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Asymptotic Behavior of Heat Kernels on Spheres of Large Dimensions

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For $n \geqslant 2$, let $(\mu_{\tau,n}^x)_{\tau \geqslant 0}$ be the distributions of the Brownian motion on the unit sphere $S^n \subset \mathbb{R}^{n+1}$ starting in some point $x \in S^n$. This paper supplements results of Saloff-Coste concerning the rate of convergence of $\mu_{\tau,n}^x$ to the uniform distribution U_n on S^n for $\tau \to \infty$ depending on the dimension n. We show that, $\lim_{n \to \infty} \|\mu_{\tau_n,n}^x - U_n\| = 2 \cdot \operatorname{erf}(e^{-s}/\sqrt{8})$ for $\tau_n := (\ln n + 2s)/(2n)$, where erf denotes the error function. Our proof depends on approximations of the measures $\mu_{\tau,n}^x$ by measures which are known explicitly via Poisson kernels on S^n, and which tend, after suitable projections and dilatations, to normal distributions on $\mathbb{R}$ for $n \to \infty$. The above result as well as some further related limit results will be derived in this paper in the slightly more general context of Jacobi-type hypergroups.

1. ASYMPTOTIC BEHAVIOR OF GAUSSIAN MEASURES ON n-SPHERES

For $n \geqslant 2$ let U_n be the uniform distribution on the n-sphere $S^n \subset \mathbb{R}^{n+1}$. If L_n is the Laplace–Beltrami operator on S^n, then $(H_{\tau,n})_{\tau \geqslant 0} := (e^{-\tau L n})_{\tau \geqslant 0}$ forms a Markovian selfadjoint semigroup of operators on $L^P(S^n, U_n)$ $(1 \leqslant p \leqslant \infty)$ which may be regarded as semigroup of operators related to Brownian motion on S^n. The semigroup $(H_{\tau,n})_{\tau \geqslant 0}$ admits a kernel $(H_{\tau,n})_{\tau \geqslant 0}$ with

$$H_{\tau,n} f(x) = \int_{S^n} h_{\tau,n}(x, y)\, f(y)\, dU_n(y) \qquad (f \in L^p(S^n, U_n)) \tag{1.1}$$

and with $0 < h_{\tau,n}(x, y) < \infty$ for $\tau > 0$, $x, y \in S^n$. In particular, for each $x \in S^n$, the functions $h_{\tau,n}^x(y) := h_{\tau,n}(x, y)$ are the L^1-densities of the semigroup $(\mu_{\tau,n}^x := h_{\tau,n}^x U_n)_{\tau \geqslant 0}$ of the distributions of the Brownian motion on S^n starting in x at time 0.

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It is well known that $\lim_{\tau \to \infty} h_{\tau, n}(x, y) = 1$ for fixed n, and it is natural to ask for quantitative estimates on how $h_{\tau, n}$ tends to 1 depending on the dimension n. Problems of this kind were studied recently in several papers where much attention was paid mainly to finite structures; see Diaconis *et al.* (1988, 1990, 1992), and references cited there. Recently, the orthogonal groups $SO(n)$ as well as the spheres S^n and projective spaces were investigated by Rosenthal (1994), Saloff-Coste (1994), and Voit (1995). In particular, Saloff-Coste (1994) used that the Laplace–Beltrami operator L_n on S^n has eigenvalues

$$\lambda_k = k(k+n-1) \qquad (k \in \mathbb{N}_0 = \{0, 1, ...\}) \tag{1.2}$$

with multiplicities

$$h_k = \frac{(k+n-2)!}{(n-1)!\, k!} (2k+n-1), \tag{1.3}$$

and he obtained the following estimations for h_τ^x (see Section 3.3 of Saloff-Coste, 1994):

(a) $$e^{-s} \leqslant \|h_{\tau, n}^x - 1\|_2 \leqslant \sqrt{5} e^{-s}$$
$$\text{for} \quad \tau := \frac{\ln(n+1) + 2s}{2n}$$

(b) $$\|h_{\tau, n}^x - 1\|_1 \leqslant \sqrt{5} e^{-s}$$
$$\text{for} \quad \tau := \frac{\ln(n+1) + 2s}{2n}$$

(c) $$\|h_{\tau, n}^x - 1\|_1 \geqslant 2 \cdot (1 - 8e^{-2s})$$
$$\text{for} \quad \tau := \frac{\ln n - 2s}{2n}, \ 0 < s < \frac{\ln n}{2n}.$$

(Notice that the L^1-norm here differs by a factor 2 from the total variation norm in [14].) The main purpose of this paper is to derive the exact asymptotic rate of $\|h_{\tau, n}^x - 1\|_1$ for $n \to \infty$ and $\tau = \tau(n)$ as given in (a). More precisely, we prove the following

1.1. THEOREM. *For each* $s \in \mathbb{R}$,

$$\lim_{n \to \infty} \|h_{\tau_n, n}^x - 1\|_1 = 2 \cdot \operatorname{erf}(e^{-s}/\sqrt{8}) \qquad \text{with} \quad \tau_n := \frac{\ln(n+1) + 2s}{2n},$$

where $\operatorname{erf}(z) = (2/\sqrt{\pi}) \int_0^z e^{-t^2}\, dt$ *denotes the error function.*

To compare Theorem 1.1 with the estimations (a), (b), and (c) above, consider two special cases. If s is positive and large, then

$$2 \cdot \operatorname{erf}(e^{-s}/\sqrt{8}) \simeq \frac{\sqrt{2}}{\sqrt{\pi}} e^{-s} \tag{1.4}$$

which means that the estimation (b) is sharp for large positive s up to some constant. Moreover, if s is negative and large, then

$$2 \cdot \operatorname{erf}(e^{-s}/\sqrt{8}) \simeq 2\left(1 - \frac{\sqrt{8} \cdot e^{s}}{\sqrt{\pi}} \cdot \exp(-e^{-2s/8})\right) \tag{1.5}$$

which tends to the maximal distance 2 extremly rapidly. In particular, this rate of convergence to 2 is much higher than the rate given in (c) above.

Theorem 1.1 is similar to asymptotic results of Diaconis (1988), Diaconis and Graham (1992), and Diaconis, Graham, and Morrison (1990) for the rate of convergence to equilibrium for random walks on hypercubes where there also the error function appears. In all cases, the error function comes in by comparing two normal distributions with different means.

We prove Theorem 1.1 by a method introduced in Voit (1996a) for random walks on hypercubes: For $s \in \mathbb{R}$, $n \geqslant 2$, and $x \in S^n$ we construct probability measures $p^x_{n,s}$ on S^n with the following properties:

(a) The measures $p^x_{n,s}$ are good approximations of the probability measures $h^s_{\tau_n, n} U_n$ on S^n (τ_n being given as in Theorem 1.1); more precisely,

$$\|h^x_{\tau_n, n} U_n - p^x_{n,s}\| = O(\ln n/n) \qquad \text{for} \quad n \to \infty, \quad s \text{ fixed}. \tag{1.6}$$

(b) The measures $p^x_{n,s}$ are explicitly known, have a simple shape, and satisfy

$$\|p^x_{n,s} - U_n\| \simeq 2 \cdot \operatorname{erf}(e^{-s}/\sqrt{8}) \qquad \text{for} \quad s \in \mathbb{R} \text{ fixed}, \quad n \to \infty. \tag{1.7}$$

(a), (b), and the triangle inequality then lead to Theorem 1.1. Notice again that the total variation norm here differs by the factor 2 from Diaconis *et al.* (1988, 1990, 1992), Saloff-Coste (1994), Voit (1996a).

