

# A Pointwise Almost Sure Bahadur-Kiefer-Type Representation for the Product Limit Estimator

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In the random censorship from the right model, we prove a strong approximation result for the Bahadur-Kiefer-type process based on the product limit estimator. We derive from this strong approximation the pointwise rate of consistency of the corresponding Bahadur-Kiefer-type statistic. © 1993 Academic Press, Inc.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let  $U_1, U_2, \dots$  be iid rv's with uniform distribution on  $(0, 1)$  and let  $Y_1, Y_2, \dots$  be iid positive rv's with distribution function  $G$ . Both sequences are assumed to be independent. For each integer  $n \geq 1$ , set  $X_n = \min(U_n, Y_n)$  and  $\delta_n = 1$  if  $U_n \leq Y_n$ ,  $\delta_n = 0$  otherwise. In the random censorship from the right model we observe the sequence  $(X_1, \delta_1), (X_2, \delta_2), \dots$ . In the following, we assume throughout that  $G$  is continuous. For each  $n \geq 1$ , the product limit estimator  $F_n$  introduced by Kaplan and Meier [8] is defined by setting

$$1 - F_n(s) = \prod_{X_{(i)} \leq s} (1 - \delta_{(i)} / (n - i + 1)) \quad \text{for } 0 \leq s \leq \infty, \quad (1)$$

where  $X_{(1)} \leq \dots \leq X_{(n)}$  are the order statistics of the  $X_i$ ,  $1 \leq i \leq n$ , and  $\delta_{(i)}$ ,  $1 \leq i \leq n$ , are the corresponding  $\delta$ 's. In the following, we denote by  $Q_n(s) = \inf\{t : F_n(t) \geq s\}$  for  $0 < t < 1$  the product limit quantile function, by  $\alpha_n(s) = n^{1/2}(F_n(s) - s)$  and  $\beta_n(s) = n^{1/2}(Q_n(s) - s)$  for  $0 < s < 1$  the product limit and the quantile product limit processes. In this paper we are concerned with the Bahadur-Kiefer-type process  $R_n$  defined as follows. Let

$$R_n(s) = \alpha_n(s) + \beta_n(s) \quad \text{for } 0 < s < 1. \quad (2)$$

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In the uncensored case, i.e., when  $G(1)=0$ , the process  $R_n$  has been introduced by Bahadur [2] and later studied by Kiefer [9, 10] and others (see, e.g., Deheuvels [6] and Deheuvels and Mason [7]). In the censored case the process  $R_n$  has been studied, among others, by Cheng [4], Aly, Csörgő, and Horváth [1], and Beirlant and Einmahl [3]. In the sequel, we use the following notation. Let  $\tau = \inf\{s : G(s) = 1\}$ ,  $T = \min(1, \tau)$ , and  $h$  be the function defined on  $(0, T)$  by setting

$$h(s) = (1-s)^{1/2} (1-G(s))^{-1/2} \times \left( \int_0^s (1-t)^{-2} (1-G(t))^{-1} dt \right)^{1/4} \quad \text{for } 0 < s < T. \quad (3)$$

Beirlant and Einmahl [3] have shown that for any  $0 < s < \theta < T$ , almost surely

$$\limsup_{n \rightarrow \infty} n^{1/4} (\log \log n)^{-3/4} |R_n(s)| \leq 2^{3/4} h(s) \quad (4)$$

and

$$\begin{aligned} \limsup_{n \rightarrow \infty} n^{1/4} (\log n)^{-1/2} (\log \log n)^{-1/4} \sup_{0 < t < \theta} |R_n(t)| \\ = 2^{1/4} \left( \sup_{0 < t < \theta} h^2(t) \right)^{1/2}. \end{aligned} \quad (5)$$

