Psi_{\tilde{i}}(s)=0, s\ge u, X_i>u)$ should be smaller than $(\tilde{X}_i \mid \Psi_{\tilde{i}}(s)=1, 0\le s<u, \Psi_{\tilde{i}}(s)=0, s\ge u, \tilde{X}_i>u)$ in a stochastic order sense. Clearly, by reversing this relationship, we can generate negative dependency. In this case, the conditions on the parameter functions should be changed as follows:

- $0<\alpha_i(s,w)\le 1$, for all $s,w\ge 0$ (‘*fr’* class);
- $\rho_i(s,w)$ is increasing in w, satisfying $0<\rho_i(s,w)\le w$, $\rho_i(s,0)=0$, for all $s,w\ge 0$ (‘*st’* class);
- $\beta_i(s,w)$ is an increasing function of w for all $s>0$ (‘*lr’* class),

$i=1,2$, respectively.

In this case, the distributional formulas, such as $S(x_1,x_2)$, in each class are the same as before. For notational convenience, we will denote by C_{FR-}, C_{ST-} and C_{LR-} the corresponding classes for negative dependency, respectively, whereas, by C_{FR}, C_{ST} and C_{LR}, the classes for positive dependency defined in subsections 2.1-2.3, respectively. In Section 4, we will deal with a model with negative dependency to illustrate the generating methodology.

3. Relationships Between the Classes

In Section 2, we have suggested three classes of bivariate distributions for modeling positive dependency. In this section we will characterize the relationships among these classes. Similar arguments can also be applied to the classes for negative dependency.

As defined in subsection 2.4, C_{FR}, C_{ST} and C_{LR} represent the classes of bivariate distributions defined in subsections 2.1-2.3, respectively. Note that, for two random variables Z_1 and Z_2 (see Shaked and Shanthikumar (2007)),

$$Z_1 \le_{lr} Z_2 \Rightarrow Z_1 \le_{fr} Z_2 \Rightarrow Z_1 \le_{st} Z_2 .$$

Therefore, obviously, $C_{LR}\subset C_{FR}\subset C_{ST}$. More detailed relationships between the members in the classes are given in the following theorem.

Theorem 3. *The following relationship holds.*
(i)Let $S(x_1,x_2)\in C_{LR}$ *be given by (10) and (11). Then* $S(x_1,x_2)$ *belongs to* C_{FR} *and it can be expressed by (3) and (4), respectively, with the corresponding parameter functions*

$$\alpha_i(u,t)=\frac{\beta_i(u,t)\int_t^\infty f_i(u+s)ds}{\int_t^\infty \beta_i(u,s)\, f_i(u+s)\, ds},\ \textit{for}\ \ i=1,2. \tag{12}$$

(ii)Let $S(x_1,x_2)\in C_{FR}$ *be given by (3) and (4). Then* $S(x_1,x_2)$ *belongs to* C_{ST} *and it can be expressed by (7) and (8), respectively, with the corresponding parameter functions*

$$\rho_i(u,t) = \Lambda_i^{-1}\left(\int_0^t \alpha_i(u,s)\lambda_i(u+s)\,ds + \Lambda_i(u)\right) - u\,,\ for\ \ i=1,2\,, \tag{13}$$

where $\Lambda_i(u) \equiv \int_0^u \lambda_i(s)\,ds$ *and* $\Lambda_i^{-1}(u)$ *is the corresponding inverse function.*

Proof. (i)From (9), under the model based on the likelihood ratio order, the pdf of the residual lifetime of component 1 given that component 2 has failed first at time u is

$$\frac{\beta_1(u,t)\,f_1(u+t)}{S_1(u)}\,,\ \ t \ge 0\,.$$

Then, the corresponding failure rate function can be obtained as

$$\frac{\beta_1(u,t)\,f_1(u+t)/S_1(u)}{\int_t^\infty \beta_1(u,w)\,f_1(u+w)dw/S_1(u)} = \left(\frac{\beta_1(u,t)\int_t^\infty f_1(u+s)ds}{\int_t^\infty \beta_1(u,w)\,f_1(u+w)dw}\right)\cdot \lambda_1(u+t)\,, \quad t \ge 0\,. \tag{14}$$

On the other hand, under the model based on the failure rate order, the failure rate of the residual lifetime of component 1 given that component 2 has failed first at time u is, from (2),

$$\alpha_1(u,t)\lambda_1(u+t)\,,\ \ t \ge 0\,. \tag{15}$$

Now comparing (14) with (15), the survival function with parameter function $\beta_1(u,t)$ belonging to C_{LR} corresponds to the survival function with parameter function (12) with $i=1$ belonging to C_{FR}. For component 2, the results can be obtained symmetrically.

