Journal of Multivariate Analysis 119 (2013) 144–162

Contents lists available at SciVerse ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Asymptotic cumulants of ability estimators using fallible item parameters

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ARTICLE INFO

Article history:
Received 28 October 2012
Available online 27 April 2013

AMS subject classifications:
62E20
62F15
62G20

Keywords:
Item response theory
Asymptotic cumulants
Bayes modal
Pseudo maximum likelihood
Higher-order asymptotic variance
Model misspecification

ABSTRACT

The asymptotic cumulants of ability estimators using fallible or estimated item parameters in an ability test based on item response theory are given up to the fourth order with higher-order asymptotic variance. The ability estimators cover those obtained by maximum likelihood, Bayes, and pseudo Bayes modal estimation. For estimation of item parameters, the marginal maximum likelihood and Bayes methods are used. Asymptotic cumulants with higher-order asymptotic variance are given with and without model misspecification, and before and after studentization. Three conditions for the relative size of the number of items for ability estimation to that of examinees for item parameter calibration are presented; two of them give some justification for neglecting sampling variation of estimated item parameters. Numerical illustration with simulations is shown using the two-parameter logistic model.

1. Introduction

When the proficiency level or ability of an examinee is estimated using test items based on item response theory (IRT), and especially when the item parameters concerned have been estimated separately from the examinee, it is a common practice to regard the item parameter estimates as fixed ones. This is partially due to the intractableness of considering their stochastic properties in this situation. When the item parameters have been estimated based on a large sample for item calibration, the practice is quite reasonable, since the item parameter estimates can be seen, in a practical sense, as population values.

When population or fixed item parameters are available, the estimation of ability is carried out in various ways. Maximum likelihood (ML) estimation [18], [6, Section 20.3] assuming a correctly specified IRT model is a standard one. The Bayes methods of estimation have also been developed with various advantages over ML estimation. The maximum a posteriori (MAP) estimator, also known as the Bayes modal (BM) estimator [38, Chapter 2], [7], and the expected a posteriori (EAP) estimator, also known as the posterior mean [4,5], are familiar ones. For an informative prior distribution, the standard normal is typically used [21]. In this paper, the term BM estimator (BME) refers to the estimator using the standard normal prior unless otherwise specified. On the other hand, non-informative prior distributions, e.g., the Jeffreys [14], [15, Section 3.10] prior, can also be used.

Warm [51] coined a weighted likelihood (WL) method, which is seen as a weighted score and consequently a pseudo Bayes method. The WL method was developed to remove the asymptotic bias of the ability estimator by ML. It is known that, in the case of the two-parameter logistic model (2PLM), the WL estimator (WLE) is equal to the Jeffreys Bayes modal

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0047-259X/$ – see front matter
http://dx.doi.org/10.1016/j.jmva.2013.04.008

estimator (JME). The asymptotic unbiasedness of the WLE when the 2PLM holds stems from the general result that the JME based on a distribution in the exponential family with canonical parametrization has no asymptotic bias [10].

Lord [20] and Lord [21] gave the asymptotic biases of the ML estimator (MLE) and the BME of ability under correct model specification (c.m.s.) in the three-parameter logistic model (3PLM), respectively. Ogasawara [28] gave the asymptotic cumulants of the MLE in a general model like the four-parameter logistic model (4PLM) up to the fourth order with higher-order asymptotic variance under possible model misspecification (p.m.m.) with and without studentization of the MLE. Ogasawara [30] dealt with the ML, BM, WL and JM estimators as special cases of the (pseudo) Bayes estimator with general weight, where the weight is null for the MLE, and derived results corresponding to those of the MLE in [28].

The above asymptotic results are derived with the assumption of known item parameters and known probabilities of correct responses to items when the IRT model fitted is misspecified. In this paper, the asymptotic properties corresponding to Ogasawara [30] will be derived under the condition that the item parameters have been estimated by using a sample of examinees for item calibration independent of the examinee whose ability is to be estimated.

As there are several estimators of ability, there are several ways to estimate item parameters including the Bayes method (see [3,42,11]). Joint maximum likelihood (JML) estimation gives simultaneously estimates of abilities and item parameters [22,19]. Since in order to see the asymptotic properties of the whole set of estimators by JML a special condition of both the numbers of examinees and items being large is required [46], the JML method is not dealt with in this paper. On the other hand, standard methods of estimation are marginal maximum likelihood (MML) [7,8] and its Bayes versions. Bayesian estimation of item parameters has been developed by Swaminathan and Gifford [43–45], Mislevy and Bock [24], Mislevy [23], Shigemasu and Fujimori [39], Tsutakawa and Lin [49], Tsutakawa [47], and Zeng [54], among others, for the one-parameter logistic model (1PLM), 2PLM, and 3PLM. Ogasawara [32] gave the asymptotic properties of the MML estimators (MMLEs) and marginal BMEs (MBMEs) of item parameters corresponding to the properties given by Ogasawara [30] using a general weight similar to that of Ogasawara [30]. In this paper, the MMLEs and MBMEs of the item parameters are assumed to be used for item calibration.

In Appendix A.1.1 of the Appendix, a review of the works on the ability estimators using fallible or estimated item parameters will be given. In the following sections, the asymptotic expansions of the ability estimators dealt with by Ogasawara [30] will be derived, where three conditions of the asymptotic relative sizes between the samples in item calibration and ability estimation are presented. Under some conditions, the asymptotic justification of neglecting sampling variability of item parameter estimators will be shown. Numerical illustration will also be given using the 2PLM. Technical details not presented here are available in the supplements to this paper [33,34].

2. Stochastic expansion of the ability estimator using fallible item parameters

2.1. Orders of the relative sample sizes

Let θ be the fixed ability parameter in an IRT model, e.g., the 3PLM and the 4PLM. Denote its estimator and true (population) value by $\hat{\theta}$ and θ_0 with $-\infty < \theta_0 < \infty$, respectively, where $\hat{\theta}$ is the generic estimator given by the ML or (pseudo) Bayes method based on responses by an examinee to n dichotomously scored items. When the item parameters in the IRT model are unknown, as is usual in practice, they are assumed to be estimated by a separate sample of size N for item calibration. As is addressed in Appendix A.1.1 of the Appendix, N is usually (much) larger than n, since large sample sizes are required for accurate estimation in item calibration. So, the assumptions $O(n^{3/2}/N) = O(1)$ and $O(n^2/N) = O(1)$ will be used as well as $O(n/N) = O(1)$. The values of n in ability tests based on IRT are in many cases less than 100, and N is typically required to be as large as 1000.

The following (approximate) functional values may be helpful to see the reasonableness of the above assumptions.

n :	30	50	70	100	200	300
$N(n^{3/2})$:	164	354	586	1000	2828	5196
$N(n^2)$:	900	2500	4900	10,000	40,000	90,000
$n(N^{1/2})$:	14	22	26	32	45	55
$n(N^{2/3})$:	34	63	79	100	159	208
N :	200	500	700	1000	2000	3000

In the above cases, for example, $n = 30$ with $N(n^2) = 900$ and $n(N^{2/3}) = 100$ with $N = 1000$ may be typical values in practice, though it is to be noted that in asymptotics, for example, $O(n^2/N) = O(1)$ is a limiting property. Further, assume for comparison that $O(n^{5/2}/N) = O(1)$. When $n = 30$, $N(n^{5/2})$ becomes 4930, which is seldom satisfied in practice.

2.2. Stochastic expansion of $\hat{\theta}$

Let $\boldsymbol{\alpha}$ be the q-dimensional vector of item parameters, with $\hat{\boldsymbol{\alpha}}$ and $\boldsymbol{\alpha}_0$ being its sample and population counterparts, respectively. In this paper, $\hat{\boldsymbol{\alpha}}$ is assumed to be obtained by MML or MBM using a sample of size N for item calibration that maximizes

$$\frac{N!}{\prod_{j=1}^{2^n} r_j!}\left[\prod_{j=1}^{2^n}\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n}\Psi_k^{X_{jk}}(1-\Psi_k)^{1-X_{jk}}\right\}^{r_j}\varphi(\theta)d\theta\right]p_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}),$$
$$\Psi_k=\Psi(\theta,\boldsymbol{\alpha}_{(k)})=c_k+\frac{1-c_k}{1+\exp\{-1.7a_k(\theta-b_k)\}},$$
$$\boldsymbol{\alpha}_{(k)}=(a_k,b_k,c_k)'\quad(k=1,\ldots,n),\qquad \boldsymbol{\alpha}=(\boldsymbol{\alpha}'_{(1)},\ldots,\boldsymbol{\alpha}'_{(n)})',\tag{2.1}$$

where 2^n is the number of response patterns for n dichotomously scored items; r_j is the number of examinees having the jth response pattern; Ψ_k is the probability of a correct response to the kth item by an examinee with ability θ in the case of the 3PLM; non-stochastic X_{jk} is 1 (correct response) and 0 (incorrect response) for the kth item in the jth response pattern; and $p_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})$ is the prior density of $\boldsymbol{\alpha}$ for MBM, which is 1 for MML. Actually, the integral in (2.1) is numerically obtained by Gaussian quadrature. In the case of the 2PLM, c_k in Ψ_k is 0 and $\boldsymbol{\alpha}_{(k)}=(a_k,b_k)'$ $(k=1,\ldots,n)$.

In the second stage, $N=1$, $\boldsymbol{\alpha}$ is regarded as a known fixed vector equal to $\hat{\boldsymbol{\alpha}}$, θ is regarded as an unknown fixed parameter, and $\hat{\theta}$ is obtained by maximizing the weighted likelihood as follows. Let $\bar{l}_{W\theta}$ be the mean of the weighted log-likelihood:

$$\bar{l}_{W\theta}=n^{-1}\{\log L+\log p(\theta)\}=\bar{l}_\theta+n^{-1}\log p(\theta)\quad\text{with } L=\prod_{k=1}^{n}P_k^{U_k}Q_k^{1-U_k},\tag{2.2}$$

where U_k is a dichotomous variable taking 0 and 1 for incorrect and correct responses to the kth item by an examinee with ability θ, and $P_k=P_k(\theta)=\Pr(U_k=1|\theta,\boldsymbol{\alpha}_{(k)})$ $(k=1,\ldots,n)$ is the probability of the correct response under c.m.s. with $Q_k\equiv 1-P_k$ $(k=1,\ldots,n)$. When the model is misspecified, the true probability is denoted by

$$P_{\mathrm{T}k}\equiv \mathrm{E}_{\mathrm{T}\theta}(U_k|\theta)\quad\text{with } Q_{\mathrm{T}k}\equiv 1-P_{\mathrm{T}k}\ (k=1,\ldots,n),\tag{2.3}$$

where $\mathrm{E}_{\mathrm{T}\theta}(\cdot)$ indicates that the expectation of U_k is taken using the true distribution of U_k given θ, which can be independent of θ. When $\theta=\theta_0$, the notation $\mathrm{E}_{\mathrm{T}\theta_0}(\cdot)\equiv\mathrm{E}_{\mathrm{T}\theta}(\cdot|\theta_0)$ is also used.

The quantity $p(\theta)$ in (2.2) is the prior density of θ when the Bayes method is used. In the case of WL estimation, $p(\theta)$ is not necessarily available in explicit form. However, $\eta_\theta\equiv\partial\log p(\theta)/\partial\theta$ or its counterpart is given with $\eta_{\theta_0}\equiv\partial\log p(\theta)/\partial\theta|_{\theta=\theta_0}$. That is, for ML, BM, WL, and JM estimation, $\eta_\theta=0,\ -1,\ \bar{j}/(2\bar{i})$, and $\bar{i}^{(\mathrm{D1})}/(2\bar{i})$, respectively, where $\bar{i}$ is the Fisher information averaged over items $\bar{i}=n^{-1}\sum_{k=1}^{n}\left(\frac{\partial P_k}{\partial\theta}\right)^2\frac{1}{P_kQ_k}=O(1)$, $\bar{j}=n^{-1}\sum_{k=1}^{n}\frac{\partial P_k}{\partial\theta}\frac{\partial^2P_k}{\partial\theta^2}\frac{1}{P_kQ_k}$, and $\bar{i}^{(\mathrm{D1})}\equiv\frac{\partial\bar{i}}{\partial\theta}$.

When $\boldsymbol{\alpha}_0$ is known, Ogasawara [30, Eq. (3.2), Appendix A.1] gave the following stochastic expansion under p.m.m.:

$$\hat{\theta}-\theta_0=\sum_{k=1}^{3}\boldsymbol{\gamma}_{\theta_0}^{(k)\prime}\mathbf{l}_{\theta_0}^{(k)}-n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0}+O_p(n^{-2}),\tag{2.4}$$

where $\boldsymbol{\gamma}_{\theta_0}^{(k)}=O(1)$, $\mathbf{l}_{\theta_0}^{(k)}=O_p(n^{-k/2})$ $(k=1,2,3)$,

$$\begin{aligned}
\boldsymbol{\gamma}_{\theta_0}^{(1)\prime}\mathbf{l}_{\theta_0}^{(1)}&=\gamma_{\theta_0}^{(1)}l_{\theta_0}^{(1)}=-\lambda_{\theta_0}^{-1}\partial\,\bar{l}_{\theta_0}/\partial\theta_0,\\
\boldsymbol{\gamma}_{\theta_0}^{(2)\prime}\mathbf{l}_{\theta_0}^{(2)}&=\left\{\lambda_{\theta_0}^{-2},-\frac{\lambda_{\theta_0}^{-3}}{2}\mathrm{E}_{\mathrm{T}\theta_0}(j_0^{(3)})\right\}\left\{m\frac{\partial\bar{l}_{\theta_0}}{\partial\theta_0},\left(\frac{\partial\bar{l}_{\theta_0}}{\partial\theta_0}\right)^2\right\}',\\
\boldsymbol{\gamma}_{\theta_0}^{(3)\prime}\mathbf{l}_{\theta_0}^{(3)}&=\Bigg[-\lambda_{\theta_0}^{-3},\ \frac{3}{2}\lambda_{\theta_0}^{-4}\mathrm{E}_{\mathrm{T}\theta_0}(j_0^{(3)}),\ -\frac{\lambda_{\theta_0}^{-3}}{2},\ -\frac{\lambda_{\theta_0}^{-5}}{2}\{\mathrm{E}_{\mathrm{T}\theta_0}(j_0^{(3)})\}^2+\frac{\lambda_{\theta_0}^{-4}}{6}\mathrm{E}_{\mathrm{T}\theta_0}(j_0^{(4)}),\\
&\qquad\lambda_{\theta_0}^{-2}\eta_{\theta_0},\ \lambda_{\theta_0}^{-2}\frac{\partial\eta_{\theta_0}}{\partial\theta_0}-\lambda_{\theta_0}^{-3}\mathrm{E}_{\mathrm{T}\theta_0}(j_0^{(3)})\eta_{\theta_0}\Bigg]\\
&\quad\times\left[m^2\frac{\partial\bar{l}_{\theta_0}}{\partial\theta_0},\ m\left(\frac{\partial\bar{l}_{\theta_0}}{\partial\theta_0}\right)^2,\ \{j_0^{(3)}-\mathrm{E}_{\mathrm{T}}(j_0^{(3)})\}\left(\frac{\partial\bar{l}_{\theta_0}}{\partial\theta_0}\right)^2,\ \left(\frac{\partial\bar{l}_{\theta_0}}{\partial\theta_0}\right)^3,\ n^{-1}m,\ n^{-1}\frac{\partial\bar{l}_{\theta_0}}{\partial\theta_0}\right]'
\end{aligned}\tag{2.5}$$

(for full comprehension of (2.5), see also [28,31]).

In (2.5), $\partial\bar{l}_{\theta_0}/\partial\theta_0=\partial\bar{l}_\theta/\partial\theta|_{\theta=\theta_0}$ with other partial derivatives similarly defined, $\lambda_{\theta_0}\equiv\mathrm{E}_{\mathrm{T}\theta_0}\left(\frac{\partial^2\bar{l}_{\theta_0}}{\partial\theta_0^2}\right)$, $j_0^{(k)}\equiv\frac{\partial^k\bar{l}_{\theta_0}}{\partial\theta_0^k}=O_p(1)$ $(k=3,4)$, and $m=\frac{\partial^2\bar{l}_{\theta_0}}{\partial\theta_0^2}-\lambda_{\theta_0}=O_p(n^{-1/2})$. Under c.m.s., $\lambda_{\theta_0}=-\bar{i}_0$, with $\bar{i}_0$ being $\bar{i}$ evaluated at $\theta=\theta_0$.

