

# Parameter Estimation with Exact Distribution for Multidimensional Ornstein–Uhlenbeck Processes

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It is shown that the suitably normalized maximum likelihood estimators of some parameters of multidimensional Ornstein–Uhlenbeck processes with coefficient matrix of a special structure have exactly a normal distribution. This result provides a generalization to an arbitrary dimension of the well-known behavior of the estimator of the period of a complex AR(1) process. © 1996 Academic Press, Inc.

## 1. INTRODUCTION

Consider the complex-valued stationary autoregressive process  $\xi(t) = \xi_1(t) + i\xi_2(t)$ ,  $t \geq 0$ , given by the stochastic differential equation (SDE)

$$d\xi(t) = -\gamma\xi(t) dt + dw(t),$$

where  $w(t) = w_1(t) + iw_2(t)$ ,  $t \geq 0$ , is a standard complex Wiener process (i.e.,  $w_1(t)$  and  $w_2(t)$  are independent standard real-valued Wiener processes) and  $\gamma = \lambda - i\omega$  with  $\lambda > 0$ ,  $\omega \in \mathbb{R}$ .

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Consider the statistics

$$s_{\xi}^2(t) = \int_0^t |\xi(u)|^2 du, \quad r_{\xi}(t) = \int_0^t |\xi(u)|^2 d\theta(u),$$

where  $\theta(t)$ ,  $t \geq 0$ , is defined by

$$\xi(t) = |\xi(t)| e^{i\theta(t)}.$$

The process

$$r_{\xi}(t) = \int_0^t (\xi_1(u) d\xi_2(u) - \xi_2(u) d\xi_1(u)), \quad t \geq 0,$$

is called Lévy's stochastic area process. (It is interesting to remark that in case  $\gamma = 0$ , i.e.,  $\xi(t) = w(t)$ ,  $r_w(t) = \int_0^t (w_1(u) dw_2(u) - w_2(u) dw_1(u))$  the process  $(w_1(t), w_2(t), r_w(t))$ ,  $t \geq 0$ , is just the standard Wiener process on the Heisenberg group: see e.g. [6, 11].)

It is known that the maximum likelihood estimate (MLE) of the period  $\omega$  is

$$\hat{\omega}_{\xi}(t) = \frac{r_{\xi}(t)}{s_{\xi}^2(t)},$$

and

$$s_{\xi}(t)(\hat{\omega}_{\xi}(t) - \omega) \stackrel{\mathcal{D}}{=} \mathcal{N}(0, 1) \quad \text{for all } t > 0,$$

where  $\stackrel{\mathcal{D}}{=}$  denotes equality in distribution. Surprisingly, we have an exact distribution, not only an asymptotic property! This result was first formulated and applied in astronomy in Arató *et al.* [5]. Complicated proofs can be found in Novikov [10], Liptser and Shirayev [9], and Arató [1]–[3]. Recently, Arató [4] gave an elegant new proof using Novikov's method.

The statement can be reformulated also in the following way. Let us consider the two-dimensional real-valued stationary autoregressive process  $X(t)$ ,  $t \geq 0$ , given by the SDE

$$\begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \begin{pmatrix} -\lambda & -\omega \\ \omega & -\lambda \end{pmatrix} \begin{pmatrix} X_1(t) dt \\ X_2(t) dt \end{pmatrix} + \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}, \quad (1)$$

where  $W(t)$ ,  $t \geq 0$ , is a standard two-dimensional Wiener process, and  $\lambda > 0$ ,  $\omega \in \mathbb{R}$ . Consider the statistics

$$s_X^2(t) = \int_0^t (X_1^2(u) + X_2^2(u)) du, \quad r_X(t) = \int_0^t (X_1(u) dX_2(u) - X_2(u) dX_1(u)).$$

The maximum likelihood estimate of the period  $\omega$  is

$$\hat{\omega}_X(t) = \frac{r_X(t)}{s_X^2(t)}$$

and

$$s_X(t)(\hat{\omega}_X(t) - \omega) \stackrel{\mathcal{D}}{=} \mathcal{N}(0, 1) \quad \text{for all } t > 0.$$

The following natural question can be formulated. Consider the  $q$ -dimensional process  $X(t)$ ,  $t \geq 0$ , given by the SDE

$$dX(t) = AX(t) dt + dW(t), \quad t \geq 0,$$

where  $W(t)$ ,  $t \geq 0$ , is a standard  $q$ -dimensional Wiener process and  $A$  is a  $q \times q$  matrix. Which conditions should be imposed on the matrix  $A$  and on the distribution of the initial value  $X(0)$  in order that the suitably normalized MLE of some of its entries will have exactly a normal distribution?

