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Study of some measures of dependence between order statistics and systems

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ABSTRACT

Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be a random vector, and denote by $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ the corresponding order statistics. When $X_1, X_2, \ldots, X_n$ represent the lifetimes of n components in a system, the order statistic $X_{n-k+1:n}$ represents the lifetime of a k-out-of-n system (i.e., a system which works when at least k components work). In this paper, we obtain some expressions for the Pearson's correlation coefficient between $X_{i:n}$ and $X_{j:n}$. We pay special attention to the case $n = 2$, that is, to measure the dependence between the first and second failure in a two-component parallel system. We also obtain the Spearman's rho and Kendall's tau coefficients when the variables $X_1, X_2, \ldots, X_n$ are independent and identically distributed or when they jointly have an exchangeable distribution.

1. Introduction

There exist several results on dependence properties between order statistics from samples of independent and identically distributed (IID) random variables; see, for example, [1–11]. However, little work has been done on the dependence properties when the sample has some kind of dependency; see [12,13]. This situation is realistic and it arises naturally in several fields such as reliability theory wherein the order statistics represent the lifetimes of k-out-of-n systems (see [14,15]) and futures market (see [9]). For example, in the former, if we consider a parallel system with two components, it will be of interest to predict the failure time of the system, from the first failure among the components. This prediction will naturally be based on the dependence between $X_{1:2}$ and $X_{2:2}$.

In this paper, some new dependence properties of order statistics are obtained in the general case when $n = 2$ and also in the cases of n IID components and n exchangeable components. Specifically, we derive explicit expressions for Pearson's correlation coefficient between two order statistics (or two coherent systems). Unfortunately, Pearson's correlation coefficient does not satisfy Scarsini's [16] assumptions since it depends on the marginal distributions. Hence, we also obtain explicit expressions for other dependence measures satisfying these assumptions such as Spearman's correlation coefficient and Kendall's tau. Some numerical approximations are also obtained using a Monte Carlo procedure for Kendall's tau in the case of exchangeable dependent components.

The rest of this paper is organized as follows. The dependence between order statistics in the bivariate general case is discussed in Section 2. Explicit expressions for the correlation coefficient between two order statistics from samples of n IID

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random variables are given in Section 3. Analogous expressions for Spearman's correlation coefficient and Kendall's tau are obtained in Section 4. Finally, in Section 5, we present two methods for computing Kendall's tau between order statistics from random vectors with exchangeable joint distributions.

Throughout the paper, when a moment of random variable is considered, we assume that it is finite.

2. Bivariate case under dependence

Let (X_1, X_2) be a random vector with distribution function $F(x, y) = \Pr(X \le x, Y \le y)$ and reliability (or survival) function $R(x, y) = \Pr(X > x, Y > y)$. Let $X_{1:2} = \min(X_1, X_2)$ and $X_{2:2} = \max(X_1, X_2)$ be the corresponding order statistics (lifetimes of series and parallel systems, respectively, if X_1 and X_2 represent the lifetimes of two components in a system). The distribution functions of $X_{1:2}$ and $X_{2:2}$ are given by $F_{1:2}(t) = 1 - R(t, t)$ and $F_{2:2}(t) = F(t, t)$, respectively. It is well known that $F_{1:2} \ge F_i \ge F_{2:2}$ for $i = 1, 2$, and $F_{1:2} + F_{2:2} = F_1 + F_2$, where F_1 and F_2 are the marginal distribution functions of X_1 and X_2, respectively. Therefore, the moments readily satisfy $E(X_{1:2}^k) + E(X_{2:2}^k) = E(X_1^k) + E(X_2^k)$ for $k = 1, 2, 3, \ldots$ and, in particular, the means satisfy $\mu_{1:2} + \mu_{2:2} = \mu_1 + \mu_2$ and $\mu_{1:2} \le \mu_i \le \mu_{2:2}$ for $i = 1, 2$.

When X_1 and X_2 are IID, Bickel [17] and Esary, Proschan and Walkup [18] proved that $X_{1:2}$ and $X_{2:2}$ are non-negatively correlated and Terrell [11] proved that the maximal correlation is 1/2, i.e.,

$$\rho_{1,2:2} = \mathrm{Corr}\,(X_{1:2}, X_{2:2}) \le 1/2$$

and that the equality is attained when, and only when, X_1 and X_2 have uniform distributions in an interval. Also in the IID case, Papathanasiou [19] proved that $\mathrm{Cov}\,(X_{1:2}, X_{2:2}) \le \sigma^2/3$. In the ID (i.e., identically distributed and not necessarily independent) case, Balakrishnan and Balasubramanian [12] proved that

$$\mathrm{Cov}\,(X_{1:2}, X_{2:2}) \le (1 + \rho)\sigma^2, \tag{2.1}$$

where $\rho = \mathrm{Corr}\,(X_1, X_2)$ and $\sigma^2 = \mathrm{Var}\,(X_i)$ for $i = 1, 2$. Now, by assuming that (X_1, X_2) is a general random vector, we obtain new expressions for the correlation coefficient between $X_{1:2}$ and $X_{2:2}$.

Theorem 2.1. *If* (X_1, X_2) *is a random vector,* $\rho = \mathrm{Corr}\,(X_1, X_2)$ *and* $\rho_{1,2:2} = \mathrm{Corr}\,(X_{1:2}, X_{2:2})$*, then*

$$\rho_{1,2:2} = \rho \frac{\sigma_1 \sigma_2}{\sigma_{1:2}\sigma_{2:2}} + \frac{(\mu_1 - \mu_{1:2})(\mu_2 - \mu_{1:2})}{\sigma_{1:2}\sigma_{2:2}} \tag{2.2}$$

and

$$\rho_{1,2:2} = \rho \frac{\sigma_1 \sigma_2}{\sigma_{1:2}\sigma_{2:2}} + \frac{\sigma_1^2 + \sigma_2^2 - \sigma_{1:2}^2 - \sigma_{2:2}^2}{2\sigma_{1:2}\sigma_{2:2}}, \tag{2.3}$$

where $\mu_i = E(X_i)$, $\mu_{i:2} = E(X_{i:2})$, $\sigma_i^2 = \mathrm{Var}\,(X_i)$ *and* $\sigma_{i:2}^2 = \mathrm{Var}\,(X_{i:2}) > 0$, $i = 1, 2$.

Proof. Since $X_{1:2}X_{2:2} = X_1X_2$, we readily have $E(X_{1:2}X_{2:2}) = E(X_1X_2)$. Hence,

$$\mathrm{Cov}\,(X_{1:2}, X_{2:2}) = \mathrm{Cov}\,(X_1, X_2) + \mu_1\mu_2 - \mu_{1:2}\mu_{2:2}$$

and

$$\rho_{1,2:2} = \rho \frac{\sigma_1 \sigma_2}{\sigma_{1:2}\sigma_{2:2}} + \frac{\mu_1\mu_2 - \mu_{1:2}\mu_{2:2}}{\sigma_{1:2}\sigma_{2:2}}.$$

Using now the fact that $\mu_{2:2} = \mu_1 + \mu_2 - \mu_{1:2}$, we have

$$(\mu_1 - \mu_{1:2})(\mu_2 - \mu_{1:2}) = \mu_1\mu_2 - \mu_{1:2}\mu_{2:2}$$

which yields (2.2).

To obtain the second expression, we first note that

$$\sigma_1^2 + \sigma_2^2 - \sigma_{1:2}^2 - \sigma_{2:2}^2 = \mu_{1:2}^2 + \mu_{2:2}^2 - \mu_1^2 - \mu_2^2,$$

since $E(X_{1:2}^2) + E(X_{2:2}^2) = E(X_1^2) + E(X_2^2)$. Then, by using the fact that $\mu_1 + \mu_2 = \mu_{1:2} + \mu_{2:2}$, we get

$$\sigma_1^2 + \sigma_2^2 - \sigma_{1:2}^2 - \sigma_{2:2}^2 = 2\mu_1\mu_2 - 2\mu_{1:2}\mu_{2:2}$$

and so

$$(\mu_1 - \mu_{1:2})(\mu_2 - \mu_{1:2}) = \frac{\sigma_1^2 + \sigma_2^2}{2} - \frac{\sigma_{1:2}^2 + \sigma_{2:2}^2}{2}, \tag{2.4}$$

which yields (2.3). □

Since $\mu_{1:2} \leq \mu_i$ for $i = 1, 2$, the expression in (2.4) readily implies that

$$\frac{\sigma_1^2 + \sigma_2^2}{2} \geq \frac{\sigma_{1:2}^2 + \sigma_{2:2}^2}{2}. \tag{2.5}$$

It can also be used to obtain the following bound

$$(\mu_{2:2} - \mu_1)(\mu_{2:2} - \mu_2) \leq \frac{\sigma_1^2 + \sigma_2^2}{2},$$

which extends the well-known bound $E(X_{2:2}) \leq \mu + \sigma$ (see [12] and [20, p. 111]) in the ID case. In particular, in the ID case, we obtain the following corollary.

