

Dependence and Order in Families of Archimedean Copulas

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The copula for a bivariate distribution function $H(x, y)$ with marginal distribution functions $F(x)$ and $G(y)$ is the function C defined by $H(x, y) = C(F(x), G(y))$. C is called Archimedean if $C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v))$, where φ is a convex decreasing continuous function on $(0, 1]$ with $\varphi(1) = 0$. A copula has lower tail dependence if $C(u, u)/u$ converges to a constant γ in $(0, 1]$ as $u \rightarrow 0^+$; and has upper tail dependence if $\bar{C}(u, u)/(1 - u)$ converges to a constant δ in $(0, 1]$ as $u \rightarrow 1^-$ where \bar{C} denotes the survival function corresponding to C . In this paper we develop methods for generating families of Archimedean copulas with arbitrary values of γ and δ , and present extensions to higher dimensions. We also investigate limiting cases and the concordance ordering of these families. In the process, we present answers to two open problems posed by Joe (1993, *J. Multivariate Anal.* **46** 262–282). © 1997 Academic Press

1. INTRODUCTION

The purpose of this paper is to present some methods for generating parametric families of bivariate distribution functions which are ordered by concordance and which possess a prescribed dependence structure known as tail dependence. The concept of tail dependence for bivariate distribution functions was introduced by Joe in [11], and is a property of the copula of the distribution, i.e., the function $C: [0, 1]^2 \rightarrow [0, 1]$ satisfying $H(x, y) = C(F(x), G(y))$, where H is the bivariate distribution function of two random variables X and Y with marginal distribution functions F and G , respectively. A copula is itself a bivariate distribution function with margins uniform on $[0, 1]$. For a general discussion of copulas and their properties, see [14, Chapter 6]. We will review the concepts of lower and upper tail dependence in copulas in Section 2.

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In [11], Joe studied tail dependence and other properties of multivariate mixture families of distributions derived using Laplace transforms (also see [13]). Joe concluded his paper with the following statement:

In conclusion, some difficult open problems are:

1. Are there families of absolutely continuous multivariate copulas that have (i) simple forms, (ii) all bivariate margins in the same family, (iii) a wider range of dependence structures than those given [by multivariate mixtures], and (iv) bivariate tail dependence?
2. Is there a simple family of bivariate copulas with both upper and lower tail dependence?
3. Are there general approaches other than the mixture or Laplace transform approach? Note that for the Laplace transform approach, there does not seem to be a way to tell from a family of [Laplace transforms] whether the resulting family of copulas will interpolate between independence and the Fréchet upper bound.

A type of copula known as Archimedean can be used to address these problems, so we will begin the next section with a review of Archimedean copulas and their properties. In Sections 3 and 4 we will illustrate methods for generating families of Archimedean copulas which do interpolate between independence and the Fréchet upper bound (responding to Joe's third problem); and in Section 5 we will present methods for generating families of Archimedean copulas to answer positively the second question. In Section 6 we extend these methods to the multivariate case, and present an example satisfying three of the four conditions in Joe's first problem.

2. PRELIMINARIES

Let Φ denote the set of functions $\varphi: [0, 1] \rightarrow [0, \infty]$ which are continuous, strictly decreasing, convex, and for which $\varphi(0) = \infty$ and $\varphi(1) = 0$. Each $\varphi \in \Phi$ has an inverse $\varphi^{-1}: [0, \infty] \rightarrow [0, 1]$ which has the same properties, except $\varphi^{-1}(0) = 1$ and $\varphi^{-1}(\infty) = 0$. Each member of Φ generates a copula C , that is, a bivariate distribution function with margins uniform on $[0, 1]$ given by

$$C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v)), \quad 0 \leq u, v \leq 1. \quad (2.1)$$

These copulas are called *Archimedean*, and their properties are discussed in [6] and [7]. We call φ a *generator* of C . The support of C is $(0, 1]^2$, and C is absolutely continuous on $(0, 1)^2$. An important subclass of Φ consists of those elements φ which have two continuous derivatives on $(0, 1)$, with $\varphi'(t) < 0$ and $\varphi''(t) > 0$ for $t \in (0, 1)$. We shall denote this subclass as $\Phi \cap C^2$. Finally, if φ is a generator of C , then so is $c\varphi$ for any positive constant c .