Pap [12] and Fazekas [7] found some examples for stationary multi-dimensional Ornstein–Uhlenbeck processes which have the above property by applying the ideas in Arató [4]. Our aim is to generalize these results. In Section 2 we will study processes with nonrandom initial value. Section 3 is devoted to processes with random initial value. In Section 4 some special cases are given which cover the examples in Pap [12] and Fazekas [7].

Ornstein–Uhlenbeck processes can be considered as a generalization of Wiener processes. These processes are well studied in the literature and have been applied as a model for describing certain random phenomena, such as for instance the behavior of stock market prices.

It is clear that the present results can be used fruitfully in a statistical context such as in problems of testing hypotheses, estimating parameters, and also in constructing confidence regions for the unknown parameters.

## 2. PROCESSES WITH NONRANDOM INITIAL VALUE

Let  $X_x(t)$ ,  $t \geq 0$ , be the  $q$ -dimensional process given by the SDE

$$dX_x(t) = AX_x(t) dt + dW(t), \quad X_x(0) = x,$$

where  $W(t)$ ,  $t \geq 0$ , is a standard  $q$ -dimensional Wiener process,  $A$  is a  $q \times q$  matrix, and  $x \in \mathbb{R}^q$ . It is well known that  $X_x(t)$ ,  $t \geq 0$ , is a Gauss–Markov process and

$$X_x(t) = x + \int_0^t e^{(t-s)A} dW(s) = x + W(t) + A \int_0^t e^{(t-s)A} W(s) ds.$$

Let  $\mathbb{P}_{t, X_x}$  and  $\mathbb{P}_{t, W_x}$  be the measures generated on  $C^q[0, t]$  by the processes  $X_x(s)$ ,  $0 \leq s \leq t$ , and  $W_x(s) := x + W(s)$ ,  $0 \leq s \leq t$ , respectively. The measures  $\mathbb{P}_{t, X_x}$  and  $\mathbb{P}_{t, W_x}$  are equivalent and the Radon–Nikodym derivative has the form (see, e.g., Arató [2])

$$\frac{d\mathbb{P}_{t, X_x}}{d\mathbb{P}_{t, W_x}}(X) = \exp \left\{ \int_0^t \langle AX(u), dX(u) \rangle - \frac{1}{2} \int_0^t |AX(u)|^2 du \right\}, \quad (2)$$

where  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$  denote the inner product and the Euclidean norm in  $\mathbb{R}^q$ , respectively.

For the investigation of the distribution of functionals of integral type we shall use the following statement (cf. Gikhman and Skorokhod [8]). Let  $\psi: \mathbb{R}^q \rightarrow \mathbb{R}$  be a twice continuously differentiable function such that

$$|\psi(x)| + |\nabla \psi(x)| + |\nabla^2 \psi(x)| \leq c(1 + |x|)^2, \quad x \in \mathbb{R}^q,$$

with some  $c > 0$ . Then the function  $u(t, x)$ ,  $0 \leq t \leq T$ ,  $x \in \mathbb{R}^q$ , defined by

$$u(t, x) = \mathbb{E} \exp \left\{ i \int_0^t \psi(X_x(s)) ds \right\},$$

is the unique solution of the Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2} \Delta u + \langle Ax, \nabla u \rangle + i\psi(x)u, & 0 < t \leq T, \quad x \in \mathbb{R}^q \\ u(0, x) &= 1, & x \in \mathbb{R}^q, \end{aligned} \quad (3)$$

satisfying

$$|u(t, x)| \leq 1, \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^q.$$

For the computation of the expected value of a random variable the following simple formula holds. Let  $\xi$  and  $\eta$  be random variables with distributions  $\mathbb{P}_\xi$  and  $\mathbb{P}_\eta$  such that  $\mathbb{P}_\xi \ll \mathbb{P}_\eta$  ( $\mathbb{P}_\xi$  is absolutely continuous with respect to  $\mathbb{P}_\eta$ ). Let  $g: \mathbb{R} \rightarrow \mathbb{R}$  be a Borel-measurable function. Then

$$\mathbb{E}g(\xi) = \mathbb{E} \left( g(\eta) \frac{d\mathbb{P}_\xi}{d\mathbb{P}_\eta}(\eta) \right).$$

We shall make use of the conditional version of the above formula.

