

Multivariate weighted renewal functions

F. Mallor^{a,*}, E. Omey^b, J. Santos^c

^aDepartment of Statistics and Operations Research, Public University of Navarre, Campus Arrosadía,
31006 Pamplona, Spain

^bDepartment of Mathematics and Statistics, European Economics Hogeschool EHSAL, Stormstraat 2,
1000 Brussels, Belgium

^cDepartment of Statistics and Operations Research, Public University of Navarre, Campus Arrosadía,
31006 Pamplona, Spain

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Abstract

Let (X, Y) , (X_1, Y_1) , (X_2, Y_2) , ... denote independent positive random vectors with common distribution function $F(x, y) = P(X \leq x, Y \leq y)$ with $F(x, y) < 1$ for all x, y . Based on the X_i and the Y_j we construct the sum sequences S_n^1 and S_m^2 respectively. For a double sequence of weighting constants $\{b(n, m)\}$ we associate a weighted renewal function $G(x, y)$ defined as $G(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b(n, m) P(S_n^1 \leq x, S_m^2 \leq y)$. The function $G(x, y)$ can be expressed in terms of well-known renewal quantities. The main goal of this paper is to study asymptotic properties of $G(x, y)$. In the one-dimensional case such results have been obtained among others by Omey and Teugels [Weighted renewal functions: a hierarchical approach, *Adv. in Appl. Probab.* 34 (2002) 394–415.] and Alsmeyer [Some relations between harmonic renewal measures and certain first passage times, *Statist. Probab. Letters* 12 (1991) 19–27; On generalized renewal measures and certain first passage times, *Ann. Probab.* 20 (1992) 1229–1247]. Here we prove a multivariate version of the elementary renewal theorem and moreover we obtain a rate of convergence result in this elementary renewal theorem. We close the paper with an application and some concluding remarks. For convenience we prove and formulate the results in the two-dimensional case only.

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* Corresponding author. Fax: +34 948 16 92 04.

E-mail addresses: mallor@unavarra.es (F. Mallor), edward.omey@ehsal.be (E. Omey), javier.santos@unavarra.es (J. Santos).

1. Introduction

Let $(X, Y), (X_1, Y_1), (X_2, Y_2), \dots$ denote a sequence of independent positive random vectors with common distribution function $F(x, y) = P(X \leq x, Y \leq y)$ with $F(x, y) < 1$ for all x, y . Based on the X_i we construct the sum sequence S_n^1 as usual. We set $S_0^1 = 0$ and for $n \geq 0$, $S_{n+1}^1 = S_n^1 + X_{n+1}$. In a similar way we construct the sum sequence S_m^2 based on Y_i . The renewal processes associated with X and Y are defined for $i = 1, 2$ by $N_i(x) = \min\{n \geq 1 : S_n^i > x\}$. In renewal theory it is often more convenient to work with the quantities $M_i(x) = N_i(x) - 1$, $i = 1, 2$. The associated renewal functions are defined as $E(N_i(x))$ and $U_i(x) = E(M_i(x))$. Using these univariate quantities we can define several multivariate processes.

Following Hunter [11,12] we set $M(x, y) = \max\{n : S_n^1 \leq x, S_n^2 \leq y\} = \min(M_1(x), M_2(y))$. The vector $(M_1(x), M_2(y))$ can be called a bivariate renewal counting process. The process $M(x, y)$ is a two-dimensional renewal process. These processes have been studied among others by Hunter [11,12], Niculescu and Omei [16], Omei and Willekens [21].

For a double sequence of real weighting constants $\{b(n, m)\}$ we associate a weighted renewal function (wrf) $G(x, y)$ defined as follows:

$$G(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b(n, m) P(S_n^1 \leq x, S_m^2 \leq y).$$

In principle any sequence $\{b(n, m)\}$ can be used but, to make sure that the function $G(x, y)$ makes sense at all, we assume that $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |b(n, m)| u^n v^m < \infty$ for $|u| < 1, |v| < 1$.

Note that any wrf has a useful interpretation as an expectation. To show this, we define the function $B(x, y)$ as follows:

For $n \leq x < n+1, m \leq y < m+1$ we set $B(x, y) = \sum_{k=0}^n \sum_{l=0}^m b(k, l)$.

