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On the Dependence of Structure of Multivariate Processes and Corresponding Hitting Times

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A direct approach to derive dependence properties among the hitting times of two processes has been initiated by N. Ebrahimi (1987, *J. Appl. Probab.* **24** 115–122) and explored further by N. Ebrahimi and T. Ramalingam (1988, *J. Appl. Probab.* **25** 355–362; 1989, *J. Appl. Probab.* **26** 287–295). In this paper new results are obtained for multivariate processes, which help us to identify positive and negative dependent structure among hitting times of the processes. Furthermore, an approach to derive dependence properties among the processes is proposed and a partial solution to the question that what kinds of dependence properties, when they are imposed on processes, are reflected as analogous properties of corresponding hitting times is given.

1. INTRODUCTION

The reliability, $\bar{F}(t)$, of a system (component) is the probability that the system will preserve its characteristics within specified limits during a specified time interval $[0, t]$. If a system failure is an event in which at least one characteristics of the system shift outside certain permissible limits, and if T is the time to failure, then

$$\bar{F}(t) = P(T > t). \tag{1.1}$$

Suppose that the system reliability is determined by a finite number, k, of characteristics. For $i = 1, 2, ..., k$, denote the value of the ith characteristic at time t by $X_i(t)$ and assume that it is within permissible limits if $X_i(t) < a_i$, where $a_1, a_2, ..., a_k$ are fixed and known values. One may, for example, look upon a_i as the breaking threshold of total damages, $X_i(t)$, by time t. More general ways of defining permissible limits are clearly

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possible, but will not be pursued in this paper. Obviously, the random time, $T_i(a_i)$, at which the ith characteristic first crosses its limit is given by

$$T_i(a_i) = \begin{bmatrix} \operatorname{Inf}\{t \in \Lambda : X_i(t) \geqslant a_i\}, \\ \infty \text{ if } X_i(t) < a_i \text{ for all } t \in \Lambda, \end{bmatrix} \tag{1.2}$$

where the index set Λ is a subset of $R_+ = [0, \infty)$. In this setting, the failure time of the system, T, is given by

$$T = \min(T_1(a_1), ..., T_k(a_k)). \tag{1.3}$$

In view of (1.2) and (1.3),

$$\bar{F}(t) = P(T_i(a_i) > t,\ i = 1, ..., k). \tag{1.4}$$

Formulation of system reliability by means of Eqs. (1.2)–(1.4) is relevant to engineering disciplines relating to structural safety, mechanical vibration, etc.

In general it is possible to assess the system reliability $\bar{F}(t)$ provided that one can jointly model $X_1(t), ..., X_k(t)$. (See Ebrahimi and Ramalingam, 1993). However, such information may sometimes be unavailable or difficult to obtain. In such situations, one can seek bounds for system reliability. To obtain such bounds, information about the dependence structure of $T_i(a_1), ..., T_k(a_k)$ is essential. For example if we know that $P(\bigcap_{i=1}^k [T_i(a_i) > t_i]) \geqslant \prod_{i=1}^k P(T_i(a_i) > t_i)$, then we can assess $P(T_i(a_i) > t_i)$, $i = 1, ..., k$ and derive a bound for $\bar{F}(t)$. Besides bounds information about the dependence structure may bring forth new inequalities for stochastic processes. Ebrahimi (1987) has initiated a direct approach to study the dependence structures of hitting times for bivariate processes. His approach has been explored further by Ebrahimi and Ramalingam (1988, 1989).

Certain kinds of dependence properties, when they are imposed on processes, are reflected as analogous properties of corresponding hitting times. These results are of value as they help us to understand in what ways the hitting times for dependence structures of hitting times can be inherited from the corresponding processes. Furthermore, these results sometimes can tell us how to control the reliability of a system by controlling its characteristics.

In Section 2 of this paper, we list several concepts of dependence. In Section 3, we give several results which not only clarify some properties of dependent processes, but also help us to identify positive or negative dependent structures, both among processes and their hitting times. Finally, in Section 4 we give several examples.

2. Preliminaries

Suppose that we are given a k-dimensional ($k \geqslant 2$) stochastic vector process $\{X(t) = (X_1(t), ..., X_k(t)): t \in \Lambda\}$, where the index set Λ always be a subset of $R_+ = [0, \infty)$. The state space of $X(t)$ is the cartesian product $E = E_1 \times E_2 \times \cdots \times E_k$, which will be a subset of k-dimensional Euclidean space R^k. If the index set is $\{0, 1, 2, ...\}$, then

$$P\left(\bigcap_{i=1}^{k} (T_i(a_i) > t_i)\right) = P\left(\max_{0 \leqslant j_i \leqslant [t_i]} X_i(j_i) < a_i,\ i = 1, ..., k\right), \tag{2.1}$$

where $[b]$ is the largest integer less than or equal to b.