(ii)From (2), under the model based on the failure rate order, the survival function of the residual lifetime of component 1 given that component 2 has failed first at time u is

$$\exp\left(-\int_0^t \alpha_1(u,s)\lambda_1(u+s)\,ds\right),\ \ t \ge 0\,.$$

In order to find the appropriate parameter function $\rho_1(u,t)$ which corresponds to the survival function belonging to the class C_{ST}, set

$$\exp\left(-\int_0^t \alpha_1(u,s)\lambda_1(u+s)ds\right) = \exp\left(-\int_u^{\rho_1(u,t)+u} \lambda_1(s)\,ds\right),$$

which gives us the following equality:

$$\int_0^t \alpha_1(u,s)\lambda_1(u+s)\,ds + \Lambda_1(u) = \Lambda_1(\rho_1(u,t)+u)\,.$$

Solving the above equation with respect to the function $\rho_1(u,t)$, we obtain (13) with $i=1$. By symmetry, the relationship for component 2 can be directly obtained.

■

Remark 2.

(i)It can be easily verified that the parameter functions $\alpha_i(u,t)$ *in (12) and* $\rho_i(u,t)$ *in (13) satisfy*

the necessary conditions. That is, $\alpha_i(u,t) \geq 1$*, for all* $u,t \geq 0$*, and* $\rho_i(u,t)$ *is an increasing function of* t *satisfying* $\rho_i(u,t) \geq t$*, with* $\rho_i(u,0) = 0$*,* $i = 1,2$*.*

(ii)As $C_{LR} \subset C_{FR} \subset C_{ST}$*, class* C_{ST} *is the largest class. Therefore, mathematically, it is sufficient to consider only class* C_{ST}*. However, other classes need to be considered separately due to: (1) sometimes,* C_{ST} *might be too large to find a proper modeling; (2)* C_{FR} *and* C_{LR} *can allow a simpler and convenient formula; (3) for some simple specific families of distributions constructed from class* C_{LR} *or* C_{FR}*, it is not feasible to construct them directly from class* C_{ST}*.*

(iii)Suppose that $\lambda_i(t) = \lambda_i$*,* $t \geq 0$*,* $i = 1,2$*. It was shown in Lee and Cha (2014) that, for class* C_{FR}*, if the parameter functions* $\alpha_i(u,t)$*,* $i = 1,2$*, do not depend on the argument* u*, then* (X_1, X_2) *possesses the bivariate lack of memory property (BLMP). Similarly, it can also be shown that, for each class* C_{ST} *and* C_{LR}*, if the parameter functions,* $\rho_i(u,t)$*,* $i = 1,2$*, and* $\beta_i(u,t)$*,* $i = 1,2$*, respectively, do not depend on the argument* u*, then* (X_1, X_2) *possesses the BLMP.*

(iv)If the distributions of the original lifetimes of components 1 and 2 (X_1^* *and* X_2^* *) are identical and the parameter functions* $\alpha_i(u,t)$*,* $\rho_i(u,t)$*,* $\beta_i(u,t)$*,* $i = 1,2$*, respectively, do not depend on the argument* u*, then the bivariate distributions in* C_{FR}*,* C_{ST}*, and* C_{LR} *can be regarded as the joint distributions of pairs of sequential order statistics. For the definition of the sequential order statistics, the interested reader could refer to Kamps (1995).*

4. Specific Families of Distributions

We will now see that some specific well-known families of bivariate distributions belong to the classes defined in Section 2 and will illustrate how to generate new families of bivariate distributions. It should be stressed again that the following models are just the most typical illustrations for the application of the general methodology for constructing bivariate distributions. Numerous families of distributions can further be generated based on the parametric model suggested in Theorem 1, Corollary 1 and Theorem 2. In this sense, this paper provides a new general insight and new perspective on the modelling of the bivariate distributions.

First, it will be shown that the classes contain well-known bivariate families of distributions as special cases.