Since $P(S_n^1 \leq x, S_m^2 \leq y) = P(M_1(x) \geq n, M_2(y) \geq m)$ it easily follows that

$$G(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b(n, m) P(S_n^1 \leq x, S_m^2 \leq y) = E(B(M_1(x), M_2(y))).$$

Conversely, any function $B(x, y)$ can serve as the generator of a wrf. For a given $B(x, y)$ we set

$$b(0, 0) = B(0, 0),$$

$$b(k, 0) = B(k, 0) - B(k-1, 0) \text{ and } b(0, k) = B(0, k) - B(0, k-1) \text{ for } k \geq 1 \text{ and}$$

$$b(k, l) = B(k, l) - B(k-1, l) - B(k, l-1) + B(k-1, l-1) \text{ for } k, l \geq 1.$$

In this case we find

$$G(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b(n, m) P(S_n^1 \leq x, S_m^2 \leq y) = E(B(M_1(x), M_2(y)))$$

as required.

In the next sections we shall use the representation $G(x, y) = E(B(M_1(x), M_2(y)))$ and formulate assumptions on the function $B(x, y)$. The main goal of this paper is to study the asymptotic behaviour of the wrf $G(x, y)$. In the one-dimensional case such results have been obtained in [2,4,20]. For further use we mention here one of the main theorems of Omei

and Teugels [20]. To formulate the result we consider $G(x) = \sum_{n=0}^{\infty} a(n)P(S_n^1 \leq x) = E(A(M_1(x)))$ where $A(x) = \sum_{k=0}^n a(k)$ for $n \leq x < n+1$. Recall that a positive and measurable function $f(x)$ is regularly varying with index α (notation $f(x) \in RV(\alpha)$) if it satisfies $\lim_{t \rightarrow \infty} f(xt)/f(t) = x^\alpha$, for all $x > 0$.

Theorem 1 (Omeij and Teugels 20, Section 4). *Let $\mu = E(X) < \infty$ and assume that $A(x) \in RV(\rho)$.*

- (i) *If $0 \leq A(x) \uparrow \infty$ as $x \rightarrow \infty$ then $G(x) \sim A(x/\mu)$ as $x \rightarrow \infty$.*
- (ii) *Suppose that $A(x) \downarrow 0$ and that there exist $r \geq 1$ and $L(x) \in RV(0)$ such that $x^r L(x)P(X > x) = o(1)$ and $\liminf_{x \rightarrow \infty} x^{r-1} L(x)A(x) > 0$. Then $G(x) \sim A(x/\mu)$ as $x \rightarrow \infty$.*

Omeij and Teugels proved many other related results in [20]. In Section 2 of the present paper we shall use multivariate regular variation to prove a multivariate analogue of Theorem 1. Recall that a positive and measurable function $f(x, y)$ is a multivariate regularly varying function (notation $f(x, y) \in RVF$) if it satisfies $\lim_{t \rightarrow \infty} f(tx, ty)/f(t, t) = \lambda(x, y)$, for all $x, y > 0$ and for some positive limit function $\lambda(x, y)$. In this case it automatically follows that $f(x, x) \in RV(\rho)$ for some real number ρ and also that $\lambda(x, y)$ is a homogeneous function of order ρ , i.e. $\lambda(tx, ty) = t^\rho \lambda(x, y)$, for all $x, y, t > 0$. In this case we use the notation $f(x, y) \in RVF(\rho, \lambda)$. In what follows we say that the function $f(x, y)$ is nondecreasing (resp. nonincreasing) if it is nondecreasing (resp. nonincreasing) in each component. If $f(x, y) \in RVF(\rho, \lambda)$ is monotone it is known that the convergence $\lim_{t \rightarrow \infty} f(tx, ty)/f(t, t) = \lambda(x, y)$ holds locally uniformly (l.u.) in $x, y > 0$. More details, properties and a number of applications of multivariate regularly varying functions can be found e.g. in [9,10,17,18,21].

The paper is organized as follows. In the next section we consider a multivariate elementary renewal theorem. In Section 3 we obtain a second order result providing a multivariate elementary renewal remainder theorem. In Section 4 we apply these results in the context of reliability theory, and we close the paper with some concluding remarks.

2. A multivariate elementary renewal theorem

In this section we prove the analogue of Theorem 1 in the multivariate case. As in the univariate case we have to consider two cases. In Theorem 2 we assume that $B(x, y)$ is nondecreasing. In Theorem 3 we assume that $B(x, y)$ is nonincreasing. In what follows we use the notation $\max(x, y) = x \vee y$.