In order to describe the construction of the measures $p^x_{n,s}$ on S^n, we proceed as follows by reducing the problem on spheres to a problem on a fixed compact interval.

1.2. *Reduction to a Problem on a Compact Interval.* Fix some dimension $n \geqslant 2$ as well as some $x \in S^n$. The orthogonal group $SO(n+1)$ acts transitively on S^n, and $SO(n)$ may be identified with the stabilizer subgroup of x in $SO(n+1)$. Then, $SO(n+1)/SO(n) \simeq S^n$. Moreover, the double coset space $SO(n+1)//SO(n)$ will be identified with the compact interval $[-1, 1]$ such that the canonical projection

$$\pi_n\colon SO(n+1)/SO(n) = S^n \to SO(n+1)//SO(n) \simeq [-1, 1]$$

is given by $\pi_n(y) = \cos\angle(y, x)$ for $y \in S^n \subset \mathbb{R}^{n+1}$. Let

$$M_b(S^n \mid SO(n)) := \{\mu \in M_b(S^n)\colon A(\mu) = \mu \text{ for all } A \in SO(n)\}$$

be the Banach space of all $SO(n)$-invariant Borel measures on S^n. If the mapping of taking images of measures under π_n is again denoted by π_n, then

$$\pi_n\colon M_b(S^n \mid SO(n)) \to M_b([-1, 1])$$

becomes an isometric isomorphism of Banach spaces. In particular, the image $\pi_n(U_n)$ of the uniform distribution on S^n is given by

$$d\omega_n(x) := c_n \cdot (1-x^2)^{n/2-1}\, dx \qquad \text{with} \quad c_n = \frac{\Gamma((n+1)/2)}{\sqrt{\pi} \cdot \Gamma(n/2)}, \tag{1.8}$$

and the distributions $\mu_\tau^x \in M_b(S^n \mid SO(n))$ of the Brownian motion have images on $[-1, 1]$ whose ω_n-densities have well-known expansions in ultraspherical polynomials (see Bloom and Heyer, 1995; Hartman and Watson, 1974; Mueller and Weissler, 1982; Saloff-Coste, 1994; and Section 2.2 below). We prove that good approximations of μ_τ^x on S^n are given by the unique $SO(n)$-invariant probability measures on S^n with π_n-images

$$dp_r^{(n)} := c_n \cdot \frac{1-r^2}{(1+r^2-2rx)^{(n+1)/2}}\, (1-x^2)^{n/2-1}\, dx \qquad (x \in [-1, 1]), \tag{1.9}$$

where $r \in [0, 1[$ is related to τ and where c_n is given as in (1.8):

1.3. THEOREM. *Fix some $s \in \mathbb{R}$. If $\tau(n) := (ln(n) + 2s)/2n$ and $r(n) := e^{-n\tau(n)}$, then*

$$\|\pi_n(\mu_{\tau(n), n}^x) - p_{r(n)}^{(n)}\| = O(\ln n/n) \qquad \textit{for} \quad n \to \infty.$$

If we consider $p_{r(n)}^{(n)}$ for $r(n) := e^{-n\tau(n)}$ then

$$\|p_{r(n)}^{(n)} - N(e^{-s}/\sqrt{n-2}, 1/(n-2))\| = O(1/n), \tag{1.10}$$

$$\|\omega_n - N(0, 1/(n-2))\| = O(1/n), \tag{1.11}$$

where $N(m, \sigma^2)$ is the normal distribution on $\mathbb{R}$ with mean m and variance σ^2. Theorem 1.1 now follows from (1.10), (1.11), and Theorem 1.3; see also Theorem 2.6 below.

The well-known approach above reduces the problem on spheres to a problem on $[-1, 1]$, where for each dimension n this interval inherits some convolution structure from $SO(n+1)$ and S^n related to the ultraspherical polynomials $(P_k^{(\alpha)})_{k \geqslant 0}$ with $\alpha = n/2 - 1$. As the complete machinery works for all indices $\alpha \in \mathbb{R}$, $\alpha > -\frac{1}{2}$, we embed the theorems above into the slightly more general theory of random walks on hypergroups on $[-1, 1]$ associated with ultraspherical polynomials; see Section 2.

Before doing this, we give a brief outline of this paper: In Section 2 we first recapitulate some facts on ultraspherical polynomials and the related hypergroups (for a general approach to hypergroups see Bloom and Heyer, 1995). After having introduced Gaussian measures on $[-1, 1]$ (which are essentially equal to the measures $\pi_n(\mu^x_{\tau(n)})$ above) as well as so-called Poisson measures (cf. Eq. (1.9)), we claim in Section 2 that these Poisson measures are good approximations of Gaussian measures for large indices α. This result corresponds to Theorem 1.3(1); it is proved in Section 3 by using L^2-methods for the ultraspherical Fourier transform. Generalizations of Theorems 1.3(2) and 1.1 are stated in Theorems 2.5 and 2.6, respectively; these theorems are proved in Section 4. Finally, in Section 5 we transfer Theorem 1.1 from spheres to the real projective spaces $P_n(\mathbb{R})$.

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2. GAUSSIAN MEASURES ON ULTRASPHERICAL HYPERGROUPS

In this section we embed the results of Section 1 into the more general setting of ultraspherical hypergroups on $[-1, 1]$. We first recapitulate some facts.

2.1. *Ultraspherical Hypergroups on* $[-1, 1]$. Fix $\alpha > -\frac{1}{2}$, and consider the ultraspherical polynomials defined by

$$P_k^{(\alpha)}(x) := {}_2F_1(-k, k+2\alpha+1; \alpha+1; (1-x)/2) \qquad (x \in \mathbb{R}, k \geqslant 0) \tag{2.1}$$

which are normalized by $P_k^{(\alpha)}(1) = 1$ and which are orthogonal on $[-1, 1]$ with respect to the probability measure

$$d\omega_\alpha(x) := c_\alpha (1-x^2)^\alpha \, dx \tag{2.2}$$

with

$$c_\alpha := \left(\int_{-1}^{1} (1-x^2)^\alpha\, dx\right)^{-1} = \frac{\Gamma(2\alpha+2)}{\Gamma(\alpha+1)^2\, 2^{2\alpha+1}} = \frac{\Gamma(\alpha+3/2)}{\Gamma(\alpha+1)\sqrt{\pi}} \tag{2.3}$$