Model 1 Let $\lambda_1(t) = \lambda_1$, $\lambda_2(t) = \lambda_2$. We will consider a family of bivariate distributions belonging to C_{ST} and now we define the parameter functions $\rho_i(u,t) = p_i t$, $p_i > 1$, for all $u,t \geq 0$, $i = 1,2$. From Corollary 1,

$$f(x_1, x_2) = \lambda_1 p_2 \lambda_2 \exp\{-p_2\lambda_2 x_2 - (\lambda_1 + \lambda_2 - p_2\lambda_2)x_1\}, \quad 0 < x_1 < x_2,$$

$$f(x_1, x_2) = \lambda_2 p_1 \lambda_1 \exp\{-p_1\lambda_1 x_1 - (\lambda_1 + \lambda_2 - p_1\lambda_1)x_2\}, \quad 0 < x_2 \leq x_1.$$

Reparametrizing $\lambda_1 \equiv \alpha$, $\lambda_2 \equiv \beta$, $p_1\lambda_1 \equiv \alpha'$ and $p_2\lambda_2 \equiv \beta'$, this family of distributions becomes the bivariate survival model proposed by Freund (1961). It was also shown in Lee and Cha (2014) that the above bivariate model can also be obtained from C_{FR} by setting $\alpha_i(u,t) = \alpha_i$, $i = 1,2$, and, through a suitable reparameterization, it becomes the survival model studied by Block and Basu (1974).

Now, it will be shown that this model can also be obtained from a family of bivariate distributions belonging to C_{LR}. Let $\lambda_1(t) = \lambda_1$, $\lambda_2(t) = \lambda_2$ again. As mentioned in subsection 2.3, we need to carefully define the parameter functions $\beta_i(u,t)$, $i=1,2$, so that the conditional pdfs in (9) should be legitimate pdfs. For this, we start with simple baseline functions $\beta_{0i}(t)$, $i=1,2$, which are decreasing in t. Let $\beta_{0i}(t)=\exp\{-b_i t\}$, $i=1,2$. First, in order to find $\beta_1(u,t)$ based on $\beta_{01}(t)$, construct a function similar to the conditional pdf (9):

$$\frac{\beta_{01}(t-u)\, f_1(t)}{S_1(u)} = \frac{\exp\{-b_1(t-u)\}\lambda_1 \exp\{-\lambda_1 t\}}{\exp\{-\lambda_1 u\}}, \tag{16}$$

which is not a proper pdf yet. Integrating (16) for $t \ge u$, we have

$$\int_u^\infty \exp\{-b_1(t-u)\}\lambda_1 \exp\{-\lambda_1(t-u)\}dt = \frac{\lambda_1}{\lambda_1+b_1}.$$

Therefore, by setting $\beta_1(u,t)=((\lambda_1+b_1)/\lambda_1)\exp\{-b_1 t\}$, we can now obtain a legitimate conditional pdf of $(X_1 \mid \Psi_2(s)=1, 0\le s<u, \Psi_2(s)=0, s\ge u, X_1>u)$ in (9). Symmetrically, we can now specify $\beta_2(u,t)$ based on $\beta_{02}(t)=\exp\{-b_2 t\}$: $\beta_2(u,t)=((\lambda_2+b_2)/\lambda_2)\exp\{-b_2 t\}$. Then, from Theorem 2, we have

$$f(x_1,x_2)=\lambda_1(\lambda_2+b_2)\exp\{-(\lambda_1-b_2)x_1-(\lambda_2+b_2)x_2\},\ \ 0<x_1<x_2,$$
$$f(x_1,x_2)=\lambda_2(\lambda_1+b_1)\exp\{-(\lambda_2-b_1)x_2-(\lambda_1+b_1)x_1\},\ \ 0<x_2\le x_1.$$

It can be easily seen that, by a suitable reparameterization, we can arrive at the above Model 1. It is also clear from Theorem 3 that $\alpha_i(u,t)$, $i=1,2$, which correspond to this C_{LR} model are given by $\alpha_1(u,t)=(\lambda_1+b_1)/\lambda_1$ and $\alpha_2(u,t)=(\lambda_2+b_2)/\lambda_2$.

■

Model 2 Let $\lambda_i(t)=b_i\lambda_i^{b_i}t^{b_i-1}$, $t\ge 0$, where $\lambda_i, b_i>0$, $i=1,2$. We will now consider a family of bivariate distributions belonging to C_{ST} and define the parameter functions $\rho_i(u,t)=p_i t$, $p_i>1$, $i=1,2$. Then, from Corollary 1, we have

$$f(x_1,x_2)=p_2 b_1 b_2 \lambda_1^{b_1}\lambda_2^{b_2}x_1^{b_1-1}(p_2(x_2-x_1)+x_1)^{b_2-1}\exp\left\{-[\lambda_1 x_1]^{b_1}-[\lambda_2(p_2(x_2-x_1)+x_1)]^{b_2}\right\},$$
$$0<x_1<x_2,$$
$$f(x_1,x_2)=p_1 b_1 b_2 \lambda_1^{b_1}\lambda_2^{b_2}x_2^{b_2-1}(p_1(x_1-x_2)+x_2)^{b_1-1}\exp\left\{-[\lambda_2 x_2]^{b_2}-[\lambda_1(p_1(x_1-x_2)+x_2)]^{b_1}\right\},$$
$$0<x_2\le x_1.$$

■

In Model 1, the obtained model belongs to C_{LR}, and, accordingly, it also belongs to C_{FR} and C_{ST}. Now, in the following example, we will see a model which belongs to C_{FR} (thus also to C_{ST}) but not to C_{LR} and one which belongs to C_{ST} but not to C_{FR} (thus neither to C_{LR}).