The factors in the terms of (2.4) are rewritten as

$$\begin{aligned}
\boldsymbol{\gamma}_{\theta_0}^{(k)}&=\boldsymbol{\gamma}_{\theta}^{(k)}(\boldsymbol{\alpha}_0,\theta_0),\qquad \mathbf{l}_{\theta_0}^{(k)}=\mathbf{l}_{\theta}^{(k)}(\boldsymbol{\alpha}_0,\theta_0)\quad(k=1,2,3),\\
\lambda_{\theta_0}&=\lambda_\theta(\boldsymbol{\alpha}_0,\theta_0),\qquad \eta_{\theta_0}=\eta_\theta(\boldsymbol{\alpha}_0,\theta_0).
\end{aligned}\tag{2.6}$$

Usually, in practice, $\boldsymbol{\alpha}_0$ in (2.6) and consequently that in (2.4) is unavailable, and it is replaced by fallible or estimated $\hat{\boldsymbol{\alpha}}$. The stochastic expansion of $\hat{\boldsymbol{\alpha}}$ given by MML and MBM estimation is obtained by Ogasawara [26, Eqs. (3.1) and (A.2)]; [27, Eqs. (2.2) and (2.4)]; [32, Eq. (3.6)], which is summarized as

$$\begin{aligned}
&\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0 = \sum_{k=1}^{3} \boldsymbol{\Gamma}_{\alpha_0}^{(k)} \mathbf{l}_{\alpha_0}^{(k)} - N^{-1} \boldsymbol{\Lambda}_{\alpha_0}^{-1} \boldsymbol{\eta}_{\alpha_0} + O_p(N^{-2}), \\
&\boldsymbol{\Gamma}_{\alpha_0}^{(k)} = \boldsymbol{\Gamma}_{\alpha}^{(k)}(\boldsymbol{\alpha}_0) = O(1), \qquad \mathbf{l}_{\alpha_0}^{(k)} = \mathbf{l}_{\alpha}^{(k)}(\boldsymbol{\alpha}_0) = O_p(N^{-k/2}) \quad (k = 1, 2, 3), \\
&\boldsymbol{\Lambda}_{\alpha_0} = \boldsymbol{\Lambda}_{\alpha}(\boldsymbol{\alpha}_0) = O(1), \qquad \boldsymbol{\eta}_{\alpha_0} = \boldsymbol{\eta}_{\alpha}(\boldsymbol{\alpha}_0) = O(1).
\end{aligned} \tag{2.7}$$

Note that $\boldsymbol{\Gamma}_{\alpha_0}^{(k)}$, $\mathbf{l}_{\alpha_0}^{(k)}$, $\boldsymbol{\Lambda}_{\alpha_0}$, and $\boldsymbol{\eta}_{\alpha_0}$ correspond to $\boldsymbol{\lambda}_{\theta_0}^{(k)}$, $\mathbf{l}_{\theta_0}^{(k)}$, λ_{θ_0}, and η_{θ_0}, respectively, and are similarly defined, though for (2.7) we have q parameters in $\boldsymbol{\alpha}$ while θ is a scalar parameter. Alternatively, $\sum_{k=1}^{3} \boldsymbol{\Gamma}_{\alpha_0}^{(k)} \mathbf{l}_{\alpha_0}^{(k)}$ is expressed by using the sufficient statistic $\mathbf{p} = (r_1, \ldots, r_{2^n})'/N$, i.e., the vector of sample proportions of 2^n response patterns with $\boldsymbol{\pi}_{\mathrm{T}} \equiv \mathrm{E}_{\mathrm{T}\alpha_0}(\mathbf{p})$, which is assumed to be known under m.m., and $\mathrm{E}_{\mathrm{T}\alpha_0}(\cdot)$ is defined similarly to $\mathrm{E}_{\mathrm{T}\theta_0}(\cdot)$:

$$\sum_{k=1}^{3} \boldsymbol{\Gamma}_{\alpha_0}^{(k)} \mathbf{l}_{\alpha_0}^{(k)} = \sum_{k=1}^{3} \frac{1}{k!} \frac{\partial^k \boldsymbol{\alpha}_0}{(\partial \boldsymbol{\pi}_{\mathrm{T}}')^{\langle k \rangle}} (\mathbf{p} - \boldsymbol{\pi}_{\mathrm{T}})^{\langle k \rangle} + N^{-1} \frac{\partial \boldsymbol{\alpha}_{\Delta W}}{\partial \boldsymbol{\pi}_{\mathrm{T}}'} (\mathbf{p} - \boldsymbol{\pi}_{\mathrm{T}}), \tag{2.8}$$

where $\mathbf{x}^{\langle k \rangle}$ denotes the k-fold Kronecker product of $\mathbf{x}$, and $\partial \boldsymbol{\alpha}_{\Delta W} / \partial \boldsymbol{\pi}_{\mathrm{T}}'$ associated only with the Bayes method is given by Ogasawara [32, Eq. (3.4)].

When $\hat{\boldsymbol{\alpha}}$ is used, (2.6) with unchanged θ_0 becomes

$$\begin{aligned}
&\hat{\boldsymbol{\gamma}}_{\theta_0}^{(k)} \equiv \boldsymbol{\gamma}_{\theta}^{(k)}(\hat{\boldsymbol{\alpha}}, \theta_0) = O_p(1), \qquad \hat{\mathbf{l}}_{\theta_0}^{(k)} \equiv \mathbf{l}_{\theta}^{(k)}(\hat{\boldsymbol{\alpha}}, \theta_0) = O_p(n^{-k/2}) \quad (k = 1, 2, 3), \\
&\hat{\lambda}_{\theta_0} \equiv \lambda_{\theta}(\hat{\boldsymbol{\alpha}}, \theta_0) = O_p(1), \qquad \hat{\eta}_{\theta_0} \equiv \eta_{\theta}(\hat{\boldsymbol{\alpha}}, \theta_0) = O_p(1).
\end{aligned} \tag{2.9}$$

Now, all the terms of (2.9) are stochastic ones, and are expanded about $\hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}_0$, whose results are given in [33, Subsection A.2]. Using the notation defined there, the expansion of $\hat{\theta}$, when $N = O(n^k)$ $(k \geq 1)$, is as follows:

$$\begin{aligned}
\hat{\theta} - \theta_0 &= \sum_{k=1}^{3} \hat{\boldsymbol{\gamma}}_{\theta_0}^{(k)\prime} \hat{\mathbf{l}}_{\theta_0}^{(k)} - n^{-1} \hat{\lambda}_{\theta_0}^{-1} \hat{\eta}_{\theta_0} + \sum_{k=0}^{4} O_p(n^{-k/2} N^{-(4-k)/2}) \\
&= \{(\gamma_{\theta_0}^{(1)})_{O(1)} + (\gamma_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} + (\gamma_{\theta_0}^{(\Delta\Delta 1)})_{O_p(N^{-1})} + O_p(N^{-3/2})\}\{(l_{\theta_0}^{(1)})_{O_p(n^{-1/2})} + (l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})} \\
&\quad + (l_{\theta_0}^{(\Delta\Delta a1)})_{O_p(n^{-1/2}N^{-1/2})} + (l_{\theta_0}^{(\Delta\Delta b1)})_{O_p(N^{-1})} + (l_{\theta_0}^{(\Delta\Delta\Delta a1)})_{O_p(n^{-1/2}N^{-1})} + (l_{\theta_0}^{(\Delta\Delta\Delta b1)})_{O_p(N^{-3/2})} \\
&\quad + O_p(n^{-1/2}N^{-3/2}) + O_p(N^{-2})\} + \{(\boldsymbol{\gamma}_{\theta_0}^{(2)\prime})_{O(1)} + (\boldsymbol{\gamma}_{\theta_0}^{(\Delta 2)\prime})_{O_p(N^{-1/2})} + O_p(N^{-1})\} \\
&\quad \times \{(\mathbf{l}_{\theta_0}^{(2)})_{O_p(n^{-1})} + (\mathbf{l}_{\theta_0}^{(\Delta a2)})_{O_p(n^{-1/2}N^{-1/2})} + (\mathbf{l}_{\theta_0}^{(\Delta b2)})_{O_p(N^{-1})} + (\mathbf{l}_{\theta_0}^{(\Delta\Delta a2)})_{O_p(n^{-1}N^{-1/2})} \\
&\quad + (\mathbf{l}_{\theta_0}^{(\Delta\Delta b2)})_{O_p(n^{-1/2}N^{-1})} + (\mathbf{l}_{\theta_0}^{(\Delta\Delta c2)})_{O_p(N^{-3/2})} + O_p(n^{-1}N^{-1}) + O_p(n^{-1/2}N^{-3/2}) + O_p(N^{-2})\} \\
&\quad + \{(\boldsymbol{\gamma}_{\theta_0}^{(3)\prime})_{O(1)} + O_p(N^{-1/2})\}\{(\mathbf{l}_{\theta_0}^{(3)})_{O_p(n^{-3/2})} + (\mathbf{l}_{\theta_0}^{(\Delta a3)})_{O_p(n^{-1}N^{-1/2})} + (\mathbf{l}_{\theta_0}^{(\Delta b3)})_{O_p(n^{-1/2}N^{-1})} + (\mathbf{l}_{\theta_0}^{(\Delta c3)})_{O_p(N^{-3/2})} \\
&\quad + O_p(n^{-1}N^{-1}) + O_p(n^{-1/2}N^{-3/2}) + O_p(N^{-2})\} \\
&\quad - (n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})_{O(n^{-1})} - \{n^{-1}(\lambda_{\theta_0}^{-1}\eta_{\theta_0})^{(\Delta)}\}_{O_p(n^{-1}N^{-1/2})} + \sum_{k=0}^{4} O_p(n^{-k/2}N^{-(4-k)/2}),
\end{aligned} \tag{2.10}$$

where $(\cdot)_{O(\cdot)}$ and $(\cdot)_{O_p(\cdot)}$ indicate the orders of the values in parentheses. Note that in [33, Subsection A.2] and consequently in (2.10), the relative orders addressed in Section 2.1 are not specified. The corresponding results with such relative orders will be given in Appendix A.1.2 of the Appendix under the following conditions.

$$\text{Condition A}: \ N = O(n) \quad (n = O(N)). \tag{2.11}$$

$$\text{Condition B}: \ N = O(n^{3/2}) \quad (n = O(N^{2/3})). \tag{2.12}$$

$$\text{Condition C}: \ N = O(n^2) \quad (n = O(N^{1/2})). \tag{2.13}$$

3. Asymptotic cumulants of $\hat{\theta}$ using fallible item parameters

Define $w = n^{1/2}(\hat{\theta} - \theta_0)$, with $\beta_k^{(0)}$ $(k = 1, \ldots, 4)$ and $\beta_{\mathrm{H2}}^{(0)}$ being the kth asymptotic cumulants and the higher-order added asymptotic variance of w, respectively, each independent of n, when the item parameters are known (a constant "independent of n" in this paper refers to the constant after removing an associated power of n). Let $\bar{c} = n/N = O(1)$,

$\bar{c}^* = n^{3/2}/N = O(1)$, and $\bar{c}^{**} = n^2/N = O(1)$ under Conditions A, B, and C, respectively. Then, $\bar{\beta}_k$ $(k = 1, \ldots, 4)$ and $\bar{\beta}_{\mathrm{H2}}$ depending on $\bar{c}$, $\bar{c}^*$, or $\bar{c}^{**}$ are the counterparts of $\beta_k^{(0)}$ $(k = 1, \ldots, 4)$ and $\beta_{\mathrm{H2}}^{(0)}$, respectively, when the item parameters are estimated. Define $\bar{\beta}_k^{(\Delta)}$ $(k = 1, \ldots, 4)$ and $\bar{\beta}_{\mathrm{H2}}^{(\Delta)}$ under Conditions A–C as added terms satisfying the following equations:

$$\bar{\beta}_k = \beta_k^{(0)} + \bar{\beta}_k^{(\Delta)} \quad (k = 1, \ldots, 4), \qquad \bar{\beta}_{\mathrm{H2}} = \beta_{\mathrm{H2}}^{(0)} + \bar{\beta}_{\mathrm{H2}}^{(\Delta)}. \tag{3.1}$$

In addition, under Condition B, $\bar{\beta}_{\mathrm{h2}} = \bar{\beta}_{\mathrm{h2}}^{(\Delta)}$ $(\beta_{\mathrm{h2}}^{(0)} = 0)$ independent of n is defined as a higher-order added asymptotic variance of order $O(n^{-1/2})$ for w, which is located, in terms of order, between the usual asymptotic variance of order $O(1)$ and the higher-order added asymptotic variance of order $O(n^{-1})$ for w.

Let $\beta_k^{(\Delta)}$ $(k = 1, \ldots, 4)$, $\beta_{\mathrm{H2}}^{(\Delta)}$, and $\beta_{\mathrm{h2}}^{(\Delta)}$ be independent of $\bar{c}$, $\bar{c}^*$, and $\bar{c}^{**}$ under Conditions A, B, and C, respectively, which correspond to $\bar{\beta}_k^{(\Delta)}$ $(k = 1, \ldots, 4)$, $\bar{\beta}_{\mathrm{H2}}^{(\Delta)}$, and $\bar{\beta}_{\mathrm{h2}}^{(\Delta)}$, respectively, with a relationship such as $\bar{\beta}_1^{(\Delta)} = \bar{c}\beta_1^{(\Delta)}$. Other asymptotic cumulants given later without the bar are defined similarly. Denote the jth cumulant of a variate by $\kappa_j(\cdot)$. Then, we have the following.

(a) *Condition* A: $N = O(n)$ $(\bar{c} = n/N = O(1))$.

Theorem 1. *Under Condition* A*, the asymptotic cumulants of* w *up to the fourth order and the higher-order asymptotic variance with associated assumptions and p.m.m. are given by*

$$\begin{aligned}
\kappa_1(w) &= n^{-1/2}\bar{\beta}_1 + O(n^{-3/2}) \\
&= n^{-1/2}\beta_1^{(0)} + N^{-1/2}\bar{c}^{1/2}\beta_1^{(\Delta)} + O(n^{-3/2}), \\
\kappa_2(w) &= \bar{\beta}_2 + n^{-1}\bar{\beta}_{\mathrm{H2}} + O(n^{-2}) \\
&= (\beta_2^{(0)} + \bar{c}\beta_2^{(\Delta)}) + \{n^{-1}\beta_{\mathrm{H2}}^{(0)} + N^{-1}(\beta_{\mathrm{H2}}^{(\Delta a)} + \bar{c}\beta_{\mathrm{H2}}^{(\Delta b)})\} + O(n^{-2}), \\
\kappa_3(w) &= n^{-1/2}\bar{\beta}_3 + O(n^{-3/2}) \\
&= n^{-1/2}\beta_3^{(0)} + N^{-1/2}(\bar{c}^{1/2}\beta_3^{(\Delta a)} + \bar{c}^{3/2}\beta_3^{(\Delta b)}) + O(n^{-3/2}), \\
\kappa_4(w) &= n^{-1}\bar{\beta}_4 + O(n^{-2}) \\
&= n^{-1}\beta_4^{(0)} + N^{-1}(\beta_4^{(\Delta a)} + \bar{c}\beta_4^{(\Delta b)} + \bar{c}^2\beta_3^{(\Delta c)}) + O(n^{-2}),
\end{aligned} \tag{3.2}$$

where the expressions of undefined quantities are given in [33, *Subsection A.3*].