Corollary 2.2. *If* (X_1, X_2) *is a random vector with* X_1 *and* X_2 *being ID,* $\rho = \operatorname{Corr}(X_1, X_2)$ *and* $\rho_{1,2:2} = \operatorname{Corr}(X_{1:2}, X_{2:2})$*, then*

$$\rho_{1,2:2} = \rho \frac{\sigma^2}{\sigma_{1:2}\sigma_{2:2}} + \frac{(\mu - \mu_{1:2})^2}{\sigma_{1:2}\sigma_{2:2}} \tag{2.6}$$

and

$$\rho_{1,2:2} = (1 + \rho)\frac{\sigma^2}{\sigma_{1:2}\sigma_{2:2}} - \frac{\sigma_{1:2}^2 + \sigma_{2:2}^2}{2\sigma_{1:2}\sigma_{2:2}}, \tag{2.7}$$

where $\mu = E(X_i)$, $\mu_{i:2} = E(X_{i:2})$, $\sigma^2 = \operatorname{Var}(X_i)$ *and* $\sigma_{i:2}^2 = \operatorname{Var}(X_{i:2}) > 0$, $i = 1, 2$.

Expression (2.6) is equivalent to the expression (7) in [12]. In the general case, from (2.3), we can obtain a similar expression given by

$$\rho_{1,2:2} = (1 + \rho)\frac{\sigma_1\sigma_2}{\sigma_{1:2}\sigma_{2:2}} + \frac{(\sigma_1 - \sigma_2)^2 - \sigma_{1:2}^2 - \sigma_{2:2}^2}{2\sigma_{1:2}\sigma_{2:2}}. \tag{2.8}$$

Now, we use the preceding expressions to derive lower and upper bounds for the correlation between two order statistics.

Proposition 2.3. *If* (X_1, X_2) *is a random vector,* $\rho = \operatorname{Corr}(X_1, X_2)$ *and* $\rho_{1,2:2} = \operatorname{Corr}(X_{1:2}, X_{2:2})$*, then*

$$\rho\frac{\sigma_1\sigma_2}{\sigma_{1:2}\sigma_{2:2}} \leq \rho_{1,2:2} \leq \rho\frac{\sigma_1\sigma_2}{\sigma_{1:2}\sigma_{2:2}} + \frac{\sigma_1^2 + \sigma_2^2}{2\sigma_{1:2}\sigma_{2:2}} \leq \frac{(\sigma_1 + \sigma_2)^2}{2\sigma_{1:2}\sigma_{2:2}} \tag{2.9}$$

and

$$(1 + \rho)\frac{\sigma_1\sigma_2}{\sigma_{1:2}\sigma_{2:2}} - \frac{\sigma_{1:2}^2 + \sigma_{2:2}^2}{2\sigma_{1:2}\sigma_{2:2}} \leq \rho_{1,2:2} \leq (1 + \rho)\frac{\sigma_1\sigma_2}{\sigma_{1:2}\sigma_{2:2}} + \frac{(\sigma_1 - \sigma_2)^2}{2\sigma_{1:2}\sigma_{2:2}}, \tag{2.10}$$

where $\sigma_i^2 = \operatorname{Var}(X_i)$ *and* $\sigma_{i:2}^2 = \operatorname{Var}(X_{i:2}) > 0$, $i = 1, 2$. *Moreover, if* X_1 *and* X_2 *are positive random variables, the lower bound in* (2.9) *is attained if, and only if,* $X_1 \leq X_2$ *or* $X_1 \geq X_2$ *a.s. (almost surely).*

Proof. The lower bound in (2.9) is obtained from Eq. (2.2) by taking into account that $\mu_i - \mu_{1:2} \geq 0$ for $i = 1, 2$. The upper bounds in (2.9) are obtained from (2.3) and the fact that $\rho \leq 1$.

The bounds in (2.10) are obtained similarly from (2.8).

Moreover, if X_1 and X_2 are positive random variables, the lower bound in (2.9) is attained if, and only if, $\mu_{1:2} = \mu_i$ for $i = 1$ or $i = 2$. Since the reliability functions satisfy $R_{1:2} \leq R_i$ and

$$\mu_{1:2} = \int_0^\infty R_{1:2}(x)\mathrm{d}x = \mu_i = \int_0^\infty R_i(x)\mathrm{d}x,$$

then $R_{1:2} = R_i$ and $X_{1:2} = X_i$ a.s. Therefore, $\Pr(X_1 \leq X_2) = 1$ (whenever $i = 1$) or $\Pr(X_1 \geq X_2) = 1$ (whenever $i = 2$). $\square$

Note that the upper bound in (2.10) is equal to the first upper bound in (2.9). Also note that, if $\rho \geq 0$, then from (2.9), we have $\rho_{1,2:2} \geq 0$, that is, the order statistics of two non-negatively correlated random variables are non-negatively correlated. The reverse implication is not necessarily true (see Example 2.9). An open question is whether $\rho_{1,2:2} \geq \rho$. If $\rho \geq 0$, from (2.9), it would hold if $\sigma_1\sigma_2 \geq \sigma_{1:2}\sigma_{2:2}$. We will see later that this property is true in the ID case. Moreover, if $\mu_{1:2} \geq 0$, then from (2.2), we have

$$\rho_{1,2:2} \leq \frac{\mu_1\mu_2 + \rho\sigma_1\sigma_2}{\sigma_{1:2}\sigma_{2:2}}.$$

In particular, in the ID case, we obtain the following corollary.

Corollary 2.4. *If* (X_1, X_2) *is a random vector,* X_1 *and* X_2 *are ID,* $\rho = \mathrm{Corr}(X_1, X_2)$ *and* $\rho_{1,2:2} = \mathrm{Corr}(X_{1:2}, X_{2:2})$*, then*

$$\rho \frac{\sigma^2}{\sigma_{1:2}\sigma_{2:2}} \le \rho_{1,2:2} \le (1+\rho)\frac{\sigma^2}{\sigma_{1:2}\sigma_{2:2}} \le \frac{2\sigma^2}{\sigma_{1:2}\sigma_{2:2}} \tag{2.11}$$

and

$$(1+\rho)\frac{\sigma^2}{\sigma_{1:2}\sigma_{2:2}} - \frac{\sigma_{1:2}^2 + \sigma_{2:2}^2}{2\sigma_{1:2}\sigma_{2:2}} \le \rho_{1,2:2} \le (1+\rho)\frac{\sigma^2}{\sigma_{1:2}\sigma_{2:2}}, \tag{2.12}$$

where $\sigma^2 = \mathrm{Var}(X_i)$ *and* $\sigma_{i:2}^2 = \mathrm{Var}(X_{i:2}) > 0$, $i = 1, 2$.

From (2.5), we see that the lower bound in (2.12) is better (i.e., bigger) than the lower bound in (2.11). Moreover, from (2.5), we have $2\sigma^2 \ge \sigma_{1:2}^2 + \sigma_{2:2}^2$ and using the fact that $\sigma_{1:2}^2 + \sigma_{2:2}^2 \ge 2\sigma_{1:2}\sigma_{2:2}$, we obtain $\sigma^2 \ge \sigma_{1:2}\sigma_{2:2}$. So, if $\rho \ge 0$, then from (2.11), we obtain $\rho_{1,2:2} \ge \rho$. Analogously, if $\rho \le 0$, then from (2.6), we obtain

$$\rho_{1,2:2} \le \rho + \frac{(\mu - \mu_{1:2})^2}{\sigma_{1:2}\sigma_{2:2}}.$$

It should also be noted that in the ID case, we can use the bounds given by Rychlik [21] for $\sigma_{i:2}$ to obtain bounds based only on the parent common distribution F (see Example 2.8). Similarly, by adding some conditions on the common distribution function F of X_1 and X_2 and then using the bounds of Rychlik [21], we can obtain new bounds for $\rho_{1,2:2}$. For example, if F is DFR (decreasing failure rate), then by using Proposition 15 of Rychlik [22, p. 106], we obtain $\mu - \mu_{1:2} = \mu_{2:2} - \mu \le \sigma \ln 2$ and, so from (2.6), we have

$$\rho_{1,2:2} \le (\rho + \ln^2 2)\frac{\sigma^2}{\sigma_{1:2}\sigma_{2:2}}.$$

Similarly, if F is IFR (increasing failure rate), then from Proposition 17 of Rychlik [22, p. 109], we obtain $\mu_{2:2} - \mu \le 0.909624645\sigma$ and

$$\rho_{1,2:2} \le (\rho + 0.827416995)\frac{\sigma^2}{\sigma_{1:2}\sigma_{2:2}}.$$

The upper bound in (2.12) is improved in the following corollary.

Corollary 2.5. *If* (X_1, X_2) *is a random vector,* X_1 *and* X_2 *are ID,* $\rho = \mathrm{Corr}(X_1, X_2)$ *and* $\rho_{1,2:2} = \mathrm{Corr}(X_{1:2}, X_{2:2})$*, then*

$$\rho \frac{\sigma_{1:2}^2 + \sigma_{2:2}^2}{2\sigma_{1:2}\sigma_{2:2}} \le \rho_{1,2:2} \le (1+\rho)\frac{\sigma^2}{\sigma_{1:2}\sigma_{2:2}} - 1, \tag{2.13}$$

where $\sigma^2 = \mathrm{Var}(X_i)$ *and* $\sigma_{i:2}^2 = \mathrm{Var}(X_{i:2}) > 0$, $i = 1, 2$.