Theorem 2. *Let $\mu = E(X) < \infty$, $v = E(Y) < \infty$ and assume that $B(x, y)$ is nondecreasing with $A(x) \equiv B(x, x) \uparrow \infty$. If $B(x, y) \in RVF(\rho, \lambda)$ then*

$$\lim_{t \rightarrow \infty} G(tx, ty)/A(t) = \lambda(x/\mu, y/v).$$

Proof. To prove the result we apply Pratt's extension of Lebesgues theorem on dominated convergence, see [13,22]. First note that

$$B(M_1(tx), M_2(ty)) \leq A(M_1(tx)) + A(M_2(ty)).$$

From univariate renewal theory we know that

$$M_1(tx)/t \xrightarrow{\text{a.s.}} x/\mu \text{ and } M_2(ty)/t \xrightarrow{\text{a.s.}} y/v.$$

Using $B(x, y) \in RVF$ and l.u. convergence it follows that

$$B(M_1(tx), M_2(ty))/A(t) \xrightarrow{\text{a.s.}} \lambda(x/\mu, y/v).$$

Since $A(x) \in RV(\rho)$ it also follows that

$$A(M_1(tx))/A(t) \xrightarrow{\text{a.s.}} (x/\mu)^\rho \quad \text{and} \quad A(M_2(ty))/A(t) \xrightarrow{\text{a.s.}} (y/v)^\rho.$$

Moreover, from Theorem 1 it follows that $E(A(M_1(tx)))/A(t) \rightarrow (x/\mu)^\rho$ and $E(A(M_2(ty)))/A(t) \rightarrow (y/v)^\rho$. Pratt's lemma now applies and yields

$$E(B(M_1(tx), M_2(ty)))/A(t) \rightarrow \lambda(x/\mu, y/v).$$

This proves the result. $\square$

Theorem 3. Let $\mu = E(X) < \infty$, $v = E(Y) < \infty$ and assume that $B(x, y) \in RVF(\rho, \lambda)$ is nonincreasing with $A(x) \equiv B(x, x) \downarrow 0$. In addition assume that there exists $r \geq 1$ and $L(x) \in RV(0)$ such that $x^r L(x) P(X > x) = o(1)$, $x^r L(x) P(Y > x) = o(1)$ and $\liminf_{x \rightarrow \infty} x^{r-1} L(x) A(x) > 0$. Then

$$\lim_{t \rightarrow \infty} G(tx, ty)/A(t) = \lambda(x/\mu, y/v).$$

Proof. As before we have $B(M_1(tx), M_2(ty))/A(t) \xrightarrow{\text{a.s.}} \lambda(x/\mu, y/v)$. First we use $M(x, y) = \min(M_1(x), M_2(y))$ to see that $B(M_1(x), M_2(y)) \leq A(M(x, y))$. Next we observe that $M(tx, ty) \geq n$ if and only if $(S_n^1 \leq tx, S_n^2 \leq ty)$ and this holds if and only if $((S_n^1/x) \vee (S_n^2/y) \leq t)$. This motivates the introduction of the renewal process $K(t)$ based on $Z_i(x, y) = X_i/x \vee Y_i/y$ as follows:

$$K(t) = \max \left\{ n : \sum_{i=1}^n Z_i(x, y) \leq t \right\}.$$

Using $(S_n^1/x) \vee (S_n^2/y) \leq \sum_{i=1}^n Z_i(x, y)$ it follows that $K(t) \leq M(tx, ty)$ and consequently also that $A(M(tx, ty)) \leq A(K(t))$. Clearly $P(Z(x, y) > t) = 1 - F(tx, ty)$ and hence

$$P(Z(x, y) > t) \leq P(X > tx) + P(Y > ty).$$

We also note that $\theta := E(Z(x, y)) < \infty$. We can apply Theorem 1(ii) to see that $E(A(K(t)))/A(t) \rightarrow \theta^\rho$. Again Pratt's lemma applies and this proves the result. $\square$

In the next corollary we use the notation $U_i(x) = E(M_i(x))$ for $i = 1, 2$. The one-dimensional elementary renewal theorem states that $U_i(t) \sim t/\mu$ as $t \rightarrow \infty$. Using the regular variation of $B(x, y)$ we easily see that $B(U_1(tx), U_2(ty)) \sim \lambda(x/\mu, y/\nu)A(t)$.