Proof. See [33, Subsection A.3]. □

From (3.2) and [33, Subsection A.3], we have the following alternative expressions:

$$\begin{aligned}
\kappa_1(\hat{\theta} - \theta_0) &= n^{-1}\bar{\beta}_1 + O(n^{-2}) = n^{-1}(\beta_1^{(0)} + \bar{\beta}_1^{(\Delta)}) + O(n^{-2}) \\
&= n^{-1}\beta_1^{(0)} + N^{-1}\beta_1^{(\Delta)} + O(n^{-2}), \\
\kappa_2(\hat{\theta}) &= n^{-1}\bar{\beta}_2 + n^{-2}\bar{\beta}_{\mathrm{H2}} + O(n^{-3}) \\
&= n^{-1}\beta_2^{(0)} + N^{-1}\beta_2^{(\Delta)} + n^{-2}\beta_{\mathrm{H2}}^{(0)} + n^{-1}N^{-1}\beta_{\mathrm{H2}}^{(\Delta a)} + N^{-2}\beta_{\mathrm{H2}}^{(\Delta b)} + O(n^{-3}),
\end{aligned} \tag{3.3}$$

where $\bar{\beta}_2 = \beta_2^{(0)} + \bar{\beta}_2^{(\Delta)} = \beta_2^{(0)} + \bar{c}\beta_2^{(\Delta)} > \beta_2^{(0)}$ since $\bar{\beta}_2^{(\Delta)} = \bar{c}\beta_2^{(\Delta)} > 0$ (see [33, Subsection A.3]),

$$\begin{aligned}
\kappa_3(\hat{\theta}) &= n^{-2}\bar{\beta}_3 + O(n^{-3}) = n^{-2}(\beta_3^{(0)} + \bar{\beta}_3^{(\Delta)}) + O(n^{-3}) \\
&= n^{-2}\beta_3^{(0)} + n^{-1}N^{-1}\beta_3^{(\Delta a)} + N^{-2}\beta_3^{(\Delta b)} + O(n^{-3}), \\
\kappa_4(\hat{\theta}) &= n^{-3}\bar{\beta}_4 + O(n^{-4}) = n^{-3}(\beta_4^{(0)} + \bar{\beta}_4^{(\Delta)}) + O(n^{-4}) \\
&= n^{-3}\beta_4^{(0)} + n^{-2}N^{-1}\beta_4^{(\Delta a)} + n^{-1}N^{-2}\beta_4^{(\Delta b)} + N^{-3}\beta_3^{(\Delta c)} + O(n^{-4}).
\end{aligned} \tag{3.4}$$

From Theorem 1, it is seen that all the asymptotic cumulants derived there are different from those based on known item parameters.

(b) *Condition* B: $N = O(n^{3/2})$ $(\bar{c}^* = n^{3/2}/N = O(1))$.

Theorem 2. *Under Condition* B*, the asymptotic cumulants of* w *corresponding to those in Theorem* 1 *are given by*

$$\begin{aligned}
&\kappa_1(w) = n^{-1/2}\beta_1^{(0)} + O(n^{-1}), \\
&\kappa_2(w) = \beta_2^{(0)} + n^{-1/2}\bar{c}^*\beta_{\mathrm{h2}}^{(\Delta)} + n^{-1}\beta_{\mathrm{H2}}^{(0)} + O(n^{-3/2}), \\
&\kappa_3(w) = n^{-1/2}\beta_3^{(0)} + O(n^{-1}), \qquad \kappa_4(w) = n^{-1}\beta_4^{(0)} + O(n^{-3/2}),
\end{aligned} \tag{3.5}$$

where $\beta_{\mathrm{h2}}^{(\Delta)}$ *is algebraically equal to* $\beta_2^{(\Delta)}$ *in Theorem* 1.

Proof. See [33, Subsection A.3]. □

For $\hat{\theta}$, the following expressions are obtained:

$$\begin{aligned}
&\kappa_1(\hat{\theta}-\theta_0) = n^{-1}\beta_1^{(0)} + O(n^{-3/2}),\\
&\kappa_2(\hat{\theta}) = n^{-1}\beta_2^{(0)} + (N^{-1}\beta_{h2}^{(\Delta)})_{O(n^{-3/2})} + n^{-2}\beta_{H2}^{(0)} + O(n^{-5/2}),\\
&\kappa_3(\hat{\theta}) = n^{-2}\beta_3^{(0)} + O(n^{-5/2}), \qquad \kappa_4(\hat{\theta}) = n^{-3}\beta_4^{(0)} + O(n^{-7/2}).
\end{aligned} \tag{3.6}$$

From Theorem 2 and (3.6), we find that the asymptotic cumulants derived above are the same as those based on known item parameters, except that the intermediate higher-order terms $n^{-1/2}\bar{c}^{*}\beta_{h2}^{(\Delta)}$ in (3.5) and $(N^{-1}\beta_{h2}^{(\Delta)})_{O(n^{-3/2})}$ in (3.6) are added, which were the terms in the usual asymptotic variances in Theorem 1 and (3.3). That is, the higher-order asymptotic variance of w up to order $O(n^{-1/2})$ in Theorem 2 is unchanged from that in Theorem 1. It is to be noted that the orders of the residual terms in Theorem 2 are lower than those in Theorem 1, (3.3) and (3.4) by $n^{1/2}$.

(c) *Condition* C: $N = O(n^2)$ ($\bar{c}^{**} = n^2/N = O(1)$).

Theorem 3. *Under Condition* C*, the asymptotic cumulants of w corresponding to those in Theorems 1 and 2 are given by*

$$\begin{aligned}
&\kappa_1(w) = n^{-1/2}\beta_1^{(0)} + O(n^{-3/2}),\\
&\kappa_2(w) = \beta_2^{(0)} + n^{-1}(\beta_{H2}^{(0)} + \bar{c}^{**}\beta_{H2}^{(\Delta)}) + O(n^{-2}),\\
&\kappa_3(w) = n^{-1/2}\beta_3^{(0)} + O(n^{-3/2}), \qquad \kappa_4(w) = n^{-1}\beta_4^{(0)} + O(n^{-2}),
\end{aligned} \tag{3.7}$$

where $\beta_{H2}^{(\Delta)}$ is algebraically equal to $\beta_2^{(\Delta)}$ in Theorem 1 and $\beta_{h2}^{(\Delta)}$ in Theorem 2 (do not confuse $\beta_{H2}^{(\Delta)}$ in Theorem 3 with that in Theorem 1).

Proof. See [33, Subsection A.3]. □

For $\hat{\theta}$, we have

$$\begin{aligned}
&\kappa_1(\hat{\theta}-\theta_0) = n^{-1}\beta_1^{(0)} + O(n^{-3/2}),\\
&\kappa_2(\hat{\theta}) = n^{-1}\beta_2^{(0)} + n^{-2}\beta_{H2}^{(0)} + (N^{-1}\beta_{H2}^{(\Delta)})_{O(n^{-2})} + O(n^{-3}),\\
&\kappa_3(\hat{\theta}) = n^{-2}\beta_3^{(0)} + O(n^{-3}), \qquad \kappa_4(\hat{\theta}) = n^{-3}\beta_4^{(0)} + O(n^{-4}).
\end{aligned} \tag{3.8}$$

From Theorem 3 and (3.8), it is found that the asymptotic cumulants of w are the same as those based on known item parameters, except the higher-order asymptotic variance, where the terms $n^{-1}\bar{c}^{**}\beta_{H2}^{(\Delta)}$ in Theorem 3 and $(N^{-1}\beta_{H2}^{(\Delta)})_{O(n^{-2})}$ in (3.8) are added.

Theorems 2 and 3 indicate that the asymptotic cumulants up to the fourth order in the theorems are the same, and that their higher-order asymptotic variances up to order $O(n^{-1})$ for w are the same, and larger than that based on known item parameters by $n^{-1/2}\bar{c}^{*}\beta_{h2}^{(\Delta)}$ in Theorem 2 and by $n^{-1}\bar{c}^{**}\beta_{H2}^{(\Delta)}$ in Theorem 3, where the latter is algebraically equal to the former.

The expressions for the associated partial derivatives and expectations in the asymptotic cumulants are provided in [33, Subsection A.5] and [34, Subsection A.6], respectively.

4. Asymptotic cumulants of the studentized $\hat{\theta}$ using fallible item parameters

The asymptotic cumulants derived earlier can be used to see the properties of the ML, BM, WL, and JM estimators, denoted generically as $\hat{\theta}$. For interval estimation of θ_0, the asymptotic cumulants of the studentized $\hat{\theta}$ are required. The methods of studentization vary with the associated asymptotic variances of order $O(n^{-1})$ for $\hat{\theta}$. We use two versions of studentized $\hat{\theta}$, denoted by t under Condition A and t^* under Conditions B and C.

4.1. *Studentization under Condition A: $N = O(n)$ ($\bar{c} = n/N = O(1)$)*

Define

$$t \equiv n^{1/2}(\hat{\theta}-\theta_0)\hat{\bar{\beta}}_{2I}^{-1/2}, \tag{4.1}$$

where $\hat{\bar{\beta}}_{2I} = \hat{\beta}_{2I}^{(0)} + \hat{\bar{\beta}}_{2I}^{(\Delta)} = \hat{\beta}_{2I}^{(0)} + \bar{c}\hat{\beta}_{2I}^{(\Delta)}$; and $\hat{\beta}_{2I}^{(0)}$ and $\hat{\bar{\beta}}_{2I}^{(\Delta)}$ $(=\bar{c}\hat{\beta}_{2I}^{(\Delta)})$ are the estimators of $\beta_2^{(0)}$ and $\bar{\beta}_2^{(\Delta)}$ $(=\bar{c}\beta_2^{(\Delta)})$ under c.m.s., and include $\hat{\boldsymbol{\alpha}}$ and $\hat{\theta}$. For estimating $\bar{\beta}_2 = \beta_2^{(0)} + \bar{\beta}_2^{(\Delta)} = \beta_2^{(0)} + \bar{c}\beta_2^{(\Delta)}$, c.m.s. is assumed due to the difficulty of estimating P_{Tk} $(k = 1, \ldots, n)$, since generally only a single item response by an examinee is available for estimating each P_{Tk}. On the other hand, under c.m.s., $\beta_2^{(0)} = \beta_{2I}^{(0)} = \bar{i}_{\theta_0}^{-1}$ (the reciprocal of the average Fisher information) is well estimated. For $\bar{\beta}_2^{(\Delta)}$, estimating the vector $\boldsymbol{\pi}_{\mathrm{T}}$ of the true probabilities of 2^n response patterns is required under m.m. In order to estimate $\boldsymbol{\pi}_{\mathrm{T}}$, we can use the sample counterpart $\mathbf{p}$. However, since $\beta_2^{(\Delta)}$ includes $\beta_2^{(0)}$, we have the similar difficulty for estimating $\beta_2^{(\Delta)}$ under m.m. So, c.m.s. is assumed for estimating $\bar{\beta}_2$ by $\hat{\hat{\bar{\beta}}}_{2I}$, though the asymptotic cumulants given later will be obtained under p.m.m., yielding generally the non-unit asymptotic variance of t.

The subscript I in, for example, $\bar{\beta}_{2I}$ indicates the use of the Fisher information (matrix). That is, $\{\hat{\beta}_{2I}^{(0)}\}^{-1} = \hat{i}$ is equal to $\bar{i}_{\theta_0} = n^{-1}\sum_{k=1}^{n}\{\partial P_k(\theta_0)/\partial\theta_0\}^2/\{P_k(\theta_0)Q_k(\theta_0)\}$, where θ_0 and $\boldsymbol{\alpha}_0$ are replaced by $\hat{\theta}$ and $\hat{\boldsymbol{\alpha}}$, respectively, yielding $\hat{i} = n^{-1}\sum_{k=1}^{n}(\partial\hat{P}_k/\partial\hat{\theta})^2/(\hat{P}_k\hat{Q}_k)$, and

$$
\begin{aligned}
\hat{\beta}_{2I}^{(\Delta)} &= \hat{i}^{-2}\hat{\mathrm{E}}_{\theta_0}\left(\frac{\partial^2\bar{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}\right)\hat{\mathbf{G}}^{-1}\hat{\mathrm{E}}_{\theta_0}\left(\frac{\partial^2\bar{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0}\right) \\
&= \hat{i}^{-2}\left(n^{-1}\sum_{k=1}^{n}\frac{1}{\hat{P}_i\hat{Q}_i}\frac{\partial\hat{P}_i}{\partial\hat{\theta}}\frac{\partial\hat{P}_i}{\partial\hat{\boldsymbol{\alpha}}'}\right)\hat{\mathbf{G}}^{-1}\left(n^{-1}\sum_{k=1}^{n}\frac{1}{\hat{P}_i\hat{Q}_i}\frac{\partial\hat{P}_i}{\partial\hat{\theta}}\frac{\partial\hat{P}_i}{\partial\hat{\boldsymbol{\alpha}}}\right)
\end{aligned}
\tag{4.2}
$$

(see [34, Subsection A.6.1, Equation (a.2.1) with (a.1)]), where $\mathrm{E}_{\theta_0}(\cdot)$ indicates that the expectation is taken under c.m.s. for the distributions of U_k ($k = 1, \ldots, n$), $\hat{\mathrm{E}}_{\theta_0}(\cdot)$ is its sample counterpart, and

$$
\hat{\mathbf{G}} = N^{-1}\sum_{k=1}^{N}\frac{\partial\hat{l}_{\boldsymbol{\alpha}(k)}}{\partial\hat{\boldsymbol{\alpha}}}\frac{\partial\hat{l}_{\boldsymbol{\alpha}(k)}}{\partial\hat{\boldsymbol{\alpha}}'},
\tag{4.3}
$$

where $\hat{l}_{\boldsymbol{\alpha}(k)}$ is the log marginal likelihood of $\boldsymbol{\alpha}$ contributed by the kth subject for item calibration evaluated at $\boldsymbol{\alpha} = \hat{\boldsymbol{\alpha}}$, with $\hat{l}_{\boldsymbol{\alpha}(k)}$ seen as a function of $\hat{\boldsymbol{\alpha}}$. $\hat{\mathbf{G}}$ is employed for simplicity as one of the estimators of $\mathbf{I}_{\boldsymbol{\alpha}_0}$, the information matrix for the item parameters per observation under c.m.s.

Using the above definitions, we have

$$
\begin{aligned}
\hat{\hat{\bar{\beta}}}_{2I} &= \hat{\beta}_{2I}^{(\Delta)} + \hat{\hat{\bar{\beta}}}{}_{2I}^{(\Delta)} = \hat{\beta}_{2I}^{(\Delta)} + \bar{c}\hat{\beta}_{2I}^{(\Delta)} \\
&= \hat{i}^{-1} + \bar{c}\,\hat{i}^{-2}\hat{\mathrm{E}}_{\theta_0}\left(\frac{\partial^2\bar{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0'}\right)\hat{\mathbf{G}}^{-1}\hat{\mathrm{E}}_{\theta_0}\left(\frac{\partial^2\bar{l}_{\theta_0}}{\partial\theta_0\partial\boldsymbol{\alpha}_0}\right).
\end{aligned}
\tag{4.4}
$$

From the expansion of $\hat{\theta}$ (see (A.1)) and that of $\hat{\hat{\bar{\beta}}}{}_{2I}^{-1/2}$ given in (A.6) of Appendix A.1.3, t is expanded as

$$
\begin{aligned}
t &= n^{1/2}(\hat{\theta}-\theta_0)\hat{\hat{\bar{\beta}}}{}_{2I}^{-1/2} = w\hat{\hat{\bar{\beta}}}{}_{2I}^{-1/2} \\
&= n^{1/2}\{q^{(1)}_{O_p(n^{-1/2})} + q^{(2)}_{O_p(n^{-1})} + q^{(3)}_{O_p(n^{-3/2})} - (n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})_{O(n^{-1})} + O_p(n^{-2})\} \\
&\quad\times\left\{\bar{\beta}_{2I}^{-1/2} + b^{(1)}_{O_p(n^{-1/2})} + b^{(2)}_{O_p(n^{-1})} + \left(n^{-1}\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial\theta_0}\lambda_{\theta_0}^{-1}\eta_{\theta_0}\right)_{O(n^{-1})}\right. \\
&\quad\left.+\left(N^{-1}\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\right)_{O(N^{-1})} + O_p(n^{-3/2})\right\} \\
&= n^{1/2}q^{(1)}_{O_p(n^{-1/2})}\bar{\beta}_{2I}^{-1/2} + n^{1/2}(q^{(1)}_{O_p(n^{-1/2})}b^{(1)}_{O_p(n^{-1/2})} + q^{(2)}_{O_p(n^{-1})}\bar{\beta}_{2I}^{-1/2}) \\
&\quad+n^{1/2}\left\{q^{(1)}_{O_p(n^{-1/2})}\left(b^{(2)}_{O_p(n^{-1})} + n^{-1}\frac{\bar{\beta}_{2I}^{-1/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial\theta_0}\lambda_{\theta_0}^{-1}\eta_{\theta_0} + N^{-1}\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Lambda}_{\boldsymbol{\alpha}_0}^{-1}\boldsymbol{\eta}_{\boldsymbol{\alpha}_0}\right)\right. \\
&\quad\left.+(q^{(2)}_{O_p(n^{-1})} - n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})b^{(1)}_{O_p(n^{-1/2})} + q^{(3)}_{O_p(n^{-3/2})}\bar{\beta}_{2I}^{-1/2}\right\} - n^{-1/2}\lambda_{\theta_0}^{-1}\eta_{\theta_0}\bar{\beta}_{2I}^{-1/2} + O_p(n^{-3/2}) \\
&\equiv n^{1/2}(t^{(1)}_{O_p(n^{-1/2})} + t^{(2)}_{O_p(n^{-1})} + t^{(3)}_{O_p(n^{-3/2})} - n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0}\bar{\beta}_{2I}^{-1/2}) + O_p(n^{-3/2}).
\end{aligned}
\tag{4.5}
$$

From (4.5), the following results are obtained.