We now present three concepts of positive (negative) dependence for any k-dimensional stochastic vector process $X(t)$.

Definition 2.1. The $k \geqslant 2$ different processes $\{X_1(t); t \in \Lambda\}, ..., \{X_k(t); t \in \Lambda\}$ are positively orthant dependent (POD) (negatively orthant dependent (NOD)) if

$$P\left(\bigcap_{i=1}^{k} [X_i(t_i) > a_i]\right) \geqslant (\leqslant) \prod_{i=1}^{k} P[X_i(t_i) > a_i], \tag{2.2}$$

$$P\left(\bigcap_{i=1}^{k} [X_i(t_i) \leqslant a_i]\right) \geqslant (\leqslant) \prod_{i=1}^{k} P(X_i(t_i) \leqslant a_i), \tag{2.3}$$

for all $a_i \in E_i$ and $t_i \in \Lambda$, $i = 1, ..., k$, and $\{X_1(t); t \in \Lambda\}, ..., \{X_k(t); t \in \Lambda\}$ are each (univariate) POD (NOD). For $j = 1, ..., k$, we say that a one-dimensional process $X_j(t)$ is POD (NOD) if for any $0 \leqslant s_1 < s_2 < \cdots < s_n$, $s_i \in \Lambda$ and $a_i \in E_j$, $i = 1, ..., n$,

$$P\left(\bigcap_{i=1}^{n} [X_j(s_i) > a_i]\right) \geqslant (\leqslant) \prod_{i=1}^{n} P(X_j(s_i) > a_i)$$

and

$$P\left(\bigcap_{i=1}^{n} (X_j(s_i) \leqslant a_i)\right) \geqslant (\leqslant) \prod_{i=1}^{n} P(X_j(s_i) \leqslant a_i).$$

Also, the hitting times $T_1(a_1), ..., T_k(a_k)$ are POD (NOD) if

$$P\left(\bigcap_{i=1}^{k} (T_i(a_i) > t_i)\right) \geqslant (\leqslant) \prod_{i=1}^{k} P(T_i(a_i) > t_i), \tag{2.4}$$

and

$$P\left(\bigcap_{i=1}^{k} (T_i(a_i) \leqslant t_i)\right) \geqslant (\leqslant) \prod_{i=1}^{k} P(T_i(a_i) \leqslant t_i), \tag{2.5}$$

for every $a_i \in E_i$ and $t_i \in \Lambda$, $i = 1, ..., k$.

From Definition (2.1), it is clear that if $X_i(t_i)=X_i$, $i=1, ..., k$, then $X_1, ..., X_k$ are POD (NOD) if $P(\bigcap_{i=1}^k (X_i > x_i)) \geqslant (\leqslant) \prod_{i=1}^k P(X_i > x_i)$ and $P(\bigcap_{i=1}^k (X_i \leqslant x_i)) \geqslant (\leqslant) \prod_{i=1}^k P(X_i \leqslant x_i)$ which coincides with classical definitions of POD and NOD. (See Ebrahimi and Ghosh, 1981.)

For $k=2$, POD (NOD) will also be called positive quadrant dependent (PQD) (negative quadrant dependent (NQD)).

DEFINITION 2.2. The processes $\{X_1(t); t \in \Lambda\}, ..., \{X_k(t); t \in \Lambda\}$ are associated $(k \geqslant 2)$ if

$$\operatorname{cov}(f(X_i(t_i),\ i=1, ..., k),\ g(X_i(t_i),\ i=1, ..., k)) \geqslant 0, \tag{2.6}$$

for all non-decreasing real valued functions f and g such that the covariance exists and all $t_i \in \Lambda$, $i=1, ..., k$, and $\{X_1(t); t \in \Lambda\}, ..., \{X_k(t); t \in \Lambda\}$ are each univariate associated. For $j=1, ..., k$, a one-dimensional process $\{X_j(t); t \in \Lambda\}$ is said to be associated if for any $0 \leqslant s_1 < s_2 < \cdots s_n$, $\operatorname{cov}(f(X_j(s_i),\ i=1, ..., n),\ g(X_j(s_i),\ i=1, ..., n)) \geqslant 0$. Also, we say the hitting times $T_1(a_1), ..., T_k(a_k)$ are associated if

$$\operatorname{cov}(f(T_1(a_1), ..., T_k(a_k)),\ g(T_1(a_1), ..., T_k(a_k))) \geqslant 0, \tag{2.7}$$

for all non-decreasing functions f and g such that the covariance exists and all $a_i \in E_i$, $i=1, ..., k$.