In the uncensored case the exact constant for the lim sup in (4) has been given by Kiefer [9, 10]. In this paper we generalize this result to the censored case. We thus improve the statement (4) of Beirlant and Einmahl [3]. We present now our results. The first one gives a strong approximation of the process  $R_n$ . This strong approximation is of the same type as the one given in Deheuvels [6] (see also Deheuvels and Mason [7]) for the uncensored case. Following Deheuvels [6] (see, e.g., Deheuvels and Mason [7]), the generalization of the Kiefer result is then easily deduced from this strong approximation. Let in the sequel  $0 < \gamma < s < \theta < T$  be fixed and let  $a$  denote the function defined on  $(-\infty, \infty)$  by setting

$$\begin{aligned} a(t) &= \int_0^t (1-u)^{-2} (1-G(u))^{-1} du \quad \text{for } \gamma \leq t \leq \theta, \\ a(t) &= a(\gamma) \quad \text{for } t < \gamma, \quad \text{and} \quad a(t) = a(\theta) \quad \text{for } t > \theta. \end{aligned} \quad (6)$$

**LEMMA 1.** *Let  $0 < s < T$  be fixed. We can assume our original probability space to carry a standard Wiener process  $W$  and a sequence  $W_1, W_2, \dots$  of*

independent standard Wiener processes defined on  $(-\infty, \infty)$  and independent of  $W$  such that, almost surely as  $n \rightarrow \infty$ ,

$$\left| R_n(s) - n^{-1/2}(1-s) \sum_{i=1}^n W_i(n^{-1}(1-s)a'(s)(a(s))^{1/2}W(n)) \right| = o(n^{-3/8} \log n). \quad (7)$$

Here and in the sequel, we assume without loss of generality that each standard Wiener process  $W$  we consider is defined on  $(-\infty, \infty)$  (this amounts to set  $W(t) = \tilde{W}(-t)$  for  $t < 0$ , where  $\tilde{W}$  is a standard Wiener process independent of  $W$ ).

**THEOREM 1.** *Let  $0 < s < T$  be fixed. Then, we have, in distribution as  $n \rightarrow \infty$*

$$n^{1/4} R_n(s) \rightarrow h(s) |D|^{1/2} E, \quad (8)$$

where  $D$  and  $E$  are iid rv's with normal  $\mathcal{N}(0, 1)$  distribution. Moreover, we have, almost surely

$$\limsup_{n \rightarrow \infty} \pm n^{1/4} (\log \log n)^{-3/4} R_n(s) = 2^{5/4} 3^{-3/4} h(s). \quad (9)$$

The proof of Lemma 1 is given in Section 2. We observe that (8) is stated in Beirlant and Einmahl [3]. Result (8) is an easy consequence of (7) (see, e.g., Deheuvels [6]) and is given in our Theorem 1 for the sake of completeness. On the other hand, using the following fact, the proof of (9) follows obviously from Lemma 1.

*Fact 1.* (Deheuvels [6], see also Deheuvels and Mason [7]). Let  $W_0, W_1, \dots$  be a sequence of independent standard Wiener processes on  $(-\infty, \infty)$ . Then, for any  $\lambda > 0$ , we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \pm n^{-1/4} (\log \log n)^{-3/4} \sum_{i=1}^n W_i \left( \frac{\lambda}{n} W_0(n) \right) \\ = 2^{5/4} 3^{-3/4} \lambda^{1/2} \text{ a.s.} \end{aligned}$$

We now present an immediate generalization of Theorem 1. We assume here that  $U_1, U_2, \dots$  are iid positive rv's with distribution function  $F$ . Moreover, we assume that  $F$  is differentiable on  $(0, \infty)$  with continuous and positive derivative  $f$  and that  $G$  is continuous on  $(0, \infty)$ . We make use of the following notation. For each  $n \geq 1$ , the product limit estimator  $F_n$  and the product limit quantile function  $Q_n$  are defined exactly as in the uniform case. Let  $Q(s) = \inf\{t : F(t) \geq s\}$  for  $0 < s < 1$ ,  $\alpha_n(s) = n^{1/2}(F_n(s) - F(s))$  and  $\beta_n(s) = n^{1/2}f(Q(s))(Q_n(s) - Q(s))$  for  $n \geq 1$  and

$0 < s < 1$ . Here, for each  $n \geq 1$ , the Bahadur–Kiefer-type process  $R_n$  is defined by  $R_n(s) = \alpha_n(Q(s)) + \beta_n(s)$  for  $0 < s < F(\tau)$ . Finally, let  $h$  be the function defined on  $(0, F(\tau))$  by setting

$$h(s) = (1-s)^{1/2} (1-G(Q(s)))^{-1/2} \\ \times \left( \int_0^s (1-t)^{-2} (1-G(Q(t)))^{-1} dt \right)^{1/4} \quad \text{for } 0 < s < F(\tau).$$