Example 1. (i) In Model 2, let us take more restrictive baseline marginals : $\lambda_i(t)=b_i\lambda_i^{b_i}t^{b_i-1}$, $t\ge 0$, where $\lambda_i>0$, $b_i\ge 1$, $i=1,2$ (i.e., marginals with increasing failure rates). Clearly, the failure rate of the random variable

$$(\tilde{X}_i \mid \Psi_{\tilde{i}}(s)=1, 0\le s<u, \Psi_{\tilde{i}}(s)=0, s\ge u, \tilde{X}_i>u) \tag{17}$$

is just given by $\lambda_i(t)$, $t\ge u$, $i=1,2$. On the other hand, that of the random variable

$$(X_i \mid \Psi_{\tilde{i}}(s)=1, 0\le s<u,\ \Psi_{\tilde{i}}(s)=0, s\ge u, X_i>u) \tag{18}$$

is given by $p_i\lambda_i(u+p_i(t-u))$, $t\ge u$, $i=1,2$. It is then clear that, if $b_i\ge 1$, then $p_i\lambda_i(u+p_i(t-u))\ge\lambda_i(t)$, for all $t\ge u, u>0$. Thus, condition (5) is satisfied and this model belongs to C_{FR}.

On the other hand, let us now define the ratio of the conditional pdfs of the random variables in (17) and (18):

$$\Psi_i(t)\equiv\frac{p_i\lambda_i(u+p_i(t-u))\exp\left(-\int_0^{p_i(t-u)}\lambda_i(u+w)\,dw\right)}{\lambda_i(t)\exp\left(-\int_0^{t-u}\lambda_i(u+w)\,dw\right)},\ t\ge u,\ i=1,2.$$

Specifically, by setting $\lambda_i=1$, $b_i=3$, $p_i=2$ and $u=0.5$, it can be shown that $\Psi_i(t)$ increases for $0.5\le t<0.62201$ and then decreases for $t\ge 0.62201$. This implies that relationship (5) with $\le_{fr}$ replaced by $\le_{lr}$ is not satisfied and, accordingly, this model does not belong to C_{LR}.

(ii) In Model 2, let $\rho_i(u,t)$ be just replaced by $\rho_i(u,t)=p_i t^2$, $p_i>1$, $i=1,2$. In this case, the failure rate of the random variable in (18) is given by $2p_i(t-u)b_i\lambda_i^{b_i}(u+p_i(t-u)^2)^{b_i-1}$, $t\ge u$, $i=1,2$. Define the differences of the residual lifetime failure rates:

$$\Phi_i(t)\equiv b_i\lambda_i^{b_i}t^{b_i-1}-2p_i(t-u)b_i\lambda_i^{b_i}(u+p_i(t-u)^2)^{b_i-1},\ t\ge u,\ i=1,2.$$

Then, by setting $\lambda_i=1$, $b_i=0.5$, $p_i=2$ and $u=0.5$, it can be shown that $\Phi_i(t)>0$, for $0.5\le t<0.72771$ and $\Phi_i(t)\le 0$, for $t\ge 0.72771$. This implies that this model belongs to C_{ST} but not to C_{FR}.

In Freund (1961), the residual lifetime distributions are limited only to the same type of distributions as the original one. However, in our modeling approach, there is no such restriction as illustrated in the following model.