(where in the last equation the duplication formula was used). By Gegenbauer's product formula, the ultraspherical polynomials satisfy

$$\begin{aligned} &P_k^{(\alpha)}(\cos s)\cdot P_k^{(\alpha)}(\cos t)\\ &\quad = c_{\alpha-1/2}\int_0^\pi P_k^{(\alpha)}(\cos s\cos t + \sin s\sin t\cos z)(\sin z)^{2\alpha}\, dz \end{aligned} \tag{2.4}$$

for $s, t \in [0, \pi]$ with $c_{\alpha-1/2}$ being given according to (2.3); see Eq. (2.23) of Askey (1975). Thus, for $x, y \in [-1, 1]$ we find unique probability measures, say $\delta_x * \delta_y$, on $[-1, 1]$ with

$$P_k^{(\alpha)}(x)\cdot P_k^{(\alpha)}(y) = \int_{-1}^{1} P_k^{(\alpha)}(z)\, d(\delta_x * \delta_y)(z) \qquad \text{for all} \quad k \geqslant 0.$$

This convolution $\delta_x * \delta_y$ of point measures can be extended uniquely to a bilinear and weakly continuous convolution $*$ on the Banach space $M_b([-1, 1])$ of all (complex) Borel measures on $[-1, 1]$ such that $(M_b([-1, 1]), *)$ becomes a commutative Banach algebra. Moreover, $*$ establishes a hypergroup structure on $[-1, 1]$. For details see Bloom and Heyer (1995).

We next turn to some details concerning ultraspherical hypergroups needed later on. Let $M^1([-1, 1])$ be the set of of all probability measures. Then the orthogonality measure $\omega_\alpha \in M^1([-1, 1])$ is the normalized Haar measure of our hypergroup, i.e., $\mu * \omega_\alpha = \omega_\alpha$ for all $\mu \in M^1([-1, 1])$. (Notice that for $\alpha = n/2 - 1$, the measure ω_α is just the projection $\pi_n(U_n)$ of the uniform distribution on the sphere S^n; see Section 1.2.) If we embed $L^1([-1, 1], \omega_\alpha)$ into $M_b([-1, 1])$ in the natural way, then $L^1([-1, 1], \omega_\alpha)$ is an ideal in $M_b([-1, 1])$, and the nontrivial multiplicative linear functionals of $L^1([-1, 1], \omega_\alpha)$ are given by the polynomials $P_k^{(\alpha)}$ $(k \geqslant 0)$ via $f \mapsto \int_{-1}^{1} P_k^{(\alpha)} f\, d\omega_\alpha$. In other words, the Gelfand transform of L^1-functions is just the ultraspherical Fourier transformation

$$L^1([-1, 1], \omega_\alpha) \to c_0(\mathbb{N}_0), \qquad f \mapsto \hat{f} \qquad \text{with } \hat{f}(k) = \int_{-1}^{1} P_k^{(\alpha)}(x)\, f(x)\, d\omega_\alpha(x). \tag{2.6}$$

If we put

$$\begin{aligned} h_k^{(\alpha)} &:= \left(\int_{-1}^{1} P_k^{(\alpha)}(x)^2\, d\omega_\alpha\right)^{-1}\\ &= \frac{(2k+2\alpha+1)\cdot(2\alpha+1)_k}{(2\alpha+1)\cdot k!} \qquad (k \in \mathbb{N}_0), \end{aligned} \tag{2.7}$$

then the mapping $f \mapsto \hat{f}$ can be extended to an isometric isomorphism between $L^2([-1, 1], \omega_\alpha)$ and $l^2(\mathbb{N}_0, (h_k^{(\alpha)})_{k \in \mathbb{N}_0})$. The measure $\sum_{k=0}^{\infty} h_k^{(\alpha)} \delta_k$ on $\mathbb{N}_0$ is called the Plancherel measure of the ultraspherical hypergroup on $[-1, 1]$; see [4, 10]. We finally recapitulate that, for $\alpha = n/2 - 1$, the Plancherel weight $h_k^{(\alpha)}$ is just the multiplicity of the eigenvalue λ_k of the Laplace–Beltrami operator on S^n; see (1.2) and (1.3).

To state generalizations of the results of Section 1 properly, we next introduce two families of probability measures on ultraspherical hypergroups on $[-1, 1]$.

2.2. *Gaussian Measures.* For $\alpha > -\frac{1}{2}$, consider the function

$$q(k) := q^{(\alpha)}(k) := \frac{k(k+2\alpha+1)}{2(\alpha+1)} \qquad (k \in \mathbb{N}_0). \tag{2.8}$$

Then for all $t > 0$, the heat kernel

$$h_t(x, y) := h_t^{(\alpha)}(x, y) := \sum_{k=0}^{\infty} h_k^{(\alpha)} e^{-tq(k)/2} P_k^{(\alpha)}(x) P_k^{(\alpha)}(y) \qquad (x, y \in [-1, 1]) \tag{2.9}$$

is a positive continuous function on $[-1, 1] \times [-1, 1]$. The probability measures

$$d\mu_t(x) := d\mu_t^{(\alpha)}(x) := h_t^{(\alpha)}(x, 1) \cdot d\omega_\alpha(x) \tag{2.10}$$

on $[-1, 1]$ are called the Gaussian measures on $[-1, 1]$ with “variances” t. In view of Section 2.1 it is clear that $(\mu_t^{(\alpha)})_{t \geqslant 0}$ forms a convolution semigroup of probability measures on $[-1, 1]$ with respect to the ultraspherical convolution. This semigroup is well studied from different points of view; see Hartman and Watson (1974), Mueller and Weissler (1982), Saloff-Coste (1994), and references cited there.

2.3. *Poisson Measures.* We begin with the following generating function for ultraspherical polynomials due to Watson:

$$g_r^{(\alpha)}(x) := \frac{1-r^2}{(1+r^2-2rx)^{\alpha+3/2}} = \sum_{k=0}^{\infty} h_k^{(\alpha)} P_k^{(\alpha)}(x)\, r^k \qquad (|r| < 1, x \in [-1, 1]) \tag{2.11}$$

(see page 292 of Watson, 1933, with $x = \cos\theta$ and $\cos\varphi = 1$, and notice that we use the normalization $P_k^{(\alpha)}(1) = 1$). As

$$\int_{-1}^{1} g_r^{(\alpha)}(x)\, d\omega_\alpha(x) = \sum_{k=0}^{\infty} h_k^{(\alpha)} r^k \int_{-1}^{1} P_k^{(\alpha)}(x)\, d\omega_\alpha(x) = 1, \tag{2.12}$$

we see that the measures $p_r^{(\alpha)} := g_r^{(\alpha)} \cdot \omega_\alpha$ are probability measures on $[-1, 1]$ for $0 \leqslant r < 1$. We shall call these measures Poisson measures on the ultraspherical hypergroup, as the functions $g_r^{(\alpha)}$ were used by Muckenhoupt and Stein (1965) in connection with "Poisson kernels" associated with ultraspherical polynomials (notice that the constant factor of Eq. (2.13) in Muckenhoupt and Stein, 1965, is not correct).

We next state the main result of this paper; it will be proved in Section 3.