Theorem 4. *Under Condition* A, *the asymptotic cumulants of* t *up to the fourth order and the higher-order asymptotic variance with associated assumptions and p.m.m. are given by*

$$\begin{aligned}
\kappa_1(t) &= n^{-1/2}\bar{\beta}_{t1} + O(n^{-3/2}) = n^{-1/2}(\bar{\beta}_1\bar{\beta}_{2I}^{-1/2} + \beta_1^{(t0)}) + N^{-1/2}\bar{c}^{1/2}\beta_1^{(t\Delta)} + O(n^{-3/2}),\\
\kappa_2(t) &= \bar{\beta}_{t2} + n^{-1}\bar{\beta}_{t\,\mathrm{H}2} + O(n^{-2}) = \bar{\beta}_2\bar{\beta}_{2I}^{-1} + n^{-1}\bar{\beta}_{t\,\mathrm{H}2} + O(n^{-2}),\\
\kappa_3(t) &= n^{-1/2}\bar{\beta}_{t3} + O(n^{-3/2}) = n^{-1/2}(\bar{\beta}_3\bar{\beta}_{2I}^{-3/2} + \beta_3^{(t0)} + \bar{\beta}_3^{(t\Delta)}) + O(n^{-3/2}),\\
\kappa_4(t) &= n^{-1}\bar{\beta}_{t4} + O(n^{-2}) = n^{-1}(\bar{\beta}_4\bar{\beta}_{2I}^{-2} + \bar{\beta}_4^{(t0\Delta)}) + O(n^{-2}),
\end{aligned} \tag{4.6}$$

where the undefined quantities are defined in [33, *Subsection A.4*]. □

Note that all the asymptotic cumulants derived in (4.6) are different from those based on known item parameters. Under c.m.s., $\bar{\beta}_{t2} = \bar{\beta}_2\bar{\beta}_{2I}^{-1} = 1$ in $\kappa_2(t)$.

4.2. Studentization under Condition B: $N = O(n^{3/2})$ ($\bar{c}^* = n^{3/2}/N = O(1)$)

Define

$$t^* \equiv n^{1/2}(\hat{\theta} - \theta_0)\{\hat{\beta}_2^{(0)}\}^{-1/2} = n^{1/2}(\hat{\theta} - \theta_0)\hat{i}^{1/2}, \tag{4.7}$$

where $\hat{i}$ an estimator of $\{\beta_2^{(0)}\}^{-1}$ under c.m.s. is used due to the difficulty of estimating $\beta_2^{(0)}$ under m.m. as in t. However, the asymptotic cumulants of t^* will be derived under p.m.m. Define $\hat{i}_{\theta_0}$ as $\hat{i}$, where $\hat{\theta}$ is replaced by θ_0, with $\hat{\boldsymbol{\alpha}}$ unchanged. Let $\hat{i}_{\theta_0}^{(\mathrm{D1})} = \frac{\partial \bar{i}}{\partial\theta}\big|_{\substack{\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}\\ \theta=\theta_0}}$ and $\hat{i}_{\theta_0}^{(\mathrm{D2})} = \frac{\partial^2 \bar{i}}{\partial\theta^2}\big|_{\substack{\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}\\ \theta=\theta_0}}$. Using (A.2), expand $\hat{i}^{1/2}$ as

$$\begin{aligned}
\hat{i}^{1/2} &= \hat{i}_{\theta_0}^{1/2} + \frac{\hat{i}_{\theta_0}^{-1/2}}{2}\hat{i}_{\theta_0}^{(\mathrm{D1})}(\hat{\theta}-\theta_0) + \left(\frac{\hat{i}_{\theta_0}^{-1/2}}{4}\hat{i}_{\theta_0}^{(\mathrm{D2})} - \frac{\hat{i}_{\theta_0}^{-3/2}}{8}(\hat{i}_{\theta_0}^{(\mathrm{D1})})^2\right)(\hat{\theta}-\theta_0)^2 + O_p(n^{-3/2})\\
&\equiv \hat{i}_{\theta_0}^{1/2} + \hat{i}_{\theta_0}^{(1)}(\hat{\theta}-\theta_0) + \hat{i}_{\theta_0}^{(2)}(\hat{\theta}-\theta_0)^2 + O_p(n^{-3/2})\\
&= \hat{i}_{\theta_0}^{1/2} + \hat{i}_{\theta_0}^{(1)}\{q^{(10)}_{O_p(n^{-1/2})} + (q^{(1a)}_{O_p(N^{-1/2})})_{O_p(n^{-3/4})} + q^{(20)}_{O_p(n^{-1})} - n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0}\} + \hat{i}_{\theta_0}^{(2)}(q^{(10)}_{O_p(n^{-1/2})})^2 + O_p(n^{-5/4})\\
&= \bar{i}_{\theta_0}^{1/2} + (\bar{i}_{\theta_0}^{(1)}q^{(10)}_{O_p(n^{-1/2})})_{O_p(n^{-1/2})} + \left(\frac{\bar{i}_{\theta_0}^{-1/2}}{2}\frac{\partial\bar{i}_{\theta_0}}{\partial\boldsymbol{\alpha}_0'}(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}0}^{(1)}\mathbf{l}_{\boldsymbol{\alpha}0}^{(1)})_{O_p(N^{-1/2})} + \bar{i}_{\theta_0}^{(1)}q^{(1a)}_{O_p(N^{-1/2})}\right)_{O_p(n^{-3/4})}\\
&\quad + \left\{\bar{i}_{\theta_0}^{(1)}q^{(20)}_{O_p(n^{-1})} + \bar{i}_{\theta_0}^{(2)}(q^{(10)}_{O_p(n^{-1/2})})^2\right\}_{O_p(n^{-1})} - n^{-1}\bar{i}_{\theta_0}^{(1)}\lambda_{\theta_0}^{-1}\eta_{\theta_0} + O_p(n^{-5/4})\\
&\equiv \bar{i}_{\theta_0}^{1/2} + j^{(10)}_{O_p(n^{-1/2})} + (j^{(1a)}_{O_p(N^{-1/2})})_{O_p(n^{-3/4})} + j^{(20)}_{O_p(n^{-1})} - n^{-1}\bar{i}_{\theta_0}^{(1)}\lambda_{\theta_0}^{-1}\eta_{\theta_0} + O_p(n^{-5/4}),
\end{aligned} \tag{4.8}$$

where $\bar{i}_{\theta_0}^{(1)}$ and $\bar{i}_{\theta_0}^{(2)}$ are $\hat{i}_{\theta_0}^{(1)}$ and $\hat{i}_{\theta_0}^{(2)}$, respectively, with $\hat{\boldsymbol{\alpha}}$ being replaced by $\boldsymbol{\alpha}_0$.

Using (4.7) and (4.8) with (A.2), t^* becomes

$$\begin{aligned}
t^* &= n^{1/2}\{q^{(10)}_{O_p(n^{-1/2})} + (q^{(1a)}_{O_p(N^{-1/2})})_{O_p(n^{-3/4})} + q^{(20)}_{O_p(n^{-1})} + (q^{(2a)}_{O_p(n^{-1/2}N^{-1/2})})_{O_p(n^{-5/4})}\\
&\quad + (q^{(30)}_{O_p(n^{-3/2})} + q^{(31)}_{O_p(N^{-1})})_{O_p(n^{-3/2})} - n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0} + O_p(n^{-7/4})\}\\
&\quad \times\{\bar{i}_{\theta_0}^{1/2} + j^{(10)}_{O_p(n^{-1/2})} + (j^{(1a)}_{O_p(N^{-1/2})})_{O_p(n^{-3/4})} + j^{(20)}_{O_p(n^{-1})} - n^{-1}\bar{i}_{\theta_0}^{(1)}\lambda_{\theta_0}^{-1}\eta_{\theta_0} + O_p(n^{-5/4})\}\\
&= n^{1/2}\Big[(q^{(10)}_{O_p(n^{-1/2})}\bar{i}_{\theta_0}^{1/2})_{O_p(n^{-1/2})} + (q^{(1a)}_{O_p(N^{-1/2})}\bar{i}_{\theta_0}^{1/2})_{O_p(n^{-3/4})} + (q^{(10)}_{O_p(n^{-1/2})}j^{(10)}_{O_p(n^{-1/2})} + q^{(20)}_{O_p(n^{-1})}\bar{i}_{\theta_0}^{1/2})_{O_p(n^{-1})}\\
&\quad + (q^{(1a)}_{O_p(N^{-1/2})}j^{(10)}_{O_p(n^{-1/2})} + q^{(2a)}_{O_p(n^{-1/2}N^{-1/2})}\bar{i}_{\theta_0}^{1/2})_{O_p(n^{-5/4})} + \{q^{(10)}_{O_p(n^{-1/2})}(j^{(20)}_{O_p(n^{-1})} - n^{-1}\bar{i}_{\theta_0}^{(1)}\lambda_{\theta_0}^{-1}\eta_{\theta_0})\\
&\quad + q^{(1a)}_{O_p(N^{-1/2})}j^{(1a)}_{O_p(N^{-1/2})} + (q^{(20)}_{O_p(n^{-1})} - n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})j^{(10)}_{O_p(n^{-1/2})} + (q^{(30)}_{O_p(n^{-3/2})} + q^{(31)}_{O_p(N^{-1})})\bar{i}_{\theta_0}^{1/2}\}_{O_p(n^{-3/2})}\\
&\quad - n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0}\bar{i}_{\theta_0}^{1/2}\Big] + O_p(n^{-5/4})\\
&\equiv n^{1/2}(t^{(*1)}_{O_p(n^{-1/2})} + t^{(*1a)}_{O_p(n^{-3/4})} + t^{(*2)}_{O_p(n^{-1})} + t^{(*2a)}_{O_p(n^{-5/4})} + t^{(*3)}_{O_p(n^{-3/2})} - n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0}\bar{i}_{\theta_0}^{1/2}) + O_p(n^{-5/4}).
\end{aligned} \tag{4.9}$$

In (4.9), the terms due to the fallible item parameters are all of $t^{(*1a)}_{O_p(n^{-3/4})}$ and $t^{(*2a)}_{O_p(n^{-5/4})}$, and those with factors of order $O_p(N^{-1/2})$ and $O_p(N^{-1})$ in $t^{(*3)}_{O_p(n^{-3/2})}$. From (4.9) with this property, we have the following.

Theorem 5. *Under Condition* B, *the asymptotic cumulants of* t^* *up to the fourth order and the higher-order asymptotic variance with associated assumptions and p.m.m. are given by*

$$\begin{aligned}
\kappa_1(t^*) &= n^{-1/2}\beta_{t1}^{(0)} + O(n^{-1}) = n^{-1/2}\{\beta_1^{(0)}(\beta_{2l}^{(0)})^{-1/2} + \beta_1^{(t0)}\} + O(n^{-1})\ (\beta_{2l}^{(0)} = \bar{\mathrm{i}}_{\theta_0}) \\
\kappa_2(t^*) &= \beta_2^{(0)}\bar{\mathrm{i}}_{\theta_0} + n^{-1/2}[n^{3/2}\mathrm{E}_{\mathrm{T}\alpha_0}\{(q^{(1a)}_{O_p(N^{-1/2})}\bar{\mathrm{i}}^{1/2}_{\theta_0})^2\}]_{O(1)} + n^{-1}\beta^{(0)}_{t\,\mathrm{H}2} + O(n^{-3/2}) \\
&\equiv \beta_2^{(0)}\bar{\mathrm{i}}_{\theta_0} + n^{-1/2}\bar{\beta}^{(\Delta)}_{\mathrm{h}2}\bar{\mathrm{i}}_{\theta_0} + n^{-1}\beta^{(0)}_{t\,\mathrm{H}2} + O(n^{-3/2}) \\
&\equiv \beta_2^{(0)}\bar{\mathrm{i}}_{\theta_0} + n^{-1/2}\bar{c}^*\beta^{(\Delta)}_{\mathrm{h}2}\bar{\mathrm{i}}_{\theta_0} + n^{-1}\beta^{(0)}_{t\,\mathrm{H}2} + O(n^{-3/2}) \\
&\equiv \beta_2^{(0)}\bar{\mathrm{i}}_{\theta_0} + n^{-1/2}\bar{\beta}^{(\Delta)}_{t\,\mathrm{h}2} + n^{-1}\beta^{(0)}_{t\,\mathrm{H}2} + O(n^{-3/2}) \\
&\equiv \beta_2^{(0)}\bar{\mathrm{i}}_{\theta_0} + n^{-1/2}\bar{c}^*\beta^{(\Delta)}_{t\,\mathrm{h}2} + n^{-1}\beta^{(0)}_{t\,\mathrm{H}2} + O(n^{-3/2})
\end{aligned} \tag{4.10}$$

($\bar{\beta}^{(\Delta)}_{\mathrm{h}2}$, $\beta^{(\Delta)}_{\mathrm{h}2}$ *and* $\beta^{(0)}_{t\,\mathrm{H}2}$ *were defined earlier;* $\bar{\beta}^{(\Delta)}_{t\,\mathrm{h}2} = \bar{\beta}^{(\Delta)}_{\mathrm{h}2}\bar{\mathrm{i}}_{\theta_0} = \bar{c}^*\beta^{(\Delta)}_{\mathrm{h}2}\bar{\mathrm{i}}_{\theta_0}$, $\bar{c}^*\beta^{(\Delta)}_{t\,\mathrm{h}2} = \bar{\beta}^{(\Delta)}_{t\,\mathrm{h}2}$),

$$\kappa_3(t^*) = n^{-1/2}\beta_{t3}^{(0)} + O(n^{-1}), \qquad \kappa_4(t^*) = n^{-1}\beta_{t4}^{(0)} + O(n^{-3/2})$$

($\beta_{tk}^{(0)}$ $(k = 1, 3, 4)$ *were defined earlier).*

From Theorem 5, it is seen that the effect of fallible item parameters on the asymptotic cumulants of t^* in (4.10) is only the added higher-order asymptotic variance $n^{-1/2}\bar{c}^*\beta^{(\Delta)}_{t\,\mathrm{h}2}$, which is similar to the case of the non-studentized $\hat{\theta}$ under Condition B (see Theorem 2). Under c.m.s., the asymptotic variance $\beta_2^{(0)}\bar{\mathrm{i}}_{\theta_0}$ becomes 1.