TABLE I

Name	Generator $\varphi_\theta(t)$	Parameter Range	Copula $C_\theta(u, v)$
Ali-Mikhail-Haq [2]	$\ln \frac{1-\theta+\theta t}{t}$	$-1 \leq \theta \leq 1$	$\frac{uv}{1-\theta(1-u)(1-v)}$ (2.2)
Cook-Johnson [3]	$t^{-\theta} - 1$	$\theta > 0$	$(u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$ (2.3)
Frank [5]	$-\ln \frac{e^{-\theta t} - 1}{e^{-\theta} - 1}$	$\theta \in \mathbf{R}$	$-\frac{1}{\theta} \ln \left[1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1} \right]$ (2.4)
Gumbel [9]	$(-\ln t)^\theta$	$\theta \geq 1$	$\exp\{-[(-\ln u)^\theta + (-\ln v)^\theta]^{1/\theta}\}$ (2.5)

Before proceeding we should note that, in the definition of Φ , it is not necessary for $\varphi(0)$ to be infinite for φ to generate a copula. When $\varphi(0)$ is finite, the Archimedean copula generated by φ is given by

$$C(u, v) = \varphi^{(-1)}(\varphi(u) + \varphi(v)), \quad 0 \leq u, v \leq 1,$$

where $\varphi^{(-1)}(t) = \varphi^{-1}(t)$ for $t \in [0, \varphi(0)]$ and $\varphi^{(-1)}(t) = 0$ for $t \in (\varphi(0), \infty)$. In this case, C is absolutely continuous if and only if $\varphi(0)/\varphi'(0) = 0$; otherwise C has a singular component concentrated on the curve $\varphi(u) + \varphi(v) = \varphi(0)$ in $[0, 1]^2$ [7, Theorem 1]. Since many of our results require that φ have an inverse on $(0, \infty)$, we will only consider generators for which $\varphi(0) = \infty$.

There are many families of Archimedean copulas with one-parameter families of generators. Among the best known are the four families in Table I, each of which appears in [6], [10], and [13].

The notion of tail dependence in bivariate copulas was defined by Joe in [11]. A copula has *lower tail dependence* if $C(u, u)/u$ converges to a constant γ in $(0, 1]$ as $u \rightarrow 0^+$. We refer to γ as the lower tail dependence parameter for C ; and if $C(u, u)/u$ converges to 0 as $u \rightarrow 0^+$, we will set $\gamma = 0$. There are a number of interpretations of γ . As Joe [11] notes, if U and V are uniform $(0, 1)$ random variables with joint distribution function C , then $\gamma = \lim_{u \rightarrow 0^+} \Pr\{U \leq u \mid V \leq u\}$. Since $C(u, u)$ is the distribution function for $\max\{U, V\}$, γ represents the limiting value of the density (if it exists) of $\max\{U, V\}$ as $u \rightarrow 0^+$. Geometrically, γ is the slope (if it exists) of the (one-sided) tangent line at the origin to the graph of $y = d(u)$, where $d(u) = C(u, u)$ is the *diagonal section* of the copula C . Finally, if C is an Archimedean copula generated by φ , then $C(u, u) = \varphi^{-1}(2\varphi(u))$ and

$$\gamma = \lim_{u \rightarrow 0^+} \frac{\varphi^{-1}(2\varphi(u))}{u} = \lim_{t \rightarrow \infty} \frac{\varphi^{-1}(2t)}{\varphi^{-1}(t)}. \quad (2.6)$$

Similarly, a copula has *upper tail dependence* if $\bar{C}(u, u)/(1 - u)$ converges to a constant δ in $(0, 1]$ as $u \rightarrow 1^-$, where \bar{C} is the bivariate survival function for C given by $\bar{C}(u, v) = 1 - u - v + C(u, v)$. We will refer to δ as the upper tail dependence parameter for C , and if $\bar{C}(u, u)/(1 - u)$ converges to 0 as $u \rightarrow 1^-$, we will set $\delta = 0$. With U and V as in the preceding paragraph, $\delta = \lim_{u \rightarrow 1^-} \Pr\{U > u \mid V > u\}$. Since $1 - \bar{C}(u, u)$ is the distribution function for $\min\{U, V\}$, δ represents the limiting value of the density (if it exists) of $\min\{U, V\}$ as $u \rightarrow 1^-$. Geometrically, $2 - \delta$ is the slope (if it exists) of the (one-sided) tangent line to $y = d(u)$ at $(1, 1)$. If C is an Archimedean copula generated by φ , then $\bar{C}(u, u) = 1 - 2 + \varphi^{-1}(2\varphi(u))$ and

$$\delta = 2 - \lim_{u \rightarrow 1^-} \frac{1 - \varphi^{-1}(2\varphi(u))}{1 - u} = 2 - \lim_{t \rightarrow 0^+} \frac{1 - \varphi^{-1}(2t)}{1 - \varphi^{-1}(t)}. \quad (2.7)$$