2.4. THEOREM. *The Gaussian and Poisson measures above have the following properties*:

(1) *For each $c \in \mathbb{R}$ there exist constants $A = A(c)$ and $M = M(c)$ of the following kind*: *If*

$$t := \ln \alpha + c \geqslant 0, \qquad r := e^{-t/2},$$

then

$$\|\mu_t^{(\alpha)} - p_r^{(\alpha)}\| \leqslant M \cdot \frac{\ln \alpha + c}{\alpha + 1} \qquad \textit{for all} \quad \alpha \geqslant A.$$

In particular, *for $c \geqslant 0$ and $\alpha \geqslant 15$*, *the following explicit estimation holds*:

$$\|\mu_t^{(\alpha)} - p_r^{(\alpha)}\| \leqslant \frac{\ln \alpha + c}{4(\alpha+1)} \cdot (15e^{-2c} + 70e^{-3c} + 166e^{-4c} + 4540e^{-5c})^{1/2}.$$

(2) *For each $c \in \mathbb{R}$ there exist constants $A = A(c)$ and $M = M(c)$ as follows*: *If*

$$t := 2(\ln \alpha + c) \geqslant 0, \qquad r := e^{-t/2},$$

then the densities $h_t^{(\alpha)}$ and $g_r^{(\alpha)}$ of the Gaussian and Poisson measures respectively satisfy

$$\|h_t^{(\alpha)} - g_r^{(\alpha)}\|_\infty \leqslant M \cdot \frac{\ln \alpha + c}{\alpha + 1} \qquad \textit{for all} \quad \alpha \geqslant A.$$

More precisely, *for $c \geqslant 0$ and $\alpha \geqslant 10$*, *the following*, *explicit estimation holds*:

$$\|h_t^{(\alpha)} - g_r^{(\alpha)}\|_\infty \leqslant \frac{\ln \alpha + c}{2(\alpha+1)} \cdot (6e^{-2c} + 14e^{-3c} + 180e^{-4c}).$$

Let us discuss some consequences of Theorem 2.4(1). The first one will be a central limit theorem for the Gaussian semigroups $(\mu_t^{(\alpha)})_{t \geqslant 0}$ for $\alpha \to \infty$, where we shall obtain convergence with respect to the total variation norm for probability measures on $\mathbb{R}$.

2.5. THEOREM. *For each $s>0$ consider the dilatation $D_s\colon \mathbb{R}\to\mathbb{R}$, $x\mapsto sx$, and denote the image of a measure $\mu\in M_b(\mathbb{R})$ under D_s by $D_s(\mu)\in M_b(\mathbb{R})$. Fix some constant $c\in\mathbb{R}$. Then, for $t(\alpha):=\ln\alpha+c$, the dilated Gaussian measures $D_{\sqrt{2\alpha}}(\mu_{t(\alpha)}^{(\alpha)})$ of $\mu_{t(\alpha)}^{(\alpha)}$ satisfy*

$$\|D_{\sqrt{2\alpha}}(\mu_{t(\alpha)}^{(\alpha)})-N(\sqrt{2}e^{-c/2},1)\|=O(\ln\alpha/\alpha)\qquad \textit{for}\quad \alpha\to\infty$$

with respect to the total variation norm where $N(m,1)$ stands for the normal distribution with mean m and variance 1.

Theorem 2.5 will be proved in Section 4. We there show that 2.5 leads to the following asymptotic rate of convergence of the semigroups $(\mu_t^{(\alpha)})_{t\geqslant 0}$ to equilibrium for large α.

2.6. THEOREM. *For each constant $c\in\mathbb{R}$, the Gaussian measures $\mu_t^{(\alpha)}$ satisfy*

$$\lim_{\alpha\to\infty}\|\mu_{t(\alpha)}^{(\alpha)}-\omega_\alpha\|=\lim_{\alpha\to\infty}\|h_{t(\alpha)}^{(\alpha)}-1\|_1=2\cdot\mathrm{erf}(e^{-c/2}/2)$$

$$\textit{with}\quad t(\alpha):=\ln\alpha+c$$

where $\mathrm{erf}(z)=(2/\sqrt{\pi})\int_0^z e^{-t^2}\,dt$ denotes the error function. More precisely for $\alpha\to\infty$,

$$\|\mu_{t(\alpha)}^{(\alpha)}-\omega_\alpha\|=2\cdot\mathrm{erf}(e^{-c/2}/2)+O(\ln\alpha/\alpha).$$

Using part (2) of Theorem 2.4 instead of part (1), we also obtain the following asymptotic uniform rate of convergence; the proof will be given also in Section 4.

2.7. PROPOSITION. *For each constant $c\in\mathbb{R}$, the densities $h_t^{(\alpha)}$ of the Gaussian measures $\mu_t^{(\alpha)}$ satisfy*

$$\|h_{t(\alpha)}^{(\alpha)}-1\|_\infty=\exp(2e^{-c})-1+O(\ln\alpha/\alpha)\qquad(\alpha\to\infty)$$

with $t(\alpha):=2(\ln\alpha+c)$.

2.8. *Remarks.* (1) The theorems of Section 1 follow readily from the theorems above. More precisely, Theorem 1.1 is a consequence of Theorem 2.6 while Theorem 1.3 follows from Theorem 2.4(1). This is clear, as the notation of Section 1 is related to that of Section 2 by

$$\alpha=n/2-1,\qquad t=\tau\cdot 2n,\qquad c=2s+\ln 2.$$

In the same way, Proposition 2.7 can be translated into the language of Section 1.

(2) In [3], Bingham studies certain families of Markov processes on $S^2 \subset \mathbb{R}^3$ which depend on some parameter $\alpha > 0$ and which can also be studied by using the ultraspherical polynomials $(P_k^{(\alpha)})_{k \geqslant 0}$ and their associated hypergroup structures. In this way, the above theorems can be also applied to the Markov chains of Bingham; for a related result see also Section 2.6(5) of Voit (1995a).

(3) We expect that the above results for Gaussian semigroups on the spheres S^n can be extended to more general $SO(n+1)$-invariant Markov chains $(X_t^n)_{t \geqslant 0}$ on S^n for $n \to \infty$. The proofs, however, will become much more involved. Some results of this kind can be found in Voit [16] where a completely different method via moment problems is used. Moreover, for related results for projective spaces and more general hypergroup structures on $[-1, 1]$ associated with Jacobi polynomials we refer to Voit (1995a, 1996b, 1997). Real projective spaces will be discussed in Section 5 below.

3. APPROXIMATION OF GAUSSIAN MEASURES BY POISSON MEASURES

This section is devoted to the proof of Theorem 2.4. Main ingredients will be inequalities for the ultraspherical Fourier transform of Section 2.1. The application of such inequalities to rates of convergence was propagated by Diaconis *et al.* (1988, 1990, 1992) and others (see Rosenthal, 1994; Saloff-Coste, 1994, and references therein). We here use a slight modification as we compare arbitrary probability measures; cf. Voit (1996a, 1997); these methods obviously work for general compact commutative hypergroups. In fact, Lemma 3.1 below is a special case of Lemma 2.2 of Voit (1997). We here omit the more or less standard proof.