LEMMA 1. Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space and  $(\mathbb{X}, \mathcal{X}), (\mathbb{Y}, \mathcal{Y})$  be measurable spaces. Let  $\xi, \eta: \Omega \rightarrow \mathbb{X}$  be random elements with distributions  $\mathbb{P}_\xi$  and  $\mathbb{P}_\eta$  such that  $\mathbb{P}_\xi \ll \mathbb{P}_\eta$ . Let  $g: \mathbb{X} \rightarrow \mathbb{R}$  and  $h: \mathbb{X} \rightarrow \mathbb{Y}$  be measurable functions such that  $\mathbb{P}_{h(\eta)} \ll \mathbb{P}_{h(\xi)}$ . Then

$$\mathbb{E}(g(\xi) | h(\xi) = y) = \mathbb{E} \left( g(\eta) \frac{d\mathbb{P}_\xi}{d\mathbb{P}_\eta}(\eta) \middle| h(\eta) = y \right) \frac{d\mathbb{P}_{h(\eta)}}{d\mathbb{P}_{h(\xi)}}(y) \quad (\mathbb{P}_{h(\xi)}\text{-a.s.}).$$

*Proof.* We have to show that for any set  $B \in \mathcal{Y}$ ,

$$\begin{aligned} & \int_B \mathbb{E} \left( g(\eta) \frac{d\mathbb{P}_\xi}{d\mathbb{P}_\eta}(\eta) \middle| h(\eta) = y \right) \frac{d\mathbb{P}_{h(\eta)}}{d\mathbb{P}_{h(\xi)}}(y) \mathbb{P}_{h(\xi)}(dy) \\ &= \int_{\{\omega: h(\xi(\omega)) \in B\}} g(\xi(\omega)) \mathbb{P}(d\omega). \end{aligned}$$

The left-hand side is equal to

$$\begin{aligned} & \int_B \mathbb{E} \left( g(\eta) \frac{d\mathbb{P}_\xi}{d\mathbb{P}_\eta}(\eta) \middle| h(\eta) = y \right) \mathbb{P}_{h(\eta)}(dy) \\ &= \int_{\{\omega: h(\eta(\omega)) \in B\}} g(\eta(\omega)) \frac{d\mathbb{P}_\xi}{d\mathbb{P}_\eta}(\eta(\omega)) \mathbb{P}(d\omega) \\ &= \mathbb{E} \left( g(\eta) \chi_{h^{-1}(B)}(\eta) \frac{d\mathbb{P}_\xi}{d\mathbb{P}_\eta}(\eta) \right) = \mathbb{E}(g(\xi) \chi_{h^{-1}(B)}(\xi)) \\ &= \int_{\{\omega: h(\xi(\omega)) \in B\}} g(\xi(\omega)) \mathbb{P}(d\omega), \end{aligned}$$

where  $\chi_{h^{-1}(B)}$  denotes the indicator function of the set  $h^{-1}(B) = \{x \in \mathbb{X}: h(x) \in B\}$ . Hence the assertion. ■

Consider now the  $q$ -dimensional process  $X_x(t)$ ,  $t \geq 0$ , given by the SDE

$$dX_x(t) = \left( -\lambda I_q + \sum_{j=1}^m \omega_j C_j \right) X_x(t) dt + dW(t), \quad X_x(0) = x, \quad (4)$$

where  $I_q$  is the  $q \times q$  unit matrix,  $\lambda, \omega_1, \dots, \omega_m \in \mathbb{R}$  are unknown parameters, and  $C_1, \dots, C_m$  are fixed  $q \times q$  skew-symmetric matrices, i.e.,  $C_j' = -C_j$ ,  $j = 1, \dots, m$ . Using the formula (2), we can calculate the Radon–Nikodym derivative as

$$\begin{aligned} \frac{d\mathbb{P}_{t, X_x}}{d\mathbb{P}_{t, W_x}}(X) = \exp \left\{ -\lambda \int_0^t \langle X(s), dX(s) \rangle - \frac{1}{2} \lambda^2 \int_0^t |X(s)|^2 ds \right. \\ \left. + \sum_{j=1}^m \omega_j \int_0^t \langle C_j X(s), dX(s) \rangle \right. \\ \left. - \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m \omega_j \omega_k \int_0^t \langle C_j X(s), C_k X(s) \rangle ds \right\}, \end{aligned}$$

since skew symmetry of a matrix  $C$  implies  $\langle x, Cx \rangle = 0$  for  $x \in \mathbb{R}^q$ . Consequently the MLE of the damping parameter  $\lambda$  is

$$\hat{\lambda}_{X_x}(t) = - \frac{\int_0^t \langle X_x(s), dX_x(s) \rangle}{\int_0^t |X_x(s)|^2 ds} = - \frac{|X_x(t)|^2 - |x|^2 - qt}{2 \int_0^t |X_x(s)|^2 ds}.$$