The notion of tail dependence is useful in the theory of extreme order statistics. As noted in [11], the condition $\delta = 0$ is equivalent to the asymptotic independence of X_{\max} and Y_{\max} , where (X_i, Y_i) is a sample from a distribution with copula C (when the marginal limiting extreme distributions of X_{\max} and Y_{\max} exist). Also note that upper tail dependence in the copula is equivalent to lower tail dependence in the survival copula.

3. GENERATORS AND TAIL DEPENDENCE

In this section we study the relationship between the form of a generator of an Archimedean copula and values of the tail dependence parameters γ and δ . However, we first present three properties of Φ which enable us to construct families of generators (and consequently families of Archimedean copulas) from a single generator φ in Φ . Their proofs are trivial.

PROPERTY 1. *Let $\varphi \in \Phi$, and let $\varphi_\beta(t) = [\varphi(t)]^\beta$. Then $\varphi_\beta(t) \in \Phi$ for all $\beta \geq 1$.*

PROPERTY 2. *Let $\varphi \in \Phi$, and let $\varphi_\alpha(t) = \varphi(t^\alpha)$. Then $\varphi_\alpha(t) \in \Phi$ for all $\alpha \in (0, 1]$.*

PROPERTY 3. *Let $\varphi \in \Phi \cap C^2$ such that $t\varphi'(t)$ is nondecreasing on $(0, 1)$, and let $\varphi_\alpha(t) = \varphi(t^\alpha)$. Then $\varphi_\alpha(t) \in \Phi \cap C^2$ for all $\alpha > 0$.*

In the sequel, for any $\varphi \in \Phi$, we will refer to a family of generators $\{\varphi_\beta(t) \in \Phi \mid \varphi_\beta(t) = [\varphi(t)]^\beta\}$ as a *beta family associated with φ* , and a

family $\{\varphi_\alpha(t) \in \Phi \mid \varphi_\alpha(t) = \varphi(t^\alpha)\}$ as an *alpha family associated with φ* . For example, the alpha family associated with $\varphi(t) = t^{-1} - 1$ with $\alpha > 0$ generates the Cook and Johnson family (2.3); and the beta family associated with $\varphi(t) = -\ln t$ with $\beta \geq 1$ generates the Gumbel family (2.5).

If the copula C generated by $\varphi \in \Phi$ has lower tail dependence parameter γ and upper tail dependence parameter δ , then, as we shall see in the next two theorems, it is easy to determine the tail dependence parameters for the copulas generated by the alpha and beta families associated with φ . But we must first assume that γ and δ exist for the copula C ; for it is easy to construct a $\varphi \in \Phi$ for which γ (or δ) does not exist. For example, let φ^{-1} be the piecewise linear function joining the points $\{(x_n, y_n) \mid n = 0, 1, \dots\}$ where $(x_0, y_0) = (0, 1)$ and $(x_n, y_n) = (2^{n-1}, 2^{-\lfloor 3n/2 \rfloor})$ for $n \geq 1$. Then φ^{-1} is strictly decreasing and convex with $\varphi^{-1}(0) = 1$ and $\varphi^{-1}(\infty) = 0$, so that $\varphi \in \Phi$ and generates an Archimedean copula. However

$$\frac{\varphi^{-1}(2^{2k})}{\varphi^{-1}(2^{2k-1})} = \frac{1}{2} \quad \text{and} \quad \frac{\varphi^{-1}(2^{2k+1})}{\varphi^{-1}(2^{2k})} = \frac{1}{4} \quad \text{for } k = 1, 2, \dots,$$

so that, using (2.6), $\gamma = \lim_{t \rightarrow \infty} \varphi^{-1}(2t)/\varphi^{-1}(t)$ does not exist. Indeed, using cubic splines, one can construct a C^2 function for φ^{-1} with the same properties.