Before stating this lemma, we recapitulate that the ultraspherical Fourier–Stieltjes transform of a Borel measure $\mu \in M_b([-1, 1])$ is given by $\hat{\mu}(k) := \int_{-1}^{1} P_k^{(\alpha)}(x)\, d\mu(x) (k \in \mathbb{N}_0)$. If μ has an ω_α-density f then $\hat{\mu} = \hat{f}$ by Eq. (2.6).

3.1. Lemma. *Let μ be a signed measure on $[-1, 1]$.*

(1) *If μ has a ω_α-density f, then*

$$\|\mu\|^2 \leqslant \|f\|_2^2 = \|\hat{f}\|_2^2 = \|\hat{\mu}\|_2^2 = \sum_{k=0}^{\infty} h_k^{(\alpha)} |\hat{\mu}(k)|^2.$$

(2) *If* $\hat{\mu}$ *satisfies* $\sum_{k=0}^{\infty} h_k^{(\alpha)} |\hat{\mu}(k)| < \infty$, *then* μ *has a continuous* ω_α*-density* f *with*

$$\|\mu\| \leqslant \|f\|_\infty \leqslant \|\hat{f}\|_1 = \|\hat{\mu}\|_1 = \sum_{k=0}^{\infty} h_k^{(\alpha)} \, |\hat{\mu}(k)|.$$

We next prove Theorem 2.4. For simplicity, we often omit the superscript α.

3.2. *Proof of Part* (1). Fix $\alpha \geqslant 0$. Lemma 3.1(1) and the definition of $r = e^{-t/2}$ imply that the Gaussian measures μ_t and the Poisson measures p_r satisfy

$$\begin{aligned}
\|\mu_t - p_r\|^2 \leqslant \|\hat{\mu}_t - \hat{p}_r\|_2^2 &= \sum_{k=0}^{\infty} h_k^{(\alpha)} \cdot |\hat{\mu}_t(k) - \hat{p}_r(k)|^2 \\
&= \sum_{k=0}^{\infty} h_k^{(\alpha)} \cdot |e^{-tk/2(k+2\alpha+1)/(2\alpha+2)} - e^{-tk/2}|^2 \\
&= \sum_{k=0}^{\infty} h_k^{(\alpha)} e^{-tk} \cdot |e^{-tk(k-1)/(4(\alpha+1))} - 1|^2,
\end{aligned}$$

where the summands vanish for $k = 0$ and $k = 1$. It follows from $1 - e^{-x} \leqslant x$ for $x \geqslant 0$ and from $t = \ln \alpha + c$ that

$$\begin{aligned}
\|\mu_t - p_r\|^2 &\leqslant \sum_{k=2}^{\infty} h_k^{(\alpha)} \alpha^{-k} e^{-kc} \cdot \frac{k^2(k-1)^2 \, (\ln \alpha + c)^2}{16(\alpha+1)^2} \\
&=: \frac{(\ln \alpha + c)^2}{16(\alpha+1)^2} \sum_{k=2}^{\infty} u_k e^{-kc} \qquad (3.1)
\end{aligned}$$

with

$$u_k := \frac{2k + 2\alpha + 1}{2\alpha + 1} \cdot \frac{(2\alpha+1)_k}{k!} \cdot \frac{k^2(k-1)^2}{\alpha^k} \qquad (k \geqslant 2) \qquad (3.2)$$

Then

$$\frac{u_{k+1}}{u_k} = \frac{2k + 2\alpha + 3}{2k + 2\alpha + 1} \cdot \frac{k+1}{k-1} \cdot \frac{2\alpha + k + 1}{\alpha(k-1)}. \qquad (3.3)$$

Therefore, for $c \in \mathbb{R}$, we find $A = A(c)$ and $N = N(c)$ such that $e^{-c} \cdot u_{k+1}/u_k \leqslant \frac{1}{2}$ for all $k \geqslant N$ and $\alpha \geqslant A$. It follows that

$$\|\mu_t - p_r\|^2 \leqslant \frac{(\ln \alpha + c)^2}{16(\alpha+1)^2} \left(\sum_{k=2}^{N-1} u_k e^{-kc} + 2u_N e^{-Nc} \right) \leqslant \frac{(\ln \alpha + c)^2}{16(\alpha+1)^2} \cdot R(c)$$

for some $R(c) > 0$. The proof of the first part of Theorem 2.4(1) is now complete.

Assume now that $c \geqslant 0$ and $\alpha \geqslant 15$. Then a short computation yields that

$$\frac{2\alpha + k + 1}{\alpha(k-1)} \leqslant \frac{36}{60} \qquad \text{for all} \quad k \geqslant 5.$$

Therefore, for $k \geqslant 5$, Eq. (3.2) leads to

$$\frac{u_{k+1}}{u_k} \leqslant \frac{43}{41} \cdot \frac{6}{4} \cdot \frac{2\alpha + k + 1}{\alpha(k-1)} \leqslant \frac{43}{41} \cdot \frac{6}{4} \cdot \frac{36}{60} \leqslant 0.95.$$

Hence, by Eq. (3.1),

$$\|\mu_t - p_r\|^2 \leqslant \frac{(\ln \alpha + c)^2}{16(\alpha+1)^2} \left(\sum_{k=2}^{4} u_k \cdot e^{-kc} + 20 \cdot u_5 \cdot e^{-5c} \right).$$

A short calculation yields

$$u_2 \leqslant 15, \qquad u_3 \leqslant 70, \qquad u_4 \leqslant 166, \qquad u_5 \leqslant 227 \qquad \text{for} \quad \alpha \geqslant 15$$

which completes the proof of the second part of the proof of Theorem 2.4(1).

3.3. *Proof of Part* (2). We proceed as above. Here Lemma 3.1(2) yields

$$\begin{aligned} \|h_t - g_r\|_\infty \leqslant \|\hat{\mu}_t - \hat{p}_r\|_1 &= \sum_{k=0}^{\infty} h_k^{(\alpha)} \, |\hat{\mu}_t(k) - \hat{p}_r(k)| \\ &\leqslant \frac{\ln \alpha + c}{2(\alpha+1)} \sum_{k=2}^{\infty} h_k^{(\alpha)} \alpha^{-k} e^{-ck} k(k-1). \end{aligned}$$

The first part of the theorem now follows immediately. To complete the proof of the final assertion, assume that $\alpha \geqslant 10$. Consider

$$u_k := \frac{2k + 2\alpha + 1}{2\alpha + 1} \cdot \frac{(2\alpha+1)_k}{(k-2)! \cdot \alpha^k}.$$

Then a short calculation yields

$$u_2 \leqslant 6, \qquad u_3 \leqslant 14, \qquad u_4 \leqslant 18,$$

and $u_{k+1}/u_k \leqslant 0.9$ for $k \geqslant 4$. These estimations readily complete the proof.