Model 3 Let $\lambda_1(t)=\lambda_1$, $\lambda_2(t)=\lambda_2$. We will construct a family of bivariate distributions belonging to C_{ST} and, set $\rho_i(u,t)=p_i t(t+1)$, $p_i>1$, $i=1,2$. Then, from Theorem 1, we have

$$f(x_1,x_2)=\lambda_1\lambda_2p_2(2x_2-2x_1+1)\exp\{-\lambda_2p_2(x_2-x_1)(x_2-x_1+1)-(\lambda_1+\lambda_2)x_1\},\ 0<x_1<x_2,$$
$$f(x_1,x_2)=\lambda_1\lambda_2p_1(2x_1-2x_2+1)\exp\{-\lambda_1p_1(x_1-x_2)(x_1-x_2+1)-(\lambda_1+\lambda_2)x_2\},\ 0<x_2\le x_1.$$

Observe that the original lifetimes are exponential distributions but, for instance, the residual lifetime of component 1 is now given by

$$P(X_1 > x \mid \Psi_2(s) = 1, 0 \le s < u, \Psi_2(s) = 0, s \ge u, X_1 > u) \quad = \exp\left(-\int_0^{\rho_1(u,x-u)} \lambda_1(u+w)dw \right)$$

$$= \exp\{-\lambda_1 p_1(x-u)(x-u+1)\}, \quad 0 < u < x,$$

which is non-exponential distribution. In this case, for the residual lifetime distribution, the linear failure rate is used (see Bain (1974) and Lawless (2003) for application of this type of failure rate in reliability and biological contexts).

■

By similar arguments as those described in Example 1, it can be easily shown that Model 3 also belongs to C_{FR} but not to C_{LR}.

The following model is constructed based on the approach described in subsection 2.3 and it belongs to the class C_{LR}.

Model 4 Let $\lambda_1(t) = \lambda_1$, $\lambda_2(t) = \lambda_2$. We start with simple baseline decreasing functions $\beta_{0i}(t) = b_i(t+1)\exp(-t)$, $b_i > 0$, $i = 1,2$. Following similar procedure as those described in Model 1, $\beta_i(u,t)$, $i = 1,2$, can be specified as

$$\beta_i(u,t) = \frac{(\lambda_i + 1)^2}{\lambda_i(\lambda_i + 2)}(t+1)\exp(-t), \quad i = 1,2,$$

which is independent of b_i, $i = 1,2$. Then, from Theorem 2, we have

$$f(x_1, x_2) = \frac{\lambda_1(\lambda_2 + 1)^2}{(\lambda_2 + 2)}(x_2 - x_1 + 1)\exp\{-(\lambda_1 - 1)x_1 - (\lambda_2 + 1)x_2\}, \; 0 < x_1 < x_2,$$

$$f(x_1, x_2) = \frac{\lambda_2(\lambda_1 + 1)^2}{(\lambda_1 + 2)}(x_1 - x_2 + 1)\exp\{-(\lambda_2 - 1)x_2 - (\lambda_1 + 1)x_1\}, \; 0 < x_2 \le x_1.$$

Note that, from Theorem 3, it can be seen that Model 4 can also be generated from the class C_{FR} by setting

$$\alpha_i(u,t) = \frac{(1+\lambda_i)^2(t+1)}{\lambda_i[(1+\lambda_i)(t+1)+1]}, \quad i = 1,2.$$

■

Until now, the families of bivariate distributions have been constructed based on simple original baseline distributions such as exponential and Weibull. Other families of bivariate distributions could also be constructed from more general original baseline distributions such as Gompertz and the two-parameter Pareto (Lomax) distributions, etc.

In the previous models, we have been interested in the construction of positively dependent bivariate distributions. Now we discuss the construction of the bivariate distributions with negative dependency.

Model 5 Let $\lambda_1(t) = \lambda_1$, $\lambda_2(t) = \lambda_2$. In order to construct a family of bivariate distributions with negative dependency belonging to C_{LR-}, we now start with simple increasing baseline functions

$\beta_{0i}(t) = b_i t + 1$, $i = 1,2$. Following similar procedure as those described in Models 1 and 4, $\beta_i(u,t)$, $i = 1,2$, can be specified as $\beta_i(u,t) = (\lambda_i /(\lambda_i + b_i))(b_i t + 1)$, $i = 1,2$. Then, from Theorem 2, we have

$$f(x_1, x_2) = \lambda_1 \lambda_2 \frac{\lambda_2}{(\lambda_2 + b_2)}[b_2(x_2 - x_1) + 1]\exp\{-\lambda_1 x_1 - \lambda_2 x_2\},\ 0 < x_1 < x_2,$$

$$f(x_1, x_2) = \lambda_1 \lambda_2 \frac{\lambda_1}{(\lambda_1 + b_1)}[b_1(x_1 - x_2) + 1]\exp\{-\lambda_1 x_1 - \lambda_2 x_2\},\ 0 < x_2 \le x_1.$$