Corollary 4. *Under the conditions of Theorem 2 or 3, we have*

$$\lim_{t \rightarrow \infty} G(tx, ty)/B(U_1(tx), U_2(ty)) = 1.$$

From these results we can obtain several special cases and examples.

Example 5. Taking $b(n, m) \equiv 1$ we have $B(x, y) = ([x] + 1)([y] + 1)$ and $G_1(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(S_n^1 \leq x, S_m^2 \leq y)$. Since $B(x, y) \in RVF(\rho = 2, \lambda(x, y) = xy)$ we see that $G_1(tx, ty)/t^2 \rightarrow xy/\mu\nu$. This result was obtained by Omev and Willekens in [21].

Example 6. Taking $b(n, m) = 0$ for $n \neq m$ and $b(n, n) = a(n)$ we have $G_2(x, y) = \sum_{n=0}^{\infty} a(n)P(S_n^1 \leq x, S_n^2 \leq y)$ which can be rewritten as $G_2(x, y) = E(A(M(x, y)))$ where, as before, $A(x) = \sum_{k=0}^n a(k)$ for $n \leq x < n+1$ and where $M(x, y) = \min(M_1(x), M_2(y))$. It is easy to see that $B(x, y) = A(\min(x, y))$. Assuming $A(x) \in RV(\rho)$ our Theorems 2 and 3 provide conditions under which $G_2(tx, ty)/A(t) \rightarrow (\min(x/\mu, y/\nu))^\rho$. The special case where $a(n) \equiv 1$ has been considered in [21].

Example 7. If in Example 6 we take $a(0) = 0$ and $a(n) = 1/n$ for $n \geq 1$ we obtain the bivariate harmonic renewal function

$$G_3(x, y) = \sum_{n=1}^{\infty} \frac{1}{n} P(S_n^1 \leq x, S_n^2 \leq y).$$

Theorem 2 can be used and we obtain that $G_3(tx, ty)/\log(t) \rightarrow 1$. The univariate harmonic renewal function has been studied among others in [1,2,6–8].

3. Multivariate renewal remainder theorem

In Corollary 4 we provided conditions under which $G(tx, ty) \sim B(U_1(tx), U_2(ty))$. The purpose of this section is to find an upper bound for

$$|G(tx, ty) - B(U_1(tx), U_2(ty))|.$$

It seems to be very hard to obtain exact asymptotic estimates. Even in the univariate case such second order results are difficult to prove, cf. [20, Theorem 13].

To obtain an upper bound we will have to assume more about (X, Y) and about $B(x, y)$. Throughout this section we shall assume that $\text{Var}(X) + \text{Var}(Y) < \infty$. This condition assures that the central limit theorem for $(M_1(x), M_2(y))$ holds, cf. [11,12], and that $\text{Var}(M_i(tx)) = O(1)t$. As to $B(x, y)$, we shall assume that the function $B(x, y)$ has nonnegative and monotone nondecreasing second derivatives which will be denoted, as usual, by $B_{1,1}(x, y)$, $B_{1,2}(x, y) = B_{2,1}(x, y)$ and $B_{2,2}(x, y)$. We shall assume that the second derivatives are

multivariate regularly varying functions. The first derivatives will be denoted by $B_1(x, y)$ and $B_2(x, y)$. In our approach we define R as follows:

$$R = B(x, y) - B(u, v) - (x - u)B_1(u, v) - (y - v)B_2(u, v).$$

Using the first terms of a Taylor expansion we have $|R| \leq A + B + C$ where

$$\begin{aligned} A &= B_{1,1}(x \vee u, y \vee v)(x - u)^2/2, \\ B &= B_{2,2}(x \vee u, y \vee v)(y - v)^2/2, \\ C &= B_{1,2}(x \vee u, y \vee v)|x - u||y - v|. \end{aligned}$$

Now we replace (x, y) by $(M_1(tx), M_2(ty))$ and (u, v) by $(U_1(tx), U_2(ty))$. In this case we see that $E(R) = G(tx, ty) - B(U_1(tx), U_2(ty))$. To estimate $E(R)$ we have to find upper bounds for the 3 terms $E(A)$, $E(B)$ and $E(C)$.