3.4. *Remark.* We do not know whether the order $O(\ln\alpha/\alpha)$ of convergence in Theorem 2.4(1) is sharp for $\alpha\to\infty$ and $c\in\mathbb{R}$ fixed. If we apply the inequality

$$\|\mu_t - p_r\| \geqslant \|\hat{\mu}_t - \hat{p}_r\|_\infty = \max_{k\in\{0,1,...\}} e^{-tk/2}\cdot |e^{-tk(k-1)/(4(\alpha+1))} - 1|$$

in the setting of Theorem 2.4(1), then $t=\ln\alpha + c$ and $k:=2$ lead to the lower bound

$$\|\mu_t - p_r\| \geqslant O(\ln\alpha/\alpha^2)$$

which is considerably weaker than the upper bound $O(\ln\alpha/\alpha)$.

4. APPROXIMATIONS BY NORMAL DISTRIBUTIONS

This section is devoted to proofs of Theorems 2.5 and 2.6 as well of Proposition 2.7.

4.1. *Proof of Theorem* 2.5. Fix $c\in\mathbb{R}$ and put

$$t(\alpha) := \ln\alpha + c, \qquad r(\alpha) := e^{-t(\alpha)/2} \qquad \text{for} \quad \alpha\geqslant e^{-c}.$$

As dilatations of bounded Borel measures on $\mathbb{R}$ form isometric isomorphisms on $M_b(\mathbb{R})$, Theorem 2.4(1) implies that the Gaussian measures $\mu_{t(\alpha)}^{(\alpha)}$ and the measures $p_{r(\alpha)}^{(\alpha)}$ satisfy

$$\|D_{\sqrt{2\alpha}}(\mu_{t(\alpha)}^{(\alpha)}) - D_{\sqrt{2\alpha}}(p_{r(\alpha)}^{(\alpha)})\| = O(\ln\alpha/\alpha) \qquad \text{for} \quad \alpha\to\infty. \tag{4.1}$$

In order to complete the proof, we now compare the dilated Poisson measures $D_{\sqrt{2\alpha}}(p_{r(\alpha)}^{(\alpha)})$ with the normal distributions $N(\sqrt{2}\cdot e^{-c/2}, 1)$ and check that

$$M_\alpha := \|D_{\sqrt{2\alpha}}(p_{r(\alpha)}^{(\alpha)}) - N(\sqrt{2}\cdot e^{-c/2}, 1)\| = O(1/\alpha). \tag{4.2}$$

To do this, we first introduce some abbreviations. Let

$$A(x) := e^{-x^2/2}, \qquad A_\alpha(x) := \left(1-\frac{x^2}{2\alpha}\right)^\alpha,$$

$$B(x) := \exp(e^{-c} - \sqrt{2}\cdot e^{-c/2}x), \qquad B_\alpha(x) := (1 + e^{-c}/\alpha - \sqrt{2}\cdot xe^{-c/2}/\alpha)^{\alpha+3/2}$$

and

$$d_\alpha := \frac{c_\alpha}{\sqrt{\alpha}}\cdot(1-e^{-c}/\alpha) \qquad \text{with} \quad c_\alpha \text{ as in Section 2.1.}$$

If we use the definition of $r(\alpha)$ and if we carry out the substitution $x \mapsto x/\sqrt{2\alpha}$ in the definition of the Poisson measures in Section 2.3, then we obtain that

$$M_\alpha = \frac{1}{\sqrt{2\pi}} \int_{\{|x| \geqslant \sqrt{2\alpha}\}} \frac{A(x)}{B(x)}\, dx + \int_{-\sqrt{2\alpha}}^{\sqrt{2\alpha}} \left| \frac{1}{\sqrt{2\pi}} \cdot \frac{A(x)}{B(x)} - d_\alpha \cdot \frac{A_\alpha(x)}{B_\alpha(x)} \right| dx. \tag{4.3}$$

As the first integral obviously has order $O(1/\alpha)$, two applications of the triangle inequality lead to

$$\begin{aligned} M_\alpha \leqslant O(1/\alpha) + \left| \frac{1}{\sqrt{2\pi}} - d_\alpha \right| \cdot \int_{-\sqrt{2\alpha}}^{\sqrt{2\alpha}} \frac{A(x)}{B(x)}\, dx + d_\alpha \int_{-\sqrt{2\alpha}}^{\sqrt{2\alpha}} \frac{|A(x) - A_\alpha(x)|}{B(x)}\, dx \\ + d_\alpha \int_{-\sqrt{2\alpha}}^{\sqrt{2\alpha}} A_\alpha(x) \cdot \left| \frac{1}{B(x)} - \frac{1}{B_\alpha(x)} \right| dx. \end{aligned} \tag{4.4}$$

By the definition of c_α and an asymptotic formula for $\Gamma(x)$ (see 6.1.47 in [1]), we have

$$\frac{c_\alpha}{\sqrt{2\alpha}} - \frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \left(\frac{\Gamma(\alpha + 3/2)}{\Gamma(\alpha+1)\sqrt{\alpha}} - 1 \right) = O(1/\alpha) \qquad \text{for} \quad \alpha \to \infty.$$

Therefore, as the integral of the second term of the right-hand side of (4.4) remains bounded, we may estimate this second term by $O(1/\alpha)$. In order to deal with the two remaining terms of the right-hand side of (4.4), we use the following well-known inequality:

$$0 \leqslant e^{-t} - \left(1 - \frac{t}{a}\right)^a \leqslant t^2 e^{-t}/a \qquad \text{for all} \quad a \geqslant 1,\ |t| \leqslant a. \tag{4.5}$$

This and $d_\alpha \to 1/\sqrt{2\pi}$ imply that

$$\begin{aligned} & d_\alpha \int_{-\sqrt{2\alpha}}^{\sqrt{2\alpha}} \frac{|A(x) - A_\alpha(x)|}{B(x)}\, dx \\ & \qquad \leqslant \frac{d_\alpha}{\alpha} \int_{-\sqrt{2\alpha}}^{\sqrt{2\alpha}} \frac{x^2}{4} \cdot \exp(-(x - \sqrt{2} \cdot e^{-c/2})^2/2)\, dx = O\left(\frac{1}{\alpha}\right). \end{aligned} \tag{4.6}$$

To deal with the last term of (4.4), we conclude from (4.5) that

$$A_\alpha(x) \leqslant A(x), \qquad 0 \leqslant B(x) - B_\alpha(x) \leqslant \frac{B(x)}{\alpha} \cdot (e^{-c} - \sqrt{2} \cdot e^{-c/2} x)^2 \tag{4.7}$$

for all $|x| \leqslant \sqrt{2\alpha}$ and all α which are sufficiently large. Hence

$$\begin{aligned}
&\int_{-\sqrt{2\alpha}}^{\sqrt{2\alpha}} A_\alpha(x) \cdot \left| \frac{1}{B(x)} - \frac{1}{B_\alpha(x)} \right| dx \\
&\quad \leqslant \int_{-\sqrt{2\alpha}}^{\sqrt{2\alpha}} A(x) \cdot |B(x) - B_\alpha(x)| \cdot \frac{1}{B_\alpha(x)^2}\, dx \\
&\quad \leqslant \frac{1}{\alpha} \int_{-\sqrt{2\alpha}}^{\sqrt{2\alpha}} \frac{A(x)}{B(x)} \cdot \frac{B(x)^2}{B_\alpha(x)^2} \cdot (e^{-c} - \sqrt{2} \cdot e^{-c/2} x)^2\, dx. \qquad (4.8)
\end{aligned}$$

As $B(x)/B_\alpha(x) = O(1)$ uniformly for $x \in [-\sqrt{2\alpha}, \sqrt{2\alpha}]$ and $\alpha \to \infty$ by (4.7), and as

$$\frac{A(x)}{B(x)} = \exp(-(x - \sqrt{2} \cdot e^{-c/2})^2/2),$$

it follows that the last term of the right-hand side of (4.4) also has order $O(1/\alpha)$. In summary, we obtain $M_\alpha = O(1/\alpha)$ which completes the proof of Theorem 2.5.