Then, in this case, using Lemma 1 of Beirlant and Einmahl [3], it is obvious that (8) and (9) remain valid if we assume that for all  $0 < s < \tau$

$$\lim_{\varepsilon \rightarrow 0} \sup_{|t-s| \leq \varepsilon} (\varepsilon^{-1/2} |f(s) - f(t)|) = 0.$$

## 2. PROOF OF LEMMA 1

Let  $0 < \gamma < s < \theta < T$  be fixed. We make use of the following facts:

*Fact 2.* (Aly, Csörgő, and Horváth [1] p. 193–194; see also Sander [13]). We have, a.s. as  $n \rightarrow \infty$

$$|R_n(s) - [\alpha_n(s) - \alpha_n(s + n^{-1/2}\beta_n(s))]| = O(n^{-1/2}). \quad (10)$$

*Fact 3.* (Major and Rejtő [11]). We can assume our original probability space to carry a sequence  $W'_1, W'_2, \dots$  of independent standard Wiener processes such that, a.s. as  $n \rightarrow \infty$

$$\sup_{\gamma < t < \theta} |\alpha_n(t) - n^{-1/2}(1-t) S_n(t)| = O(n^{-1/2}(\log n)^2),$$

$$\text{where } S_n(t) = \sum_{i=1}^n W'_i(a(t)) \quad \text{for } -\infty < t < \infty. \quad (11)$$

*Fact 4.* (The law of the iterated logarithm for partial sums). We have, a.s.

$$\limsup_{n \rightarrow \infty} \pm n^{-1/2}(\log \log n)^{-1/2} S_n(s) = 2^{1/2}(a(s))^{1/2} \quad (12)$$

*Fact 5.* (Lemma 1.1.1 of Csörgő and Révész [5]). Let  $(W(t), 0 \leq t \leq A)$  be a standard Wiener process. Then, for any  $\varepsilon > 0$ , there exists a  $C(\varepsilon) > 0$  such that for all  $v \geq 0$  and all  $0 < \eta \leq A$

$$P\left(\sup_{0 \leq t \leq A-\eta} \sup_{0 \leq u \leq \eta} |W(t+u) - W(t)| \geq v\eta^{1/2}\right) \\ \leq C(\varepsilon) A\eta^{-1} e^{-v^2/(2+\varepsilon)}. \quad (13)$$

In the sequel we apply Fact 5 with  $\varepsilon = 1$  and set  $C_1 = C(1)$ .

We now present the proof of Lemma 1. First, by (5), (11), and (12), it follows that, a.s. as  $n \rightarrow \infty$

$$\begin{aligned} |\beta_n(s)| &\leq |R_n(s)| + |\alpha_n(s) - n^{-1/2}(1-s)S_n(s)| + n^{-1/2}(1-s)|S_n(s)| \\ &= O((\log \log n)^{1/2}). \end{aligned} \quad (14)$$

The definitions of  $a$  and  $S_n$ , and a simple application of (13), show that

$$P(n^{-1/2} \sup_{-\infty \leq t \leq \infty} |S_n(t)| \geq (6 \log n)^{1/2} (a(\theta))^{1/2}) \leq C_1 n^{-2}. \quad (15)$$

Thus, (15) and the Borel–Cantelli lemma imply that, a.s. as  $n \rightarrow \infty$

$$|n^{-1/2}S_n(s + n^{-1/2}\beta_n(s))| = O((\log n)^{1/2}). \quad (16)$$

Moreover, it follows from (11), (14), and (16), that, a.s. as  $n \rightarrow \infty$

$$\begin{aligned} &|[\alpha_n(s) - \alpha_n(s + n^{-1/2}\beta_n(s))] - n^{-1/2}(1-s)[S_n(s) - S_n(s + n^{-1/2}\beta_n(s))]| \\ &\leq |n^{-1/2}\beta_n(s)S_n(s + n^{-1/2}\beta_n(s))| + |\alpha_n(s) - n^{-1/2}(1-s)S_n(s)| \\ &\quad + |\alpha_n(s + n^{-1/2}\beta_n(s)) - n^{-1/2}(1-s - n^{-1/2}\beta_n(s))S_n(s + n^{-1/2}\beta_n(s))| \\ &= O(n^{-1/2}(\log n)^2). \end{aligned} \quad (17)$$

