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Bivariate Extension of Lomax and Finite Range Distributions through Characterization Approach

Dilip Roy

Burdwan University, Burdwan, India

and

R. P. Gupta

Dalhousie University, Halifax, Canada

In the univariate setup the Lomax distribution is being widely used for stochastic modelling of decreasing failure rate life components. It also serves as a useful model in the study of labour turnover, queueing theory, and biological analysis. A bivariate extension of the Lomax distribution given in Lindley and Singpurwalla (1986) fails to cover the case of independence. Our present attempt is to obtain the unique determination of a bivariate Lomax distribution through characterization results. In this process we also obtain bivariate extensions of the exponential and a finite range distributions. The bivariate Lomax distribution thus obtained is a member of the Arnold (1990) flexible family of Pareto distributions and the bivariate exponential distribution derived here is identical with that of Gumbel (1960). Various properties of the proposed extensions are presented.

1. INTRODUCTION

For a stochastic description of the life of a decreasing failure rate (DFR) component or a system the Lomax distribution with survival function $S(x) = (1 + x/a)^{-q}$, $a > 0$, $q > 0$ has been widely used by practitioners. It also works as a suitable model for describing completed length of service (CLS) and is known as Silcock's distribution or Pearsonian Type XI distribution (Bartholomew (1982)). Following Muth (1977) it can be easily shown that the Lomax distribution has constant negative memory. The other two distributions which have constant memories are the exponential

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distribution with zero memory and a finite range beta type distribution (to be referred to as FR distribution) with positive memory. Survival functions of the FR distribution is $S(x) = (1 - x/a)^q$, $0 \leqslant x \leqslant a$, $q > 0$, $a > 0$. These three distributions are in many ways interrelated as may be seen from the characterization results in Ferguson (1967), Wang and Srivastava (1980), and Mukherjee and Roy (1986).

A multivariate generalization of the Lomax distribution has been proposed in Lindley and Singpurwalla (1986) and has been studied in detail by Nayak (1987). It has a built-in structure of dependency. As a result, for no choice of the parameters can one obtain the situation where component lives are independently distributed. In fact, in this model, there is no scope of examining the hypothesis that the component lives are independent. This is a major limitation when used for stochastic modelling.

Arnold (1990) has used multivariate geometric minima of univariate Pareto random variables to generate a standard multivariate Pareto distribution and proposed therefrom a flexible family of multivariate Pareto distributions. His family of distributions includes the case of independent marginals, the Mardia (1962) family and the Durling–Owen–Drane (1970) family as special cases. Arnold (1990) has also examined related properties and estimation issues of his class in two dimension case. Our problem then arises in selecting a particular model from amongst the members of Arnold's (1990) flexible class so that the chosen model can retain interrelationships with a bivariate exponential and a bivariate FR distribution.

To solve this problem we consider a bivariate generalization of a univariate characterizing property that links up Lomax distribution with exponential and FR distributions. We next try to identify the bivariate Lomax distribution for which this generalization is a unique property. The characterization result we present also provides answers to the problem of bivariate generalizations of the exponential and the FR distributions. The bivariate Lomax distribution thus obtained is a particular member of the Arnold (1990) class with suitable reparametrization. A bivariate exponential distribution due to Gumbel (1960) happens to be identical with what we have obtained here. We also examine the suitability of our proposed distributions in respect of other properties including constant memory.

2. SOME BASIC CONCEPTS

In the univariate case it has been observed in Mukherjee and Roy (1986) that for absolutely continuous life distributions with $C(u)$ as the coefficient of variation of the residual life after an elapsed time u, $C(u) = k$, a constant, characterizes the exponential distribution for $k = 1$, the Lomax distribution for $k>1$ and the FR distribution for $k<1$. To generalize this property in the bivariate set up we consider nonnegative vector variable $\mathbf{X}=(X_1, X_2)$ with survival function $S(x_1, x_2)$, hazard function $R(x_1, x_2)$ and hazard rates $r_i(x_l, x_2)$, $i=1, 2$, where

$$R(x_1, x_2) = -\log S(x_1, x_2), \qquad r_i(x_1, x_2) = \frac{\partial}{\partial x_i} R(x_1, x_2). \tag{2.1}$$