The considerations of Section 4.1 make also sense for the limit case $c = \infty$, i.e., $e^{-c/2} = 0$. Here, the Gaussian measure on $[-1, 1]$ is the Haar measure ω_α. Hence:

4.2. Lemma. *If $D_{\sqrt{2\alpha}}(\omega_\alpha)$ is the dilatation of the Haar measure ω_α, then*

$$\|D_{\sqrt{2\alpha}}(\omega_\alpha) - N(0,1)\| = O(1/\alpha) \qquad \textit{for} \quad \alpha \to \infty.$$

We now use Theorem 2.5 and Lemma 4.1 in order to prove Theorem 2.6.

4.3. *Proof of Theorem* 2.6. Fix $c \in \mathbb{R}$ and put $t(\alpha) := \ln \alpha + c$ and $r(\alpha) := e^{-t(\alpha)/2}$ as above. As dilatations of measures are isometric, we have

$$\|\mu_{t(\alpha)}^{(\alpha)} - \omega_\alpha\| = \|D_{\sqrt{2\alpha}}(\mu_{t(\alpha)}^{(\alpha)}) - D_{\sqrt{2\alpha}}(\omega_\alpha)\|$$

for $\alpha > 0$. If now follows from Theorem 2.5 and Lemma 4.2 that

$$\|\mu_{t(\alpha)}^{(\alpha)} - \omega_\alpha\| = \|N(\sqrt{2} \cdot e^{-c/2}, 1) - N(0,1)\| + O(\ln \alpha/\alpha). \qquad (4.10)$$

A short computation yields

$$\|N(m,1) - N(0,1)\| = 2 \cdot \operatorname{erf}(m/\sqrt{8}) \qquad \text{for all} \quad m \geqslant 0$$

(see, for instance, p. 59 of [7]) which immediately implies Theorem 2.6.

4.4. *Remark.* It is possible to derive Theorem 2.6 directly from Theorem 2.4(1) without Theorem 2.5. In this case the complicated estimations of Section 4.1 are not needed. We used Theorem 2.5 in the proof of 2.6 in order to illustrate the close connection between central limit theorems and the asymptotic rate of convergence to equilibrium.

4.5. *Proof of Proposition* 2.7. Fix $c \in \mathbb{R}$ and put

$$t(\alpha) := 2(\ln \alpha + c), \qquad r(\alpha) := e^{-t(\alpha)/2} \qquad \text{for} \quad \alpha \geqslant e^{-c}.$$

As $\|h_{t(\alpha)}^{(\alpha)} - g_{r(\alpha)}^{(\alpha)}\|_\infty = O(\ln \alpha/\alpha)$ for $\alpha \to \infty$ by Theorem 2.4(2), it suffices to check that

$$\|g_{r(\alpha)}^{(\alpha)} - 1\|_\infty = \exp(e^{-c}) - 1 + O(1/\alpha) \qquad \text{for} \quad \alpha \to \infty \tag{4.11}$$

in order to conclude that $\|h_{t(\alpha)}^{(\alpha)} - 1\|_\infty = \exp(e^{-c}) - 1 + O(\ln \alpha/\alpha)$ as claimed in Proposition 2.7. To prove (4.11), we use the definition of the densities g_α of Section 2.3 and observe that

$$x \mapsto \frac{1-r^2}{(1-2rx+r^2)^{\alpha+3/2}} \qquad \text{for} \quad x \in [-1, 1]$$

takes its maximum and minimum in $x = \pm 1$, respectively. Hence, (4.11) follows immediately from the following three estimations:

$$\begin{aligned} \frac{1-r(\alpha)^2}{(1-2r(\alpha)+r(\alpha)^2)^{\alpha+3/2}} &= \frac{1-r(\alpha)^2}{(1-r(\alpha))^{2\alpha+3}} \\ &= \frac{1+e^{-c}/\alpha}{(1-e^{-c}/\alpha)^{2\alpha+2}} = \exp(e^{-c}) + O(1/\alpha), \\ \frac{1-r(\alpha)^2}{(1+2r(\alpha)+r(\alpha)^2)^{\alpha+3/2}} &= \frac{1-e^{-c}/\alpha}{(1+e^{-c}/\alpha)^{2\alpha+2}} = \exp(-e^{-c}) + O(1/\alpha), \end{aligned}$$

and $|\exp(-e^{-c}) - 1| \leqslant |\exp(e^{-c}) - 1|$. The proof of Proposition 2.6 is now complete.

4.6. *Remark.* An analysis of proofs in Sections 4.1 and 4.3 shows that it is possible to obtain explicit bounds of a reasonable size for the error term in Eq. (4.11) whenever α is sufficiently large. Combining this with the explicit bounds of Theorem 2.4(1), one obtains very good upper and lower bounds for $\|\mu_{t(\alpha)}^{(\alpha)} - \omega_\alpha\| = \|h_{t(\alpha)}^{(\alpha)} - 1\|_1$ of the form

$$2 \cdot \operatorname{erf}(c^{-c/2}/2) \pm \left(M_{1,c} \cdot \frac{\ln \alpha + c}{\alpha} + \frac{M_{2,c}}{\alpha} \right)$$

for certain constants $M_{i,c}$ depending on c. Different bounds are derived in Saloff-Coste (1994) by completely different methods.

5. PROJECTIVE SPACES OVER THE REAL NUMBERS

The results of the previous sections can be used to study the asymptotic behavior of Gaussian measures on the real projective spaces $\mathbb{P}_n = \mathbb{P}_n(\mathbb{R})$ which appear as quotient of S^n by identifying antipodal points. In this way, Gaussian semigroups on $\mathbb{P}_n$ starting in some $x \in \mathbb{P}_n$ can be lifted to Gaussian semigroups on S^n starting in $\pm x \in S^n$ with the probability $\frac{1}{2}$. Transfering these semigroups to $[-1, 1] \simeq SO(n+1)//SO(n)$, we land up with symmetrizations of the Gaussian measures on $[-1, 1]$ above. Hence, Theorems 1.3 and 1.1 for spheres lead to corresponding results for $\mathbb{P}_n$. The following basic lemma leads to the asymptotic rate of convergence of the Brownian motion on $\mathbb{P}_n$.