THEOREM 1. *Let $\varphi \in \Phi$ generate the copula C with lower tail dependence parameter $\gamma \in [0, 1]$ and upper tail dependence parameter $\delta \in [0, 1]$. Let $\varphi_\alpha(t) = \varphi(t^\alpha)$, and further suppose that each φ_α generates a copula C_α . Then C_α has lower and upper tail dependence parameters γ_α and δ_α , respectively, given by $\gamma_\alpha = \gamma^{1/\alpha}$ and $\delta_\alpha = \delta$.*

Proof. Since $\varphi_\alpha^{-1}(t) = [\varphi^{-1}(t)]^{1/\alpha}$, (2.6) yields

$$\gamma_\alpha = \lim_{t \rightarrow 0^+} \frac{\varphi_\alpha^{-1}(2\varphi_\alpha(t))}{t} = \lim_{t \rightarrow 0^+} \frac{[\varphi^{-1}(2\varphi(t^\alpha))]^{1/\alpha}}{[t^\alpha]^{1/\alpha}} = \gamma^{1/\alpha}.$$

Similarly, using (2.7), we have

$$\begin{aligned} 2 - \delta_\alpha &= \lim_{t \rightarrow 1^-} \frac{1 - \varphi_\alpha^{-1}(2\varphi_\alpha(t))}{1 - t} \\ &= \lim_{t \rightarrow 1^-} \frac{1 - [\varphi^{-1}(2\varphi(t^\alpha))]^{1/\alpha}}{1 - t} \\ &= \lim_{t \rightarrow 1^-} \frac{1 - [\varphi^{-1}(2\varphi(t))]^{1/\alpha}}{1 - t^{1/\alpha}}. \end{aligned}$$

Hence

$$\begin{aligned}\frac{2 - \delta_\alpha}{2 - \delta} &= \lim_{t \rightarrow 1^-} \frac{1 - [\varphi^{-1}(2\varphi(t))]^{1/\alpha}}{1 - t^{1/\alpha}} \cdot \frac{1 - t}{1 - [\varphi^{-1}(2\varphi(t))]} \\ &= \lim_{u \rightarrow 1^-} \frac{1 - u^{1/\alpha}}{1 - u} \cdot \lim_{t \rightarrow 1^-} \frac{1 - t}{1 - t^{1/\alpha}} = 1.\end{aligned}$$

Thus $\delta_\alpha = \delta$ for $\alpha > 0$. ■

THEOREM 2. *Let $\varphi \in \Phi$ generate the copula C with lower tail dependence parameter $\gamma \in [0, 1]$ and upper tail dependence parameter $\delta \in [0, 1]$. Let $\varphi_\beta(t) = [\varphi(t)]^\beta$, $\beta \geq 1$. Then $\varphi_\beta(t)$ generates a copula C_β with lower tail dependence parameter $\gamma_\beta = \gamma^{1/\beta}$ and upper tail dependence parameter $\delta_\beta = 2 - (2 - \delta)^{1/\beta}$.*

Proof. Since $\varphi_\beta^{-1}(t) = \varphi^{-1}(t^{1/\beta})$, (2.6) yields

$$\gamma_\beta = \lim_{t \rightarrow 0^+} \frac{\varphi_\beta^{-1}(2\varphi_\beta(t))}{t} = \lim_{t \rightarrow 0^+} \frac{\varphi^{-1}(2^{1/\beta}\varphi(t))}{t} = \lim_{u \rightarrow \infty} \frac{\varphi^{-1}(2^{1/\beta}u)}{\varphi^{-1}(u)}.$$

Now let $h(x) = \lim_{u \rightarrow \infty} \varphi^{-1}(2^x u) / \varphi^{-1}(u)$ for $x \in [0, 1]$. Then

$$\begin{aligned}h(x+y) &= \lim_{u \rightarrow \infty} \frac{\varphi^{-1}(2^{x+y}u)}{\varphi^{-1}(u)} = \lim_{u \rightarrow \infty} \frac{\varphi^{-1}(2^x 2^y u)}{\varphi^{-1}(2^y u)} \cdot \frac{\varphi^{-1}(2^y u)}{\varphi^{-1}(u)} \\ &= \lim_{v \rightarrow \infty} \frac{\varphi^{-1}(2^x v)}{\varphi^{-1}(v)} \cdot \lim_{u \rightarrow \infty} \frac{\varphi^{-1}(2^y u)}{\varphi^{-1}(u)} = h(x) h(y).\end{aligned}$$

This is a variant of Cauchy's equation on the triangle $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq x+y \leq 1$, the general solution of which is $h(x) = e^{kx}$ for some k [1]. But $\gamma_{1/x} = h(x)$ for $x \in (0, 1]$, hence $\gamma = \gamma_1 = h(1) = e^k$ so that $h(x) = \gamma^x$. Thus $\gamma_\beta = h(1/\beta) = \gamma^{1/\beta}$, as required.