The MLE of the parameter  $\omega = (\omega_1, \dots, \omega_m)'$  is

$$\hat{\omega}_{X_x}(t) = \Sigma_{X_x}^{-1}(t) \mathbf{r}_{X_x}(t),$$

where  $\Sigma_{X_x}(t)$  is the  $m \times m$  matrix

$$\Sigma_{X_x}(t) = \left( \int_0^t \langle C_j X_x(s), C_k X_x(s) \rangle ds \right)_{1 \leq j, k \leq m}$$

and  $\mathbf{r}_{X_x}(t)$  is the  $m$ -dimensional column vector

$$\mathbf{r}_{X_x}(t) = \left( \int_0^t \langle C_j X_x(s), dX_x(s) \rangle \right)'_{1 \leq j \leq m}.$$

We shall consider the following conditions:

- (C1)  $C'_i = -C_i$ ,  $i = 1, \dots, m$  (i.e.,  $C_1, \dots, C_m$  are skew-symmetric matrices),
- (C2)  $(C_i C_j + C_j C_i) C_k = C_k (C_i C_j + C_j C_i)$ ,  $i, j, k = 1, \dots, m$ ,
- (C3)  $(C_i C_j + C_j C_i) (C_k C_\ell + C_\ell C_k) \in \mathcal{L}(C_u C_v, 1 \leq u \leq v \leq m)$ ,  $i, j, k, \ell = 1, \dots, m$ , where  $\mathcal{L}(C_u C_v, 1 \leq u \leq v \leq m)$  denotes the linear hull of the matrices  $C_u C_v$ ,  $1 \leq u \leq v \leq m$ .

**THEOREM 1.** *Let  $X_x(t)$ ,  $t \geq 0$ , be the process given by (4). Let us suppose that the conditions (C1)–(C3) are satisfied. Then*

$$\Sigma_{X_x}^{1/2}(t)(\hat{\omega}_{X_x}(t) - \omega) \stackrel{\mathcal{D}}{=} \mathcal{N}(0, I_m) \quad \text{for all } t > 0.$$

*Proof.* First we investigate the distribution of the statistic  $\Sigma_{X_x}(t)$ .

LEMMA 2. *If the conditions (C1)–(C3) are satisfied then the distribution of the random matrix  $\Sigma_{X_x}(t)$  does not depend on the parameter  $\mathbf{\omega} = (\omega_1, \dots, \omega_m)'$  (it depends only on the parameter  $\lambda$ ).*

*Proof of Lemma 2.* First we examine the characteristic function

$$u(t, x) = \mathbb{E} \exp \left\{ i \sum_{1 \leq j \leq k \leq m} \alpha_{jk} \int_0^t \langle C_j X_x(s), C_k X_x(s) \rangle ds \right\}.$$

Applying (3) we obtain that for any  $T > 0$  the function  $u(t, x)$ ,  $0 \leq t \leq T$ ,  $x \in \mathbb{R}^q$ , is the unique solution of the Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + \left\langle \left( \lambda I + \sum_{j=1}^m \omega_j C_j \right) x, \nabla u \right\rangle + i \sum_{1 \leq j \leq k \leq m} \alpha_{jk} \langle C_j x, C_k x \rangle u, \\ 0 < t \leq T, \quad x \in \mathbb{R}^q, \end{aligned} \quad (5)$$

with  $u(0, x) = 1$  for  $x \in \mathbb{R}^q$ , satisfying  $|u(t, x)| \leq 1$  for  $0 \leq t \leq T$ ,  $x \in \mathbb{R}^q$ .

It is sufficient to show that there is a function  $v(t; y_{ij}, 1 \leq i \leq j \leq m)$ ,  $t \in [0, \infty)$ ,  $y_{ij} \in \mathbb{R}$ , such that

$$u(t, x) = v(t; \langle C_i x, C_j x \rangle, 1 \leq i \leq j \leq m),$$

where the function  $v$  does not depend on the parameter  $\mathbf{\omega} = (\omega_1, \dots, \omega_m)'$ .