5.1. LEMMA. *If $N(m, 1)$ is the normal distribution on $\mathbb{R}$ with mean m and variance* 1, *then, for $m \geqslant 0$*

$$\left\| N(0, 1) - \frac{1}{2}(N(m, 1) + N(-m, 1)) \right\|$$
$$= 2 \cdot \operatorname{erf}(x_0/\sqrt{2}) - \operatorname{erf}\left(\frac{x_0 + m}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{x_0 - m}{\sqrt{2}}\right),$$

where erf *denotes the error function and where $x_0 = x_0(m)$ is given by*

$$e^{x_0 m} = e^{m^2/2} + \sqrt{e^{m^2} - 1}.$$

Proof. A short computation yields that the zeros of the Lebesgue density of the signed measure $N(0, 1) - \frac{1}{2}(N(m, 1) + N(-m, 1))$ are given by $\pm x_0$ with x_0 as above. Therefore, symmetry arguments lead to the claim as

$$\left\| N(0, 1) - \frac{1}{2}(N(m, 1) + N(-m, 1)) \right\|$$
$$= \frac{4}{\sqrt{2\pi}} \int_0^{x_0} \left(e^{-x^2/2} - \frac{1}{2} \cdot e^{-(x-m)^2/2} - \frac{1}{2} \cdot e^{-(x+m)^2/2} \right) dx$$
$$= 2 \cdot \operatorname{erf}(x_0/\sqrt{2}) - \operatorname{erf}\left(\frac{x_0 + m}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{x_0 - m}{\sqrt{2}}\right).$$

In the following theorem we use the time-normalization of the heat semigroup on $\mathbb{P}_n$ as studied in Section 4 of Saloff-Coste (1994). This theorem follows immediately from Theorem 2.5, together with Lemma 5.1.

5.2. THEOREM. *Let $\tilde{L}_n$ be the Laplace–Beltrami operator on $\mathbb{P}_n$ with eigenvalues*

$$\lambda_k(\mathbb{P}_n) = 2k(2k+n-1) \qquad (k \in \mathbb{N}_0).$$

Let $(\tilde{h}^n_\tau)_{\tau \geqslant 0}$ be the kernel of the Brownian semigroup $(e^{-\tau \tilde{L}_n})_{\tau \geqslant 0}$ on $\mathbb{P}_n$; fix $s \in \mathbb{R}$. If

$$\tau_n := \frac{\ln n + 2s}{2n} \qquad \text{for} \quad n \in \mathbb{N},$$

then

$$\lim_{n \to \infty} \|\tilde{h}^n_{\tau_n} - 1\|_1 = 2 \cdot \mathrm{erf}(x_0/\sqrt{2}) - \mathrm{erf}\left(\frac{x_0 + e^{-s}}{\sqrt{2}}\right) - \mathrm{erf}\left(\frac{x_0 - e^{-s}}{\sqrt{2}}\right)$$

with respect to $\tilde{U}_n$ on $\mathbb{P}_n$, where $x_0 = x_0(e^{-s})$ is given according to Lemma 5.1.

5.3. *Remark.* We expect that results similar to Theorem 5.2 are available for the projective spaces $\mathbb{P}_n(\mathbb{C})$ and $\mathbb{P}_n(\mathbb{H})$. In fact, this is indicated by central limit theorems on Jacobi-type hypergroups on $[-1, 1]$ with indices (α, β), where β is fixed and α tends to infinity; see Voit (1996b). The limit distributions there are noncentral χ^2—distributions whose degrees of freedom depend on β. We also mention that Voit (1995b) contains a related two-dimensional central limit theorem for the double coset hypergroups $U(n)//U(n-1) \simeq \{z \in \mathbb{C}\colon |z| \leqslant 1\}$ for $n \to \infty$. If one has analogues of the Poisson measures for these hypergroups on $[-1, 1]$ and $\{z \in \mathbb{C}\colon |z| \leqslant 1\}$ (which can be handled in a sufficiently easy way), then the methods above carry over to these examples.

REFERENCES

[1] Abramowitz, M., and Stegun, I. A. (1984). *Handbook of Mathematical Functions.* National Bureau of Standards, Washington, DC.

[2] Askey, R. (1975). *Orthogonal Polynomials and Special Functions.* SIAM, Philadelphia.

[3] Bingham, N. H. (1972). Random walks on spheres. *Z. Wahrsch. Verw. Gebiete* **22** 169–192.

[4] Bloom, W. R., and Heyer, H. (1995). *Harmonic Analysis of Probability Measures on Hypergroups.* De Gruyter, Berlin.

[5] Diaconis, P. (1988). Group Representations in Probability and Statistics. Institute of Mathematical Statistics, Hayward, California.

[6] Diaconis, P., and Graham, R. L. (1992). An affine walk on the hypercube. *J. Comp. Appl. Math.* **41** 215–235.

[7] Diaconis, P., Graham, R. L., and Morrison, J. A. (1990). Asymptotic analysis of a random walk on a hypercube with many dimensions. *Random Struct. Alg.* **1** 51–72.

[8] Gasper, G. (1971). Positivity and convolution structure for Jacobi series. *Ann. Math.* **93** 112–118.

[9] Hartman, P., and Watson, G. S. (1974). "Normal" distribution functions on spheres and the modified Bessel functions. *Ann. Probab.* **2** 593–607.

[10] Lasser, R. (1983). Orthogonal polynomials and hyperproups. *Rend. Math. Appl.* **3** 185–209.

[11] Muckenhoupt, B., and Stein, E. M. (1965). Classical expansions and their relation to conjugate harmonic functions. *Trans. Amer. Math. Soc.* **118** 17-92.

[12] Mueller, C., and Weissler, F. B. (1982). Hypercontractivity for the heat semigroup for ultraspherical polynomials and on the n-sphere. *J. Funct. Anal.* **48** 252–283.

[13] Rosenthal, J. (1994). Random rotations: Characters and random walks on $SO(N)$. *Ann. Probab.* **22** 398–423.

[14] Saloff-Coste, L. (1994). Precise estimates on the rate of which certain diffusions tend to equilibrium. *Math. Z.* **217** 641–677.

[15] Szegö, G. (1959). Orthogonal Polynomials. *Am. Math. Soc. Coll. Publ.* **23**. Providence, R.I.: *Amer. Math. Soc.*.

[16] Voit, M. (1995a). A central limit theorem for isotropic random walks on n-spheres for $n \to \infty$. *J. Math. Anal. Appl.* **189** 215–224.

[17] Voit, M. (1995b). Limit theorems for random walks on the double coset spaces $U(n)//U(n-1)$ for $n \to \infty$. *J. Comp. Appl. Math.* **65** 449–459.

[18] Voit, M. (1996a). Asymptotic distributions for the Ehrenfest urn and related random walks. *J. Appl. Probab.* **33**(3) 340–356.

[19] Voit, M. (1996b). Limit theorems for compact two-point homogeneous spaces of larpe dimensions. *J. Theoret. Probab* **9** 353–370.

[20] Voit, M. (1997). Rate of convergence to Gaussian measures on n-spheres and Jacobi hyperproups. *Ann. Probab.* **25**(1), to appear.

[21] Watson, G. N. (1933). Notes on generating functions of polynomials III. *J. London Math. Soc.* **8** 289–292.