Corresponding means, variances and coefficient of variations of the residual lives after an elapsed time (x_l, x_2) be denoted by $M_i(x_l, x_2)$, $V_i(x_1, x_2)$ and $C_i(x_1, x_2)$, $i=1, 2$, respectively, where

$$\begin{aligned} &M_i(x_1, x_2 = E(X_i - x_i \mid X_1 > x_1, X_2 > x_2) \\ &V_i(x_1, x_2) = \mathrm{Var}(X_i - x_i \mid X_1 > x_1, X_2 > x_2) \\ &C_i(x_1, x_2) = \{V_i(x_1, x_2)\}^{1/2} / M_i(x_1, x_2). \end{aligned} \tag{2.2}$$

Then the condition $C(u)=k, \forall u \geqslant 0$ of the univariate set up may be generalized as

$$C_i(x_1, x_2) = k_i, \qquad i=1, 2, \quad \forall x_1 \geqslant 0, \quad \forall x_2 \geqslant 0. \tag{2.3}$$

Now considering (2.3) as a bivariate characterizing property we look for unique determination of the survival function under various choices of (k_1, k_2). Obviously, to obtain the bivariate extension of the exponential distribution we need to choose $k_1=k_2=1$, that of the Lomax distribution we need to choose $k_1>1, k_2>1$, and that of the FR distribution we need to choose $k_1<1, k_2<1$. It will be observed in Lemma 2.4 that $k_1 \neq k_2$ will lead to independence of X_1 and X_2, and hence to make (2.3) meaningful we need to consider $k_1=k_2$. As a result other combinations of (k_1, k_2) are not of much interest. For example, if $k_1>1$ and $k_2<1$ then X_1 and X_2 are necessarily independent and hence from the univariate characterization result we get the Lomax distribution for X_1, and the FR distribution for X_2.

We next present a few results which will be used in the characterization theorem to be presented subsequently.

LEMMA 2.1. *When hazard rates are continuous, two alternative expressions for the survival function are*

$$S(x_1, x_2) = \exp\left[-\int_0^{x_1} r_1(u, 0)\, du - \int_0^{x_2} r_2(x_1, v)\, dv\right]$$

$$S(x_1, x_2) = \exp\left[-\int_0^{x_1} r_1(u, x_2)\, du - \int_0^{x_2} r_2(0, v)\, dv\right].$$

LEMMA 2.2. *When hazard rates exist*

$$r_i(x_1, x_2) = \left[1 + \frac{\partial}{\partial x_i} M_i(x_1, x_2)\right] \Big/ M_i(x_1, x_2), \qquad i=1, 2. \tag{2.4}$$

LEMMA 2.3. *When the hazard rates are continuous and $V_i(x_1, x_2)$ exist the condition* (2.3) *is equivalent to the condition*

$$\frac{\partial}{\partial x_i} M_i(x_1, x_2) = \frac{k_i^2 - 1}{k_i^2 + 1}, \qquad i = 1, 2. \tag{2.5}$$

Proof. It follows from (2.2) and usual integration by parts

$$V_1(x_1, x_2) = 2[S(x_1, x_2)]^{-1} \int_0^\infty (x_1 + t)\, S(x_1 + t, x_2)\, dt$$

$$- [M_1(x_1, x_2)]^2 - 2x_1 M_1(x_1, x_2)$$

$$V_2(x_1, x_2) = 2[S(x_1, x_2)]^{-1} \int_0^\infty (x_2 + t)\, S(x_1, x_2 + t)\, dt$$

$$- [M_2(x_1, x_2)]^2 - 2x_2 M_2(x_1, x_2).$$

Under condition (2.3), after simplification we get

$$(k_1^2 + 1)\, S(x_1, x_2)\, M_1^2(x_1, x_2) = 2 \int_{x_1}^\infty uS(u, x_2)\, du - 2x_1 S(x_1, x_2)\, M_1(x_1, x_2) \tag{2.6}$$

$$(k_2^2 + 1)\, S(x_1, x_2)\, M_2^2(x_1, x_2) = 2 \int_{x_2}^\infty uS(x_1, u)\, du - 2x_2 S(x_1, x_2)\, M(x_1, x_2). \tag{2.7}$$

Differentiating (2.6) with respect to x_1 and (2.7) with respect to x_2 and simplifying the resultant expressions using Lemma 2.2 we get (2.5). The converse will be ensured once Theorem 3.1 is proved. ▮

LEMMA 2.4. *If hazard rates are continuous, condition* (2.3) *implies independence of X_1 and X_2 whenever $k_1 \neq k_2$.*