Condition (C3) implies that there exist constants  $\beta_{ijkl}^{pq}$  such that

$$(C'_i C_j + C'_j C_i)' (C'_k C_l + C'_l C_k) = \sum_{1 \leq p \leq q \leq m} \beta_{ijkl}^{pq} C'_p C_q. \quad (6)$$

Let  $v$  be the unique solution of the Cauchy problem

$$\begin{aligned} \frac{\partial v}{\partial t} = \frac{1}{2} \sum_{1 \leq i \leq j \leq m} \sum_{1 \leq k \leq l \leq m} \sum_{1 \leq p \leq q \leq m} \beta_{ijkl}^{pq} y_{pq} \frac{\partial^2 v}{\partial y_{ij} \partial y_{kl}} \\ + \sum_{1 \leq i \leq j \leq m} (\text{Tr}(C'_i C_j) + 2\lambda y_{ij}) \frac{\partial v}{\partial y_{ij}} + i \sum_{1 \leq j \leq k \leq m} \alpha_{jk} y_{jk} v \end{aligned}$$

with  $v(0; y_{ij}, 1 \leq i \leq j \leq m) = 1$  for  $y_{ij} \in \mathbb{R}$ , satisfying  $|v(t; y_{ij})| \leq 1$  for  $0 \leq t \leq T$ ,  $y_{ij} \in \mathbb{R}$ . The function  $v$  clearly does not depend on the parameter  $\mathbf{\omega} = (\omega_1, \dots, \omega_m)'$ .

Consider the function  $\tilde{u}(t, x) = v(t; \langle C_i x, C_j x \rangle, 1 \leq i \leq j \leq m)$ . Using the formula

$$\frac{\partial \langle C_i x, C_j x \rangle}{\partial x_k} = \langle (C'_i C_j + C'_j C_i) x, e_k \rangle$$

we obtain

$$\nabla \tilde{u} = \sum_{1 \leq i \leq j \leq m} (C'_i C_j + C'_j C_i) x \frac{\partial v}{\partial y_{ij}},$$

hence conditions (C1) and (C2) imply

$$\left\langle \left( \lambda I + \sum_{i=1}^m \omega_i C_i \right) x, \nabla \tilde{u} \right\rangle = 2\lambda \sum_{1 \leq i \leq j \leq m} y_{ij} \frac{\partial v}{\partial y_{ij}}.$$

Applying

$$\begin{aligned} \sum_{k=1}^q \frac{\partial^2 \langle C_i x, C_j x \rangle}{\partial x_k^2} &= \sum_{k=1}^q \langle (C'_i C_j + C'_j C_i) e_k, e_k \rangle \\ &= \text{Tr}(C'_i C_j + C'_j C_i) = 2 \text{Tr}(C'_i C_j) \end{aligned}$$

and using (6) we get

$$\begin{aligned} \Delta \tilde{u} &= \sum_{1 \leq i \leq j \leq m} \sum_{1 \leq k \leq l \leq m} \sum_{1 \leq p \leq q \leq m} \beta_{ijkl}^{pq} y_{pq} \frac{\partial^2 v}{\partial y_{ij} \partial y_{kl}} \\ &\quad + 2 \sum_{1 \leq i \leq j \leq m} \text{Tr}(C'_i C_j) \frac{\partial v}{\partial y_{ij}}. \end{aligned}$$

Hence we can conclude that the function  $\tilde{u}$  is the unique solution of the Cauchy problem (5) satisfying  $|\tilde{u}(t, x)| \leq 1$  for  $0 \leq t \leq T$ ,  $x \in \mathbb{R}^q$ . Thus  $\tilde{u} = u$  and the assertion in Lemma 2 follows. ■

*Proof of Theorem 1.* Let  $\mathbf{b} = (b_1, \dots, b_m)' \in \mathbb{R}^m$ . Let us consider the  $q$ -dimensional process  $Y_x(t)$ ,  $t \geq 0$ , given by the SDE

$$dY_x(t) = \left( -\lambda I + \sum_{j=1}^m (\omega_j - b_j) C_j \right) Y_x(t) dt + dW(t), \quad Y_x(0) = x,$$

driven by the same Wiener process  $W(t)$ ,  $t \geq 0$ . Then the measures  $\mathbb{P}_{t, X_x}$  and  $\mathbb{P}_{t, Y_x}$  are equivalent and