4.3. Studentization under Condition C: $N = O(n^2)$ ($\bar{c}^{**} = n^2/N = O(1)$)

Under Condition C, the same studentization as under Condition B is used, i.e., $t^* = n^{1/2}(\hat{\theta} - \theta_0)\hat{\mathrm{i}}^{1/2}$. Using (A.3), expand $\hat{\mathrm{i}}^{1/2}$ as

$$\begin{aligned}
\hat{\mathrm{i}}^{1/2} &= \hat{\mathrm{i}}^{1/2}_{\theta_0} + \hat{\mathrm{i}}^{(1)}_{\theta_0}(\hat{\theta} - \theta_0) + \hat{\mathrm{i}}^{(2)}_{\theta_0}(\hat{\theta} - \theta_0)^2 + O_p(n^{-3/2}) \\
&= \hat{\mathrm{i}}^{1/2}_{\theta_0} + \hat{\mathrm{i}}^{(1)}_{\theta_0}\{q^{(10)}_{O_p(n^{-1/2})} + q^{(20)}_{O_p(n^{-1})} + q^{(21)}_{O_p(N^{-1/2})} - n^{-1}\lambda^{-1}_{\theta_0}\eta_{\theta_0}\} + \hat{\mathrm{i}}^{(2)}_{\theta_0}(q^{(10)}_{O_p(n^{-1/2})})^2 + O_p(n^{-3/2}) \\
&= \bar{\mathrm{i}}^{1/2}_{\theta_0} + (\bar{\mathrm{i}}^{(1)}_{\theta_0}q^{(10)}_{O_p(n^{-1/2})})_{O_p(n^{-1/2})} + \Bigg[\left\{\bar{\mathrm{i}}^{(1)}_{\theta_0}q^{(20)}_{O_p(n^{-1})} + \bar{\mathrm{i}}^{(2)}_{\theta_0}(q^{(10)}_{O_p(n^{-1/2})})^2\right\}_{O_p(n^{-1})} \\
&\quad + \left(\frac{\bar{\mathrm{i}}^{-1/2}_{\theta_0}}{2}\frac{\partial\bar{\mathrm{i}}_{\theta_0}}{\partial\boldsymbol{\alpha}'_0}(\boldsymbol{\Gamma}^{(1)}_{\alpha_0}\mathbf{l}^{(1)}_{\alpha_0})_{O_p(N^{-1/2})} + \bar{\mathrm{i}}^{(1)}_{\theta_0}q^{(21)}_{O_p(N^{-1/2})}\right)_{O_p(N^{-1/2})}\Bigg]_{O_p(n^{-1})} - n^{-1}\bar{\mathrm{i}}^{(1)}_{\theta_0}\lambda^{-1}_{\theta_0}\eta_{\theta_0} + O_p(n^{-3/2}) \\
&\equiv \bar{\mathrm{i}}^{1/2}_{\theta_0} + j^{(10)}_{O_p(n^{-1/2})} + (j^{(20)}_{O_p(n^{-1})} + j^{(21)}_{O_p(N^{-1/2})})_{O_p(n^{-1})} - n^{-1}\bar{\mathrm{i}}^{(1)}_{\theta_0}\lambda^{-1}_{\theta_0}\eta_{\theta_0} + O_p(n^{-3/2}) \\
&\equiv \bar{\mathrm{i}}^{1/2}_{\theta_0} + j^{(1)}_{O_p(n^{-1/2})} + j^{(2)}_{O_p(n^{-1})} - n^{-1}\bar{\mathrm{i}}^{(1)}_{\theta_0}\lambda^{-1}_{\theta_0}\eta_{\theta_0} + O_p(n^{-3/2})\ (j^{(1)}_{O_p(n^{-1/2})} = j^{(10)}_{O_p(n^{-1/2})}).
\end{aligned} \tag{4.11}$$

Using (4.7) and (4.11) with (A.3), t^* is expanded as

$$\begin{aligned}
t^* &= n^{1/2}\{q^{(1)}_{O_p(n^{-1/2})} + q^{(2)}_{O_p(n^{-1})} + q^{(3)}_{O_p(n^{-3/2})} - n^{-1}\lambda^{-1}_{\theta_0}\eta_{\theta_0} + O_p(n^{-2})\} \\
&\quad \times \{\bar{\mathrm{i}}^{1/2}_{\theta_0} + j^{(1)}_{O_p(n^{-1/2})} + j^{(2)}_{O_p(n^{-1})} - n^{-1}\bar{\mathrm{i}}^{(1)}_{\theta_0}\lambda^{-1}_{\theta_0}\eta_{\theta_0} + O_p(n^{-3/2})\} \\
&= n^{1/2}\Big[q^{(1)}_{O_p(n^{-1/2})}\bar{\mathrm{i}}^{1/2}_{\theta_0} + (q^{(1)}_{O_p(n^{-1/2})}j^{(1)}_{O_p(n^{-1/2})} + q^{(2)}_{O_p(n^{-1})}\bar{\mathrm{i}}^{1/2}_{\theta_0}) + \{q^{(1)}_{O_p(n^{-1/2})}(j^{(2)}_{O_p(n^{-1})} - n^{-1}\bar{\mathrm{i}}^{(1)}_{\theta_0}\lambda^{-1}_{\theta_0}\eta_{\theta_0}) \\
&\quad + (q^{(2)}_{O_p(n^{-1})} - n^{-1}\lambda^{-1}_{\theta_0}\eta_{\theta_0})j^{(1)}_{O_p(n^{-1/2})} + q^{(3)}_{O_p(n^{-3/2})}\bar{\mathrm{i}}^{1/2}_{\theta_0}\} - n^{-1}\lambda^{-1}_{\theta_0}\eta_{\theta_0}\bar{\mathrm{i}}^{1/2}_{\theta_0}\Big] + O_p(n^{-3/2}) \\
&\equiv n^{1/2}(t^{(*1)}_{O_p(n^{-1/2})} + t^{(*2)}_{O_p(n^{-1})} + t^{(*3)}_{O_p(n^{-3/2})} - n^{-1}\lambda^{-1}_{\theta_0}\eta_{\theta_0}\bar{\mathrm{i}}^{1/2}_{\theta_0}) + O_p(n^{-3/2}) \\
&= n^{1/2}q^{(10)}_{O_p(n^{-1/2})}\bar{\mathrm{i}}^{1/2}_{\theta_0} + n^{1/2}\{q^{(10)}_{O_p(n^{-1/2})}j^{(10)}_{O_p(n^{-1/2})} + (q^{(20)}_{O_p(n^{-1})} + q^{(21)}_{O_p(N^{-1/2})})\bar{\mathrm{i}}^{1/2}_{\theta_0}\} \\
&\quad + n^{1/2}\{q^{(10)}_{O_p(n^{-1/2})}(j^{(20)}_{O_p(n^{-1})} + j^{(21)}_{O_p(N^{-1/2})} - n^{-1}\bar{\mathrm{i}}^{(1)}_{\theta_0}\lambda^{-1}_{\theta_0}\eta_{\theta_0}) + (q^{(20)}_{O_p(n^{-1})} + q^{(21)}_{O_p(N^{-1/2})} - n^{-1}\lambda^{-1}_{\theta_0}\eta_{\theta_0})j^{(10)}_{O_p(n^{-1/2})} \\
&\quad + (q^{(30)}_{O_p(n^{-3/2})} + q^{(31)}_{O_p(n^{-1/2}N^{-1/2})})\bar{\mathrm{i}}^{1/2}_{\theta_0}\} - n^{-1/2}\lambda^{-1}_{\theta_0}\eta_{\theta_0}\bar{\mathrm{i}}^{1/2}_{\theta_0} + O_p(n^{-3/2}).
\end{aligned} \tag{4.12}$$

In (4.12), the terms due to fallible item parameters are $q^{(21)}_{O_p(N^{-1/2})}\bar{\mathrm{i}}^{1/2}_{\theta_0}$ in $t^{(*2)}_{O_p(n^{-1})}$, and $q^{(10)}_{O_p(n^{-1/2})}j^{(21)}_{O_p(N^{-1/2})}$, $q^{(21)}_{O_p(N^{-1/2})}j^{(10)}_{O_p(n^{-1/2})}$ and $q^{(31)}_{O_p(n^{-1/2}N^{-1/2})}\bar{\mathrm{i}}^{1/2}_{\theta_0}$ in $t^{(*3)}_{O_p(n^{-3/2})}$. From (A.4) with this property, the following results are obtained.

Theorem 6. *Under Condition* C, *the asymptotic cumulants of* t^* *up to the fourth order and the higher-order asymptotic variance with associated assumptions and p.m.m. are given by*

$$\begin{aligned}
\kappa_1(t^*) &= n^{-1/2}\beta_{t1}^{(0)} + O(n^{-3/2}) = n^{-1/2}\{\beta_1^{(0)}(\beta_{2I}^{(0)})^{-1/2} + \beta_1^{(t0)}\} + O(n^{-3/2})\ (\beta_{2I}^{(0)} = \bar{i}_{\theta_0}) \\
\kappa_2(t^*) &= \beta_2^{(0)}\bar{i}_{\theta_0} + n^{-1}[\beta_{t\,\mathrm{H2}}^{(0)} + n^2 \mathrm{E}_{\mathrm{T}\alpha_0}\{(q^{(21)}_{O_p(N^{-1/2})}\bar{i}_{\theta_0}^{1/2})^2\}]_{O(1)} + O(n^{-2}) \\
&\equiv \beta_2^{(0)}\bar{i}_{\theta_0} + n^{-1}(\beta_{t\,\mathrm{H2}}^{(0)} + \bar{\beta}_{\mathrm{H2}}^{(t*\Delta)}) + O(n^{-2}), \\
&\equiv \beta_2^{(0)}\bar{i}_{\theta_0} + n^{-1}(\beta_{t\,\mathrm{H2}}^{(0)} + \bar{c}^{**}\beta_{\mathrm{H2}}^{(t*\Delta)}) + O(n^{-2}), \\
\kappa_3(t^*) &= n^{-1/2}\beta_{t3}^{(0)} + O(n^{-3/2}),\ \kappa_4(t^*) = n^{-1}\beta_{t4}^{(0)} + O(n^{-2}).
\end{aligned} \tag{4.13}$$

From Theorem 6, it is found that the effect of fallible item parameters on (4.13) is only the added term $n^{-1}\bar{\beta}_{\mathrm{H2}}^{(t*\Delta)}$ in the higher-order asymptotic variance, where $\beta_{\mathrm{H2}}^{(t*\Delta)} = \beta_{t\,\mathrm{h2}}^{(\Delta)}$ in Theorem 5. Under c.m.s., the asymptotic variance $\beta_2^{(0)}\bar{i}_{\theta_0}$ becomes 1.

5. Numerical illustration

In this section, the asymptotic cumulants derived in the previous sections are numerically illustrated with simulations for true values. The 2PLM is used with and without m.m. The m.m. is specified only when estimating ability such that the logit of P_k ($k = 1, \ldots, n$) is perturbed using a random term following the normal distribution [28]. In the perturbation, the initial item parameters are set to satisfy the likelihood equations using $P_{\mathrm{T}k}$ for U_k ($k = 1, \ldots, n$). For item calibration, the 2PLM is used as a true model without m.m. Under m.m. in ability estimation, the correlations of $P_{\mathrm{T}k}$ and P_k over items are 0.56, 0.55, 0.48, and 0.52 when $\theta = -1$, 0, 1, and 2, respectively.

Currently, in ability tests based on IRT, the 3PLM may be the standard one rather than the 2PLM. In this section, the latter is used for simplicity and for the availability of the MMLEs for item parameters without difficulty. The item parameters are randomly generated using a uniform distribution with the range [0.3, 1.3] and $N(0.2,\ 1^2)$ for a_k and b_k with $P_k = 1/[1 + \exp\{-1.7a_k(\theta - b_k)\}]$ ($k = 1, \ldots, n$), respectively. The ML, BM, and WL estimators are used, where the WLE is equal to the JME in the 2PLM. For estimating the item parameters without m.m., MML and the Bayes method with the independent log-normal priors for a-parameters i.e., $\log a_k \sim N(0,\ 0.5^2)$ ($k = 1, \ldots, n$) (see e.g., [2]) are used. For the distribution of abilities to be integrated out in MML and MBM estimation, the standard normal is assumed.

The sample sizes $n = 30, 50, 70$ and $N = 500, 1000$, considering typical ones encountered in practice, are employed (the small size $N = 200$ is also partially used for comparison). The proficiency levels $\theta = -1$, 0, 1, and 2 are adopted for illustration, since the levels $\theta = -3, -2$ and 3 tend to give unstable or non-converged estimation of ability. In order to have the asymptotic cumulants of $\hat{\theta}$ based on estimated item parameters under c.m.s., several expectations $\mathrm{E}_{\boldsymbol{\alpha}_0}(\cdot)$ are required. In principle, the expectations can be computed using the probabilities of 2^n response patterns given by the 2PLM. When n is large, however, the amount of computation becomes excessive. In this section, an imputation method of randomly generated $N^{\#}$(=2000) patterns according to the multinomial distribution with the vector $\boldsymbol{\pi}_{\mathrm{T}}$ of probabilities is used, each with the pseudo probability $1/N^{\#}$ for calculating the expectations ($N^{\#}$ should not be confused with N; the results with $N^{\#} = 10{,}000$ are almost the same). On the other hand, simulated cumulants are given by randomly generating N sets of item responses for item calibration followed by estimation of ability using the estimated item parameters and randomly generated n item responses of an examinee with θ. The cases of known item parameters without the estimation of item parameters are also used for comparison. The whole process is replicated 1000 and 10,000 times for the cases with and without estimation of item parameters, respectively.

Tables 1–8 show the simulated and asymptotic cumulants. The tables include the numbers of deleted cases until 1000 or 10,000 regular cases were obtained. The deleted cases are due to non-convergence in estimating $\boldsymbol{\alpha}_0$ and θ_0, including some cases of perfect score when $\theta = 2$, where the finite value of the MLE is not available.

Tables 1–3 show the various standard errors under c.m.s. The SD values are simulated ones while the (H)ASE values are asymptotic ones. Note that $\mathrm{HASE}^{(0)}$ is the higher-order asymptotic standard error of $\hat{\theta}$ with known item parameters [28], while that of $\hat{\theta}$ with estimated item parameters under Condition A is missing in the tables due to the complicated expression (Theorem 1). $\mathrm{HASE}^{(1)}$ is that up to order $O(n^{-2})$ under Conditions B and C (Theorems 2 and 3), which is given by $\mathrm{HASE}^{(0)}$ and the difference of the squared $\mathrm{ASE}^{(0)} = (n^{-1}\beta_2^{(0)})^{1/2}$ and $\mathrm{ASE}^{(1)} = \{n^{-1}(\beta_2^{(0)} + \bar{c}\beta_{2I}^{(\Delta)})\}^{1/2}$, i.e., $\mathrm{HASE}^{(1)} = \{(n^{-1}\beta_2^{(0)} + n^{-2}\beta_{\mathrm{H2}}^{(0)}) + n^{-1}\bar{c}\beta_{2I}^{(\Delta)}\}^{1/2} = \{(\mathrm{HASE}^{(0)})^2 + n^{-1}\bar{c}\beta_{2I}^{(\Delta)}\}^{1/2}$, where $\beta_2^{(0)} = \beta_{2I}^{(0)} = \bar{i}_{\theta_0}^{-1}$ under c.m.s. (recall that $n^{-1}\bar{c} = n^{-3/2}\bar{c}^* = n^{-2}\bar{c}^{**}$ when seeing the tables under Conditions A–C).

From Tables 1 and 2, it is seen that the $\mathrm{SD}^{(0)}$ values (and $\mathrm{SD}^{(1)}$ values) are somewhat different for the ML, BM, and WL estimators, and that the increases of $\mathrm{SD}^{(1)}$, $\mathrm{ASE}^{(1)}$ and $\mathrm{HASE}^{(1)}$ over $\mathrm{SD}^{(0)}$, $\mathrm{ASE}^{(0)}$ and $\mathrm{HASE}^{(0)}$, respectively, are small. That is, the different $\mathrm{SD}^{(1)}$ values are similar to the corresponding $\mathrm{HASE}^{(1)}$ values or $\mathrm{HASE}^{(0)}$ values rather than the $\mathrm{ASE}^{(1)}$ values, which suggests the reasonableness of the assumptions in Conditions B and C in these cases. However, when n is as large as 70 in Table 3, the increase of the $\mathrm{SD}^{(1)}$ values over the corresponding $\mathrm{SD}^{(0)}$ values becomes substantial, and the approximations

Table 1

Simulated and asymptotic standard errors of $\hat{\theta}$ when the 2PLM holds, where the item parameters are known or estimated by MML ($n = 30$).