Proof. Under condition (2.3) we get from Lemma 2.3 that

$$M_i(x_1, x_2) = \frac{k_i^2 - 1}{k_i^2 + 1} x_i + a_i'(x_{3-i}), \qquad i = 1, 2, \tag{2.8}$$

where $a_i'(\cdot)$ is a constant of integration with respect to x_i and may be a function of x_{3-i}. Simplification of (2.8) using lemma 2.2 results in

$$r_i(x_1, x_2) = \begin{cases} \dfrac{2k_i^2}{k_i^2-1} \dfrac{1}{x_i + a_i(x_{3-i})} & \text{when} \quad k_i \neq 1 \\ \dfrac{1}{a_i'(x_{3-i})} & \text{when} \quad k_i = 1 \end{cases} \tag{2.9}$$

for $i = 1, 2$, where $a_i(x_{3-i}) = a_i'(x_{3-i})(k_i^2+1)/(k_i^2-1)$.

Case 1. $k_1 = 1, k_2 \neq 1$. Using (2.9) in Lemm 2.1 we obtain two alternative expressions for $S(x_1, x_2)$ as

$$S(x_1, x_2) = \exp\left[-\frac{x_1}{a_1'(0)} - q \log \frac{x_2 + a_2(x_1)}{a_2(x_1)} \right], \qquad \forall x_1, x_2 \geqslant 0$$

$$S(x_1, x_2) = \exp\left[-\frac{x_1}{a_1'(x_2)} - q \log \frac{x_2 + a_2(0)}{a_2(0)} \right], \qquad \forall x_1, x_2 \geqslant 0,$$

where $q = 2k_2^2/(k_2^2-1)$. Comparing them we get the identity

$$\frac{x_1}{a_1'(0)} + q \log \frac{x_2 + a_2(x_1)}{a_2(x_1)} = \frac{x_1}{a_1'(x_2)} + q \log \frac{x_2 + a_2(0)}{a_2(0)} \qquad \forall x_1, x_2 \geqslant 0. \tag{2.10}$$

The right-hand side of (2.10) being linear in x_1 the left-hand side of it must be linear in x_1 and hence

$$\log \frac{x_2 + a_2(x_1)}{a_2(x_1)} = \alpha(x_2) + \beta(x_2)\, x_1, \qquad \text{say.}$$

This implies that

$$a_2(x_1) = [\exp\{\alpha(1) + \beta(1)\, x_1\} - 1]^{-1}$$

and hence

$$\log[x_2 \exp\{\alpha(1) + \beta(1)\, x_1\} + (1 - x_2)]$$

must be linear in x_1 for all choices of $x_2 \geqslant 0$. Differentiating the same twice with respect to x_1 and equating the resultant expression with zero for all choices of $x_2 \geqslant 0$ we get $\beta(1) = 0$, i.e., $a_2(x_1)$ is independent of x_1. This implies independence of X_1 and X_2.

Case 2. $k_1 \neq 1, k_2 = 1$. This can be similarly proved.

Case 3. $k_1 \neq k_2 \neq 1$. Using (2.9) in Lemma 2.1 we get two alternative expressions for $S(x_1, x_2)$, as

$$S(x_1, x_2) = \left[\frac{x_1 + a_1(0)}{a_1(0)}\right]^{-q_1} \left[\frac{x_2 + a_2(x_1)}{a_2(x_1)}\right]^{-q_2} \qquad \forall x_1, x_2 \geqslant 0 \tag{2.11}$$

$$S(x_1, x_2) = \left[\frac{x_1 + a_1(x_2)}{a_1(x_2)}\right]^{-q_1} \left[\frac{x_2 + a_2(0)}{a_2(0)}\right]^{-q_2} \qquad \forall x_2, x_2 \geqslant 0 \tag{2.12}$$

where $q_1 = 2k_1^2/(k_1^2 - 1)$ and $q_2 = 2k_2^2/(k_2^2 - 1)$.

Since $k_1 \neq k_2$ we note that $q_1/q_2 \neq 1$. Now, comparing (2.11) and (2.12) we obtain the identity

$$\left[\frac{x_1 + a_1(0)}{a_1(0)}\right]\left[\frac{x_2 + a_2(x_1)}{a_2(x_1)}\right]^{q_2/q_1} = \left[\frac{x_1 + a_1(x_2)}{a_1(x_2)}\right]\left[\frac{x_2 + a_2(0)}{a_2(0)}\right]^{q_2/q_1}; \qquad \forall x_1, x_2 \geqslant 0. \tag{2.13}$$