For further use we define for $\varepsilon > 0$ the integers $w_1 = \lceil xt/(\mu - \varepsilon) \rceil + 1$ and $w_2 = \lceil ty/(v - \varepsilon) \rceil + 1$. Note that $U_1(tx)/w_1 \rightarrow (\mu - \varepsilon)/\mu < 1$ and $U_2(ty)/w_2 \rightarrow (v - \varepsilon)/v < 1$.

First we discuss the term (A) in full details. Since $B_{1,1}(x, y)$ is monotone we have

$$\begin{aligned} 2A &\leq (B_{1,1}(M_1(tx), M_1(tx)) + B_{1,1}(M_2(ty), M_2(ty)) \\ &\quad + B_{1,1}(U_1(tx), U_1(tx)) + B_{1,1}(U_2(ty), U_2(ty)))(M_1(tx) - U_1(tx))^2. \end{aligned} \quad (1)$$

Now we take expectations and have to consider the four terms in $E(2A)$.

The first term is given by $E(A_1) = E(B_{1,1}(M_1(tx), M_1(tx))(M_1(tx) - U_1(tx))^2)$. We consider two cases.

(i) If $M_1(tx) \leq w_1$ then $A_1 \leq B_{1,1}(w_1, w_1)(M_1(tx) - U_1(tx))^2$ and consequently also that

$$E(A_1 I\{M_1(tx) \leq w_1\}) \leq B_{1,1}(w_1, w_1) \text{Var}(M_1(tx)),$$

where $I\{\cdot\}$ denotes the indicator function. Using the regular variation of $B_{1,1}(x, y)$ we obtain that $E(A_1 I\{M_1(tx) \leq w_1\}) = O(1)B_{1,1}(t, t)t$.

(ii) If $M_1(tx) > w_1$ we have $A_1 \leq B_{1,1}(M_1(tx), M_1(tx))M_1^2(tx)$. Using Cauchy–Schwarz we see that

$$E^2(A_1 I\{M_1(tx) > w_1\}) \leq E(B_{1,1}^2(M_1(tx), M_1(tx))M_1^4(tx))P(M_1(tx) > w_1).$$

We can use Theorem 1(i) to see that

$$E(B_{1,1}^2(M_1(tx), M_1(tx))M_1^4(tx)) = O(1)B_{1,1}^2(t, t)t^4.$$

For the second term, we have $P(M_1(tx) > w_1) = P(S_{w_1}^1 < tx)$. By our choice of w_1 it follows that $P(M_1(tx) > w_1) = P(S_{w_1}^1 < tx) \leq P(S_{w_1}^1 \leq w_1(\mu - \varepsilon))$.

By a classical result of Chernoff, see [23], we have that there exists a $\rho(\varepsilon) \in (0, 1)$ such that $P(S_k \leq k(\mu - \varepsilon)) \leq \rho^k(\varepsilon)$ for all $k \geq 1$. We conclude that $P(M_1(tx) > w_1) = O(1)\rho^{w_1}(\varepsilon)$.

It follows that $E^2(A_1 I\{M_1(tx) > w_1\}) = O(1)B_{1,1}^2(t, t)t^4\rho^{w_1}(\varepsilon)$. Since $\rho^{w_1}(\varepsilon)$ tends to 0 exponentially fast, we find that

$$E(A_1 I\{M_1(tx) > w_1\}) = o(1)B_{1,1}(t, t)t.$$

The second term in right-hand side of inequality (1) is given by

$$E(A_2) = E(B_{1,1}(M_2(ty), M_2(ty))(M_1(tx) - U_1(tx))^2).$$

Again we consider several cases.

(i) If $M_2(ty) \leq w_2$, then

$$A_2 \leq B_{1,1}(w_2, w_2)(M_1(tx) - U_1(tx))^2$$

and

$$E(A_2 I\{M_2(ty) \leq w_2\}) \leq B_{1,1}(w_2, w_2) \text{Var}(M_1(tx)).$$

Using the regular variation of $B_{1,1}(x, y)$ we obtain that

$$E(A_2 I\{M_2(ty) \leq w_2\}) = O(1)B_{1,1}(t, t)t.$$

(ii) If $M_2(ty) > w_2$ and $M_1(tx) \leq w_1$ we have

$$(M_1(tx) - U_1(tx))^2 = O(1)(w_1^2 + 2w_1U_1(tx) + U_1^2(tx)) = O(1)t^2$$

and hence

$$\begin{aligned} E(A_2 I\{M_2(ty) > w_2, M_1(tx) \leq w_1\}) \\ = O(1)t^2 E(B_{1,1}(M_2(ty), M_2(ty)) I\{M_2(ty) > w_2, M_1(tx) \leq w_1\}). \end{aligned}$$