Standard error (number of cases deleted)		Known item parameters			Estimated item parameters					
		$SD^{(0)}$	$ASE^{(0)}$	$HASE^{(0)}$	$N = 500$			$N = 1000$		
					$SD^{(1)}$	$ASE^{(1)}$	$HASE^{(1)}$	$SD^{(1)}$	$ASE^{(1)}$	$HASE^{(1)}$
	ML	0.379	0.356	0.374	0.379	0.361	0.378	0.393	0.359	0.376
$\theta = -1(0/1/0)$	BM	0.305	*	0.300	0.309	*	0.305	0.316	*	0.302
	WL	0.358	*	0.360	0.360	*	0.364	0.372	*	0.362
	ML	0.302	0.296	0.303	0.326	0.299	0.305	0.305	0.298	0.304
$\theta = 0(0/3/0)$	BM	0.276	*	0.276	0.297	*	0.279	0.278	*	0.278
	WL	0.295	*	0.297	0.319	*	0.299	0.298	*	0.298
	ML	0.325	0.312	0.326	0.335	0.317	0.330	0.347	0.314	0.328
$\theta = 1(0/1/0)$	BM	0.282	*	0.281	0.291	*	0.286	0.300	*	0.284
	WL	0.313	*	0.314	0.324	*	0.318	0.335	*	0.316
	ML	0.457	0.423	0.448	0.485	0.433	0.458	0.461	0.428	0.453
$\theta = 2(33/2/2)$	BM	0.305	*	0.227	0.317	*	0.245	0.301	*	0.236
	WL	0.417	*	0.431	0.439	*	0.441	0.417	*	0.436

Note. n = the number of items, N = the number of examinees when item parameters are estimated, $SD^{(0)}$ ($SD^{(1)}$) = the standard deviation of $\hat{\theta}$ from simulations, $ASE^{(0)} = (n^{-1}\beta_2^{(0)})^{1/2}$, $ASE^{(1)} = (n^{-1}\bar{\beta}_{2l})^{1/2} = \{n^{-1}(\beta_2^{(0)} + \bar{c}\beta_{2l}^{(\Delta)})\}^{1/2} = \{(ASE^{(0)})^2 + n^{-1}\bar{c}\beta_{2l}^{(\Delta)}\}^{1/2}$, $HASE^{(0)} = (n^{-1}\beta_2^{(0)} + n^{-2}\beta_{H2}^{(0)})^{1/2}$, $HASE^{(1)} = (n^{-1}\bar{\beta}_{2l} + n^{-2}\beta_{H2}^{(0)})^{1/2} = \{(HASE^{(0)})^2 + n^{-1}\bar{c}\beta_{2l}^{(\Delta)}\}^{1/2}$, ML = maximum likelihood, BM = Bayes modal, WL = weighted likelihood. The asterisks denote that the corresponding values by ML hold. (#1/#2/#3) indicates that #1, #2 and #3 cases for $SD^{(0)}$, $SD^{(1)}$ ($N = 500$) and $SD^{(0)}$ ($N = 1000$) were deleted until 10,000, 1000, and 1000 regular cases were obtained in the simulations, respectively. Under c.m.s., $\beta_2^{(0)} = \beta_{2l}^{(0)} = \bar{i}_{\theta_0}^{-1}$.

Table 2

Simulated and asymptotic standard errors of $\hat{\theta}$ when the 2PLM holds, where the item parameters are known or estimated by MML ($n = 50$).

Standard error (number of cases deleted)		Known item parameters			Estimated item parameters					
		$SD^{(0)}$	$ASE^{(0)}$	$HASE^{(0)}$	$N = 500$			$N = 1000$		
					$SD^{(1)}$	$ASE^{(1)}$	$HASE^{(1)}$	$SD^{(1)}$	$ASE^{(1)}$	$HASE^{(1)}$
	ML	0.334	0.330	0.334	0.364	0.333	0.337	0.345	0.331	0.335
$\theta = -1(0/0/0)$	BM	0.287	*	0.279	0.312	*	0.283	0.295	*	0.281
	WL	0.329	*	0.331	0.359	*	0.334	0.340	*	0.333
	ML	0.280	0.270	0.275	0.304	0.272	0.277	0.287	0.271	0.276
$\theta = 0(0/0/0)$	BM	0.259	*	0.255	0.281	*	0.257	0.265	*	0.256
	WL	0.275	*	0.270	0.300	*	0.272	0.282	*	0.271
	ML	0.270	0.263	0.268	0.289	0.266	0.270	0.274	0.264	0.269
$\theta = 1(0/0/0)$	BM	0.248	*	0.245	0.264	*	0.248	0.250	*	0.246
	WL	0.265	*	0.263	0.284	*	0.266	0.269	*	0.265
	ML	0.348	0.326	0.345	0.389	0.333	0.352	0.391	0.329	0.349
$\theta = 2(0/1/0)$	BM	0.271	*	0.259	0.295	*	0.268	0.299	*	0.263
	WL	0.329	*	0.329	0.368	*	0.337	0.370	*	0.333

Note. See the footnote of Table 1.

Table 3

Simulated and asymptotic standard errors of $\hat{\theta}$ when the 2PLM holds, where the item parameters are known or estimated by MML ($n = 70$).

Standard error (number of cases deleted)		Known item parameters			Estimated item parameters					
		$SD^{(0)}$	$ASE^{(0)}$	$HASE^{(0)}$	$N = 500$			$N = 1000$		
					$SD^{(1)}$	$ASE^{(1)}$	$HASE^{(1)}$	$SD^{(1)}$	$ASE^{(1)}$	$HASE^{(1)}$
	ML	0.222	0.213	0.218	0.278	0.214	0.219	0.261	0.214	0.219
$\theta = -1(0/21/11)$	BM	0.206	*	0.203	0.255	*	0.204	0.238	*	0.203
	WL	0.217	*	0.214	0.273	*	0.215	0.256	*	0.214
	ML	0.190	0.186	0.188	0.247	0.186	0.188	0.235	0.186	0.188
$\theta = 0(0/23/10)$	BM	0.183	*	0.181	0.237	*	0.182	0.224	*	0.182
	WL	0.188	*	0.186	0.246	*	0.186	0.233	*	0.186
	ML	0.226	0.218	0.223	0.281	0.219	0.224	0.265	0.219	0.224
$\theta = 1(0/18/9)$	BM	0.209	*	0.207	0.256	*	0.208	0.242	*	0.207
	WL	0.222	*	0.219	0.277	*	0.220	0.261	*	0.220
	ML	0.327	0.313	0.322	0.387	0.316	0.325	0.390	0.315	0.323
$\theta = 2(0/27/16)$	BM	0.259	*	0.237	0.300	*	0.242	0.297	*	0.240
	WL	0.318	*	0.317	0.377	*	0.320	0.379	*	0.319

Note. See the footnote of Table 1.

Table 4

Simulated and asymptotic standard errors of $\hat{\theta}$ when the 2PLM does not hold in estimating $\hat{\theta}$, where the item parameters are known or estimated by MML, while the 2PLM holds when the item parameters are estimated ($n = 50$).

Standard error under m.m. (# of cases deleted)		Known item parameters			Estimated item parameters					
		$SD^{(0)}$	$ASE^{(0)}$	$HASE^{(0)}$	$N = 500$			$N = 1000$		
					$SD^{(1)}$	$ASE^{(1)}$	$HASE^{(1)}$	$SD^{(1)}$	$ASE^{(1)}$	$HASE^{(1)}$
	ML	0.326	0.322	0.325	0.341	0.329	0.331	0.347	0.325	0.328
$\theta = -1(0/4/0)$	BM	0.281	$*$	0.272	0.292	$*$	0.279	0.298	$*$	0.276
	WL	0.322	$*$	0.322	0.337	$*$	0.329	0.343	$*$	0.326
	ML	0.259	0.252	0.257	0.275	0.257	0.261	0.271	0.254	0.259
$\theta = 0(0/4/0)$	BM	0.240	$*$	0.238	0.254	$*$	0.243	0.250	$*$	0.241
	WL	0.255	$*$	0.253	0.271	$*$	0.257	0.266	$*$	0.255
	ML	0.246	0.243	0.247	0.268	0.248	0.252	0.264	0.246	0.249
$\theta = 1(0/2/0)$	BM	0.226	$*$	0.226	0.245	$*$	0.231	0.241	$*$	0.229
	WL	0.241	$*$	0.243	0.264	$*$	0.248	0.259	$*$	0.245
	ML	0.312	0.297	0.311	0.350	0.306	0.320	0.353	0.301	0.315
$\theta = 2(0/2/0)$	BM	0.245	$*$	0.231	0.269	$*$	0.243	0.271	$*$	0.237
	WL	0.295	$*$	0.297	0.332	$*$	0.306	0.335	$*$	0.301

Note. See the footnote of Table 1.

Table 5

Simulated standard errors of $\hat{\theta}$ when the 2PLM holds, where the item parameters are estimated by the Bayes method with the log-normal priors for a-parameters.

Standard error by Bayes method (# of cases deleted)		$SD^{(1)}$, estimated item parameters					
		$N = 500$			$N = 1000$		
		$n = 30$	$n = 50$	$n = 70$	$n = 30$	$n = 50$	$n = 70$
	ML	0.379	0.353	0.271	0.392	0.339	0.259
$\theta = -1(0/1/7/0/0/10)$	BM	0.309	0.305	0.248	0.316	0.292	0.237
	WL	0.361	0.348	0.266	0.372	0.335	0.254
	ML	0.328	0.296	0.235	0.306	0.283	0.233
$\theta = 0(0/1/13/0/0/11)$	BM	0.298	0.274	0.225	0.278	0.262	0.223
	WL	0.321	0.292	0.233	0.299	0.278	0.231
	ML	0.337	0.282	0.266	0.349	0.270	0.264
$\theta = 1(0/0/6/0/0/11)$	BM	0.291	0.259	0.243	0.300	0.247	0.241
	WL	0.325	0.278	0.262	0.336	0.265	0.259
	ML	0.485	0.372	0.377	0.462	0.383	0.389
$\theta = 2(2/2/11/2/0/16)$	BM	0.318	0.289	0.290	0.301	0.296	0.296
	WL	0.442	0.353	0.367	0.420	0.363	0.378

Note. (#1/#2/#3/#4/#5/#6) indicates that these numbers of cases were deleted under the six conditions in each line until regular 1000 cases were obtained, respectively. See also the footnote of Table 1.

Table 6

Simulated and asymptotic biases of $\hat{\theta}$ when the 2PLM holds, where the item parameters are known or estimated by MML ($n = 50$).

Bias		Known item parameters		Estimated item parameters			
		Sim.	Th.	$N = 500$		$N = 1000$	
				Sim.	Th.	Sim.	Th.
	ML	−1.38	−1.11	−1.08	−0.10	−1.82	−0.60
$\theta = -1$	BM	3.84	4.33	4.32	5.33	3.69	4.83
	WL	−0.21	0	0.07	1.00	−0.66	0.50
	ML	−0.51	−0.58	−0.85	−0.67	−0.48	−0.62
$\theta = 0$	BM	−0.40	−0.58	−0.71	−0.67	−0.37	−0.62
	WL	0.05	0	−0.28	−0.10	0.09	−0.05
	ML	0.45	0.38	1.85	−1.09	2.74	−0.36
$\theta = 1$	BM	−2.89	−3.08	−1.80	−4.55	−1.02	−3.81
	WL	0.07	0	1.46	−1.47	2.34	−0.73
	ML	2.00	1.76	4.26	−0.90	6.76	0.43
$\theta = 2$	BM	−8.11	−8.85	−6.65	−11.50	−4.73	−10.18
	WL	0.06	0	2.26	−2.65	4.68	−1.33

Note. Sim. = n times the simulated bias, Th. = $\bar{\beta}_1$ (n times the theoretical or asymptotic bias). See also the footnote of Table 1.

by $ASE^{(1)}$ and $HASE^{(1)}$ are poor, although the $HASE^{(1)}$ values retain their relative sizes similar to the simulated ones among the ML, BM, and WL estimators.

Table 7
Simulated and asymptotic third cumulants of $\hat{\theta}$ when the 2PLM holds, where the item parameters are known or estimated by MML ($n = 50$).

Third cumulant		Known item parameters		Estimated item parameters			
		Sim.	Th.	$N = 500$		$N = 1000$	
				Sim.	Th.	Sim.	Th.
$\theta = -1$	ML	−27.8	−24.0	−30.6	−25.7	−33.6	−24.9
	BM	−10.8	*	−8.3	*	−14.4	*
	WL	−23.5	*	−24.7	*	−29.1	*
$\theta = 0$	ML	−12.2	−8.4	−12.3	−8.6	−4.0	−8.5
	BM	−8.0	*	−7.8	*	−1.4	*
	WL	−12.1	*	−12.5	*	−4.2	*
$\theta = 1$	ML	7.3	5.2	14.3	5.2	5.1	5.4
	BM	3.6	*	8.4	*	1.6	*
	WL	6.7	*	13.2	*	4.6	*
$\theta = 2$	ML	60.1	37.2	122.2	38.8	72.6	38.8
	BM	10.8	*	20.9	*	6.1	*
	WL	44.1	*	92.0	*	52.4	*

Note. Sim. = n^2 times the simulated third cumulant, Th. = $\bar{\beta}_3$ (n^2 times the theoretical or asymptotic third cumulant). See also the footnote of Table 1.

Table 8
Simulated and asymptotic fourth cumulants of $\hat{\theta}$ when the 2PLM holds, where the item parameters are known or estimated by MML ($n = 50$).

Fourth cumulant		Known item parameters		Estimated item parameters	
		Sim.	Th.	$N = 500$	$N = 1000$
				Sim.	Sim.
$\theta = -1$	ML	463	225	595	224
	BM	80	*	−129	−20
	WL	275	*	278	69
$\theta = 0$	ML	167	100	3	196
	BM	89	*	−32	120
	WL	167	*	19	191
$\theta = 1$	ML	130	73	239	189
	BM	62	*	136	107
	WL	110	*	209	165
$\theta = 2$	ML	1660	813	5988	2742
	BM	47	*	405	1747
	WL	1004	*	4023	1828

Note. Sim. = n^3 times the simulated fourth cumulant, Th. = $\bar{\beta}_4$ (n^3 times the theoretical or asymptotic fourth cumulant). See also the footnote of Table 1.

Table 4 gives the results with $n = 50$ under m.m. when estimating ability. Although the values of the standard errors are somewhat smaller than those in Table 2, similar results are found. The reduction of the values can be seen as a kind of regression effect under m.m. [28].

Table 5 shows the simulated standard errors or $\mathrm{SD}^{(1)}$ values when item parameters are estimated by the Bayes method with log-normal priors for a-parameters. The values correspond to those in Tables 1–3. Note that the asymptotic values are unchanged as far as those in the tables are concerned. It is seen that, when $n = 30$, the values are almost the same as in Table 1, while, when $n = 50$ and 70, they are slightly smaller than those in Tables 2 and 3.

In Tables 6–8, the results of biases, the third cumulants, and the fourth ones independent of n when $n = 50$ under c.m.s. are shown. In Table 8, the asymptotic fourth cumulants with estimated item parameters are not given, due to its complicated formula under Condition A, while those common to the case with known item parameters, and the cases based on estimated item parameters under Conditions B and C, are shown. The tables show that the cumulants with estimated item parameters are in a crude sense similar to those with known item parameters, with some exceptions for the BME in Table 8.

Table 9 gives the results with $n = 50$ when N is as small as 200, which is an inappropriate one for stable item calibration. The Bayesian method for estimating item parameters is used, since, when MML was used, 4% or 5% of the cases had to be discarded at each level of ability due mainly to non-convergence in the estimation of item parameters while only less than 1% is discarded at each ability level in Table 9. It is found that the simulated and asymptotic standard errors have not increased so much (compare with Tables 1 and 5). It is ironical that, when N is as small as 200, the $\mathrm{ASE}^{(1)}$ values in Table 9 are reasonable, especially when θ is not 0. Note that the set of sample sizes in Table 9 was also used by Hoshino and Shigemasu [13, Fig. 1], with similar patterns repeated in Table 9. The results of bias in Table 9 does not correspond to those in Table 6 given by MML, since $\boldsymbol{\eta}_{\alpha_0}$, due to the log-normal priors, influences the biases in Table 9, which seem to be different from those in Table 6. The third and fourth cumulants in Table 9 are similar to the corresponding ones in Tables 7 and 8, respectively.

Table 9
Simulated and asymptotic cumulants of $\hat{\theta}$ when the 2PLM holds, where the item parameters are estimated by the Bayes method with log-normal priors for a-parameters and N is small ($n = 50$).

Small N, Bayes method (# of cases deleted)		$N = 200$, estimated item parameters except ASE$^{(0)}$								
		Standard error				Bias		Third cumulant		Fourth cumulant
		SD$^{(1)}$	ASE$^{(1)}$	ASE$^{(0)}$	HASE$^{(1)}$	Sim.	Th.	Sim.	Th.	Sim.
$\theta = -1(2)$	ML	0.343	0.338	0.330	0.342	1.99	4.38	−31.4	−27.6	553
	BM	0.300	*	*	0.289	6.54	9.81	−13.0	*	143
	WL	0.338	*	*	0.339	3.13	5.49	−26.2	*	349
$\theta = 0(6)$	ML	0.289	0.275	0.270	0.280	−0.69	−3.13	−8.7	−8.7	121
	BM	0.269	*	*	0.260	−0.58	−3.13	−5.4	*	53
	WL	0.286	*	*	0.275	−0.17	−2.55	−8.7	*	124
$\theta = 1(8)$	ML	0.294	0.270	0.263	0.275	−1.45	−10.19	2.0	2.1	2
	BM	0.271	*	*	0.253	−4.55	−13.65	−0.5	*	−13
	WL	0.289	*	*	0.270	−1.82	−10.57	1.6	*	−4
$\theta = 2(8)$	ML	0.365	0.344	0.326	0.362	−1.19	−16.71	64.3	30.2	2153
	BM	0.292	*	*	0.281	−10.55	−27.32	11.8	*	247
	WL	0.350	*	*	0.347	−3.02	−18.47	49.4	*	1524

Note. (#) indicates that the number # of cases were deleted until regular 1000 cases were obtained. Th. of bias = $\bar{\beta}_1$, Th. of third cumulant = $\bar{\beta}_3$, Sim. = the simulated values corresponding to Th. independent of n. See also the footnote of Table 1.