Next, by (5) and (11) we obtain readily that, a.s. as  $n \rightarrow \infty$

$$\begin{aligned} &|\beta_n(s) + n^{-1/2}(1-s)S_n(s)| \\ &\leq |R_n(s)| + |\alpha_n(s) - n^{-1/2}(1-s)S_n(s)| \\ &= O(n^{-1/4}(\log n)^{1/2}(\log \log n)^{1/4}). \end{aligned} \quad (18)$$

Set  $\eta_n = n^{-3/4}(\log n)^{3/4}$ . We note that  $a$  is a Lipschitz function over  $(-\infty, \infty)$ , so that the definitions of  $a$  and  $S_n$ , and an application of (13), show that

$$\begin{aligned} &P\left(\sup_{-\infty < t < \infty} \sup_{0 \leq u \leq \eta_n} n^{-1/2}|S_n(t+u) - S_n(t)| \geq (6 \log n)^{1/2} (a'(\theta) \eta_n)^{1/2}\right) \\ &\leq C_1 (a'(\theta))^{-1} a(\theta) n^{-5/4} (\log n)^{-3/4}. \end{aligned} \quad (19)$$

Thus, (19), the Borel–Cantelli lemma and (18) imply that, a.s. as  $n \rightarrow \infty$

$$\begin{aligned} & |n^{-1/2}[S_n(s + n^{-1/2}\beta_n(s)) - S_n(s - n^{-1}(1-s)S_n(s))]| \\ &= o(n^{-3/8} \log n). \end{aligned} \quad (20)$$

Finally, (10), combined with (17) and (20), yields the following strong approximation of  $R_n$ :

$$\begin{aligned} & |R_n(s) - n^{-1/2}(1-s)[S_n(s) - S_n(s - n^{-1}(1-s)S_n(s))]| \\ &= o(n^{-3/8} \log n). \end{aligned} \quad (21)$$

We now show that (21) implies the result of Lemma 1. We make use of the following notation. Fix  $\lambda > 2^{1/2}(1-s)(a(s))^{1/2}$  and a decreasing sequence  $\gamma_1, \gamma_2, \dots$  such that  $\gamma_n = n^{-1/2}(\log \log n)^{1/2}$  for  $n \geq 6$ . Moreover, let  $W_{1,1}, W_{1,2}, \dots$  and  $W_{2,1}, W_{2,2}, \dots$  be two independent sequences of independent standard Wiener processes. Both sequences are assumed to be independent from  $W'_1, W'_2, \dots$ . For each  $i \geq 1$ , set  $W'_{1,i}(t) = W_{1,i}(t)$  and  $W'_{2,i}(t) = W'_i(t)$  if  $t \leq a(s - \lambda\gamma_i)$ ,  $W'_{1,i}(t) = W_{1,i}(a(s - \lambda\gamma_i)) + W'_i(t) - W'_i(a(s - \lambda\gamma_i))$  and  $W'_{2,i}(t) = W'_i(a(s - \lambda\gamma_i)) + W_{2,i}(t) - W_{2,i}(a(s - \lambda\gamma_i))$  if  $t > a(s - \lambda\gamma_i)$ . Furthermore, set  $S'_n(t) = \sum_{i=1}^n W'_{1,i}(a(t))$  and  $S''_n(t) = \sum_{i=1}^n W'_{2,i}(a(t))$  for  $-\infty < t < \infty$ . Note for further use that  $W'_{1,1}, W'_{1,2}, \dots$  and  $W'_{2,1}, W'_{2,2}, \dots$  are two independent sequences of independent standard Wiener processes.