The right-hand side of (2.13) being linear in x_1, the left-hand side of it must be linear in x_1, say, $\gamma(x_2) + \delta(x_2)\, x_1$. This implies for a choice of $x_2 = 1$

$$a_2(x_1) = \left[\left\{\left(\frac{a_1(0)}{x_1 + a_1(0)}\right)(\gamma(1) + \delta(1)\, x_1)\right\}^{q_1/q_2} - 1\right]^{-1}.$$

Substituting the same in the left-hand side-of (2.13) we have, after necessary simplification,

$$\left[(1 - x_2)\left\{\frac{x_1 + a_1(0)}{a_1(0)}\right\}^{q_1/q_2} + x_2\{\gamma(1) + \delta(1)\, x_1\}^{q_1/q_2}\right]^{q_2/q_1}$$

must be linear in x_1 for all choices of $x_2 \geqslant 0$.

Differentiating the same twice with respect to x_1 and equating the resultant expression with zero for all choices of $x_2 \geqslant 0$ we get

$$\delta(1)/\gamma(1) = 1/a_1(0)$$

which in turn implies independence of x_1 and x_2. This completes the proof of the lemma. ▮

Thus for obtaining nontrivial extensions of the Lomax distribution and the FR distribution we have to necessarily confine ourselves in the domain of $k_1 = k_2 > 1$ and $k_1 = k_2 < 1$, respectively.

3. MAIN RESULTS

We are now in a position to present a characterization result and provide with unique determinations of a bivariate exponential distribution, a bivariate Lomax distribution and a bivariate FR distribution.

THEOREM 3.1. *For a nonnegative vector variable* $\mathbf{X}=(X_1, X_2)$ *with continuous hazard rates let the condition* (2.3) *be true. Then*

(i) $k_1=k_2=1$ *if and only if* $\mathbf{X}$ *follows bivariate exponential distribution due to Gumbel* (BVED-G) *with survival function*

$$S(x_1, x_2)=\exp(-\sigma_1 x_1-\sigma_2 x_2-\sigma_3 x_1 x_2), \tag{3.1}$$

where $\sigma_1>0$, $\sigma_2>0$, $0\leqslant\sigma_3\leqslant\sigma_1\sigma_2$.

(ii) $k_1=k_2>1$ *if and only if* $\mathbf{X}$ *follows a bivariate Lomax distribution* (*BVLD*) *with survival function*

$$S(x_1, x_2)=(1+\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_1 x_2)^{-q}, \tag{3.2}$$

where $\lambda_1>0$, $\lambda_2>0$, $0\leqslant\lambda_3\leqslant\lambda_1\lambda_2(1+q)$, $q=2k_1^2/(k_1^2-1)$.

(iii) $1/\sqrt{3}\leqslant k_1=k_2<1$ *if and only if* $\mathbf{X}$ *follows bivariate* FR *distribution* (BVFRD) *with survival function*

$$S(x_1, x_2)=(1-\theta_1 x_1-\theta_2 x_2-\theta_3 x_1 x_2)^p, \tag{3.3}$$

where $\theta_1>0$, $\theta_2>0$, $p-1\geqslant\theta_3/(\theta_1\theta_2)\geqslant-1$, $p=2k_1^2/(1-k_1^2)$

$$0\leqslant x_1\leqslant\theta_1^{-1}; \qquad 0\leqslant x_2\leqslant(1-\theta_1 x_1)/(\theta_2+\theta_3 x_1).$$

(iv) $0<k_1=k_2<1/\sqrt{3}$ *if and only if* X_1 *and* X_2 *are independently distributed having survival function as in* (3.3) *with* $\theta_3=-\theta_1\theta_2$.

(v) $k_1\neq k_2$ *if and only if* X_1 *and* X_2 *are independently distributed with marginal distributions determined from the univariate characterization result* (*Mukherjee and Roy* (1986)) *from amongst the class of exponential, Lomax and FR distributions.*

Proof. (i) Under the given condition we obtain via Lemmas 2.2 and 2.3

$$r_i(x_1, x_2)=1/a_i'(x_{3-i}), \qquad i=1, 2. \tag{3.4}$$

Thus hazard rates are locally constant. Hence from Johnson and Kotz (1975) we note that $\mathbf{X}$ follows BVED-G (1960). The converse is easy to establish.

(ii) Under the given condition $k_1 = k_2 > 1$ we obtain via Lemma 2.2 and 2.1

$$r_i(x_1, x_2) = q/[x_i + a_i(x_{3-i})], \qquad i = 1, 2, \tag{3.5}$$

where $q = 2k_1^2/(k_1^2 - 1) > 2$.