$$\begin{aligned} \frac{d\mathbb{P}_{t, X_x}}{d\mathbb{P}_{t, Y_x}}(X) &= \exp \left\{ \sum_{j=1}^m b_j \int_0^t \langle C_j X(u), dX(u) \rangle \right. \\ &\quad \left. + \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m (b_j b_k - \omega_j b_k - \omega_k b_j) \int_0^t \langle C_j X(u), C_k X(u) \rangle du \right\} \\ &= \exp \left\{ \langle \mathbf{b}, \mathbf{r}_X(t) \rangle + \frac{1}{2} \langle \mathbf{b}, \Sigma_X(t) \mathbf{b} \rangle - \langle \mathbf{b}, \Sigma_X(t) \boldsymbol{\omega} \rangle \right\}. \end{aligned}$$



With the help of Lemmas 1 and 2 we can calculate the conditional moment generating function

$$\begin{aligned}
 & \mathbb{E}(\exp\{-\langle \mathbf{b}, \mathbf{r}_{X_x}(t) - \Sigma_{X_x}(t) \boldsymbol{\omega} \rangle\} \mid \Sigma_{X_x}(t) = \boldsymbol{\sigma}) \\
 &= \mathbb{E}\left(\exp\left\{\frac{1}{2}\langle \mathbf{b}, \Sigma_{Y_x}(t) \mathbf{b} \rangle\right\} \mid \Sigma_{Y_x}(t) = \boldsymbol{\sigma}\right) \frac{d\mathbb{P}_{\Sigma_{Y_x}(t)}(\boldsymbol{\sigma})}{d\mathbb{P}_{\Sigma_{X_x}(t)}(\boldsymbol{\sigma})} \\
 &= \exp\left\{\frac{1}{2}\langle \mathbf{b}, \boldsymbol{\sigma} \mathbf{b} \rangle\right\}.
 \end{aligned} \tag{7}$$

Consequently we have for all  $\mathbf{c} \in \mathbb{R}^m$

$$\begin{aligned}
 & \mathbb{E}(\exp\{-\langle \mathbf{c}, \Sigma_{X_x}^{1/2}(t)(\hat{\boldsymbol{\omega}}_{X_x}(t) - \boldsymbol{\omega}) \rangle\} \mid \Sigma_{X_x}(t) = \boldsymbol{\sigma}) \\
 &= \mathbb{E}(\exp\{-\langle \mathbf{c}, \Sigma_{X_x}^{-1/2}(t)(\mathbf{r}_{X_x}(t) - \Sigma_{X_x}(t) \boldsymbol{\omega}) \rangle\} \mid \Sigma_{X_x}(t) = \boldsymbol{\sigma}) \\
 &= \mathbb{E}(\exp\{(-\langle \boldsymbol{\sigma}^{-1/2} \mathbf{c}, \mathbf{r}_{X_x}(t) - \Sigma_{X_x}(t) \boldsymbol{\omega} \rangle) \mid \Sigma_{X_x}(t) = \boldsymbol{\sigma}) \\
 &= \exp\left\{\frac{1}{2}\langle \boldsymbol{\sigma}^{-1/2} \mathbf{c}, \boldsymbol{\sigma} \boldsymbol{\sigma}^{-1/2} \mathbf{c} \rangle\right\} = \exp\left\{\frac{1}{2}\langle \mathbf{c}, \mathbf{c} \rangle\right\},
 \end{aligned}$$

since we can apply (7) for  $\mathbf{b} = \boldsymbol{\sigma}^{-1/2} \mathbf{c}$ . Hence we obtain for the unconditional moment generating function

$$\mathbb{E} \exp\{-\langle \mathbf{c}, \Sigma_{X_x}^{1/2}(t)(\hat{\boldsymbol{\omega}}_{X_x}(t) - \boldsymbol{\omega}) \rangle\} = \exp\left\{\frac{1}{2}\langle \mathbf{c}, \mathbf{c} \rangle\right\}$$

for all  $\mathbf{c} \in \mathbb{R}^m$ , which proves the assertion. ■

### 3. PROCESSES WITH RANDOM INITIAL VALUE

Let  $\xi$  be an absolutely continuous random variable in  $\mathbb{R}^q$  with density function  $f: \mathbb{R}^q \rightarrow \mathbb{R}$ . Consider the  $q$ -dimensional process  $X(t)$ ,  $t \geq 0$ , given by the SDE