Proof. From (2.7) and the fact that

$$\frac{\sigma_{1:2}^2 + \sigma_{2:2}^2}{2\sigma_{1:2}\sigma_{2:2}} \ge 1, \tag{2.14}$$

we obtain the upper bound in (2.13). Next, from (2.4) and (2.14), we have

$$2\sigma^2 \ge \sigma_{1:2}^2 + \sigma_{2:2}^2 \ge 2\sigma_{1:2}\sigma_{2:2}.$$

Then, using (2.7) and that $1 + \rho \ge 0$, we get the lower bound in (2.13). $\square$

The upper bound in (2.11) is worse than the upper bound in (2.13). If $\rho \ge 0$ (resp. $\le$), then the lower bound in (2.11) is better (resp. worse) than the lower bound in (2.13) since $2\sigma^2 \ge \sigma_{1:2}^2 + \sigma_{2:2}^2$. Note that if $\rho \ge 0$, then from (2.13), we have $\rho \le \rho_{1,2:2}$. It would be interesting to obtain more bounds depending only on the distribution of (X_1, X_2) (moments, dependence parameters, etc.) but we are not able to do that. However, note that they can be obtained from (2.13) whenever $\sigma_{1:2}\sigma_{2:2}$ is bounded.

Now, we shall consider some specific kind of dependence between the components to obtain some new properties. Specifically, we shall use the following well known notion of quadrant (orthant) dependence.

Definition 2.6 (*Lehmann [23]*). A random vector (X_1, X_2) is positive (negative) quadrant dependent *PQD* (*NQD*) iff

$$R(x_1, x_2) \ge (\le)\, R_1(x_1)R_2(x_2) \quad \text{for all } x_1, x_2, \tag{2.15}$$

where $R(x_1, x_2) = \Pr(X_1 > x_1, X_2 > x_2)$ and $R_i(x_i) = \Pr(X_i > x_i)$, $i = 1, 2$.

It is well known that (2.15) is equivalent to

$$F(x_1, x_2) \geq (\leq)\, F_1(x_1)F_2(x_2) \quad \text{for all } x_1, x_2,$$

and is also equivalent to

$$E(\phi_1(X_1)\phi_2(X_2)) \geq (\leq)\, E(\phi_1(X_1))E(\phi_2(X_2))$$

for every pair of non-negative both increasing or both decreasing functions whose expected values exist. Hence, if (X_1, X_2) is PQD (NQD), then $\rho \geq (\leq)\, 0$.

Theorem 2.7. *If (X_1, X_2) is a non-negative PQD (NQD) random vector, $\rho = \mathrm{Corr}\,(X_1, X_2)$ and $\rho_{1,2:2} = \mathrm{Corr}\,(X_{1:2}, X_{2:2})$, then*

$$\rho_{1,2:2} \leq (\geq)\, \rho \frac{\sigma_1\sigma_2}{\sigma_{1:2}\sigma_{2:2}} + \frac{(\mu_1 - \mu^I_{1:2})(\mu_2 - \mu^I_{1:2})}{\sigma_{1:2}\sigma_{2:2}}, \tag{2.16}$$

where $\mu^I_{1:2} = \int_0^\infty R_1^2(t)\mathrm{d}t$ denotes the expected lifetime of the series system obtained from independent components with the same marginal distributions as X_1 and X_2. Moreover, if X_1 and X_2 have exponential distributions, then

$$\rho_{1,2:2} \leq (\geq)\, \rho \frac{\mu_1\mu_2}{\sigma_{1:2}\sigma_{2:2}} + \frac{\mu_1^2\mu_2^2}{(\mu_1+\mu_2)^2\sigma_{1:2}\sigma_{2:2}}. \tag{2.17}$$

Proof. Navarro and Lai [13, Proposition 2.1] proved that, if (X_1, X_2) is a non-negative PQD (NQD) random vector, then $\mu_{1:2} \geq (\leq)\, \mu^I_{1:2}$. Therefore, from (2.2), we obtain (2.16). Moreover, in the particular case of exponential marginal distributions with means μ_1 and μ_2, the variances satisfy $\sigma_i^2 = \mu_i^2$ for $i = 1, 2$, and since

$$\mu^I_{1:2} = \frac{\mu_1\mu_2}{\mu_1+\mu_2},$$

(2.17) holds. □

In particular, in the ID case, if (X_1, X_2) is PQD (NQD), we have

$$\rho_{1,2:2} \leq (\geq)\, \rho \frac{\sigma^2}{\sigma_{1:2}\sigma_{2:2}} + \frac{(\mu - \mu^I_{1:2})^2}{\sigma_{1:2}\sigma_{2:2}},$$

and if X_1 and X_2 have a common exponential distribution, then

$$\rho_{1,2:2} \leq (\geq)\, \frac{\mu^2(\rho + 1/4)}{\sigma_{1:2}\sigma_{2:2}}.$$

The following example shows that bounds for $\rho_{1,2:2} = \mathrm{Corr}\,(X_{1:2}, X_{2:2})$ can be obtained by using the preceding results and the bounds for the variance of Rychlik [21].

Example 2.8. If (X_1, X_2) has an exchangeable distribution with uniform marginal distributions in $(0, 2\mu)$, then from Rychlik [21, Example 3], we have $\mathrm{Var}\,(X_{i:2}) \geq \mu^2/12$. Thus, from (2.6), if $\rho \geq 0$, we have

$$\rho_{1,2:2} \leq 4\rho + 12\frac{(\mu - \mu_{1:2})^2}{\mu^2}.$$

In this case, the expression in (2.13) gives

$$\rho_{1,2:2} \leq (1+\rho)\frac{\sigma^2}{\sigma_{1:2}\sigma_{2:2}} - 1 \leq 4(1+\rho)\frac{\mu^2}{\mu^2} - 1 = 3 + 4\rho.$$

Note that the upper bound $3 + 4\rho$ is not a useful bound when $\rho \geq -1/2$. This bound can also be obtained from the first expression by using the fact that $\mu_{i:2} \geq \mu/2$ for $i = 1, 2$ (see Rychlik [21, Example 1]). ◁

In the following examples, the preceding results are used to obtain the exact value of $\rho_{1,2:2} = \mathrm{Corr}\,(X_{1:2}, X_{2:2})$ for some specific models. First, we discuss the case of an exchangeable bivariate normal distribution.

Example 2.9. If (X_1, X_2) has an exchangeable normal distribution, then it is easy to obtain from [24] that

$$\mu_{1:2} = \mu - \sigma\sqrt{\frac{1-\rho}{\pi}}$$

and

$$\sigma_{1:2} = \sigma_{2:2} = \sigma\sqrt{1 - \frac{1-\rho}{\pi}}.$$

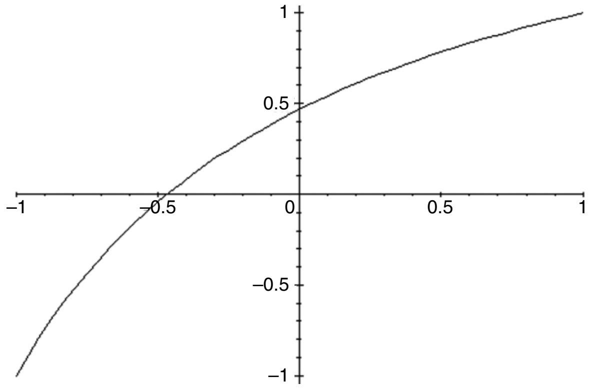

Fig. 1. $\rho_{1,2:2} = \mathrm{Corr}\,(X_{1:2}, X_{2:2})$ as a function of $\rho = \mathrm{Corr}\,(X_1, X_2)$ in the bivariate exchangeable normal distribution. Note that $\rho_{1,2:2} \geq \rho$ and that $\rho_{1,2:2}$ can be positive when ρ is negative (e.g., when $\rho = -0.2$).

So, from (2.6), we obtain

$$\rho_{1,2:2} = \frac{1 + (\pi - 1)\rho}{\pi - 1 + \rho},$$

i.e., $\rho_{1,2:2}$ is an increasing function of ρ from -1 (when $\rho = -1$) to 1 (when $\rho = 1$). A plot of $\rho_{1,2:2}$ is presented in Fig. 1. In this case, $\rho_{1,2:2}$ does not depend either on μ or on σ. $\triangleleft$

A similar result is obtained in the following example for the bivariate Pareto distribution used by Lindley and Singpurwalla [25] to model the behavior of two units in a system sharing a common environment.