Since  $S'_n(s) - S'_n(s+u) = S_n(s) - S_n(s+u)$  for  $u \geq -\lambda\gamma_n$ , by (12), our choice of  $\lambda$  and (21), it follows that, a.s. as  $n \rightarrow \infty$

$$\begin{aligned} & |R_n(s) - n^{-1/2}(1-s)[S'_n(s) - S'_n(s - n^{-1}(1-s)S_n(s))]| \\ &= o(n^{-3/8} \log n). \end{aligned} \quad (22)$$

On the other hand, set  $M_0 = 0$ ,  $M_n = \sum_{i=1}^n [W'_i(a(s)) - W'_i(a(s - \lambda\gamma_i))]$  and  $A_n = \sum_{i=1}^n E[(M_i - M_{i-1})^2 / M_{i-1}] = \sum_{i=1}^n (a(s) - a(s - \lambda\gamma_i))$  for  $n \geq 1$ . We observe that  $M_1, M_2, \dots$  is a square integrable martingale such that  $\lim_{n \rightarrow \infty} A_n = \infty$ . Moreover, we have for  $n \geq 1$

$$\begin{aligned} A_n &\leq \lambda a'(s)(\log \log n)^{1/2} \sum_{i=1}^n i^{-1/2} \\ &\leq 2\lambda a'(s) n^{1/2}(\log \log n)^{1/2}. \end{aligned}$$

Thus, by using Proposition VII-2-4 of Neveu [12], we obtain readily that, a.s. as  $n \rightarrow \infty$

$$\begin{aligned}
n^{-1} \left| S_n(s) - \sum_{i=1}^n W'_i(a(s - \lambda \gamma_i)) \right| &= n^{-1} |M_n| \\
&= o(n^{-1} A_n^{1/2} (\log A_n)^{3/4}) \\
&= o(n^{-3/4} (\log n)^{3/4} (\log \log n)^{1/4}). \tag{23}
\end{aligned}$$

Since  $W'_i(a(s - \lambda \gamma_i)) = W'_{2,i}(a(s - \lambda \gamma_i))$ , it follows from (23), and the obvious fact that (23) remains valid with the formal replacement of  $S_n(s)$  and  $W'_i(a(s - \lambda \gamma_i))$  by  $S''_n(s)$  and  $W'_{2,i}(a(s - \lambda \gamma_i))$ , that, a.s. as  $n \rightarrow \infty$

$$n^{-1} |S_n(s) - S''_n(s)| = o(n^{-3/4} (\log n)^{3/4} (\log \log n)^{1/4}). \tag{24}$$

Setting  $\delta_n = n^{-3/4} (\log n)^{3/4} (\log \log n)^{1/4}$ , the definitions of  $a$  and  $S'_n$ , and a simple application of (13) then show that

$$\begin{aligned}
P\left(\sup_{-\infty < t < \infty} \sup_{0 \leq u \leq \delta_n} n^{-1/2} |S'_n(t+u) - S'_n(t)| \geq (6 \log n)^{1/2} (a'(\theta) \delta_n)^{1/2}\right) \\
\leq C_1 (a'(\theta))^{-1} a(\theta) n^{-5/4} (\log n)^{-3/4} (\log \log n)^{-1/4}. \tag{25}
\end{aligned}$$

Thus, (25) and the Borel-Cantelli lemma imply that, a.s. as  $n \rightarrow \infty$

$$\begin{aligned}
&\sup_{-\infty < t < \infty} \sup_{0 \leq u \leq \delta_n} n^{-1/2} |S'_n(t+u) - S'_n(t)| \\
&= O(n^{-3/8} (\log n)^{7/8} (\log \log n)^{1/8}),
\end{aligned}$$

which in turn, when combined with (22) and (24), implies that, a.s. as  $n \rightarrow \infty$

$$\begin{aligned}
&|R_n(s) - n^{-1/2} (1-s) [S'_n(s) - S'_n(s - n^{-1} (1-s) S''_n(s))]| \\
&= o(n^{-3/8} \log n). \tag{26}
\end{aligned}$$

The proof of (7) is then completed by setting  $W(n) = (a(s))^{-1/2} S''_n(s)$  and  $W_i(u) = W_{1,i}(a(s)) - W_{1,i}(a(s) - u)$  for  $-\infty < u < \infty$ ,  $i \geq 1$ , and  $n \geq 1$ .

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