$$dX(t) = AX(t) dt + dW(t), \quad X(0) = \xi,$$

where  $A$  is a  $q \times q$  matrix. Let  $\mathbb{P}_{t, X}$  be the measure generated on  $C^q[0, t]$  by the process  $X(s)$ ,  $0 \leq s \leq t$ . We can consider the space  $C^q[0, t]$  as the product of  $\mathbb{R}$  and the space  $C_0^q[0, t]$  of continuous functions  $y: [0, t] \rightarrow \mathbb{R}^q$  such that  $y(0) = 0$  endowed with the supremum norm. Let us define the measure  $\mathbb{P}_{t, W}$  on  $C^q[0, t]$  as the product of the Lebesgue measure on  $\mathbb{R}$  and the measure generated by the process  $W(s)$ ,  $0 \leq s \leq t$ , on  $C_0^q[0, t]$ .

Then the measure  $\mathbb{P}_{t, X}$  is absolutely continuous with respect to  $\mathbb{P}_{t, W}$  and the Radon–Nikodym derivative has the form

$$\frac{d\mathbb{P}_{t, X}}{d\mathbb{P}_{t, W}}(X) = f(X(0)) \exp \left\{ \int_0^t \langle AX(u), dX(u) \rangle - \frac{1}{2} \int_0^t |AX(u)|^2 du \right\}. \quad (8)$$

Consider now the  $q$ -dimensional process  $X(t)$ ,  $t \geq 0$ , given by the SDE

$$dX(t) = \left( -\lambda I_q + \sum_{j=1}^m \omega_j C_j \right) X(t) dt + dW(t), \quad X(0) = \zeta, \quad (9)$$

where  $\lambda, \omega_1, \dots, \omega_m \in \mathbb{R}$  are unknown parameters and  $C_1, \dots, C_m$  are fixed  $q \times q$  skew-symmetric matrices. If the density function  $f$  does not depend on the parameter  $\boldsymbol{\omega} = (\omega_1, \dots, \omega_m)'$  then the MLE of  $\boldsymbol{\omega}$  has again the form

$$\hat{\boldsymbol{\omega}}_X(t) = \boldsymbol{\Sigma}_X^{-1}(t) \mathbf{r}_X(t).$$

Lemma 2 implies that under the conditions (C1)–(C3) the distribution of the random matrix  $\boldsymbol{\Sigma}_X(t)$  does not depend on the parameter  $\boldsymbol{\omega}$ , hence we obtain the following result:

**THEOREM 2.** *Let  $X(t)$ ,  $t \geq 0$ , be the process given by (9). Let us suppose that the density function  $f$  does not depend on the parameter  $\boldsymbol{\omega}$  and the conditions (C1)–(C3) are satisfied. Then*

$$\boldsymbol{\Sigma}_X^{1/2}(t)(\hat{\boldsymbol{\omega}}_X(t) - \boldsymbol{\omega}) \stackrel{\mathcal{D}}{=} \mathcal{N}(0, I_m) \quad \text{for all } t > 0. \quad (10)$$

**COROLLARY 1.** *Let  $\lambda > 0$ . Suppose that the conditions (C1)–(C3) are satisfied. Then the SDE*

$$dX(t) = \left( -\lambda I_q + \sum_{j=1}^m \omega_j C_j \right) X(t) dt + dW(t) \quad (11)$$

*has a stationary solution and (10) holds.*

*Proof.* First we show that an arbitrary eigenvalue  $\mu \in \mathbb{C}$  of the matrix

$$A = -\lambda I_q + \sum_{j=1}^m \omega_j C_j$$

has negative real part. Indeed, if  $x \in \mathbb{C}^q$  is an eigenvector corresponding to  $\mu$  then

$$\begin{aligned} \mu |x|^2 &= \langle \mu x, x \rangle = \langle Ax, x \rangle = \langle x, A'x \rangle = \left\langle x, \left( -\lambda I_q - \sum_{j=1}^m \omega_j C_j \right) x \right\rangle \\ &= \langle x, (-A - 2\lambda I_q) x \rangle = -\langle x, \mu x \rangle - 2\lambda |x|^2 = -\bar{\mu} |x|^2 - 2\lambda |x|^2 \end{aligned}$$

implies  $\operatorname{Re} \mu = -\lambda < 0$ . (Here  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$  denote the inner product and the usual norm in  $\mathbb{C}^q$ , respectively.)