Example 2.10. If (X_1, X_2) has an exchangeable Pareto distribution with reliability function

$$R(x, y) = (1 + \lambda x + \lambda y)^{-\theta}$$

for $x, y \geq 0$, where $\lambda > 0$ and $\theta > 2$, then $\mu = 1/(\lambda\theta - \lambda)$ and $0 < \rho = 1/\theta < 1/2$. Hence, the model can be reparameterized in terms of μ and ρ with $\sigma^2 = \mu^2/(1 - 2\rho)$. Moreover, from Navarro, Ruiz and Sandoval [26], we have $\mu_{1:2} = \mu/2$,

$$\sigma^2_{1:2} = \frac{\mu^2}{4(1 - 2\rho)}$$

and

$$\sigma^2_{2:2} = \frac{\mu^2(6 + 3\rho)}{4(1 - 2\rho)}.$$

From (2.6), we then obtain

$$\rho_{1,2:2} = \frac{1 + 2\rho}{\sqrt{6 + 3\rho}}$$

which is an increasing and positive function for $\rho \in (0, 1/2)$. It is of interest to observe that $\rho_{1,2:2} \geq \rho$. $\triangleleft$

Finally, we consider Freund's bivariate exponential model. Note that the order statistics from this model can be seen as generalized order statistics or sequential k-out-of-n systems (see [3,27]).

Example 2.11. Freund's bivariate exponential distribution (FBVE) was introduced to model the lifetimes of two components in a parallel system wherein the failure of one either adversely affects the other or enhances its performance. A random vector (X_1, X_2) has an exchangeable FBVE if its density function is given by

$$f(x, y) = \begin{cases} \gamma_1\gamma_2 \exp(-\gamma_2(y - x) - 2\gamma_1 x) & \text{for } 0 < x < y \\ \gamma_1\gamma_2 \exp(-\gamma_2(x - y) - 2\gamma_1 y) & \text{for } 0 < y < x \end{cases}$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$; see [28] for more details. For this model, Nagaraja and Baggs [14] proved that the regression function of $X_{2:2}$, given $X_{1:2} = t$, is linear. They specifically showed that

$$E(X_{2:2} \mid X_{1:2} = t) = t + 1/\gamma_2.$$

So, it is of interest in this case to know the correlation coefficient to measure the linear dependency between $X_{2:2}$ and $X_{1:2}$. From the results of Nagaraja and Baggs [14] and Hutchinson and Lai [29, p. 143] and from the expression in (2.6), it is easy to show that $\mu - \mu_{1:2} = 1/(2\gamma_2)$,

$$\sigma^2_{1:2} = \frac{1}{4\gamma_1^2},$$

$$\sigma^2_{2:2} = \frac{4\gamma_1^2 + \gamma_2^2}{4\gamma_1^2\gamma_2^2},$$

and

$$\rho_{1,2:2} = \frac{\gamma_2}{\sqrt{4\gamma_1^2 + \gamma_2^2}}.$$

Note that $\rho_{1,2:2} > 0$; however, $\rho = (\gamma_2^2 - \gamma_1^2)/(\gamma_2^2 + 3\gamma_1^2)$ is restricted to the range $-1/3$ to 1; see [29, p. 143]. $\triangleleft$

3. Correlation between two order statistics

In this section, we suppose that $X_1, X_2, \ldots, X_n$ are IID positive random variables with the common continuous distribution function F and reliability function $R = 1 - F$. It is well-known that the order statistics are particular cases of coherent system lifetimes. Actually, $X_{n-k+1:n}$ represents the lifetime of the k-out-of-n:F system (i.e., a system which works when at least k of its n components work). As the results obtained in this section also hold for coherent systems, we shall present the results for coherent systems in general, and then we give the results for order statistics as particular cases.

Samaniego [30] proved that the reliability function of a coherent system can be written as a mixture of the reliability functions of order statistics associated with the component lifetimes as follows

$$R_T(t) = \sum_{i=1}^{n} s_i R_{i:n}(t).$$

The vector $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ with the coefficients in that representation was called the signature of the system. Recently, Navarro, Ruiz and Sandoval [15] proved that the reliability function R_T of a coherent system with lifetime T and IID component lifetimes can be written as

$$R_T(t) = \sum_{i=1}^{n} a_i R_{1:i}(t) = \sum_{i=1}^{n} b_i R_{i:i}(t), \tag{3.1}$$

where $R_{1:i}(t) = R^i(t)$ and $R_{i:i}(t) = 1 - F^i(t)$ are the reliability functions of series $X_{1:i}$ and parallel $X_{i:i}$ systems, respectively. The vectors of coefficients $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ were called minimal and maximal signatures, respectively. These coefficients satisfy $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 1$, but some of them can be negative. The minimal and maximal signatures can be obtained from Samaniego's signature and vice versa. The representations in (3.1) continue to hold in the exchangeable case as well (see [15,31]) but, in this case, $R_{1:i}(t)$ (resp. $R_{i:i}(t)$) is not necessarily equal to $R^i(t)$ (resp. $1-F^i(t)$). In particular, the minimal and maximal signatures for the order statistics can be obtained from the expressions (see [20, p. 46])

$$R_{r:n}(t) = \sum_{i=n-r+1}^{n} (-1)^{i+r-n-1} \binom{i-1}{n-r} \binom{n}{i} R_{1:i}(t) \tag{3.2}$$

and

$$R_{r:n}(t) = \sum_{i=r}^{n} (-1)^{i-r} \binom{i-1}{r-1} \binom{n}{i} R_{i:i}(t). \tag{3.3}$$

The following lemma gives a similar representation for the joint reliability function of two coherent systems.

Lemma 3.1. *If T and T^* are the lifetimes of two coherent systems based on the same IID component lifetimes $X_1, X_2, \ldots, X_n$ having common reliability function R, then*

$$\Pr(T > x, T^* > y) = \sum_{i=0}^{n} \sum_{j=1}^{n-i} a_{i,j} R^i(x) R^j(y) \tag{3.4}$$

for all $x \le y$, and

$$\Pr(T > x, T^* > y) = \sum_{j=0}^{n} \sum_{i=1}^{n-j} a^*_{i,j} R^i(x) R^j(y) \tag{3.5}$$

for all $x > y$, where $a_{i,j}$ and $a^*_{i,j}$, for $i, j = 1, 2, \ldots, n$, are some coefficients not depending on R.

Proof. It is well known (see [32, p. 12]) that the lifetime of a system can be written as $T = \max_{i=1,2,\dots,s} X_{P_i}$, where $X_{P_i} = \min_{j\in P_i} X_j$ and the sets $P_1, P_2, \dots, P_s$ are called minimal path sets. Thus, if T^* has minimal path sets $P_1^*, P_2^*, \dots, P_{s^*}^*$, then $T^* = \max_{i=1,2,\dots,s^*} X_{P_i^*}$ and

$$\begin{aligned}
\Pr(T > x, T^* > y) &= \Pr\left(\cup_{i=1}^{s}\{X_{P_i} > x\}, \cup_{i^*=1}^{s^*}\{X_{P_{i^*}^*} > y\}\right)\\
&= \Pr\left(\cup_{i=1}^{s}\cup_{i^*=1}^{s^*}\{X_{P_i} > x, X_{P_{i^*}^*} > y\}\right)\\
&= \sum_{i=1}^{s}\sum_{i^*=1}^{s^*}\Pr\left(X_{P_i} > x, X_{P_{i^*}^*} > y\right) - \sum_{i<j}\sum_{i^*<j^*}\Pr\left(X_{P_i\cup P_j} > x, X_{P_{i^*}^*\cup P_{j^*}^*} > y\right)\\
&\quad + \cdots \pm \Pr(X_{1:n} > x, X_{1:n} > y)\\
&= \sum_{i=1}^{s}\sum_{i^*=1}^{s^*}\Pr\left(X_{P_i - P_{i^*}^*} > x\right)\Pr\left(X_{P_{i^*}^*} > y\right)\\
&\quad - \sum_{i<j}\sum_{i^*<j^*}\Pr\left(X_{P_i\cup P_j - (P_{i^*}^*\cup P_{j^*}^*)} > x\right)\Pr\left(X_{P_{i^*}^*\cup P_{j^*}^*} > y\right) + \cdots \pm \Pr(X_{1:n} > y)\\
&= \sum_{i=1}^{s}\sum_{i^*=1}^{s^*} R^{|P_i - P_{i^*}^*|}(x)R^{|P_{i^*}^*|}(y) - \sum_{i<j}\sum_{i^*<j^*} R^{|P_i\cup P_j - (P_{i^*}^*\cup P_{j^*}^*)|}(x)R^{|P_{i^*}^*\cup P_{j^*}^*|}(y) + \cdots \pm R^n(y)
\end{aligned}$$

for $x \le y$, where $|P|$ denotes the cardinality of the set P. Therefore, we obtain the stated result for $x \le y$ taking into account that $|P - Q| + |Q| \le n$ for all $P, Q \subseteq \{1, 2, \dots, n\}$. The proof for $x > y$ follows on similar lines. □