[33, Tables A1 to A4] give the results for the studentized $\hat{\theta}$, where new $t^*_{\alpha_0}$ and $\hat{i}_{\alpha_0}$ are t^* and $\hat{i}$, respectively, with $\hat{\boldsymbol{\alpha}}$ replaced by $\boldsymbol{\alpha}_0$ and $\hat{\theta}$ being unchanged. In Table A1, HASE$^{(t1)}$ = HASE$^{(t0)}$ is by construction. In Table A2, the studentization for the MLE with known item parameters gives zero asymptotic bias [30, Corollary 1], and MLE = BME when $\theta = 0$, since $\eta_\theta = -\theta = 0$. In Tables A1 to A3, the values based on estimated item parameters are in a crude sense similar to those with known item parameters, while, in Table A4, the simulated fourth cumulants of t are different between when $N = 500$ and 1000, indicating some instability for these cases.

The computer program for the asymptotic cumulants in the numerical illustration was coded in Fortran90. Though the program is long, and is not necessarily user friendly, it is available upon request to the author for interested readers.

6. Concluding remarks

This paper gives the asymptotic cumulants up to the fourth order and the higher-order asymptotic variances of the ML, Bayes, and pseudo Bayes modal estimators of ability when the item parameters used are estimated by MML and the Bayes method under p.m.m. with and without studentization. Among them, so far, only the asymptotic bias with the assumption of uncorrelated estimators of item parameters over different items and the asymptotic variance are known under Condition A. While full results without using the assumption of uncorrelated estimators are given in this paper, the results of the higher-order asymptotic variance and the fourth asymptotic cumulant are complicated under Condition A. On the other hand, under Condition B ($N = O(n^{3/2})$) and Condition C ($N = O(n^2)$), the results become extremely simple. That is, only the higher-order asymptotic variances with estimated item parameters are different from those with known item parameters under Conditions B and C as far as the asymptotic cumulants up to the fourth order and the higher-order asymptotic variance are concerned.

Conditions B and C correspond to situations when N is relatively larger than n with different orders, which may represent typical situations in practice when item calibration is appropriately performed with large sample sizes. In many cases, the estimation of ability has been carried out using the assumption of fixed item parameters though they are actually estimated ones. Theorems 2 and 5 under Condition B, and Theorems 3 and 6 give asymptotic justification for these practices. In Section 2.1, the condition $N = O(n^{5/2})$ was mentioned as an unrealistic one. However, it can be shown that by this assumption even the higher-order asymptotic variance based on estimated item parameters becomes equal to that with known ones.

The relative effect of estimated item parameters on estimation of ability is large when n is large and when N is small. In the numerical illustration shown earlier with $n = 50$ and $N = 500$ and 1000, the relative effect of estimated item parameters was rather small. Recall that the differences between the ML, BM, and WL estimators are dominant in these cases. On the other hand, when n is as large as 70 with $N = 500$ and 1000, the relative contribution of estimated item parameters becomes substantial, where ASE$^{(1)}$ considering this sampling variation is still somewhat different from the simulated value. The last poor behavior of ASE$^{(1)}$ of order $O(n^{1/2})$ may be partially explained by the slow convergence of the (higher-order) asymptotic variances for $\hat{\boldsymbol{\alpha}}$ (see e.g., [26, Table 1]).

As explained earlier, all the asymptotic cumulants require some expectations over 2^n response patterns. The usual size n of ability tests inhibits the calculation, requiring some imputation methods. While the use of asymptotic multivariate normality for $\hat{\boldsymbol{\alpha}}$ [52] in the imputation is convenient, it is to be noted that the asymptotic cumulants higher than the second order, and the higher-order asymptotic variance under Condition A depend on the cumulants of the $\hat{\boldsymbol{\alpha}}$ being higher than the second-order ones (variances and covariances). The imputation method in this paper considers this property.

An application of the asymptotic cumulants of the non-studentized estimator $\hat{\theta}$ is to improve the point estimator by, for example, bias reduction. Under Conditions A–C, this can be done by using Theorems 1–3 as $\hat{\theta} - n^{-1}\hat{\bar{\beta}}_1$, $\hat{\theta} - n^{-1}\hat{\beta}_1^{(0)}$ and $\hat{\theta} - n^{-1}\hat{\beta}_1^{(0)}$, respectively, where $\hat{\bar{\beta}}_1$ and $\hat{\beta}_1^{(0)}$ are sample versions of $\bar{\beta}_1$ and $\beta_1^{(0)}$, respectively. Note that the formulas under Conditions B and C are identical.

On the other hand, the asymptotic cumulants of the studentized estimators, i.e. t and t^*, can be used for interval estimation. Define $\Phi(z_{\alpha^*}) = 1 - \alpha^*$ (e.g., $\alpha^* = 0.05$), where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal. Then, the endpoints of the two-sided Wald confidence interval (CI) of θ_0 under Conditions A and B (C) with first-order accuracy and asymptotic confidence level $1 - \alpha^*$ are $\hat{\theta} \pm z_{\alpha^*/2} n^{-1/2} \hat{\bar{\beta}}_{2I}^{1/2}$ and $\hat{\theta} \pm z_{\alpha^*/2} n^{-1/2} \hat{\imath}^{-1/2}$, respectively. The second-order and third-order accurate CIs can also be obtained, at least in principle, by using typically the Cornish–Fisher expansions and sample versions of the asymptotic cumulants up to the fourth order and the higher-order asymptotic variance of the studentized estimator, depending on the accuracy orders (see e.g., [29]).

Acknowledgment

This work was partially supported by a Grant-in-Aid for Scientific Research from the Japanese Ministry of Education, Culture, Sports, Science and Technology No. 23500341.

Appendix

A.1. Some known and derived results

A.1.1. A review of the estimation of ability based on estimated item parameters

Investigations on the estimation of ability based on fallible or estimated item parameters can be classified into several groups: (a) the method of expected response function, (b) Bayesian estimation, (c) an application of pseudo ML estimation, (d) logistic regression with estimated covariates, and (e) multiple imputation.

Group (a): Lewis [16] proposed the use of the expected response function (ERF), i.e., the posterior mean of the probability of a correct response to an item given an ability level, which yields a new estimator of the ability based on the ERF. Lewis [17] fully explained the above method with numerical illustration using the 1PLM. Mislevy, Wingersky, and Sheehan [25] derived a method using the ERF, where multiple imputation was used for estimation of the distribution of estimated item parameters.

Group (b): The method in this group considers the simultaneous distribution of ability and item parameter estimators in the Bayesian framework. Tsutakawa and Soltys [50] obtained an approximate mean and standard deviation of an ability estimator when the posterior mean vector and variance–covariance matrix of estimated item parameters are given for the 2PLM. Tsutakawa and Johnson [48] gave an approximate mean and variance of the posterior distribution of ability based on estimated item parameters for the 3PLM. Albert [1] obtained a posterior simultaneous distribution of ability and item parameters using Gibbs sampling for the two-parameter normal ogive model. Patz and Junker [36] derived a method of simultaneous estimation of ability and item parameters for the 2PLM using the Markov chain Monte Carlo (MCMC) method.

Group (c): Pseudo ML (PML) estimation deals with two sets of parameters, where the first set of parameters is estimated by a non-ML method, e.g., least squares and moment methods, and the second set, usually of primary interest, is estimated by ML given the estimates in the first set. Gong and Samaniego [12, Eq. (2.6)] introduced PML estimation and derived the asymptotic variance of the estimator of the second set (PMLE), when the asymptotic variance of the estimator of the first set and the asymptotic covariance Σ_{12} of the estimator of the first set and the mean log-likelihood derivative of the PMLE are given. Note that $\Sigma_{12} = 0$ when the non-ML estimation is asymptotically equal to the MLE. Parke [35, Eq. (2.1)] relaxed the condition for $\Sigma_{12} = 0$ to any consistent estimators of the first set using the results of Pierce [37, Section 4]. The condition $\Sigma_{12} = 0$ corresponds to the independent condition for ability and item parameter estimation employed in this paper, which comes from two independent samples, one from item calibration and the other for ability estimation.

Yuan and Jennrich [53] generalized PML estimation using a generalized estimation equation, and obtained the asymptotic distribution of the PMLE and its conditions. Hoshino and Shigemasu [13] gave the asymptotic variance of the ability estimator based on estimated item parameters for the 1PLM, the 2PLM, and the 3PLM as an application of Parke's [35] formula mentioned above. Cheng and Yuan [9] rediscovered Hoshino and Shigemasu's [13] result, and showed in the case of the 2PLM that the effect of estimated item parameters is relatively small in the central area of ability, i.e., around zero. The results of Hoshino and Shigemasu [13, Fig. 1] also show a similar pattern, though they did not mention it. The results of the current paper are seen as extensions of those of Hoshino and Shigemasu [13].

Group (d): Note that the situation of ability estimation based on logistic models can be seen as a special case of estimating the unknown regression coefficient corresponding to ability, where the covariates are known or estimated item parameters. Stefanski and Carroll [41] derived the asymptotic expansion for the regression coefficient and a method of asymptotic bias correction when the covariates are subject to sampling variation. Zhang, Xie, Song, and Lu [55] gave a similar method for the MLE and WLE of ability based on estimated item parameters for the 3PLM. Their result is based on the assumption that the asymptotic covariances between item parameters of different items are zero, which is removed in the current paper.

Let n^* be the sample size in logistic regression, which corresponds to n, the number of items in the case of ability estimation, and let N^* be the sample size in estimating the values of covariates, which corresponds to N, the sample size for item calibration. In logistic regression, as illustrated by Stefanski and Carroll [41], n^* is much larger than N^*. This is opposite in the case of ability estimation. They deal with the case of $O(n^*/N^{*2}) = O(1)$, while Stefanski [40] used a more extreme case of $O(n^*/N^{*4}) = O(1)$. Since these cases are unrealistic in ability estimation, the reversed conditions, i.e., $O(n^{3/2}/N) = O(1)$ and $O(n^2/N) = O(1)$, are dealt with in the current paper.

Group (e): Recently, Yang, Hanson, and Cai [52] proposed a method of imputation with the assumption of asymptotic multivariate normality for estimators of item parameters, which gave corrected values of estimates of ability, its asymptotic variance, test information, and reliability with their approximate percentiles. They stress the points that their results are not restricted to dichotomous models and that the non-zero asymptotic covariances between item parameter estimators of different items are considered. Mislevy et al.'s [25] method in Group (a) can also be classified in this group.

A.1.2. Stochastic expansions of $\hat{\theta}$ under three conditions on the relative sample size

(a) *Condition* A: $N = O(n)$.

$$\begin{aligned}
\hat{\theta} - \theta_0 &= \{(\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(1)})_{O_p(n^{-1/2})} + (\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})}\}_{O_p(n^{-1/2})} + \{(\boldsymbol{\gamma}_{\theta_0}^{(2)\prime} \mathbf{l}_{\theta_0}^{(2)})_{O_p(n^{-1})} + (\boldsymbol{\gamma}_{\theta_0}^{(2)\prime} \mathbf{l}_{\theta_0}^{(\Delta a2)} \\
&\quad + \gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta\Delta a1)} + \gamma_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(1)})_{O_p(n^{-1/2}N^{-1/2})} + (\boldsymbol{\gamma}_{\theta_0}^{(2)\prime} \mathbf{l}_{\theta_0}^{(\Delta b2)} + \gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta\Delta b1)} + \gamma_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1})}\}_{O_p(n^{-1})} \\
&\quad + [\, (\boldsymbol{\gamma}_{\theta_0}^{(3)\prime} \mathbf{l}_{\theta_0}^{(3)})_{O_p(n^{-3/2})} + (\boldsymbol{\gamma}_{\theta_0}^{(3)\prime} \mathbf{l}_{\theta_0}^{(\Delta a3)})_{O_p(n^{-1}N^{-1/2})} + (\boldsymbol{\gamma}_{\theta_0}^{(3)\prime} \mathbf{l}_{\theta_0}^{(\Delta b3)})_{O_p(n^{-1/2}N^{-1})} + (\boldsymbol{\gamma}_{\theta_0}^{(3)\prime} \mathbf{l}_{\theta_0}^{(\Delta c3)})_{O_p(N^{-3/2})} \\
&\quad + (\boldsymbol{\gamma}_{\theta_0}^{(2)\prime} \mathbf{l}_{\theta_0}^{(\Delta\Delta a2)} + \boldsymbol{\gamma}_{\theta_0}^{(\Delta 2)\prime} \mathbf{l}_{\theta_0}^{(2)})_{O_p(n^{-1}N^{-1/2})} + (\boldsymbol{\gamma}_{\theta_0}^{(2)\prime} \mathbf{l}_{\theta_0}^{(\Delta\Delta b2)} + \boldsymbol{\gamma}_{\theta_0}^{(\Delta 2)\prime} \mathbf{l}_{\theta_0}^{(\Delta a2)})_{O_p(n^{-1/2}N^{-1})} \\
&\quad + (\boldsymbol{\gamma}_{\theta_0}^{(2)\prime} \mathbf{l}_{\theta_0}^{(\Delta\Delta c2)} + \boldsymbol{\gamma}_{\theta_0}^{(\Delta 2)\prime} \mathbf{l}_{\theta_0}^{(\Delta b2)})_{O_p(N^{-3/2})} + (\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta\Delta\Delta a1)} + \gamma_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(\Delta\Delta a1)} + \gamma_{\theta_0}^{(\Delta\Delta 1)} l_{\theta_0}^{(1)})_{O_p(n^{-1/2}N^{-1})} \\
&\quad + (\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta\Delta\Delta b1)} + \gamma_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(\Delta\Delta b1)} + \gamma_{\theta_0}^{(\Delta\Delta 1)} l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-3/2})} - \{n^{-1}(\lambda_{\theta_0}^{-1}\eta_{\theta_0})^{(\Delta)}\}_{O_p(n^{-1}N^{-1/2})}\,]_{O_p(n^{-3/2})} \\
&\quad - (n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})_{O(n^{-1})} + O_p(n^{-2}) \\
&\equiv (q_{O_p(n^{-1/2})}^{(10)} + q_{O_p(N^{-1/2})}^{(11)})_{O_p(n^{-1/2})} + (q_{O_p(n^{-1})}^{(20)} + q_{O_p(n^{-1/2}N^{-1/2})}^{(21)} + q_{O_p(N^{-1})}^{(22)})_{O_p(n^{-1})} \\
&\quad + [\, q_{O_p(n^{-3/2})}^{(30)} + q_{O_p(n^{-1}N^{-1/2})}^{(31)} + q_{O_p(n^{-1/2}N^{-1})}^{(32)} + q_{O_p(N^{-3/2})}^{(33)} \\
&\quad - \{n^{-1}(\lambda_{\theta_0}^{-1}\eta_{\theta_0})^{(\Delta)}\}_{O_p(n^{-1}N^{-1/2})}\,]_{O_p(n^{-3/2})} - (n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})_{O(n^{-1})} + O_p(n^{-2}) \\
&\equiv q_{O_p(n^{-1/2})}^{(1)} + q_{O_p(n^{-1})}^{(2)} + q_{O_p(n^{-3/2})}^{(3)} - (n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})_{O(n^{-1})} + O_p(n^{-2}).
\end{aligned} \tag{A.1}$$

(b) *Condition* B: $N = O(n^{3/2})$.