Let us now investigate the effect of parameters $b_i, i = 1,2$, on the degree of dependency. For b_1, obviously, $b_1 = 0$ corresponds to the independent case. As the effect of b_1 on the residual lifetime could be more conveniently interpreted via the corresponding failure rate function rather than via the corresponding conditional pdf, it would now be better to analyze the effect of b_1 on the parameter function $\alpha_1(u,t)$ which corresponds to $\beta_1(u,t)$. From Theorem 3, the parameter function $\alpha_1(u,t)$ which corresponds to $\beta_1(u,t)$ is obtained by

$$\alpha_1(u,t) = \frac{(b_1 t + 1)\int_t^\infty f_1(u+s)ds}{\int_t^\infty (b_1 s + 1) f_1(u+s)\, ds} \le 1, \text{ for all } u,t \ge 0.$$

Suppose that $b_{11} < b_{12}$ and define $\alpha_1(u,t;b_{11})$ and $\alpha_1(u,t;b_{12})$ as the function $\alpha_1(u,t)$ in which b_1 is replaced by b_{11} and b_{12}, respectively. Observe that

$$\alpha_1(u,t;b_{11}) - \alpha_1(u,t;b_{12}) = \frac{\left[\int_t^\infty \int_t^\infty [(b_{11}t+1)(b_{12}s+1) - (b_{12}t+1)(b_{11}s+1)] f_1(u+w)\, f_1(u+s)dwds\right]}{\int_t^\infty (b_{11}s+1)\, f_1(u+s)\, ds \cdot \int_t^\infty (b_{12}s+1)\, f_1(u+s)\, ds}.$$

Note that

$$\frac{(b_{12}t+1)}{(b_{11}t+1)} = 1 + \frac{(b_{12} - b_{11})t}{(b_{11}t+1)}$$

is increasing function of t and thus

$$\frac{(b_{12}t+1)}{(b_{11}t+1)} < \frac{(b_{12}s+1)}{(b_{11}s+1)}, \text{ for all } s > t,$$

which implies that $[(b_{11}t+1)(b_{12}s+1) - (b_{12}t+1)(b_{11}s+1)] > 0$, for all $s > t$, and finally we have $1 \ge \alpha_1(u,t;b_{11}) > \alpha_1(u,t;b_{12})$, for all $u,t \ge 0$. Therefore, the degree of the negative dependency would become stronger as the parameter b_1 increases. Symmetrical interpretation applies to the parameter b_2.

■

Example 2. In the above Model 5, we specify the parameters as follows: $\lambda_1 = 1$, $\lambda_2 = 1$, $b_i = b \ge 0$. The values of $Cov(X_1, X_2)$ corresponding to each value of b are shown in Table 1.

b	0	1	2	3	4	5

$Cov(X_1, X_2)$	0	-0.3125	-0.4445	-0.5156	-0.5600	-0.5903

Table 1. The values of $Cov(X_1, X_2)$

5. Illustration

Before analyzing a real bivariate data set, we will briefly discuss how to narrow the possible choices of the baseline distributions and provide a guideline for this. In the proposed classes, the change point, after which the residual lifetime distribution of the remaining component changes, is $\min\{X_1^*, X_2^*\} = \min\{X_1, X_2\}$ and the corresponding failure rate of $\min\{X_1, X_2\}$ is given by $\lambda_1(t) + \lambda_2(t)$. Note that, given a bivariate data set, $\min\{X_1, X_2\}$ can be observed from each observation of (X_1, X_2). Thus, we may obtain the observable failure rate (or empirical failure rate) of $\min\{X_1, X_2\}$ as follows:

$$[n(t) - n(t + \Delta t)]/[n(t)\Delta t], \quad (19)$$

where $n(t)$ is the number of data points of $\min\{X_1, X_2\}$ observed after t (see, e.g., Rausand and Høyland (2004)). Then, by interpreting the shape of the empirical failure rate, we can narrow the possible choices of the baseline distributions. Thus, the value of this approach would be especially to rule out implausible baseline distributions. In the following, this methodology is illustrated in analyzing a bivariate data set.

We analyze the bivariate failure time data set given in Reliability Edge (2002). The data set was obtained from a life test on 18 identical parallel systems. Each parallel system is constituted by two motors, Motor A and Motor B. The life times X_1 (Motor A) and X_2 (Motor B) were measured in days. The parallel system is in a redundant configuration, that is, the system properly operates if at least one of the two motors functions. In redundant systems with two components (like hydraulic systems), the full load of the system is frequently shared by the two components in the system. In this situation, when the system starts to operate, each component takes a portion of the total load. When one of the components fails, the remaining component must take on the full load, which shortens its remaining lifetime. Thus, the bivariate distributions developed in this paper would be suitable for this data set. Figure 1 plots the failure times (x_{1i}, x_{2i}) of the i-th system $i = 1,\ldots,18$. There clearly seems to be a strong positive correlation between X_1 and X_2, because the correlation coefficient is $\rho = 0.669$. Obviously, among the models developed in Section 4, Mode 5 is not suitable for this data set and we thus choose the best model among Models 1-4.