Note that the proof gives a constructive method to obtain the coefficients in these representations. An analogous expression can be obtained in term of the values $F(x)$ and $F(y)$ of the component distribution function. In particular, for the order statistics, we obtain the expression in (2.2.4) of David and Nagaraja [20, p. 12] in terms of $F(x)$ and $F(y)$. By using a similar method, we can also obtain the following expression in terms of $R(x)$ and $R(y)$:

$$\Pr(X_{r:n} > x, X_{s:n} > y) = \sum_{j=0}^{r-1}\sum_{i=0}^{s-j-1}\sum_{t=0}^{i}\sum_{\ell=0}^{j}\frac{(-1)^{j+i-\ell-t}n!}{(n-j-i)!(j-\ell)!(i-t)!\ell!t!}R^{t+j-\ell}(x)R^{n-j-t}(y) \tag{3.6}$$

for $x \le y$ and $\Pr(X_{r:n} > x, X_{s:n} > y) = \Pr(X_{r:n} > x)$ for $x > y$ whenever $r < s$. Hence, the coefficients $a_{i,j}$ and $a_{i,j}^*$ for $(X_{r:n}, X_{s:n})$ in (3.4) and (3.5) can be obtained from (3.6) and (3.2), respectively. Expression (3.6) can also be rewritten as

$$\Pr(X_{r:n} > x, X_{s:n} > y) = \sum_{\ell=0}^{r-1}\sum_{j=\ell}^{r-1}\sum_{i=j}^{s-1}\frac{(-1)^{s+j+i-\ell-1}n!}{(n-i)(n-s)!(s-i-1)!(j-\ell)!(i-j)!\ell!}R^{i-\ell}(x)R^{n-i}(y) \tag{3.7}$$

for $x \le y$ and $r < s$.

The representations in terms of the reliability functions of series systems given in Lemma 3.1 can be used effectively to compute the correlation coefficient between two coherent systems and, in particular, two order statistics. For this purpose, we need the expressions presented in the following lemma for positive random variables with finite moments. These expressions were given by Jones and Balakrishnan [33] and Navarro, Ruiz and del Aguila [34].

Lemma 3.2. *If (X, Y) is a positive random vector, then*
(i) $E(X) = \int_0^\infty R_1(x)\mathrm{d}x$,
(ii) $E(X^2) = \int_0^\infty 2xR_1(x)\mathrm{d}x$ *and*
(iii) $E(XY) = \int_0^\infty\int_0^\infty R(x, y)\mathrm{d}x\mathrm{d}y$,
where $R(x, y) = \Pr(X > x, Y > y)$ and $R_1(x) = \Pr(X > x)$.

The proof is straightforward. Now, we present the main result of this section which gives the expressions needed for computing the correlation coefficient between two coherent systems and, in particular, the correlation coefficient between two order statistics.

Theorem 3.3. *If T and T^* are the lifetimes of two coherent systems based on the same IID component lifetimes $X_1, X_2, \dots, X_n$ having the common continuous reliability function R, then*

$$E(T) = \sum_{i=1}^{n} a_i \int_0^\infty R^i(x)\mathrm{d}x, \tag{3.8}$$

$$E(T^2) = \sum_{i=1}^{n} a_i \int_0^\infty 2xR^i(x)\mathrm{d}x \tag{3.9}$$

and

$$E(TT^*) = \sum_{i=0}^{n}\sum_{j=1}^{n-i}(a_{i,j} + a^*_{j,i})\int_0^\infty R^i(x)\int_x^\infty R^j(y)\mathrm{d}y\mathrm{d}x, \tag{3.10}$$

where $(a_1, a_2, \dots, a_n)$ *is the minimal signature of* T *and* $a_{i,j}$ *and* $a^*_{i,j}$ *are the coefficients in (3.4) and (3.5).*

Proof. Expressions in (3.8) and (3.9) are obtained from (3.1) and Lemma 3.2. To obtain (3.10), note that, from Lemma 3.2, we have

$$E(TT^*) = \int_0^\infty\int_x^\infty R_{T,T^*}(x, y)\mathrm{d}y\mathrm{d}x + \int_0^\infty\int_y^\infty R_{T,T^*}(x, y)\mathrm{d}x\mathrm{d}y.$$

Then, upon using (3.4) and (3.5), we get

$$E(TT^*) = \sum_{i=0}^{n}\sum_{j=1}^{n-i}a_{i,j}\int_0^\infty R^i(x)\int_x^\infty R^j(y)\mathrm{d}y\mathrm{d}x + \sum_{j=0}^{n}\sum_{i=1}^{n-j}a^*_{i,j}\int_0^\infty R^j(y)\int_y^\infty R^i(x)\mathrm{d}x\mathrm{d}y,$$

from which (3.10) follows. □

Note that $\mathrm{Var}(T)$, $\mathrm{Cov}(T, T^*)$ and $\mathrm{Corr}(T, T^*)$ can all be computed from (3.8)–(3.10). Also, note that $E(TT^*)$ can be rewritten as

$$E(TT^*) = \sum_{i=0}^{n}\sum_{j=1}^{n-i}(a_{i,j} + a^*_{j,i})\int_0^\infty R^{i+j}(x)m_{1:j}(x)\mathrm{d}x,$$

where

$$m_{1:j}(x) = E(X_{1:j} - x|X_{1:j} > x) = \frac{1}{R^j(x)}\int_x^\infty R^j(y)\mathrm{d}y$$

is the mean residual life function of the series system $X_{1:j}$.

In particular, if the component lifetimes $X_1, X_2, \dots, X_n$ have exponential distributions with common mean μ, then $m_{1:j}(x) = E(X_{1:j}) = \mu/j$ for $j = 1, 2, \dots, n$, and so

$$E(T) = \mu\sum_{i=1}^{n}a_i/i, \tag{3.11}$$

$$E(T^2) = 2\mu^2\sum_{i=1}^{n}a_i/i^2, \tag{3.12}$$

and

$$E(TT^*) = \sum_{i=0}^{n}\sum_{j=1}^{n-i}(a_{i,j} + a^*_{j,i})\frac{\mu^2}{j(i+j)}. \tag{3.13}$$

Example 3.4. Let us consider the order statistics $X_{1:3}$ and $X_{2:3}$ in the case when X_i's all have an exponential distribution with mean μ ($i = 1, 2, 3$). Then, it is easy to obtain from (3.2) that the minimal signatures of $X_{1:3}$ and $X_{2:3}$ are $(0, 0, 1)$ and $(0, 3, -2)$, respectively. From (3.6), we similarly have

$$\Pr(X_{1:3} > x, X_{2:3} > y) = 3R(x)R^2(y) - 2R^3(y)$$

for $x < y$, and

$$\Pr(X_{1:3} > x, X_{2:3} > y) = R^3(x)$$

for $x \geq y$. Thus, the non-zero coefficients are $a_{1,2} = 3, a_{0,3} = -2$ and $a^*_{3,0} = 1$. Therefore, from (3.11)–(3.13), we obtain $E(X_{1:3}) = \mu/3, E(X_{2:3}) = 5\mu/6, E(X^2_{1:3}) = 2\mu^2/9, E(X^2_{2:3}) = 19\mu^2/18, \mathrm{Var}(X_{1:3}) = \mu^2/9, \mathrm{Var}(X_{2:3}) = 13\mu^2/36, E(X_{1:3}X_{2:3}) = 7\mu^2/18, \mathrm{Cov}(X_{1:3}, X_{2:3}) = \mu^2/9$, and $\mathrm{Corr}(X_{1:3}, X_{2:3}) = 2/\sqrt{13}$. ◁

Example 3.5. If we consider the system $T = \min(X_1, \max(X_2, X_3))$ and its dual system $T_D = \max(X_1, \min(X_2, X_3))$ with minimal signatures $(0, 2, -1)$ and $(1, 1, -1)$, respectively, then

$$\Pr(T > x, T_D > y) = 2R(x)R(y) + R(x)R^2(y) - R^3(y) - R^2(x)R(y)$$

for $x < y$, and

$$\Pr(T > x, T_D > y) = \Pr(T > x) = 2R^2(x) - R^3(x)$$

for $x \geq y$. Thus, the non-zero coefficients are $a_{0,3} = -1$, $a_{1,1} = 2$, $a_{1,2} = 1$, $a_{2,1} = -1$, $a^*_{2,0} = 2$ and $a^*_{3,0} = -1$. Therefore, if we suppose that X_i's have an exponential distribution with mean μ ($i = 1, 2, 3$), then $E(T) = 2\mu/3$, $E(T_D) = 7\mu/6$, $E(T^2) = 7\mu^2/9$, $E(T_D^2) = 41\mu^2/18$, $\mathrm{Var}\,(T) = \mu^2/3$, $\mathrm{Var}\,(T_D) = 11\mu^2/12$, $E(TT_D) = 10\mu^2/9$, $\mathrm{Cov}\,(T, T_D) = \mu^2/3$, and $\mathrm{Corr}\,(T, T_D) = 2/\sqrt{11}$. ◁

In particular, if we apply (3.11)–(3.13) to order statistics from IID exponential random variables, we can compute the means, variances and correlation coefficients of order statistics. The correlations values are presented in Table 1 for sample sizes 2–6. Alternatively, we can use the following well known (see [35]) expressions

$$E(X_{r:n}) = \sum_{j=n-r+1}^{n} \frac{1}{j},$$

$$\mathrm{Var}\,(X_{r:n}) = \sum_{j=n-r+1}^{n} \frac{1}{j^2}$$

for $r = 1, 2, \dots, n$,

$$\mathrm{Cov}\,(X_{r:n}, X_{s:n}) = \mathrm{Var}\,(X_{r:n}) = \sum_{j=n-r+1}^{n} \frac{1}{j^2}$$

and

$$\mathrm{Corr}\,(X_{r:n}, X_{s:n}) = \sqrt{\frac{\mathrm{Var}\,(X_{r:n})}{\mathrm{Var}\,(X_{s:n})}} = \sqrt{\frac{\sum_{j=n-r+1}^{n} \frac{1}{j^2}}{\sum_{j=n-s+1}^{n} \frac{1}{j^2}}}$$

for $1 \leq r < s \leq n$.