Using Cauchy–Schwarz, Theorem 1, and the result of Chernoff, it follows that

$$\begin{aligned} E(A_2 I\{M_2(ty) > w_2, M_1(tx) \leq w_1\}) \\ = O(1)t^2 E(B_{1,1}(M_2(ty), M_2(ty)) I\{M_2(ty) > w_2\}) = o(1)tB_{1,1}(t, t). \end{aligned}$$

(iii) If $M_2(ty) > w_2$ and $M_1(tx) > w_1$ we have

$$(M_1(tx) - U_1(tx))^2 \leq M_1^2(tx)$$

and

$$\begin{aligned} E(A_2 I\{M_2(ty) > w_2, M_1(tx) > w_1\}) \\ = E(B_{1,1}(M_2(ty), M_2(ty))M_1^2(tx) I\{M_2(ty) > w_2, M_1(tx) > w_1\}). \end{aligned}$$

Using Cauchy–Schwarz and Theorem 1(i) again it follows that:

$$\begin{aligned} E^2(A_2 I\{M_2(ty) > w_2, M_1(tx) > w_1\}) &= O(1)B_{1,1}^2(t, t)t^4 P(M_2(ty) > w_2) \\ &= o(1)tB_{1,1}(t, t). \end{aligned}$$

The last two terms in inequality (1) are easier to bound. We see that

$$E(A_3 + A_4) = (B_{1,1}(U_1(tx), U_1(tx)) + B_{1,1}(U_2(ty), U_2(ty)))\text{Var}(M_1(tx)).$$

Using the regular variation of $B_{1,1}(x, y)$ it easily follows that:

$$E(A_3 + A_4) = O(1)B_{1,1}(t, t)t.$$

We conclude that $E(A) = O(1)B_{1,1}(t, t)t$.

In a completely similar way the other terms can be treated and we find that

$$E(B) = O(1)tB_{2,2}(t, t) \quad \text{and} \quad E(C) = O(1)tB_{1,2}(t, t).$$

Summarizing our estimates we have proved the following result.

Theorem 8. Suppose that $\text{Var}(X) + \text{Var}(Y) < \infty$ and that for $i, j = 1, 2$ we have $0 \leq B_{i,j}(x, y) \uparrow$ and $B_{i,j}(x, y) \in RVF$, then

$$|G(tx, ty) - B(U_1(tx), U_2(ty))| = O(1)t(B_{1,1}(t, t) + B_{2,2}(t, t) + B_{1,2}(t, t)).$$

Remark 9. A similar result can be proved in the case where $|B_{i,j}(x, y)|$ is nonincreasing or nondecreasing and where $|B_{i,j}(x, y)| \in RVF$. Depending on the case we deal with, we use part(i) or part(ii) of Theorem 1. It may be necessary to strengthen the conditions on (X, Y) when applying Theorem 1(ii).

Remark 10. In many cases we have $B_{i,j}(t, t) = O(1)B(t, t)/t^2$ and then the estimate simplifies to $|G(tx, ty) - B(U_1(tx), U_2(ty))| = O(1)B(t, t)/t$.

We can use a similar approach to obtain a second order result for $G(x, y) = E(A(M(x, y)))$ where $M(x, y) = \min(M_1(x), M_2(y))$. For convenience we set $U(x, y) = \min(U_1(x), U_2(y))$ where for $i = 1, 2$, $U_i(x) = E(M_i(x))$. Now we suppose that the function $A(x)$ has a nondecreasing second derivative. We denote the first and the second derivative of $A(x)$ by $A_1(x)$ and $A_2(x)$. We also assume that $A_2(x) \in RV$. Now define R as follows:

$$R = A(M(tx, ty)) - A(U(tx, ty)) - A_1(U(tx, ty))(M(tx, ty) - U(tx, ty)).$$

Using the first terms of a Taylor expansion we obtain

$$|R| \leq (A_2(M(tx, ty)) + A_2(U(tx, ty)))(M(tx, ty) - U(tx, ty))^2/2.$$

Clearly $|M(tx, ty) - U(tx, ty)| \leq |M_1(tx) - U_1(tx)| + |M_2(ty) - U_2(ty)|$. Proceeding as in the proof of Theorem 8, we use Theorem 1 and Cauchy–Schwarz to obtain $|E(R)| = O(1)A_2(t)t$. It follows that:

$$\begin{aligned} & |G(tx, ty) - A(U(tx, ty)) - A_1(U(tx, ty))E(M(tx, ty) - U(tx, ty))| \\ &= O(1)tA_2(t). \end{aligned}$$

In contrast to Theorem 8, the term $E(M(tx, ty)) - U(tx, ty)$ does not vanish automatically here. In general, the contribution of this term is of the form $E(M(tx, ty)) - U(tx, ty) = O(1)\sqrt{t}$, cf. [21, Section 4].

4. Application and concluding remarks

4.1. Application

We can apply our results in the context of multivariate shock models as follows: Suppose that a system consists of two components and that the system works in cycles of a fixed or a variable length. During each cycle the components are subject to stress and can be damaged. Suppose that the i th cycle causes a random damage (X_i, Y_i) to the system. Here X_i (resp. Y_i) represents the damage for component 1 (resp. component 2). We suppose that the working cycles have the same independent behaviour. We suppose that the damage is accumulated and that the system works if the cumulative damage is less than a fixed threshold (x, y) . In this context $M_1(x)$ denotes the number of working cycles until the first component fails and $M_2(y)$ denotes the number of working cycles until the second component fails. In a series system, the lifetime of the system is given by $M(x, y) = \min(M_1(x), M_2(y))$. In other systems the lifetime of the system may be represented as a function $L(M_1(x), M_2(y))$ where $L(., .)$ is an integer-valued function giving the number of working cycles. Next we let $Z(k)$ denote the profit or the cost that can be realized if the lifetime of the system is $L = k$. The total cost or profit during the lifetime is given by $B(M_1(x), M_2(y)) = Z(L(M_1(x), M_2(y)))$. In this case the wrf $G(x, y) = E(B(M_1(x), M_2(y)))$ represents the expected cost or profit during the lifetime of the system. Our results can be used to estimate $G(tx, ty)$ for large thresholds (tx, ty) .

4.2. Concluding remarks

(1) In this paper we assumed that X and Y have a finite mean. In the infinite-means case another approach is needed to obtain the asymptotic behaviour of $G(x, y)$. In [14] the authors use Laplace-transforms to obtain estimates of the form $\lim_{t \rightarrow \infty} G(r(t)x, s(t)y)/h(t) = \theta(x, y)$ for appropriate auxiliary functions $r(t)$, $s(t)$ and $h(t)$.

(2) It is clear that our results can be extended to d -dimensional random vectors and to d -dimensional weighted renewal functions. For example, the function $b(.)$ is obtained from the function $B(.)$ as

$$\begin{aligned} b(n_1, n_2, \dots, n_d) &= B(n_1, n_2, \dots, n_d) - \sum_i B(n_i - 1; \mathbf{n}) + \sum_{i < j} B(n_i - 1, n_j - 1; \mathbf{n}) \\ &\quad - \dots (-1)^{d+1} B(\mathbf{n} - \mathbf{1}). \end{aligned}$$

(3) A special case of wrf appears when the weights $b(n, m)$ have a probabilistic meaning. Suppose that $b(n, m) = P(U = n, V = m)$ where (U, V) denotes a vector of integer-valued random variables. In this case $G(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b(n, m) P(S_n^1 \leq x, S_m^2 \leq y) = P(S_U^1 \leq x, S_V^2 \leq y)$ is the d.f. of a vector that consists of two random sums of random variables. In this case we trivially have $G(\infty, \infty) = 1$ and it is appropriate to study the asymptotic behaviour of the tail $1 - G(x, y)$. In this case Omey in [19], and Santos in [24], have used multivariate subexponential distributions to relate the asymptotic behaviour of

$1 - F(x, y)$ to that of $1 - G(x, y)$. In the case where $U = V$ it turns out that in many cases one has $1 - G(tx, ty) \sim E(U)(1 - F(tx, ty))$. Results of this type will be the content of a forthcoming paper [15].

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