Hence the SDE (11) has a stationary solution  $X(t)$ ,  $t \geq 0$ , which is a Gaussian process with

$$\mathbb{E}X(t) = 0, \quad \mathbb{E}X(s+t) X'(s) =: R(t) = e^{tA} R(0),$$

where  $R(0) = \mathbb{E}X(s) X'(s)$  is the unique solution of the matrix equation

$$AR(0) + R(0) A' = -I_q \quad (12)$$

(see Arató [2]). Obviously the solution of (12) is

$$R(0) = \frac{1}{2\lambda} I_q,$$

thus  $X(0) \stackrel{\mathcal{D}}{=} \mathcal{N}(0, (2\lambda)^{-1} I_q)$ , hence the conditions of Theorem 2 are satisfied. ■

#### 4. SPECIAL CASES

Some corollaries of the above general statements should be mentioned.

**COROLLARY 2.** *Consider the  $q$ -dimensional process  $X(t)$ ,  $t \geq 0$ , given by*

$$dX(t) = \left( -\lambda I_q + \sum_{i=1}^m \omega_i C_i \right) X(t) dt + dW(t),$$

*and either  $X(0) = x$  where  $x \in \mathbb{R}^q$ , or  $X(0) = \xi$  where  $\xi$  is an absolutely continuous random variable in  $\mathbb{R}^q$  with density function  $f: \mathbb{R}^q \rightarrow \mathbb{R}$  which does not depend on the parameter  $\omega = (\omega_1, \dots, \omega_m)'$ . Suppose that*

$$(C1') \quad C'_i = -C_i, \quad i = 1, \dots, m$$

$$(C2') \quad C_i C_j = -C_j C_i, \quad 1 \leq i < j \leq m$$

$$(C3') \quad C_i^2 C_j^2 \in \mathcal{L}(C_1^2, \dots, C_m^2), \quad 1 \leq i \leq j \leq m.$$

*Then the MLE of the parameters  $\omega_1, \dots, \omega_m$  are given by*

$$\hat{\omega}_X^{(i)}(t) = \frac{r_X^{(i)}(t)}{(s_X^{(i)}(t))^2},$$

where

$$r_X^{(i)}(t) = \int_0^t \langle C_i X(s), dX(s) \rangle, \quad (s_X^{(i)}(t))^2 = \int_0^t |C_i X(s)|^2 ds,$$

and

$$(s_X^{(1)}(t)(\hat{\omega}_X^{(1)}(t) - \omega_1), \dots, s_X^{(m)}(t)(\hat{\omega}_X^{(m)}(t) - \omega_m)) \stackrel{\mathcal{D}}{=} \mathcal{N}(0, I_m)$$

for all  $t > 0$ .

*Remark 1.* Condition (C2') can be replaced by the assumption that  $C_i C_j$  is skew-symmetric for all  $1 \leq i < j \leq m$  or  $\langle C_i x, C_j x \rangle = 0$  for all  $x \in \mathbb{R}^q$  and  $1 \leq i < j \leq m$  (which is some kind of orthogonality).

*Remark 2.* One can easily show that Corollary 2 applies to the models treated in Pap [12] and Fazekas [7].

**COROLLARY 3.** Consider the  $q$ -dimensional process  $X(t)$ ,  $t \geq 0$ , given by

$$dX(t) = (-\lambda I_q + \omega C) X(t) dt + dW(t),$$

and either  $X(0) = x$ , where  $x \in \mathbb{R}^q$ , or  $X(0) = \xi$ , where  $\xi$  is an absolutely continuous random variable in  $\mathbb{R}^q$  with density function  $f: \mathbb{R}^q \rightarrow \mathbb{R}$  which does not depend on the parameter  $\omega$ . Suppose that

$$(C1'') \quad C' = -C$$

$$(C2'') \quad C^4 \in \mathcal{L}(C^2) \text{ (or equivalently, } \exists \alpha \in \mathbb{R} \text{ such that } C^4 = \alpha C^2).$$

Then the MLE of the parameter  $\omega$  is

$$\hat{\omega}_X(t) = \frac{r_X(t)}{s_X^2(t)},$$

where

$$r_X(t) = \int_0^t \langle CX(s), dX(s) \rangle, \quad s_X^2(t) = \int_0^t |CX(s)|^2 ds,$$

and

$$s_X(t)(\hat{\omega}_X(t) - \omega) \stackrel{\mathcal{D}}{=} \mathcal{N}(0, 1) \quad \text{for all } t > 0.$$

*Remark 3.* Corollary 3 applies to the model (1), as well as to the three-dimensional AR process in Pap [12].

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