From Definition (2.2) it is clear that if $X_i(t_i)=X_i$, the sequence of random variables $X_1, ..., X_k$ are associated if $\operatorname{cov}(f(X_1, ..., X_k), g(X_1, ..., X_k)) \geqslant 0$ which coincides with the Definition given by Barlow and Proschan (1981).

DEFINITION 2.3. The processes $\{X_1(t): t \in \Lambda\}, ..., \{X_k(t); t \in \Lambda\}$ $(k \geqslant 2)$ are negatively associated (NA) if for every disjoint subsets B_1 and B_2 of $\{1, 2, ..., k\}$,

$$\operatorname{cov}(f(X_i(t_i),\ i \in B_1),\ g(X_i(t_i),\ i \in B_2)) \leqslant 0, \tag{2.8}$$

for all real valued non-decreasing functions f and g. Furthermore, each $\{X_i(t); t \in \Lambda\}$, $i=1, ..., k$, are NA. For $j=1, ..., k$ we say that a one-dimensional process $\{X_j(t); t \in \Lambda\}$ is NA if for any $0 \leqslant s_1 < s_2 < \cdots s_n$ and any two disjoint subsets A_1 and A_2 of $\{1, 2, ..., n\}$, $\operatorname{cov}(f(X_j(s_i), i \in A_1), g(X_j(s_i), i \in A_2))) \leqslant 0$.

Also, the hitting times $T_1(a_1), ..., T_k(a_k)$ are NA if for every pair of disjoint subsets B_1 and B_2 of $\{1, 2, ..., k\}$,

$$\operatorname{cov}(f(T_i(a_i),\ i \in B_1),\ g(T_i(a_i),\ i \in B_2)) \leqslant 0, \tag{2.9}$$

for all non-decreasing real valued functions f and g for which the covariance exists and all $a_i \in E_i$, $i = 1, ..., k$.

In Definition (2.3) if $X_i(t_i) = X_i$, then the $X_1, ..., X_k$ are NA if $\operatorname{cov}(f(X_i, i \in B_1), g(X_i, i \in B_2)) \leqslant 0$ for any two disjoint subsets B_1 and B_2 of $\{1, ..., k\}$.

It is clear that if the univariate process has stationary independent increments and if for every $t \in \Lambda$ the density function of $X(t)$, $f_{X(t)}$, is positive frequency of order $2(PF_2)$, then $\{X(t); t \in \Lambda\}$ is associated. (A function $h(x)$ is said to be PF_2 if $\log h(x)$ is a concave function.) Also if the process $\{X(t); t \in \Lambda\}$ is a Markov process and $P[X(t) = y \mid X(s) = x]$ is totally positive of order $2(TP_2)$ in x and y for all $s \leqslant t$, then $\{X(t); t \in \Lambda\}$ is associated. (A function $h(s, t)$ is said to be TP_2 if

$$\begin{vmatrix} h(s_1, t_1) & h(s_1, t_2) \\ h(s_2, t_2) & h(s_2, t_2) \end{vmatrix} \geqslant 0$$

for all $s_1 < s_2$, $t_1 < t_2$.) For more properties of PF_2 and TP_2, see Barlow and Proschan (1981).

Having laid down some concepts of dependence, several comments regarding independence among the components of $X(t)$ are in order. First, we will say that $X_1(t), ..., X_k(t)$ are independent if for all $t_i \in \Lambda$, $i = 1, ..., k$, $X_1(t_1), ..., X_k(t_k)$ are independent in the usual sense of independence among k random variables. Second, we say that hitting times $T_i(a_i)$, $i = 1, ..., k$, are independent if for all $a_i \in E_i$, $i = 1, ..., k$, $T_1(a_1), ..., T_k(a_k)$ are independent.

Now, we list several properties of POD (NOD) and associated (NA) stochastic processes. Here Δ is either POD, NOD, associated, or NA.

(a) If $X_1(t), ..., X_k(t)$ are Δ, then any subset is also Δ;

(b) If $X_1(t), ..., X_k(t)$ are Δ, then $g_1(X_1(t)), ..., g_k(X_k(t))$ are Δ for all non-decreasing (non-increasing) functions $g_1, ..., g_k$.

(c) If $X(t) = (X_1(t), ..., X_k(t))$ and $Y(t) = (Y_1(t), ..., Y_k(t))$ are POD (NOD), $X(t)$ and $Y(t)$ are independent, and $X(t)$ has independent components, then $X(t) + Y(t)$ is POD (NOD).

(d) If $X(t) = (X_1(t), ..., X_k(t))$ and $Y(t) = (Y_1(t), ..., Y_k(t))$ are associated (NA), $X(t)$ and $Y(t)$ are independent, then $X(t) + Y(t)$ is associated (NA).