Similar results can also be obtained for other distributions. For example, if the component lifetimes $X_1, X_2, \dots, X_n$ have uniform distributions in the interval (0, 1), then from Theorem 3.3, we obtain

$$E(T) = \sum_{i=1}^{n} a_i/(i+1),$$

$$E(T^2) = 2\sum_{i=1}^{n} a_i/(i+1)(i+2)$$

and

$$E(TT^*) = \sum_{i=0}^{n}\sum_{j=1}^{n-i}(a_{i,j} + a^*_{j,i})\frac{\mu^2}{(j+1)(i+j+2)}.$$

Finally, we note that, from [31], the results in this section can also be applied to systems with different sizes and, in particular, to the order statistics $X_{i:n}$ and $X_{j:m}$ with $n \neq m$.

4. Spearman's correlation and Kendall's tau between two order statistics

Some other useful measures of dependence are Spearman's ρ and Kendall's τ. The main advantage of these measures is that they are invariant under monotone transformations. Hence, when we apply them to coherent systems or to order statistics, they are distribution-free as they do not depend on the common distribution function F. These two measures are closely related; see Fredericks and Nelsen [36] and the references therein. Spearman's $\rho_S(X, Y)$ of two random variables X and Y is simply the Pearson's correlation coefficient between $F_X(X)$ and $F_Y(Y)$ which have uniform distributions in (0, 1), where F_X and F_Y are the corresponding distribution functions. It is well-known that

$$\rho_S(X, Y) = -3 + 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_X(x)F_Y(y)\mathrm{d}F(x, y),$$

where F is the joint distribution of (X_1, X_2); see [37, p. 32]. It can also be rewritten as

$$\rho_S(X, Y) = 3 - 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(x)F_Y(y)\mathrm{d}R(x, y), \tag{4.1}$$

where $R_X = 1 - F_X$ and R is the joint reliability function of (X_1, X_2).

Spearman's ρ of extreme order statistics (i.e., $X_{1:n}$ and $X_{n:n}$) were studied in [1,4,8,10]. Chen [4], in particular, proved that $\rho_S(X_{1:n}, X_{n:n}) = 3(1 - 4p_{1,n})$, where

$$p_{1,n} = n(n-1)\int_0^1\int_0^t (1-s)^n t^n (t-s)^{n-2}\mathrm{d}s\mathrm{d}t = E((1-X_{1:n})^n X_{n:n}^n).$$

In the following theorem, which is the main result of this section, we extend these results for any pair of order statistics, and in fact, it is stated in a more general way for any pair of coherent systems.

Theorem 4.1. *If T and T^* are the lifetimes of two coherent systems based on the same IID component lifetimes $X_1, X_2, \ldots, X_n$ having a common uniform distribution in the interval $(0, 1)$ and $\Pr(T = T^*) = 0$, then*

$$\rho_S(T, T^*) = 3(1 - 4p),$$

where

$$p = \sum_{i=1}^n \sum_{j=1}^n a_i b_j^* \int_0^1 \int_0^1 (1-x)^i y^j f_{T,T^*}(x, y)\mathrm{d}x\mathrm{d}y, \tag{4.2}$$

$(a_1, a_2, \ldots, a_n)$ is the minimal signature of T, $(b_1^*, b_2^*, \ldots, b_n^*)$ is the maximal signature of T^*, and f_{T,T^*} is the joint density of (T, T^*).

Proof. If $\Pr(T = T^*) = 0$, then (T, T^*) has an absolutely continuous joint distribution and hence, from (4.1), we have

$$p = \int_0^1 \int_0^1 R_T(x) F_{T^*}(y) f_{T,T^*}(x, y)\mathrm{d}x\mathrm{d}y,$$

where f_{T,T^*} is the joint probability density function. Then, upon using the facts that $R_{1:i}(x) = (1-x)^i$ and $F_{i:i}(x) = x^i$, from (3.1), we obtain (4.2). □

Note that if T and T^* are the lifetimes of two coherent systems based on the same IID component lifetimes $X_1, X_2, \ldots, X_n$ having a common absolutely continuous distribution, then the random vector (T, T^*) has an absolutely continuous distribution if and only if $\Pr(T = T^*) = 0$ holds. Also note that

$$p = \sum_{i=1}^n \sum_{j=1}^n a_i b_j^* E((1-T)^i (T^*)^j)$$

and that f_{T,T^*} in (4.2) can be replaced by the expressions obtained from (3.4) and (3.5). Also, from (iii) of Lemma 3.2, $E((1-T)^i(T^*)^j)$ can be expressed as

$$E((1-T)^i(T^*)^j) = \int_0^1 \int_0^1 \Pr(T < 1 - x^{1/i}, T^* > y^{1/j})\mathrm{d}x\mathrm{d}y.$$

We can obtain representations for $\Pr(T < s, T^* > t)$ similar to those for R_{T,T^*} in (3.4) and (3.5). We can also obtain analogous expressions using only minimal (or maximal) signatures. For example, (4.1) can be written as

$$\rho_S(X, Y) = -3 + 12\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(x) R_Y(y)\mathrm{d}R(x, y),$$

and hence

$$\rho_S(T, T^*) = -3 + 12\sum_{i=1}^n \sum_{j=1}^n a_i a_j^* \int_0^1 \int_0^1 (1-x)^i (1-y)^j f_{T,T^*}(x, y)\mathrm{d}x\mathrm{d}y. \tag{4.3}$$

Then, using the representations in Lemma 3.1, we obtain the following result.

Theorem 4.2. *If T and T^* are the lifetimes of two coherent systems based on the same IID component lifetimes $X_1, X_2, \ldots, X_n$ having a common uniform distribution in the interval $(0, 1)$ and $\Pr(T = T^*) = 0$, then*

$$\rho_S(T, T^*) = -3 + 12\sum_{i=1}^n \sum_{j=1}^n \sum_{r=1}^n \sum_{s=1}^{n-r} (a_i a_j^* a_{r,s} + a_j a_i^* a_{s,r}^*)\frac{rs}{(j+s)(i+j+r+s)}, \tag{4.4}$$

where $(a_1, a_2, \ldots, a_n)$ and $(a_1^*, a_2^*, \ldots, a_n^*)$ are the minimal signatures of T and T^*, respectively, and $a_{r,s}$ and $a_{s,r}^*$ are the coefficients in (3.4) and (3.5) for (T, T^*).

Proof. From (3.4), (3.5) and (4.3), we have

$$\rho_S(T,T^*) = -3 + 12\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n-r} rs(a_i a_j^* a_{r,s} + a_j a_i^* a_{s,r}^*)\int_0^1\int_x^1 (1-x)^{i+r-1}(1-y)^{j+s-1}\mathrm{d}y\mathrm{d}x.$$

Then, the expression in (4.4) is obtained by carrying out the required integrations. □

In particular, we obtain the following expressions for order statistics.

Theorem 4.3. *If $X_{r:n}$ and $X_{s:n}$ $(r < s)$ are two order statistics from an IID sample, then $\rho_S(X_{r:n}, X_{s:n}) = 3(1-4p_{r,s})$, where*

$$p_{r,s} = \sum_{i=n-r+1}^{n}\sum_{j=s}^{n}(-1)^{i+j+r-s-n-1}\binom{i-1}{n-r}\binom{n}{i}\binom{j-1}{s-1}\binom{n}{j}\frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$
$$\times\int_0^1\int_0^y (1-x)^i y^j x^{r-1}(y-x)^{s-r-1}(1-y)^{n-s}\mathrm{d}x\mathrm{d}y \tag{4.5}$$
$$= \sum_{i=n-r+1}^{n}\sum_{j=s}^{n}\sum_{k=0}^{i}(-1)^{i+j+r+k-s-n-1}\frac{r\binom{i-1}{n-r}\binom{n}{i}\binom{j-1}{s-1}\binom{n}{j}\binom{i}{k}\binom{n}{s}\binom{s}{r}}{(k+s+j)\binom{k+s-1}{s-r}\binom{n+k+j}{n-s}}. \tag{4.6}$$

Proof. From Eqs. (3.2) and (3.3) and the expression for the joint density of two order statistics, the general formula in (4.2) reduces to the one in (4.5). To obtain (4.6) from (4.5), we replace $(1-x)^i$ by $\sum_{k=0}^{i}(-1)^k\binom{i}{k}x^k$, and make the change $z = x/y$ in the integral, obtaining two beta-type integrals, leading to the expression in (4.6). □

In particular, when $r = 1$ and $s = n$ in (4.5) and (4.6), we obtain formulas 1 and 2 in Theorem 1 of Chen [4], respectively. An alternative expression is presented in the following theorem.