$$\begin{aligned}
\hat{\theta} - \theta_0 &= (\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(1)})_{O_p(n^{-1/2})} + \{(\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})}\}_{O_p(n^{-3/4})} + (\boldsymbol{\gamma}_{\theta_0}^{(2)\prime} \mathbf{l}_{\theta_0}^{(2)})_{O_p(n^{-1})} + \{(\boldsymbol{\gamma}_{\theta_0}^{(2)\prime} \mathbf{l}_{\theta_0}^{(\Delta a2)} + \gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta\Delta a1)} \\
&\quad + \gamma_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(1)})_{O_p(n^{-1/2}N^{-1/2})}\}_{O_p(n^{-5/4})} + \{(\boldsymbol{\gamma}_{\theta_0}^{(3)\prime} \mathbf{l}_{\theta_0}^{(3)})_{O_p(n^{-3/2})} + (\boldsymbol{\gamma}_{\theta_0}^{(2)\prime} \mathbf{l}_{\theta_0}^{(\Delta b2)} + \gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta\Delta b1)} \\
&\quad + \gamma_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1})}\}_{O_p(n^{-3/2})} - (n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})_{O(n^{-1})} + O_p(n^{-7/4}) \\
&\equiv q_{O_p(n^{-1/2})}^{(10)} + (q_{O_p(N^{-1/2})}^{(1a)})_{O_p(n^{-3/4})} + q_{O_p(n^{-1})}^{(20)} + (q_{O_p(n^{-1/2}N^{-1/2})}^{(2a)})_{O_p(n^{-5/4})} \\
&\quad + (q_{O_p(n^{-3/2})}^{(30)} + q_{O_p(N^{-1})}^{(31)})_{O_p(n^{-3/2})} - (n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})_{O(n^{-1})} + O_p(n^{-7/4}) \\
&\equiv q_{O_p(n^{-1/2})}^{(1)} + q_{O_p(n^{-3/4})}^{(1a)} + q_{O_p(n^{-1})}^{(2)} + q_{O_p(n^{-5/4})}^{(2a)} + q_{O_p(n^{-3/2})}^{(3)} - (n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})_{O(n^{-1})} + O_p(n^{-7/4}) \\
&\quad (q_{O_p(n^{-1/2})}^{(10)} = q_{O_p(n^{-1/2})}^{(1)},\ q_{O_p(n^{-1})}^{(20)} = q_{O_p(n^{-1})}^{(2)},\ q_{O_p(n^{-1/2}N^{-1/2})}^{(2a)} = q_{O_p(n^{-5/4})}^{(2a)}).
\end{aligned} \tag{A.2}$$

(c) *Condition* C: $N = O(n^2)$.

$$\begin{aligned}
\hat{\theta} - \theta_0 &= (\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(1)})_{O_p(n^{-1/2})} + \{(\boldsymbol{\gamma}_{\theta_0}^{(2)\prime} \mathbf{l}_{\theta_0}^{(2)})_{O_p(n^{-1})} + (\gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta 1)})_{O_p(N^{-1/2})}\}_{O_p(n^{-1})} + \{\, (\boldsymbol{\gamma}_{\theta_0}^{(3)\prime} \mathbf{l}_{\theta_0}^{(3)})_{O_p(n^{-3/2})} \\
&\quad + (\boldsymbol{\gamma}_{\theta_0}^{(2)\prime} \mathbf{l}_{\theta_0}^{(\Delta a2)} + \gamma_{\theta_0}^{(1)} l_{\theta_0}^{(\Delta\Delta a1)} + \gamma_{\theta_0}^{(\Delta 1)} l_{\theta_0}^{(1)})_{O_p(n^{-1/2}N^{-1/2})}\,\}_{O_p(n^{-3/2})} - (n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})_{O(n^{-1})} + O_p(n^{-2}) \\
&\equiv q_{O_p(n^{-1/2})}^{(10)} + (q_{O_p(n^{-1})}^{(20)} + q_{O_p(N^{-1/2})}^{(21)})_{O_p(n^{-1})} + (q_{O_p(n^{-3/2})}^{(30)} + q_{O_p(n^{-1/2}N^{-1})}^{(31)})_{O_p(n^{-3/2})} \\
&\quad - (n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})_{O(n^{-1})} + O_p(n^{-2}) \\
&\equiv q_{O_p(n^{-1/2})}^{(1)} + q_{O_p(n^{-1})}^{(2)} + q_{O_p(n^{-3/2})}^{(3)} - (n^{-1}\lambda_{\theta_0}^{-1}\eta_{\theta_0})_{O(n^{-1})} + O_p(n^{-2}) (q_{O_p(n^{-1/2})}^{(10)} = q_{O_p(n^{-1/2})}^{(1)}).
\end{aligned} \tag{A.3}$$

In the above results, the same notation is used under different conditions, for simplicity. The terms of orders $O_p(n^{-1/2}N^{-1/2})$, $O_p(N^{-1})$, and $O_p(N^{-3/2})$ are due to the fallible or estimated item parameters. The numbers of these terms is the largest in Condition A and the smallest in Condition C.

A.1.3. An expansion of $\hat{\bar{\beta}}_{2I}^{-1/2}$ for t under Condition A

Expand $\hat{\bar{\beta}}_{2I}^{-1/2}$ in (4.1) about $\hat{\theta}=\theta_0$, $\hat{\boldsymbol{\alpha}}=\boldsymbol{\alpha}_0$ and $\hat{\bar{\beta}}_{2I}^{-1/2}=\bar{\beta}_{2\mathbf{G}_0}^{-1/2}$:

$$\hat{\bar{\beta}}_{2I}^{-1/2}=\bar{\beta}_{2\mathbf{G}_0}^{-1/2}-\frac{\bar{\beta}_{2\mathbf{G}_0}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2\mathbf{G}_0}}{\partial(\theta_0,\boldsymbol{\alpha}_0')}(\hat{\theta}-\theta_0,\hat{\boldsymbol{\alpha}}'-\boldsymbol{\alpha}_0')' +\frac{1}{2}\left\{\frac{3}{4}\bar{\beta}_{2\mathbf{G}_0}^{-5/2}\left(\frac{\partial\bar{\beta}_{2\mathbf{G}_0}}{\partial(\theta_0,\boldsymbol{\alpha}_0')}\right)^{\langle 2\rangle}-\frac{\bar{\beta}_{2\mathbf{G}_0}^{-3/2}}{2}\frac{\partial^2\bar{\beta}_{2\mathbf{G}_0}}{\{\partial(\theta_0,\boldsymbol{\alpha}_0')\}^{\langle 2\rangle}}\right\}(\hat{\theta}-\theta_0,\hat{\boldsymbol{\alpha}}'-\boldsymbol{\alpha}_0')'^{\langle 2\rangle}+O_p(n^{-3/2}), \tag{A.4}$$

where $\mathbf{G}_0$ is $\hat{\mathbf{G}}$, in which $\hat{\theta}$ and $\hat{\boldsymbol{\alpha}}$ are replaced by θ_0 and $\boldsymbol{\alpha}_0$, respectively, but is still a stochastic quantity. Define

$$\begin{aligned}&\mathrm{E}_{\mathrm{T}\alpha_0}(\mathbf{G}_0)=\boldsymbol{\Gamma}_{\mathbf{G}_0},\qquad \mathrm{E}_{\alpha_0}(\mathbf{G}_0)=\boldsymbol{\Gamma}_{\mathbf{G}_0}=\mathbf{I}_{\alpha_0},\qquad \boldsymbol{\gamma}_{\mathbf{G}_0}=v(\boldsymbol{\Gamma}_{\mathbf{G}_0}),\\ &\mathbf{m}_{\mathbf{G}_0}=v\{\mathbf{G}_0-\mathrm{E}_{\mathrm{T}\alpha_0}(\mathbf{G}_0)\}=v(\mathbf{G}_0)-\boldsymbol{\gamma}_{\mathbf{G}_0}=O_p(N^{-1/2}),\end{aligned} \tag{A.5}$$

where $\mathrm{E}_{\alpha_0}(\cdot)$ is defined similarly to $\mathrm{E}_{\theta_0}(\cdot)$, and $v(\cdot)$ is the vectorizing operator taking the non-duplicated elements of a symmetric matrix. Note that, since $\mathbf{G}_0$ is stochastic, $\bar{\beta}_{2\mathbf{G}_0}$ is stochastic.

Expanding $\bar{\beta}_{2\mathbf{G}_0}$ about $\mathbf{G}_0=\mathbf{I}_{\alpha_0}$, it follows that

$$\begin{aligned}\hat{\bar{\beta}}_{2I}^{-1/2}&=\bar{\beta}_{2I}^{-1/2}-\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial\boldsymbol{\gamma}_{\mathbf{G}_0}'}\mathbf{m}_{\mathbf{G}_0}+\left\{\frac{3}{8}\bar{\beta}_{2I}^{-5/2}\left(\frac{\partial\bar{\beta}_{2I}}{\partial\boldsymbol{\gamma}_{\mathbf{G}_0}'}\right)^{\langle 2\rangle}-\frac{\bar{\beta}_{2I}^{-3/2}}{4}\frac{\partial^2\bar{\beta}_{2I}}{(\partial\boldsymbol{\gamma}_{\mathbf{G}_0}')^{\langle 2\rangle}}\right\}\mathbf{m}_{\mathbf{G}_0}^{\langle 2\rangle}\\
&\quad-\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial(\theta_0,\boldsymbol{\alpha}_0')}(\hat{\theta}-\theta_0,\hat{\boldsymbol{\alpha}}'-\boldsymbol{\alpha}_0')'+\left\{\frac{3}{4}\bar{\beta}_{2I}^{-5/2}\frac{\partial\bar{\beta}_{2I}}{\partial(\theta_0,\boldsymbol{\alpha}_0')}\otimes\frac{\partial\bar{\beta}_{2I}}{\partial\boldsymbol{\gamma}_{\mathbf{G}_0}'}\right.\\
&\quad\left.-\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial^2\bar{\beta}_{2I}}{\partial(\theta_0,\boldsymbol{\alpha}_0')\otimes\partial\boldsymbol{\gamma}_{\mathbf{G}_0}'}\right\}\{(\hat{\theta}-\theta_0,\hat{\boldsymbol{\alpha}}'-\boldsymbol{\alpha}_0')'\otimes\mathbf{m}_{\mathbf{G}_0}\}\\
&\quad+\left\{\frac{3}{8}\bar{\beta}_{2I}^{-5/2}\left(\frac{\partial\bar{\beta}_{2I}}{\partial(\theta_0,\boldsymbol{\alpha}_0')}\right)^{\langle 2\rangle}-\frac{\bar{\beta}_{2I}^{-3/2}}{4}\frac{\partial^2\bar{\beta}_{2I}}{\{\partial(\theta_0,\boldsymbol{\alpha}_0')\}^{\langle 2\rangle}}\right\}\{(\hat{\theta}-\theta_0,\hat{\boldsymbol{\alpha}}'-\boldsymbol{\alpha}_0')\}'^{\langle 2\rangle}+O_p(n^{-3/2})\\
&=\bar{\beta}_{2I}^{-1/2}-\left(\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial\theta_0}q^{(10)}_{O_p(n^{-1/2})}\right)_{O_p(n^{-1/2})}-\left[\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial(\boldsymbol{\gamma}_{\mathbf{G}_0}',\theta_0,\boldsymbol{\alpha}_0')}\{\mathbf{m}_{\mathbf{G}_0}',q^{(11)}_{O_p(N^{-1/2})},(\boldsymbol{\Gamma}^{(1)}_{\alpha_0}\mathbf{I}^{(1)}_{\alpha_0})'\}'\right]_{O_p(N^{-1/2})}\\
&\quad+\left[-\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial\theta_0}q^{(20)}_{O_p(n^{-1})}+\left\{\frac{3}{8}\bar{\beta}_{2I}^{-5/2}\left(\frac{\partial\bar{\beta}_{2I}}{\partial\theta_0}\right)^2-\frac{\bar{\beta}_{2I}^{-3/2}}{4}\frac{\partial^2\bar{\beta}_{2I}}{\partial\theta_0^2}\right\}(q^{(10)}_{O_p(n^{-1/2})})^2\right]_{O_p(n^{-1})}\\
&\quad+\left[-\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial\theta_0}q^{(21)}_{O_p(n^{-1/2}N^{-1/2})}+\left\{\frac{3}{4}\bar{\beta}_{2I}^{-5/2}\frac{\partial\bar{\beta}_{2I}}{\partial\theta_0}\frac{\partial\bar{\beta}_{2I}}{\partial(\boldsymbol{\gamma}_{\mathbf{G}_0}',\boldsymbol{\alpha}_0')}\right.\right.\\
&\quad\left.\left.-\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial^2\bar{\beta}_{2I}}{\partial\theta_0\partial(\boldsymbol{\gamma}_{\mathbf{G}_0}',\boldsymbol{\alpha}_0')}\right\}q^{(10)}_{O_p(n^{-1/2})}\{\mathbf{m}_{\mathbf{G}_0}',(\boldsymbol{\Gamma}^{(1)}_{\alpha_0}\mathbf{I}^{(1)}_{\alpha_0})'\}'\right]_{O_p(n^{-1/2}N^{-1/2})}\\
&\quad+\left[\left\{\frac{3}{8}\bar{\beta}_{2I}^{-5/2}\left(\frac{\partial\bar{\beta}_{2I}}{\partial(\boldsymbol{\gamma}_{\mathbf{G}_0}',\theta_0,\boldsymbol{\alpha}_0')}\right)^{\langle 2\rangle}-\frac{\bar{\beta}_{2I}^{-3/2}}{4}\frac{\partial^2\bar{\beta}_{2I}}{\{\partial(\boldsymbol{\gamma}_{\mathbf{G}_0}',\theta_0,\boldsymbol{\alpha}_0')\}^{\langle 2\rangle}}\right\}\{\mathbf{m}_{\mathbf{G}_0}',q^{(11)}_{O_p(N^{-1/2})},(\boldsymbol{\Gamma}^{(1)}_{\alpha_0}\mathbf{I}^{(1)}_{\alpha_0})'\}'^{\langle 2\rangle}\right.\\
&\quad\left.-\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial(\theta_0,\boldsymbol{\alpha}_0')}\{q^{(22)}_{O_p(N^{-1})},(\boldsymbol{\Gamma}^{(2)}_{\alpha_0}\mathbf{I}^{(2)}_{\alpha_0})'\}\right]_{O_p(N^{-1})}\\
&\quad+\left(n^{-1}\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial\theta_0}\lambda_{\theta_0}^{-1}\eta_{\theta_0}\right)_{O(n^{-1})}+\left(N^{-1}\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Lambda}_{\alpha_0}^{-1}\boldsymbol{\eta}_{\alpha_0}\right)_{O(N^{-1})}+O_p(n^{-3/2})\\
&\equiv\bar{\beta}_{2I}^{-1/2}+(b^{(10)}_{O_p(n^{-1/2})}+b^{(11)}_{O_p(N^{-1/2})})+(b^{(20)}_{O_p(n^{-1})}+b^{(21)}_{O_p(n^{-1/2}N^{-1/2})}+b^{(22)}_{O_p(N^{-1})})\\
&\quad+\left(n^{-1}\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial\theta_0}\lambda_{\theta_0}^{-1}\eta_{\theta_0}\right)_{O(n^{-1})}+\left(N^{-1}\frac{\bar{\beta}_{2I}^{-3/2}}{2}\frac{\partial\bar{\beta}_{2I}}{\partial\boldsymbol{\alpha}_0'}\boldsymbol{\Lambda}_{\alpha_0}^{-1}\boldsymbol{\eta}_{\alpha_0}\right)_{O(N^{-1})}+O_p(n^{-3/2})\end{aligned}$$

$$\equiv \bar{\beta}_{2I}^{-1/2} + b^{(1)}_{O_p(n^{-1/2})} + b^{(2)}_{O_p(n^{-1})} + \left(n^{-1} \frac{\bar{\beta}_{2I}^{-3/2}}{2} \frac{\partial \bar{\beta}_{2I}}{\partial \theta_0} \lambda_{\theta_0}^{-1} \eta_{\theta_0} \right)_{O(n^{-1})}$$

$$+ \left(N^{-1} \frac{\bar{\beta}_{2I}^{-3/2}}{2} \frac{\partial \bar{\beta}_{2I}}{\partial \boldsymbol{\alpha}_0'} \boldsymbol{\Lambda}_{\alpha_0}^{-1} \boldsymbol{\eta}_{\alpha_0} \right)_{O(N^{-1})} + O_p(n^{-3/2}). \tag{A.6}$$

For Subsections A.2 to A.6 of the appendix, see [33,34].

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