(e) If $X(t) = (X_1(t), ..., X_k(t))$ and $Y(t) = (Y_1(t), ..., Y_k(t))$ are both non-negative and associated (NA) processes and if $X(t)$ and $Y(t)$ are independent, then $Z(t) = X(t)\, Y(t) = (X_1(t)\, Y_1(t), ..., X_k(t)\, Y_k(t))$ is associated (NA).

(f) If $X_1(t), ..., X_k(t)$ are Δ and $X_i(t)$ has continuous sample path, $i=1, ..., k$, then $Y_1(t), ..., Y_k(t)$ are Δ, where $Y_i(t)=\int_0^t g_i(u)\, X_i(u)\, du$, $i=1, ..., k$, and $g_i(u)$'s are non-negative continuous functions.

Proof. The proof is straightforward and it is omitted.

3. Theoretical Results

In this section we will assume that the index set $\Lambda=\{0, 1, 2, ...\}$. Similar results can be obtained for a general multi-dimensional process provided that the process has continuous sample paths.

THEOREM 1. *Consider a one-dimensional process $\{X_1(t);\, t \in \Lambda\}$ such that $X_1(t)$ is Δ, where Δ is either POD, NOD, associated, or NA. Then $T(a_1), ..., T(a_k)$ are Δ. Here $T(a_i)=\operatorname{Inf}\{n\colon X_1(n) \geqslant a_i\}$.*

Proof. We will prove this theorem for $k=2$, and Δ is either POD or associated.

(a) Suppose Δ is POD, then we need to show that for $a_1 \leqslant a_2$,

$$P\left(\bigcap_{i=1}^{2} [T(a_i)>t_i]\right) \geqslant \prod_{i=1}^{2} P(T(a_i)>t_i).$$

Now,

$$\begin{aligned} P\left(\bigcap_{i=1}^{2} [T(a_i)>t_i]\right) &= P(\max_{0 \leqslant j \leqslant [t_i]} X_1(j)<a_i,\ i=1, 2) \\ &= P(X_1(j)<a_1, 0 \leqslant j \leqslant t[1], X_1(j)<a_2, \\ &\quad [t_1]+1 \leqslant j \leqslant [t_2])\, I(t_1 \leqslant t_2) \\ &\quad + P(X_1(j)<a_1, 0 \leqslant j \leqslant t[1])\, I(t_1>t_2). \end{aligned}$$

(Here I is the usual indicator function.)

$$\begin{aligned} &\geqslant P(X_1(j)<a_1, 0 \leqslant j \leqslant [t_1])\, P(X_1(j)<a_2, [t_1]+1 \leqslant j \leqslant [t_2])\, I(t_1 \leqslant t_2) \\ &\quad + P(X_1(j)<a_1, 0 \leqslant j \leqslant t[1])\, I(t_1>t_2) \\ &\geqslant (T(a_1)>t_1)\, P(T(a_2)>t_2)\, I(t_1 \leqslant t_2) \\ &\quad + P(T(a_1)>t_1)\, P(T(a_2)>t_2)\, I(t_1>t_2)=P(T(a_1)>t_1)\, P(T(a_2)>t_2). \end{aligned}$$

(b) Let f and g be two non-decreasing functions, and let Δ be associated, then

$$\begin{aligned}
&\operatorname{cov}(f(T(a_1), T(a_2)), g(T(a_1), T(a_2)) \\
&\quad = \operatorname{cov}(f(\min\{n: X_1(n) \geqslant a_1\}, \min\{n: X_1(n) \geqslant a_2\}), \\
&\qquad g(\min\{n: X_1(n) > a_1\}, \min\{n: X_1(n) \geqslant a_2\}) \geqslant 0. \qquad (3.1)
\end{aligned}$$

The last inequality in (3.1) comes from the fact that both f and g are nondecreasing functions of $\{X_1(n); n \in \{0, 1, 2, \ldots\}\}$. Similar arguments can be used to prove for Δ being NOD or NA.

In order to prove our next result we need the following notations. Let $X = (X_1, \ldots, X_d)$ be a d-dimensional vector with distribution function F and the marginal distribution functions F_j, $1 \leqslant j \leqslant d$. The dependence function of X (or of F) is defined by

$$D_F(u_1, \ldots, u_d) = P(F_j(X_j) \leqslant u_j,\ 1 \leqslant j \leqslant d). \qquad (3.2)$$