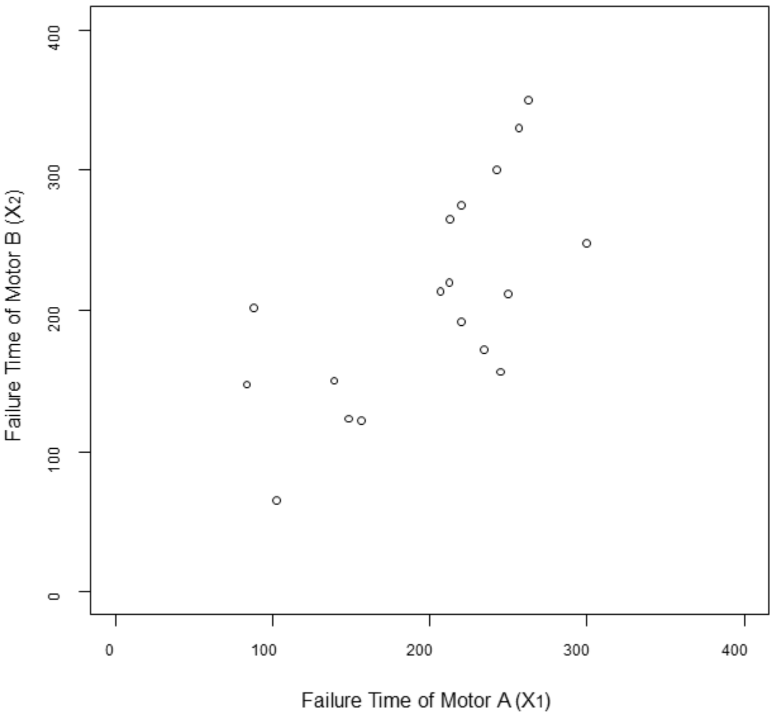

Figure 1. Scatter Plot

First of all, taking into account the guideline suggested above, we obtained the empirical failure rate of $\min\{X_1, X_2\}$ in (19) by setting the length of interval $\Delta t \equiv 40$, starting from 65 (see Table 2).

t	0-65	65-105	105-145	145-185	185-225	225-265
Failure Rate	0	0.0042	0.0050	0.0042	0.0150	0.0250

Table 2. Empirical Failure Rate

Overall, the empirical failure rate of $\min\{X_1, X_2\}$ represents a fairly increasing pattern. However, for Model 1 (the model by Freund (1961) and Block and Basu (1974)) and Models 3-4, $\min\{X_1, X_2\}$ has a constant failure rate $\lambda_1 + \lambda_2$. Thus, we presume that the performances of estimations for these models would be rather poor. On the other hand, for Model 2, $\min\{X_1, X_2\}$ has an increasing failure rate when $b_i > 1$ and the performance of the estimation is expected to be much better. In order to verify it, the bivariate data set was fitted to Models 1-4 and the results are given in Table 3. For all the models, MLE's were used for the parameter estimation and they were obtained numerically.

MODEL	Estimated parameters	Log-likelihood	AIC	BIC
MODEL 1 (Freund (1961), Block and Basu (1974))	$\hat{\lambda}_1 = 0.0031$, $\hat{\lambda}_2 = 0.0025$ $\hat{p}_1 = 7.0353$, $\hat{p}_2 = 7.5719$	-211.9717	431.9434	435.5049

MODEL 2	$\hat{\lambda}_1 = 0.0041$, $\hat{\lambda}_2 = 0.0037$ $\hat{b}_1 = 3.3332$, $\hat{b}_2 = 2.5146$ $\hat{p}_1 = 2.0200$, $\hat{p}_2 = 2.1100$	-198.1599	408.3198	413.662
MODEL 3	$\hat{\lambda}_1 = 0.0031$, $\hat{\lambda}_2 = 0.0025$ $\hat{p}_1 = 0.1265$, $\hat{p}_2 = 0.1007$	-208.5076	425.0152	428.5767
MODEL 4	$\hat{\lambda}_1 = 0.0028$, $\hat{\lambda}_2 = 0.0021$	-966.7475	1937.495	1939.276

Table 3. Estimation Results

Based on the result described in Table 3, all the measures Log-likelihood, AIC and BIC support the assertion that Models 2 and 3 outperform the conventional model of Freund (1961) and Block and Basu (1974). Especially, as we presumed, Model 2 is the best among the considered models in fitting the above bivariate data.