Theorem 4.4. *If $X_{r:n}$ and $X_{s:n}$ $(r < s)$ are two order statistics from an IID sample, then $\rho_S(X_{r:n}, X_{s:n}) = -3 + 12q_{r,s}$, where*

$$q_{r,s} = \sum_{i=n-r+1}^{n}\sum_{j=n-s+1}^{n}\sum_{\ell=0}^{r-1}\sum_{k=\ell}^{r-1}\sum_{t=k}^{s-1}(-1)^{r+i+j+k+t-\ell-1}\binom{i-1}{n-r}\binom{n}{i}\binom{j-1}{n-s}\binom{n}{j}$$
$$\times\frac{(t-\ell)n!}{(j+n-t)(i+j+n-\ell)(n-s)!(s-t-1)!(k-\ell)!(t-k)!\ell!}.$$

Proof. From (3.2) and (4.4), we obtain

$$q_{r,s} = \sum_{i=n-r+1}^{n}\sum_{j=n-s+1}^{n}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n-\alpha}(-1)^{r+s+i+j}\binom{i-1}{n-r}\binom{n}{i}\binom{j-1}{n-s}\binom{n}{j}\frac{\alpha\beta a_{\alpha,\beta}}{(j+\beta)(i+j+\alpha+\beta)}$$

since the coefficients $a_{i,j}^*$ in (3.5) for $(X_{r:n}, X_{s:n})$ are zero for $j = 1, 2, \ldots, n$. Then, by using (3.7), we obtain the required result. □

The values for Spearman's ρ between order statistics from samples of sizes 2–6 are presented in Table 1. Similar developments can be made for Kendall's tau using the expression

$$\tau(X,Y) = -1 + 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(x,y)\mathrm{d}R(x,y), \tag{4.7}$$

where R is the joint reliability function of (X, Y) and X and Y have uniform distributions in $(0, 1)$. For example, the general result obtained from the representations in Lemma 3.1 can be stated as follows.

Theorem 4.5. *If T and T^* are the lifetimes of two coherent systems based on the same IID component lifetimes $X_1, X_2, \ldots, X_n$ having a common uniform distribution in the interval $(0, 1)$ and $\Pr(T = T^*) = 0$, then*

$$\tau(T,T^*) = -1 + 4\sum_{i=1}^{n}\sum_{j=1}^{n-i}\sum_{r=1}^{n}\sum_{s=1}^{n-r}(a_{i,j}a_{r,s} + a_{j,i}^* a_{s,r}^*)\frac{rs}{(j+s)(i+j+r+s)},$$

where $a_{r,s}$ and $a_{s,r}^*$ are the coefficients in (3.4) and (3.5) for (T, T^*).

In particular, for the order statistics from (3.7), we obtain

$$\tau(X_{r:n}, X_{s:n}) = -1 + 4 \sum_{a=0}^{r-1} \sum_{b=a}^{r-1} \sum_{c=b}^{s-1} \sum_{i=0}^{r-1} \sum_{j=i}^{r-1} \sum_{k=j}^{s-1} \frac{(-1)^{s+j+k-i-1} n!}{(n-k)(n-s)!(s-k-1)!(j-i)!(k-j)!i!} \times \frac{(-1)^{s+b+c-a-1} n!}{(n-c)(n-s)!(s-c-1)!(b-a)!(c-b)!a!} \cdot \frac{(n-k)(k-i)}{(n-c+n-k)(n+n-a-i)}$$

for $1 \le r < s \le n$. Alternative (more simple) expressions were obtained by Schmitz [10] (between $X_{1:n}$ and $X_{n:n}$) and Avérous, Genest and Kochar [1]. The values for Kendall's tau between order statistics from samples of sizes 2–6 are presented in Table 1. Note that Kendall's tau is symmetric, that is, $\tau(X_{r:n}, X_{s:n}) = \tau(X_{n-r+1:n}, X_{n-s+1:n})$. Also note that $\tau(X_{r:n}, X_{s:n})$ decreases as $s-r$ increases. In the next section, we show how Kendall's tau can be computed in the case of dependent samples.

Finally, it needs to be mentioned that expressions similar to those given in the preceding result can be obtained for $X_{r:n}$ and $X_{s:m}$ when $n \neq m$ by using the representations in [31].

5. Dependence measures for order statistics from dependent samples

In this section, we suppose that $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$ is a random vector of (possibly dependent) identically distributed random variables with common continuous distribution function F. Then, it is well known that from Sklar's theorem (see [38, p. 18]), the joint distribution function can be written as

$$F_{\boldsymbol{X}}(x_1, x_2, \dots, x_n) = C(F(x_1), F(x_2), \dots, F(x_n)),$$

where the copula C is the joint distribution function of $\boldsymbol{U} = (U_1, U_2, \dots, U_n)$ and where $U_i = F(X_i)$ has a uniform distribution in $(0, 1)$ for $i = 1, 2, \dots, n$.

If $(X_{1:n}, X_{2:n}, \dots, X_{n:n})$ is the random vector of order statistics obtained from $\boldsymbol{X}$ and $(U_{1:n}, U_{2:n}, \dots, U_{n:n})$ is the random vector of order statistics obtained from $\boldsymbol{U}$, then $U_{i:n} = F(X_{i:n})$. Moreover, if G_i is the distribution function of $U_{i:n}$ and C^* is the copula of $(U_{1:n}, U_{2:n}, \dots, U_{n:n})$, then

$$\begin{aligned} \Pr(X_{1:n} \le x_1, \dots, X_{n:n} \le x_n) &= \Pr(U_{1:n} \le F(x_1), \dots, U_{n:n} \le F(x_n)) \\ &= \Pr(G_1(U_{1:n}) \le G_1(F(x_1)), \dots, G_n(U_{n:n}) \le G_n(F(x_n))) \\ &= C^*(G_1(F(x_1)), \dots, G_n(F(x_n))), \end{aligned}$$

that is, $(X_{1:n}, X_{2:n}, \dots, X_{n:n})$ and $(U_{1:n}, U_{2:n}, \dots, U_{n:n})$ share the same copula (C^*) and $G_i \circ F$ is the distribution function of $X_{i:n}$, for $i = 1, 2, \dots, n$.

Therefore, $(X_{i:n}, X_{j:n})$ and $(U_{i:n}, U_{j:n})$ also share the same copula and $d(X_{i:n}, X_{j:n}) = d(U_{i:n}, U_{j:n})$, where $d(Y, Z)$ stands for any measure of concordance of real-valued random variables Y and Z in the sense of Scarsini [16] including Spearman's rho, Kendall's tau and Gini's coefficient of association. Further, from Capéraà and Genest [39] (see also [40, p. 300]), we have Spearman's correlation coefficient to be greater (smaller) than Kendall's tau for positively (negatively) dependent random variables. Hence, we can compute Kendall's tau which will then provide a bound for Spearman's rho under some dependence properties. Moreover, Nelsen [38, p. 153] proved that three times Kendall's tau is an upper bound for Spearman's correlation coefficient (see [40, p. 300]) for positively quadrant dependent (PQD) random variables. Other relationships between Spearman's rho and Kendall's tau have been given by Fredricks and Nelsen [36]. These results lead to the following lemma.

Lemma 5.1. *If $\rho_S(Y, Z)$ and $\tau(Y, Z)$ are Spearman's rho and Kendall's tau, respectively, then*

$$3\tau(Y, Z) \ge \rho_S(Y, Z) \ge \tau(Y, Z) \ge 0$$

whenever one of Y or Z is simultaneously left-tail decreasing (LTD) and right-tail increasing (RTI) in the other variable. Moreover, the inequalities can be reversed whenever one of Y or Z is simultaneously left-tail increasing (LTI) and right-tail decreasing (RTD) in the other variable.

The proof is immediate since LTD (LTI) implies PQD (NQD); see [2]. It is well-known that (see [2]) two random variables are LTD and RTI when they are TP_2 (i.e., when their joint density is totally positive of order 2). It is also easy to show, using the properties of multivariate likelihood ratio order (see [41]), that if a random vector is MTP_2 (multivariate totally positive of order 2 or positively likelihood ratio dependent), then any pair of its random variables is TP_2. Therefore, if $(U_{1:n}, U_{2:n}, \dots, U_{n:n})$ is MTP_2, then

$$\rho_S(X_{i:n}, X_{j:n}) \ge \tau(U_{i:n}, U_{j:n}) \ge 0,$$

i.e., Kendall's tau is a lower bound for Spearman's correlation of two order statistics, which does not depend on the parent distribution function.