It is clear that D_F is the distribution function on $[0, 1]^d$, and it has uniform marginal distributions if the F_j's are continuous. The marginal distributions together with the dependence function determine F, since $F(x_1, \ldots, x_d) = D_F(F_1(x_1), \ldots, F_d(x_d))$. Furthermore, a dependence function D_F is said to be an "extreme dependence function" if all the marginals are nondegenerative, and for each $n \geqslant 1$,

$$D_F^n(u_1, \ldots, u_d) = D_F(u_1^n, \ldots, u_d^n),\ (u_1, \ldots, u_d) \in [0, 1]^d.$$

Deheuvels (1983) and Hsing (1987) have excellent papers about the concept of "dependence function." Hsing (1987) showed that D_F is "extreme dependence function" if and only if

$$D_{F^n}(u_1, \ldots, u_d) = D_F(u_1, \ldots, u_d),\ (u_1, \ldots, u_d) \in [0, 1]^d. \qquad (3.3)$$

It is clear that if $(X_{i1}, \ldots, X_{id})$, $1 \leqslant i \leqslant n$ be independent random vectors all having distribution F, then F^n is the distribution function of $(\max_{1 \leqslant i \leqslant n} X_{i1}, \ldots, \max_{1 \leqslant i \leqslant n} X_{id})$, and hence (3.3) is equivalent to

$$D_{F^n}(u_1, \ldots, u_d) = P(G_j(\max_{1 \leqslant i \leqslant n} X_{ij}) \leqslant u_j, j = 1, \ldots, d), \qquad (3.4)$$

where G_j is the distribution function of $\max_{1 \leqslant i \leqslant n} X_{ij}$ which is F_j^n.

Now we define similar concepts for one-dimensional and k-dimensional processes. For a one-dimensional process $\{X(t); t \in \Lambda\}$, $\{W(t_1, \ldots, t_m): t_1, \ldots, t_m \in \Lambda,\ m \in \{0, 1, \ldots\}\}$ is said to be a "dependence function" if for a given m, and $0 \leqslant t_1 < t_2 < \cdots < t_m$,

$$W(t_1, \ldots, t_m) = D_{F_{X(t_1), \ldots, X(t_m)}}. \qquad (3.5)$$

Here from (3.2), $D_{F_{X(t_1),\ldots,X(t_m)}}(u_1, \ldots, u_m) = P(F_i(X(t_i)) \leqslant u_i, \quad i=1, \ldots, m)$, where $F_i(x) = P(X(t_i) \leqslant x)$. For a k-dimensional process $\{X_1(t); t \in \Lambda\}, \ldots, \{X_k(t); t \in \Lambda\}$, $\{V(t_1, \ldots, t_k): t_1, \ldots, t_k \in \Lambda\}$ is said to be a "dependence function" if for any $t_1, \ldots, t_k \in \Lambda$,

$$V(t_1, \ldots, t_k) = D_{F_{X_1(t_1),\ldots,X_k(t_k)}}, \tag{3.6}$$

where $D_{F_{X_1(t_1),\ldots,X_k(t_k)}}(u_1, \ldots, u_k) = P(F_i(X_i(t_i)) \leqslant u_i, \ i=1, \ldots, k)$. Here $F_i(x) = P(X_i(t_i) \leqslant x)$.

We are now ready to prove our result.

THEOREM 2. *Consider d-different processes* $\{X_1(t); \ t \in \Lambda\}, \ldots, \{X_d(t); t \in \Lambda\}$ *such that* (a) $X_1(t), \ldots, X_d(t)$ *are* Δ (*Here* Δ *is either POD, NOD, associated, or NA*), (b) $X_i(t)$ *is strictly stationary*, $i=1, \ldots, d$, (*The process* $Y(t)$ *is said to be strictly stationary if for* $0 \leqslant t_1 < t_2 < \cdots < t_n$ *and* $h > 0$, $(Y(t_1+h), \ldots, Y(t_n+h)) =^d (Y(t_1), \ldots, Y(t_n))$), (c) *for* $h>0$, $(X_1(t+h), \ldots, X_d(t+h)) =^d (X_1(t), \ldots, X_d(t))$, *and* (d) $D_{G_{n_1,n_2,\ldots,n_d}}(u_1, u_2, \ldots, u_d) = D_F(u_1, \ldots, u_d)$ *for all* $n_1, \ldots, n_d \in \{0, 1, \ldots\}$ *and* $(u_1, \ldots, u_d) \in [0,1]^d$, *this condition is equivalent to Condition* (3.3) *for the case that* X_{ij}*'s are not i.i.d., where* $D_{G_{n_1,n_2,\ldots,n_d}}(u_1, u_2, \ldots, u_d) = P(G_i(\max_{1 \leqslant j \leqslant n_i} X_{ij}(j)) \leqslant u_i, \quad i=1, \ldots, d)$, and $D_F(u_1, \ldots, u_d) = P(F_i(X_i(0)) \leqslant u_i, \ i=1, \ldots, d)$, *then* $T_1(a_1), \ldots, T_k(a_k)$ *are* Δ.