Similarly, using (2.7), we have

$$\begin{aligned}2 - \delta_\beta &= \lim_{t \rightarrow 1^-} \frac{1 - \varphi_\beta^{-1}(2\varphi_\beta(t))}{1 - t} = \lim_{t \rightarrow 1^-} \frac{1 - \varphi^{-1}(2^{1/\beta}\varphi(t))}{1 - t} \\ &= \lim_{u \rightarrow 0^+} \frac{1 - \varphi^{-1}(2^{1/\beta}u)}{1 - \varphi^{-1}(u)}.\end{aligned}$$

Now let $h(x) = \lim_{u \rightarrow 0^+} [1 - \varphi^{-1}(2^x u)] / [1 - \varphi^{-1}(u)]$ for $x \in [0, 1]$ and repeat the above argument to show that $h(x) = (2 - \delta)^x$. But $2 - \delta_{1/x} = h(x)$, so that $2 - \delta_\beta = h(1/\beta) = (2 - \delta)^{1/\beta}$. ■

4. LIMITING CASES AND ORDER

Let M , Π , and W denote the copulas of the Fréchet upper bound, independence, and the Fréchet lower bound, respectively; i.e., let $M(u, v) = \min\{u, v\}$, $\Pi(u, v) = uv$, and $W(u, v) = \max\{u + v - 1, 0\}$. In the next theorem, which follows directly from Propositions 4.2 and 4.3 in [6], we present sufficient conditions under which the family of copulas generated by an alpha family of generators includes Π as a limiting case; and sufficient conditions under which the family of copulas generated by a beta family of generators includes M as a limiting case.

THEOREM 3. (i) Let $\varphi \in \Phi \cap C^2$ such that $\varphi'(1) \neq 0$. Set $\varphi_\alpha(t) = \varphi(t^\alpha)$ for $\alpha \in (0, 1)$ and let C_α denote the copula generated by φ_α . Then $\lim_{\alpha \rightarrow 0^+} C_\alpha(u, v) = \Pi(u, v)$. (ii) Let $\varphi \in \Phi \cap C^2$, set $\varphi_\beta(t) = [\varphi(t)]^\beta$ for $\beta \geq 1$, and let C_β denote the copula generated by φ_β . Then $\lim_{\beta \rightarrow \infty} C_\beta(u, v) = M(u, v)$.

In passing we note that M , although a limiting case in a family of Archimedean copulas, is not itself an Archimedean copula [7, 14].

Theorem 3 provides an answer to Joe's third open problem. Let $\varphi \in \Phi \cap C^2$ and consider the two-parameter family of generators $\varphi_{\alpha, \beta}(t) = \varphi_\beta \circ \varphi_\alpha(t) = [\varphi(t^\alpha)]^\beta$ for $\alpha \in (0, 1)$ and $\beta \geq 1$. Then if $C_{\alpha, \beta}$ denotes the copula generated by $\varphi_{\alpha, \beta}$, it follows that $C_{0, 1} = \Pi$ (where $C_{0, \beta}(u, v) = \lim_{\alpha \rightarrow 0^+} C_{\alpha, \beta}(u, v)$) and $C_{\alpha, \infty} = M$. We present an explicit example in the next section.

Note that since Theorem 3 only concerns bivariate copulas, it is in a sense only a partial answer to Joe's third problem, and that one still does not know from the form of the generators of an *arbitrary* Archimedean family whether M and Π are limiting members of the family.

There does not seem to be a corresponding general result for W as a member or limiting case in families of copulas generated by either an alpha or beta family of generators.

A copula C_2 is *more concordant than* (or *more positively quadrant dependent than*) C_1 if $C_2(u, v) \geq C_1(u, v)$ for all $u, v \in [0, 1]$, in which case we write $C_2 \geq C_1$. The following two properties (which readily follow from Theorem 3.1 in [6]) show that all beta families of generators, and many alpha families, generate families of copulas which are ordered by concordance. In these cases, we have a natural interpretation of the parameter for the family.