We now perform the goodness of fit test for Model 2. For this, we separated the support $\{(x_1, x_2) \mid 0 \le x_1 < \infty, 0 \le x_2 < \infty\}$ into 9 rectangles:

$$\{(x_1, x_2) \mid v_i \le x_1 < v_{i+1}, w_j \le x_2 < w_{j+1}\}, \quad i, j = 1,2,3,$$

$$v_1 = w_1 = 0, \quad v_2 = w_2 = 135, \quad v_3 = w_3 = 245, \quad v_4 = w_4 = 345,$$

and 'the remaining region' (total 10 non-overlapping regions), so that each region can include balanced number of observations. Then we performed the chi-squared goodness-of-fit test with the degree of freedom 3, which is obtained by 10 (number of regions) - 6 (number of estimated parameters)-1. The test statistic was obtained by $\chi^2 = 2.83 < \chi^2(\alpha,3)$, $\alpha = 0.10, 0.05, 0.01$, where $\chi^2(\alpha,3)$ is the critical value of the chi-squared test with the degree of freedom 3 under the significance level α. Therefore, this test result would support the view that the data are from the parametric model of Model 2.

6. Concluding Remarks

In this paper, new general classes of bivariate distributions have been proposed and several specific families of distributions have been generated for illustrations. It has been seen that the obtained formulas for the joint distribution are very simple and easy to apply. The relationships among the three classes have been characterized and this characterization has shown to be very useful in constructing specific family of distributions and interpreting the degree of dependency based on the model parameters. Furthermore, bivariate lack of memory property has been briefly discussed and the subclasses in which all the bivariate distributions possess the bivariate lack of memory property have been characterized. Based on the proposed classes, numerous bivariate distributions can further be generated and new issues on the estimation and testing of the model parameters should be discussed in the future studies.

As $C_{LR} \subset C_{FR} \subset C_{ST}$, class C_{ST} is the largest one among the three classes. Therefore, mathematically, it is sufficient to consider only class C_{ST}. However, there were several illustrations where some families of distributions cannot be easily constructed from a larger class, which justifies the consideration of each class separately. In order to show the practical relevance of the developed classes and to provide a guideline for narrowing the choices of the baseline distributions, the families of bivariate distributions obtained in Section 4 have been applied to a real bivariate failure time data set. It has been observed that the shape of the failure rate of $\min\{X_1, X_2\}$ can help narrow the choices of baseline distributions.

It would be interesting to show that the dependence in C_{ST} increases when the functions $\rho_i(s,w), i=1,2$ increase. For example, let (X_1, X_2) and (Y_1, Y_2) be two elements of the class C_{ST} with same marginal distributions, that is, $X_1 =_d Y_1$ and $X_2 =_d Y_2$ and parameter functions $\rho_i(s,w), i=1,2$, and $\tilde{\rho}_i(s,w), i=1,2$, respectively. It will be useful to show that

$$\rho_i(s,w) \le \tilde{\rho}_i(s,w) \quad \Rightarrow \quad (X_1, X_2) \prec_C (Y_1, Y_2)$$

where $\prec_C$ denotes the concordance order (see, e.g., Joe (1997)). Furthermore, another interesting issue to explore would be to see whether there exists an extreme random pair $(Z_1^*, Z_2^*) \in C_{ST}$ with suitably characterized parameter functions $\rho_i^*(s,w)$, $i=1,2$, such that, for all $(X_1, X_2) \in C_{ST}$, $(X_1, X_2) \prec_C (Z_1^*, Z_2^*)$. However, by a counter example, it can be shown that the condition $\rho_i(s,w) \le \tilde{\rho}_i(s,w)$ does not imply $(X_1, X_2) \prec_C (Y_1, Y_2)$ and, accordingly, there does not exist such an extreme random pair (Z_1^*, Z_2^*). Thus, in the future studies, such monotonicity in dependency could be studied with respect to some other bivariate order which is weaker than the concordance order. This interesting issue was originally proposed by one of the reviewers of this paper.

Acknowledgements

The authors would like to thank the associate editor and the referees for valuable comments and constructive suggestions, which have improved the presentation of this paper. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2011-0017338). This work was also supported by Priority Research Centers Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2009-0093827). This work was also supported by the BK21 Plus Project through the National Research Foundation of Korea (KRF) funded by the Ministry of Education (22A20130011003).

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