Using (3.5) in Lemma 2.1 we get two alternative expressions for $S(x_1, x_2)$,

$$S(x_1, x_2) = \left[\left\{ \frac{x_1 + a_1(0)}{a_1(0)} \right\} \left\{ \frac{x_2 + a_2(x_1)}{a_2(x_1)} \right\} \right]^{-q} \qquad \forall x_1, x_2 \geqslant 0 \tag{3.6}$$

$$S(x_1, x_2) = \left[\left\{ \frac{x_1 + a_1(x_2)}{a_1(x_2)} \right\} \left\{ \frac{x_2 + a_2(0)}{a_2(0)} \right\} \right]^{-q} \qquad \forall x_1, x_2 \geqslant 0. \tag{3.7}$$

Comparing (3.6) and (3.7) we get, as in the proof of Lemma 2.4,

$$\frac{x_1 + a_1(0)}{a_2(x_1)} = \sigma + \psi x_1, \tag{3.8}$$

where σ and ψ are constants independent of x_1 and x_2. Combining (3.8) with (3.6) we get

$$S(x_1, x_2) = [1 + (1/a_1(0))\, x_1 + (\sigma/a_1(0))\, x_2 + (\psi/a_1(0))\, x_1 x_2]^{-q},$$

which is of the form (3.2). It is easy to verify from the properties of survival function that $\lambda_1 > 0$, $\lambda_2 > 0$ since $q > 0$. Further $S(x, x)$, being the survival function of $\mathrm{Min}(X_1, X_2)$, implies that $\lambda_3 \geqslant 0$ from the properties of survival function. Finally from the nonnegativity condition of the joint density one has $\lambda_3 \leqslant \lambda_1 \lambda_2 (1 + q)$. These conditions are sufficient also to make (3.2) a survival function.

To prove the converse we note from (3.2) that the conditional distributions $\{X_i \mid X_{3-i} > x_{3-i}\}$, $i = 1, 2$, are of Lomax form with identical shape parameter as q. Hence from the univariate properties of the Lomax distribution we get $C_i^2(x_1, x_2) = q/(q-2)$, $i = 1, 2$.

(iii) Proof is similar to that of (ii) except for the fact that $q = 2k_1^2/(k_1^2 - 1) \leqslant -1$. Writing $p = -q (\geqslant 1)$ we get as in the earlier case

$$S(x_1, x_2) = [1 + (1/a_1(0))\, x_1 + (\sigma/a_1(0))\, x_2 + (\psi/a_1(0))\, x_1 x_2]^{p}. \tag{3.9}$$

As $p > 1$, we get from the properties of survival function that $1/a_1(0) < 0$, $\sigma/a_1(0) < 0$. Writing $-1/a_1(0) = \theta_1$, $-\sigma/a_1(0) = \theta_2$, and $-\psi/a_1(0) = \theta_3$ we have (3.3), where $\theta_1 > 0$, $\theta_2 > 0$. This implies that $0 < x_1 < \theta_1^{-1}$, $0 \leqslant x_2 \leqslant (1 - \theta_1 x_1)/(\theta_2 + \theta_3 x_1)$. Further nonnegativity of the probability density function implies $-1 \leqslant \theta_3/(\theta_1 \theta_2) \leqslant p - 1$. These conditions, taken together, are also sufficient to define (3.3) as a survival function.

To prove the converse we proceed as in the case (ii) of above.

(iv) Proof is as in (iii) except for the fact that $0 < p < 1$. This condition implies that $\theta_3 = -\theta_1\theta_2$ because otherwise the probability density function may become negative. Now $\theta_3 = -\theta_1\theta_2$ in (3.3) implies independence of X_1 and X_2. The converse follows from the univariate result because of the independence of X_1 and X_2.

(v) From Lemma 2.4 we have independence of X_1 and X_2 and hence univariate characterization results apply. ∎

The above characterization result provides bivariate extensions of exponential, Lomax, and FR distributions. As noted earlier, bivariate exponential distribution due to Gumbel is identical with what we have obtained under coefficient of variations of residual lives as unity. No other bivariate exponential distribution proposed in the literature so far (Marshall and Olkin (1967), Block (1977), Sarker (1987)) can have this property, as our determination is unique. Also, from Lemma 2.3 we can obtain uniquely the BVED-G from the local constancy of the mean residual lives (Zahedi (1985)).