PROPERTY 4. Let $\varphi \in \Phi$, set $\varphi_\beta(t) = [\varphi(t)]^\beta$, and let C_β denote the copula generated by φ_β for $\beta \geq 1$. Then $\beta_2 \geq \beta_1$ implies $C_{\beta_2} \geq C_{\beta_1}$.

PROPERTY 5. Let $\varphi \in \Phi$, and let $\varphi_\alpha(t) = \varphi(t^\alpha)$. Let $A \subseteq (0, \infty)$ such that if $\alpha \in A$ then (i) $\varphi_\alpha \in \Phi$ and (ii) $\varphi([\varphi^{-1}(t)]^\theta)$ is subadditive for all $\theta \in (0, 1)$. Then for $\alpha_1, \alpha_2 \in A$, $\alpha_2 \geq \alpha_1$ implies $C_{\alpha_2} \geq C_{\alpha_1}$.

An alternative form for condition (ii) in Property 5 above is: $\varphi'_\alpha/\varphi_\alpha$ is nonincreasing in α .

However, not all alpha families of generators generate families of copulas ordered by concordance. For example, if $\varphi(t) = \ln((2/t) - 1)$, then no two members of the family of copulas generated by $\varphi_\alpha(t) = \ln((2/t^\alpha) - 1)$ for $\alpha \in (0, 1]$ are comparable; for it is easy to show that $C_\alpha(u, v) \geq C_{\alpha/2}(u, v)$ if and only if $u^{\alpha/2} + v^{\alpha/2} \leq 1$.

5. AN EXAMPLE

As an application, consider the generator $\varphi(t) = t^{-1} - 1$. This function generates the copula $C(u, v) = uv/(u + v - uv)$, which is a member of several well-known families of copulas, including those of Ali, Mikhail, and Haq (2.2) and Cook and Johnson (2.3). Using the properties and theorems in the preceding sections, we can construct a two-parameter family of copulas with generators given by $\varphi_{\alpha, \beta}(t) = \varphi_\beta \circ \varphi_\alpha(t) = (t^{-\alpha} - 1)^\beta$, $\alpha \geq 0$, $\beta \geq 1$. Explicitly, we have

$$C_{\alpha, \beta}(u, v) = \{[(u^{-\alpha} - 1)^\beta + (v^{-\alpha} - 1)^\beta]^{1/\beta} + 1\}^{-1/\alpha}, \quad u, v \in [0, 1].$$

The tail dependence parameters for $C(u, v) = uv/(u + v - uv)$ are $\gamma = 1/2$ and $\delta = 0$; and so, by Theorems 1 and 2, the tail parameters for the copulas $C_{\alpha, \beta}$ generated by $\varphi_{\alpha, \beta}$ are given by $\gamma_{\alpha, \beta} = 2^{-1/(\alpha\beta)}$ and $\delta_{\alpha, \beta} = 2 - 2^{1/\beta}$. This pair of equations is invertible, thus to find a copula with a predetermined lower tail dependence parameter γ and upper tail dependence parameter δ , set $\alpha = -\ln(2 - \delta)/\ln \gamma$ and $\beta = \ln 2/\ln(2 - \delta)$. The one-parameter subfamily $C_{0, \beta}$ is the Gumbel family (2.5), while the one-parameter subfamily $C_{\alpha, 1}$ is the Cook and Johnson family (2.3). This family also appears in [12] as the family of survival copulas for a "random hazards" bivariate Weibull model.

As noted earlier, this family of copulas interpolates between Π and M , since $C_{0, 1} = \Pi$ and $C_{\alpha, \infty} = M$. From Properties 4 and 5 in the preceding section, it is easy to show that this family of copulas is concordance ordered by both α and β ; i.e., if $\alpha_2 \geq \alpha_1$ and $\beta_2 \geq \beta_1$; then $C_{\alpha_2, \beta_2} \geq C_{\alpha_1, \beta_1}$. Hence every member of this family is positively quadrant dependent, since $C_{0, 1} = \Pi$. Furthermore, the family is also concordance ordered by the tail parameters γ and δ , since $\alpha_2 \geq \alpha_1$ and $\beta_2 \geq \beta_1$ imply that the corresponding values of γ and δ satisfy $\gamma_2 \geq \gamma_1$ and $\delta_2 \geq \delta_1$, and conversely.