Proof. We will prove this theorem for $d=2$ and Δ is POD. Similar approach can be used to prove the theorem for other cases.

$$\begin{aligned}
&P(T_1(a_1) > n_1, T_2(a_2) > n_2) \\
&\quad = P(\max_{1 \leqslant j \leqslant n_1} X_1(j) \leqslant a_1, \max_{1 \leqslant j \leqslant n_2} X_2(j) \leqslant a_2) \\
&\quad = P(G_1(\max_{1 \leqslant j \leqslant n_1} X_1(j)) \leqslant G_1(a_1), G_2(\max_{1 \leqslant j \leqslant n_2} X_2(j)) \leqslant G_2(a_2)) \\
&\quad = D_G(G_1(a_1), G_2(a_2)) = D_F(G_1(a_1), G_2(a_2)) \\
&\quad \geqslant D_{F_1}(G_1(a_1)) \, D_{F_2}(G_2(a_2)) \\
&\quad = D_{G_1}(G_1(a_1)) \, D_{G_2}(G_2(a_2)) \\
&\quad = P(T_1(a_2) > n_1) \, P(T_2(a_2) > n_2).
\end{aligned}$$

Here G_1 and G_2 are the distribution functions of $\max_{1 \leqslant j \leqslant n_1} X_1(j)$ and $\max_{1 \leqslant j \leqslant n_2} X_2(j)$, respectively, G is the joint distribution function of $\max_{1 \leqslant j \leqslant n_1} X_1(j)$ and $\max_{1 \leqslant j \leqslant n_2} X_2(j)$, $F(x_1, x_2) = P(X_1(0) \leqslant x_1, X_2(0) \leqslant x_2)$, and $F_i(x_i) = P(X_i(0) \leqslant x_i)$, $i=1, 2$.

Clearly, if $\{X(n): n \in \{0, 1, \ldots\}\}$ and $\{Y(n): n \in \{0, 1, \ldots\}\}$ are such that $(X(n), Y(n))$, $n=1, 2, \ldots$, are i.i.d. and $(X(n), Y(n))$ is Δ, then these two processes satisfy the conditions of Theorem 2.

THEOREM 3. *Suppose* $\{X(t); t \in \Lambda\}$ *and* $\{Y(t); t \in \Lambda\}$ *satisfy a linear regression relationship of the form* $X(t) = aY(t) + Z(t)$, $a > 0$, *where* $Z(t)$ *is a white noise process (i.e., purely random process) independent of* $\{Y(t); t \in \Lambda\}$, *and* $\{Y(t); t \in \Lambda\}$ *is associated (NA). Then* $X(t)$ *and* $Y(t)$ *are associated. Furthermore, the corresponding hitting times* $T_1(a)$ *and* $T_2(b)$ *are associated. Here* $T_1(a) = \mathrm{Inf}\{n, X(n) \geqslant a\}$, $T_2(b) = \mathrm{Inf}\{n\colon Y(n) \geqslant b\}$.

Proof. We shall prove this theorem when $Y(t)$ is associated. First, it is clear that $\{X(t); t \in \Lambda\}$ is associated. Now, for any non-decreasing functions f and g and $n_1, n_2 \in \{0, 1, 2, ...\}$,

$$\begin{aligned} &\operatorname{cov}(f(X(n_1), Y(n_2)), g(X(n_1), Y(n_2)) \\ &\quad = \operatorname{cov}(f(Y(n_1) + Z(n_1), Y(n_2)), g(Y(n_1) + Z(n_1), Y(n_2)) \geqslant 0. \end{aligned} \tag{3.7}$$

The last inequality comes from the fact that $\{Y(t); t \in \Lambda\}$ and $\{Z(t); t \in \Lambda\}$ are associated and they are independent. Consequently $\{X(t); t \in \Lambda\}$ and $\{Y(t); t \in \Lambda\}$ are associated.

Furthermore, one can show that for any $n_{11}, ..., n_{1m}, n_{21}, ..., n_{2l} \in \{0, 1, 2, ...\}$,

$$\begin{aligned} &\operatorname{cov}(f(X(n_{11}), ..., X(n_{1m}), Y(n_{21}), ..., Y(n_{2l})), \\ &\quad g(X(n_{11}), ..., X(n_{1m}), Y(n_{21}), ..., Y(n_{2l}))) \geqslant 0, \end{aligned} \tag{3.8}$$

and consequently

$$\operatorname{cov}(f(T_1(a), T_2(b)), g(T_1(a), T_2(b)) \geqslant 0.$$

The inequality comes from the fact that both f and g are non-decreasing functions of $\{X(n)\colon n \in \Lambda\}$ and $\{Y(n)\colon n \in \Lambda\}$.