The bivariate Lomax distribution obtained by us is a member of the Arnold (1990) class and is interesting from both stochastic modelling and characterization points of view. In general for the BVLD we consider (3.2) as its survival function with $q > 0$. This admits Lomax marginals in general and independent Lomax marginals for $\lambda_3 = \lambda_1\lambda_2$. Conditional distribution of X_i given $\{X_{3-i} > x_{3-i}\}$ is again of Lomax form for $i = 1, 2$. The following are some important properties of BVLD:

(P1) BVLD is a member of the bivariate decreasing hazard (failure) rate class of life distributions (Roy (1994)). This is consistent with the DFR property of the univariate Lomax distribution.

(P2) Hazard rates are locally harmonic.

(P3) In case $q > 1$, bivariate mean residual lives exist and are locally linear and increasing.

(P4) It has a constant negative bivariate memory defined in the following sense (Muth (1977)):

$$1 - r_1(x_1, x_2)\, M_1(x_1, x_2) = 1 - r_2(x_1, x_2)\, M_2(x_1, x_2) = -1/(q-1) < 0.$$

(P5) If the hazard function of a dependent system of two components following BVED-G is scaled up (or down) by an environmental factor, following gamma distribution, then the resultant mixture distribution follows BVLD. Thus, it is a member of the dependent EM_k class envisaged in Roy and Mukherjee (1988).

(P6) If $q \to \infty$, such that $q\lambda_i \to \sigma_i$, a constant, for $i=1, 2, 3$, the BVLD reduces to BVED-G with survival function (3.1).

The proposed bivariate FR distribution is the same as that obtained at (3.3) with $p>1$. In the univariate case FR distribution is a special case of beta distribution. From our proposed BVFRD one can obtain Dirichlet's form for $\theta_3=0$. BVFRD admits marginal FR distributions in general and independent FR marginals for $\theta_3=-\theta_1\theta_2$. Conditional distribution of X_i given $\{X_{3-i}>x_{3-i}\}$ is again of FR form for $i=1, 2$. The following are some important properties of BVFRD:

(P1)′ BVFRD is a member of bivariate increasing hazard (failure) rate class of life distributions (Roy (1994)). This is consistent with IFR property of the univariate FR distribution.

(P2)′ Hazard rates are locally harmonic in the support of X.

(P3)′ Bivariate mean residual lives are locally linear and decreasing.

(P4)′ It has a constant positive bivariate memory in the following sense:

$$1-r_1(x_1, x_2)\, M_1(x_1, x_2)=1-r_2(x_1, x_2)\, M_2(x_1, x_2)=1/(p+1)>0.$$

The following theorem is a general version of the characterizing properties mentioned in (P4) and (P4)′. The proof follows from the fact that condition (3.10) presented below can be simplified to (2.5) via (2.4) of Lemma 2.2 and an application of Theorem 3.1.

THEOREM 3.2. *For a nonnegative bivariate random variable* $\mathbf{X}=(X_1, X_2)'$ *with continuous hazard rates let the condition*

$$1-r_i(x_1, x_2)\, M_i(x_1, x_2)=k_i, \qquad i=1, 2 \tag{3.10}$$

be true. Then

(i) $k_1=k_2=0$ *if and only if* $\mathbf{X}$ *follows* BVED-G;

(ii) $k_1=k_2<0$ *if and only if* $\mathbf{X}$ *follows* BVLD;

(iii) $0<k_1=k_2\leqslant 1/2$ *if and only if* $\mathbf{X}$ *follows* .BVFRD;

(iv) $\frac{1}{2}<k_1=k_2<1$ *if and only if* X_1, X_2 *have independent* FR *distributions*;

(v) $k_1\neq k_2$ *if and only if* X_1, X_2 *have independent distributions characterized by the univariate result* (*Theorem* 4.2 *of Mukherjee and Roy* (1986)).

Remarks. Multivariate generalizations of BVLD and BVFRD are straightforward. The condition (2.3) can be generalized as

$$C_i(x_1, x_2, ..., x_p) = k_i, \qquad i = 1, 2, ..., p.$$

The only combinations which will be of interest to us are

$$k_1 = k_2 = \cdots k_p = 1, \quad k_1 = k_2 = \cdots = k_p > 1, \quad \text{and} \quad k_1 = k_2 = \cdots = k_p < 1.$$

The first combination will lead to multivariate exponential distribution due to Gumbel (1960) (MVEDG), the second combination will determine a multivariate Lomax distribution (MVLD), and the third combination will determine a multivariate FR distribution (MVFRD).

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