Thus this family provides a positive answer to the second of Joe's open problems [11] from Section 1: Is there a simple family of bivariate copulas

with both upper and lower tail dependence? Indeed, there are one-parameter subfamilies which accomplish the same end—simply set $\beta = \alpha + 1$, $\alpha \in [0, \infty)$; or $\beta = 1/(1 - \alpha)$, $\alpha \in [0, 1)$, for example.

Of course, other families can be readily constructed using other generators in Φ . Examples include $\varphi(t) = \ln(1 - \ln t)$, $\varphi(t) = 1/t - t$, $\varphi(t) = \exp\{(1/t) - 1\} - 1$, and so on. Many such generators generate copulas for which both γ and δ are zero; but note that even in this event, Theorem 2 guarantees that members of the family of copulas generated by a beta family of generators will have positive upper tail dependence.

6. MULTIVARIATE EXTENSIONS

The natural extension of (2.1) to n dimensions is

$$C^n(u_1, u_2, \dots, u_n) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2) + \dots + \varphi(u_n)),$$

$$0 \leq u_1, u_2, \dots, u_n \leq 1. \quad (6.1)$$

[In this section, an integer superscript such as n on C , γ , or δ will denote dimension rather than exponentiation.] Of course, C^n may not be an n -dimensional copula (briefly, n -copula), i.e., an n -dimensional distribution function whose univariate margins are uniform on $(0, 1)$. What properties of φ (or φ^{-1}) will insure that C^n given by (6.1) is an n -copula? The answer is given in the following theorem.

THEOREM 4 [14, Theorem 6.3.6]. *Let $\varphi \in \Phi$. Then C^n given by (6.1) is an n -copula for all $n \geq 2$ if and only if φ^{-1} is completely monotonic in $(0, \infty)$, i.e., φ^{-1} is real-analytic and satisfies*

$$(-1)^k \frac{d^k}{dt^k} \varphi^{-1}(t) \geq 0$$

for all $t \in (0, \infty)$ and $k = 0, 1, 2, \dots$

This theorem can be easily extended to cover the cases in which φ^{-1} is m -monotonic in $(0, \infty)$, i.e., the cases in which only the first m derivatives of φ^{-1} alternate in sign. In such cases, C^n given by (6.1) is an n -copula for $2 \leq n \leq m$.

Note that the functions C^n in (6.1) are the *serial iterates* [14] of C ; that is, if we set $C^2(u_1, u_2) = C(u_1, u_2) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2))$, then $C^n(u_1, u_2, \dots, u_n) = C(C^{n-1}(u_1, u_2, \dots, u_{n-1}), u_n)$ for $n \geq 3$. However, we must mention that this technique for creating multivariate distribution functions—endowing a bivariate copula with multidimensional margins—generally fails [8]. For

example, $W(W(u_1, u_2), u_3) = \max\{u_1 + u_2 + u_3 - 2, 0\}$ is not a distribution function.

We are now in a position to demonstrate that the 2-copulas generated by beta families of generators in Section 3 extend, via (6.1), to n dimensions. The following theorem, which is an immediate consequence of Criterion 2 in [4, Section XIII.4], accomplishes this.

THEOREM 5. *Let $\varphi \in \Phi$ such that φ^{-1} is completely monotonic in $(0, \infty)$. Let $\varphi_\beta(t) = [\varphi(t)]^\beta$ for $\beta \geq 1$. Then φ_β^{-1} is completely monotonic in $(0, \infty)$.*

Again consider the two-parameter family of generators $\varphi_{\alpha, \beta}(t) = \varphi_\beta \circ \varphi_\alpha(t) = [\varphi(t^\alpha)]^\beta$ for $\alpha > 0$ and $\beta \geq 1$. If φ_α^{-1} is completely monotonic in $(0, \infty)$, then Theorem 5 guarantees that $\varphi_{\alpha, \beta}^{-1}$ is completely monotonic in $(0, \infty)$, and hence, by Theorem 4, $\varphi_{\alpha, \beta}$ generates an n -copula of the form given by (6.1). For example, if we again set $\varphi(t) = t^{-1} - 1$, it is easy to show that for $\alpha > 0$, $\varphi_\alpha^{-1}(t) = (1+t)^{-1/\alpha}$ is completely monotonic in $(0, \infty)$, and thus $\varphi_{\alpha, \beta}(t) = (t^{-\alpha} - 1)^\beta$ generates a two-parameter family of n -copulas