Remark 3.1. In Theorem 3, one can assume $X(t)$ and $Y(t)$ satisfy a linear regression relationship but with a "delay" of d time units, that is, $X(t) = aY(t-d) + Z(t)$ or in general $X(t) = \sum_{d=0}^{r} dY(t-d) + Z(t)$.

Consider a simple form of econometrical model relating the investment and capital gain. Let $X(t)$ and $Y(t)$, $t \in \Lambda$, denote respectively the investment and capital gain at time t. The model is

$$\begin{aligned} Y(t) &= aX(t-1) + Z_1(t) \\ X(t) &= bY(t) + Z_2(t), \end{aligned} \tag{3.9}$$

where $a, b > 0$, $Z_1(t)$ and $Z_2(t)$ are both noise processes and $(Z_1(t), Z_2(t))$ are a sequence of independent random vectors. The following theorem gives information about dependent structure of $X(t)$ and $Y(t)$ and their hitting times.

THEOREM 4. *The bivariate process* $\{(Y(n), X(n)) \colon n \in \{0, 1, 2, ...\}\}$ *is associated. Furthermore,* $T_1(a) = \text{Inf}\{n \colon Y(n) \geqslant a\}$, $T_1(b) = \text{Inf}\{n \colon X(n) \geqslant b\}$ *is associated.*

Proof. From Eq. (3.9) one can write

$$\begin{aligned} Y(n) &= \sum_{i=0}^{n} (ab)^{n-i} Z_1(i) + \sum_{i=0}^{n-1} (a)^{n-i} b^{n-i-1} Z_2(i), \qquad n \geqslant 1, \\ Y(0) &= Z_1(0). \end{aligned} \tag{3.10}$$

For any $0 < n_1 < n_2 < \cdots < n_l$, since $Y(n_1), ..., Y(n_l)$ are non-decreasing functions of $(Z_1(i), Z_2(i))$, $i = 0, ..., n_l$, we get that the process $\{Y(n);\ n \in \{0, 1, ...\}\}$ is associated. Similarly, from Eq. (3.9) we get that

$$\begin{aligned} X(0) &= bZ_1(0) + Z_2(0) \\ X(n) &= \sum_{i=0}^{n} (a^{n-i} b^{n-i+1} Z_1(i)) + \sum_{i=0}^{n} (ab)^{n-i} Z_2(i), \end{aligned} \tag{3.11}$$

and consequently $\{X(n);\ n \in \{0, 1, ...\}\}$ process is also associated.

Now, for $0 < n_1 < n_2$, and non-decreasing functions f and g,

$$\begin{aligned} &\text{cov}(f(X(n_1), Y(n_2)), g(X(n_1), Y(n_2))) \\ &\quad = \text{cov}(f(bY(n_1) + Z_2(n_1), Y(n_2)), g(bY(n_1) + Z_2(n_1), Y(n_2))) \geqslant 0. \end{aligned}$$

The last inequality comes from the fact that f and g are non-decreasing functions of $Y(n_1)$, $Z_2(n_1)$ and $Y(n_2)$.

The proof of the second part is similar to arguments used in the second part of Theorem 3 and it is omitted.

4. EXAMPLES

EXAMPLE 1. Consider two different repair policies. The first policy is that we replace a failed unit with a new identical unit. We let $N(t)$ denote the number of replacements up to time t. The second policy consists of repairing the unit to its condition just prior to failure, that is, a minimal repair. We denote the number of minimal repairs up to time t by $W(t)$. Suppose the sequences $\{X_n; n \geqslant 1\}$, $\{Y_n; n \geqslant 1\}$ denote the interarrival times for renewal process $N(t)$ and minimal repair process $W(t)$ respectively. Then, it is clear that $X_1 = Y_1$,

$$P(X_n > t \mid X_i = Y_i,\ i = 1, ..., n-1) = \bar{F}(t),$$

and

$$P(Y_n > t \mid X_1 = Y_1 = y_1,\ Y_i = y_i,\ i = 2, ..., n-1) = \frac{\bar{F}(t + y_1 + \cdots + y_{n-1})}{\bar{F}(y_1 + \cdots + y_{n-1})}.$$

Suppose $\bar{F}(t)$ is a new worse than used (NWU), $\bar{F}$ is said to be NWU if $\bar{F}(x+y) \geqslant (\bar{F}(x))(\bar{F}(y))$ for all $x, y \geqslant 0$, then one can easily show that for any $n_1, n_2 \in \{1, 2, ...\}$, $X_1, ..., X_{n_1}, Y_1, ..., Y_{n_2}$ are POD. Using this fact we get that

$$\begin{aligned} P(N(t_1) > n_1, W(t_2) > n_2) &= P(X_1 + \cdots + X_{n_1} \leqslant t_1, ..., Y_1 + \cdots + Y_{n_2} \leqslant t_2) \\ &\geqslant P(N(t_1) > n_1)\, P(W(t_2) > n_2). \end{aligned}$$