Note that Kendall's tau can be computed through a standard Monte Carlo procedure (see Example 5.3) by using the expression

$$\tau(X, Y) = 2 \Pr((X_1 - X_2)(Y_1 - Y_2) > 0) - 1$$

Table 1
Pearson's correlation coefficient $\rho = \mathrm{Corr}(X_{r:n}, X_{s:n})$, Spearman's correlation coefficient $\rho_S = \rho_S(X_{r:n}, X_{s:n})$ and Kendall's tau $\tau = \tau(X_{r:n}, X_{s:n})$ of order statistics from IID exponential samples of sizes 2 to 6.

n	r	s	ρ	ρ_S	τ
2	1	2	0.447214	0.466667	0.333333
3	1	2	0.554700	0.550000	0.400000
3	1	3	0.285714	0.292857	0.200000
3	2	3	0.515079	0.550000	0.400000
4	1	2	0.600000	0.584416	0.428571
4	1	3	0.384111	0.372468	0.257143
4	1	4	0.209529	0.211775	0.142857
4	2	3	0.640184	0.650909	0.485714
4	2	4	0.349215	0.372468	0.257143
4	3	4	0.545491	0.584416	0.428571
5	1	2	0.624695	0.603175	0.444444
5	1	3	0.432731	0.411303	0.285714
5	1	4	0.293733	0.280481	0.190476
5	1	5	0.165317	0.165517	0.111111
5	2	3	0.692708	0.693164	0.523810
5	2	4	0.470203	0.474811	0.333333
5	2	5	0.264636	0.280481	0.190476
5	3	4	0.678789	0.693164	0.523810
5	3	5	0.382031	0.411303	0.285714
5	4	5	0.562813	0.603175	0.444444
6	1	2	0.640184	0.614973	0.454545
6	1	3	0.461757	0.434492	0.303030
6	1	4	0.339227	0.317354	0.216450
6	1	5	0.237759	0.224699	0.151515
6	1	6	0.136475	0.135746	0.090909
6	2	3	0.721288	0.716387	0.545455
6	2	4	0.529889	0.525079	0.372294
6	2	5	0.371391	0.372018	0.255411
6	2	6	0.213181	0.224699	0.151515
6	3	4	0.734643	0.739056	0.567100
6	3	5	0.514900	0.525079	0.372294
6	3	6	0.295556	0.317354	0.216450
6	4	5	0.700884	0.716387	0.545455
6	4	6	0.402312	0.434492	0.303030
6	5	6	0.574007	0.614973	0.454545

(see [38, p. 159]), where (X_1, Y_1) and (X_2, Y_2) are independent random vectors with the same distribution as (X, Y). Moreover, if $(X_1, X_2, \ldots, X_n)$ is exchangeable, then $(U_1, U_2, \ldots, U_n)$ is also exchangeable and hence, $(U_{1:n}, U_{2:n}, \ldots, U_{n:n})$ is MTP_2 if, and only if $(U_1, U_2, \ldots, U_n)$ is MTP_2. Thus, we obtain the following theorem.

Theorem 5.2. *If $(X_1, X_2, \ldots, X_n)$ is exchangeable and MTP_2, then*

$$\rho_S(X_{i:n}, X_{j:n}) \geq \tau(U_{i:n}, U_{j:n}) \geq 0,$$

where $U_{i:n} = F(X_{i:n})$ for $i = 1, 2, \ldots, n$ and F is the common marginal distribution.

A similar argument can be used in the case of negative dependence by using the results of Nelsen [38, p. 182].

Example 5.3. Let $(X_1, X_2, \ldots, X_n)$ have a Farlie–Gumbel–Morgenstern exchangeable distribution with marginal distribution F, with the joint distribution function as

$$F(x_1, x_2, \ldots, x_n) = F(x_1) \cdots F(x_n) \left(1 + \alpha \prod_{i=1}^{n} (1 - F(x_i))\right),$$

where $|\alpha| \leq 1$. It is not hard to verify that the corresponding copula

$$C(u_1, \ldots, u_n) = u_1 \cdots u_n \left(1 + \alpha \prod_{i=1}^{n} (1 - u_i)\right)$$

is absolutely continuous with density function

$$c(u_1, \ldots, u_n) = 1 + \alpha \prod_{i=1}^{n} (1 - 2u_i);$$

Table 2
Numerical approximations (using Monte Carlo simulations) of Kendall's tau values for order statistics obtained from a sample with FGM joint distribution (see Example 5.3).

α	$\tau(X_{1:2}, X_{2:2})$	$\tau(X_{1:3}, X_{2:3})$	$\tau(X_{2:3}, X_{3:3})$	$\tau(X_{1:3}, X_{3:3})$
−1.0	0.160080	0.461528	0.338744	0.198360
−0.9	0.178040	0.454704	0.344976	0.198056
−0.8	0.196536	0.448600	0.351048	0.198288
−0.7	0.214064	0.442704	0.357544	0.198000
−0.6	0.231968	0.436296	0.363536	0.197864
−0.5	0.248896	0.429608	0.369256	0.197768
−0.4	0.266560	0.423512	0.375632	0.197360
−0.3	0.284264	0.417040	0.381536	0.197440
−0.2	0.302256	0.410840	0.387800	0.197280
−0.1	0.321160	0.404464	0.393992	0.197264
0.0	0.330448	0.397560	0.398960	0.197488
0.1	0.356504	0.391544	0.406136	0.197240
0.2	0.374624	0.385456	0.412464	0.197088
0.3	0.392376	0.379432	0.418888	0.197296
0.4	0.409976	0.373008	0.425016	0.196880
0.5	0.427424	0.367048	0.431048	0.197272
0.6	0.444416	0.361296	0.437032	0.197400
0.7	0.462624	0.355432	0.443544	0.197616
0.8	0.480096	0.348872	0.449736	0.197608
0.9	0.497576	0.342512	0.455656	0.197848
1.0	0.515680	0.336472	0.461840	0.198256

see, for example, [38, p. 87]. Therefore, we can compute approximations of Kendall's tau for the order statistics based on samples from the above FGM distribution by using a standard Monte Carlo method (see Table 2). If $\alpha \geq 0$, then it is not hard to verify that F is TP_2 in pairs and hence MTP_2. Thus, Kendall's tau values are lower bounds for Spearman's rho when $\alpha \geq 0$. Numerically, we observe that $\tau(X_{1:2}, X_{2:2})$ seems to be a linear function of α. The same holds for $\tau(X_{1:3}, X_{2:3})$ and $\tau(X_{2:3}, X_{3:3})$. Moreover, it seems that $\tau(X_{1:3}, X_{3:3})$ does not depend on α. In the case of independence ($\alpha = 0$), the exact values of Kendall's tau are $\tau(X_{1:2}, X_{2:2}) = 1/3$, $\tau(X_{1:3}, X_{2:3}) = \tau(X_{2:3}, X_{3:3}) = 2/5$, and $\tau(X_{1:3}, X_{3:3}) = 1/5$. $\triangleleft$

Finally, in the following example, we show that the exact values for Kendall's tau between order statistics from an exchangeable random vector can also be computed by using (4.7) and a procedure similar to that used in Lemma 3.1 and in Theorem 4.5. As a matter of fact, this procedure can also be used in the general (not necessarily exchangeable) case.

Example 5.4. Let us consider $X_{1:2}$ and $X_{2:2}$; then, for $x \leq y$,

$$\begin{aligned}\Pr(X_{1:2} > x, X_{2:2} > y) &= \Pr(X_{1:2} > x, \{X_1 > y\} \cup \{X_2 > y\}) \\ &= \Pr(X_{1:2} > x, X_1 > y) + \Pr(X_{1:2} > x, X_2 > y) - \Pr(X_{1:2} > x, X_{1:2} > y) \\ &= R(y, x) + R(x, y) - R(y, y),\end{aligned}$$

where R is the joint reliability function of (X_1, X_2). Now if R is exchangeable, then

$$\Pr(X_{1:2} > x, X_{2:2} > y) = 2R(x, y) - R(y, y)$$

and the joint density of $X_{1:2}$ and $X_{2:2}$ is (of course) equal to $2f(x, y)$ for $x \leq y$, and zero otherwise. Then, Kendall's tau can be computed from (4.7) as

$$\tau(X_{1:2}, X_{2:2}) = -1 + 4\int_0^1 \int_x^1 (2R(x, y) - R(y, y))f(x, y)\mathrm{d}y\mathrm{d}x.$$

For example, if (X_1, X_2) has the Farlie–Gumbel–Morgenstern exchangeable distribution presented in the preceding example, a direct calculation then yields

$$\tau(X_{1:2}, X_{2:2}) = \frac{8}{45}\alpha + \frac{1}{3},$$

which coincides with the Monte Carlo approximation obtained in Table 2. Proceeding similarly, we obtain $\tau(X_{1:3}, X_{2:3}) = \frac{2}{5} - \frac{13}{210}\alpha$, $\tau(X_{2:3}, X_{3:3}) = \frac{2}{5} + \frac{13}{210}\alpha$, and $\tau(X_{1:3}, X_{3:3}) = \frac{1}{5}$. $\triangleleft$

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Appendix. Tables

See Tables 1 and 2.

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