$$C_{\alpha, \beta}^n(u_1, u_2, \dots, u_n) = \left\{ \left[\sum_{i=1}^n (u_i^{-\alpha} - 1)^\beta \right]^{1/\beta} + 1 \right\}^{-1/\alpha},$$

$$0 \leq u_1, u_2, \dots, u_n \leq 1, \quad (6.2)$$

for $\alpha > 0$, $\beta \geq 1$, and for each $n \geq 2$.

The definitions of upper and lower tail dependence extend to n dimensions. Paralleling the definitions in Section 2, we will say that an n -copula C^n has lower tail dependence parameter γ^n if $C^n(u, u, \dots, u)/u$ converges to a constant γ^n in $[0, 1]$ as $u \rightarrow 0^+$; and has upper tail dependence parameter δ^n if $\bar{C}^n(u, u, \dots, u)/(1-u)$ converges to a constant δ^n in $[0, 1]$ as $u \rightarrow 1^-$. Now let $\varphi \in \Phi$ such that φ^{-1} is completely monotonic in $(0, \infty)$, and consider the n -copulas C_β^n generated by $\varphi_\beta(t) = [\varphi(t)]^\beta$ for $\beta \geq 1$, and the n -copulas C_α^n generated by $\varphi_\alpha(t) = \varphi(t^\alpha)$ for $\alpha > 0$ (here we must assume that φ_α^{-1} is completely monotonic in $(0, \infty)$ for $\alpha > 0$). Let γ_α^n and γ_β^n denote the lower tail dependence parameters and let δ_α^n and δ_β^n denote the upper tail dependence parameters for C_α^n and C_β^n , respectively. Let γ^n and δ^n now denote the lower and upper tail dependence parameters, respectively, for the n -copula C^n generated by φ ; i.e., the n -copula given by (6.1). Using methods similar to those in the proofs of Theorems 1 and 2, it is easy to show that $\gamma_\alpha^n = (\gamma^n)^{1/\alpha}$, $\delta_\alpha^n = \delta^n$, $\gamma_\beta^n = (\gamma^n)^{1/\beta}$, and

$$\delta_\beta^n = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \left[\sum_{j=1}^k (-1)^{j+1} \binom{k}{j} \delta^j \right]^{1/\beta} \quad (6.3)$$

(where we take $\delta^1 \equiv 1$).

As a final example, let $\gamma_{\alpha, \beta}^n$ and $\delta_{\alpha, \beta}^n$ denote the lower and upper tail dependence parameters, respectively, for the two parameter family of n -copulas given by (6.2). Then $\gamma_{\alpha, \beta}^n = (\gamma^n)^{1/(\alpha\beta)}$ and $\delta_{\alpha, \beta}^n = \delta_{\beta}^n$ given by (6.3). When $\varphi(t) = t^{-1} - 1$, the k -copulas (for $k = 2, 3, \dots, n$) generated by φ have lower tail dependence parameters $\gamma^k = 1/k$ for $k = 2, 3, \dots, n$; and upper tail dependence parameters $\delta^k = 0$ for $k = 2, 3, \dots, n$. Hence the two-parameter family of n -copulas in (6.2), which have generators $\varphi_{\alpha, \beta}(t) = (t^{-\alpha} - 1)^{\beta}$, have lower tail dependence parameters $\gamma_{\alpha, \beta}^n = n^{-1/(\alpha\beta)}$ and upper tail dependence parameters $\delta_{\alpha, \beta}^n = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} k^{1/\beta}$ (using the notation of finite differences, this last result can be expressed more concisely as $\delta_{\alpha, \beta}^n = (-1)^{n+1} \Delta^n 0^{1/\beta}$). Although this family does not provide a positive answer to the first open problem in [11] (since its dependence structure is limited by the fact that it is a family with exchangeable or symmetric dependence), it is a family of absolutely continuous multivariate copulas with a simple form, all bivariate margins in the same family (indeed, here all k -variate margins are in the same family, $k = 2, 3, \dots, n-1$), and bivariate tail dependence (these in fact have k -variate lower and upper tail dependence).

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