That is, $\{N(t); t \in R_+\}$ and $\{W(t); t \in R_+\}$ are POD. If $T_1(a) = \operatorname{Inf}\{t: N(t) \geqslant a\}$ and $T_2(b) = \operatorname{Inf}\{t: W(t) \geqslant b\}$, then

$$\begin{aligned} P(T_1(a) > t_1, T_2(b) > t_2) &= P(N(t_1) \leqslant a, W(t_2) \leqslant b) \\ &\geqslant P(T_1(a) > t_1)\, P(T_2(b) > t_2). \end{aligned}$$

That is hitting times are POD.

EXAMPLE 2. Consider the non-stationary process $X(t)$ given by

$$X(t) = \alpha(t)\, Y(t), \qquad t \geqslant 0, \tag{4.1}$$

where $\alpha(t)$ is a deterministic continuous function such that $\alpha(t) \geqslant 0$ and $Y(t)$ is non-negative stationary process in the sense that

$$\operatorname{cov}(Y(t), Y(t+h)) = K(h),$$

for every $t, h > 0$. The model in (4.1) is called the uniformly modulated model (See Priestly, 1988, for more details). If $Y(t)$ is associated, then using the property (e) in Section 2, $X(t)$ is also associated. From Theorem 1, $T(a_1), ..., T(a_k)$ are also associated. Here $T(a_i) = \{n: X(n) \geqslant a_i\}$.

EXAMPLE 3. Block *et al.* (1988) proposed a bivariate exponential autoregressive model of order m, BEAR(m),

$$X(n) = \begin{cases} E(n), & n = 0, ..., m-1 \\ \sum_{q=1}^{m} A(n,q)\, X(n-q) + B(n)\, E(n), & n = m, m+1, ..., \end{cases} \tag{4.2}$$

where $E(n) = (E_1(n), E_2(n))$ is a sequence of independent bivariate exponential random vectors with mean $(\lambda_1^{-1}, \lambda_2^{-1})'$, $\lambda_1, \lambda_2 > 0$, $B(n)$ is a 2×2 diagonal matrix with $B(n) = \operatorname{diag}\{\pi_1(n), \pi_2(n)\}$, $0 < \pi_1(n), \pi_2(n) < 1$, e_j is an m-dimensional vector with component j equal to one and the other component equal to zero, $j = 1, ..., n$, 0 is the m-dimensional zero vector, $I'(n) = (I_1(n, 1), ..., I_1(n, m), I_2(n, 1), ..., I_2(n, m))$ is a sequence of $2m$-dimensional independent random vectors with components assuming values one or zero independent of all $E(n)$, and finally $A(n, q)$ is a 2×2 random diagonal matrix with $A(n, q) = \text{diag}(I_1(n, q), I_2(n, q))$, $q = 1, ..., m$. It is assumed that

$$\sum_{j=1}^{m} P\{(I_l(n, 1), ..., I_l(n, m)) = e_j'\} = 1 - \pi_l(n)$$

and

$$P\{(I_l(n, 1), ..., I_l(n, m)) = 0'\} = \pi_l(n),$$

$l = 1, 2$.

The following theorem gives the result about dependence structure of the bivariate process BEAR(m), $X(n) = (X_1(n), X_2(n))$.

THEOREM 5. *Suppose for* $j = 0, 1, ..., m-1$, *the random variables* $X_1(j)$, $X_2(j)$ *in* (4.2) *are associated. Then* $X(n)$ *is associated. Furthermore, the corresponding hitting times are also associated.*

Proof. Part one comes from Lemma (3.10) of Block *et al.* (1988). The proof of the second part of the theorem is similar to the proof of Theorem 4 and it is omitted.

Remark 4.1. The bivariate geometric autoregressive model introduced by Block *et al.* (1988) is also associated if the assumption of Lemma (3.12) of Block *et al.* holds.

EXAMPLE 4. Consider a k-dimensional multivariate stationary Gaussian process $\{X(t) = (X_1(t), ..., X_k(t)): t \in \{0, 1, ...\}\}$ such that

$$\text{cov}(X_i(t), X_i(s)) = r_i(t-s) \geqslant 0,$$

$$\text{cov}(X_i(t), X_j(s)) = h_{ij}(t-s) \geqslant 0,$$

for all $t \geqslant s$. Then, the corresponding hitting times are associated.

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