



# Moment bounds and central limit theorems for Gaussian subordinated arrays

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## ABSTRACT

A general moment bound for sums of products of Gaussian vector's functions extending the moment bound in Taqqu (1977, Lemma 4.5) [28] is established. A general central limit theorem for triangular arrays of nonlinear functionals of multidimensional non-stationary Gaussian sequences is proved. This theorem extends the previous results of Breuer and Major (1983) [5], Arcones (1994) [1] and others. A Berry–Esseen-type bound in the above-mentioned central limit theorem is derived following Nourdin et al. (2011) [20]. Two applications of the above results are discussed. The first one refers to the asymptotic behavior of a roughness statistic for continuous-time Gaussian processes and the second one is a central limit theorem satisfied by long memory locally stationary processes.

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## 1. Introduction

This paper is devoted to the proof of two new results concerning functions of Gaussian vectors. The first one (Lemma 1 of Section 2) is a moment bound for “off-diagonal” sums of products of functions of Gaussian vectors in a general frame. It is an extension of an important lemma by Taqqu [28, Lemma 4.5]. This result is useful for obtaining almost sure convergence and tightness of Gaussian subordinated functionals and statistics; see Remark 1. The proof of Lemma 1 uses the Hermitian decomposition of the  $\mathbb{L}^2$  function and the diagram formula. A related but different moment bound is proved in [25, Corollary 2.1].

The second result is a central limit theorem (CLT) for arrays of random variables that are functions of Gaussian vectors; see Theorem 1 for a precise statement. Theorem 1 generalizes and extends earlier results due to Breuer and Major [5], Giraitis and Surgailis [13] and Arcones [1, Theorem 2] to the case of non-stationary triangular arrays of Gaussian vectors. Extensions of the Breuer–Major theorem were also obtained by Chambers and Slud [6], Sanchez de Naranjo [24] and Nourdin et al. [20]. Most of the above cited papers treat the case of a single stationary Gaussian sequence and a function independent of  $n$ . Generalization to stationary or non-stationary triangular arrays is motivated by numerous statistical applications. Some examples of these applications, with a particular emphasis on strongly dependent Gaussian processes, are statistics of time

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series (see for instance [2,23]), kernel-type estimation of the regression function [14], and nonparametric estimation of the local Hurst function of a continuous-time process from a discrete grid  $i/n$ ,  $1 \leq i \leq n$  [15,3,4]. Two particular applications (limit theorems for the Increment Ratio statistic of a Gaussian process admitting a tangent process and a CLT for functions of locally stationary Gaussian processes) are discussed in Section 5.

Starting with the famous Lindeberg theorem for independent random variables, numerous studies devoted to CLT for triangular arrays under various dependence conditions had appeared. The case of martingale dependence was extensively studied in [17]. Rio [22] discussed the case of strongly mixing sequences. Some more recent papers devoted to this question are Coulon-Prieur and Doukhan [7] (with a new weak dependence condition) and Dedecker and Merlevède [11] (with a necessary and sufficient condition for stable convergence of normalized partial sums). The CLT for linear triangular arrays was discussed in detail in [21] for several forms of dependence conditions.

The case of Gaussian subordinated variables (functions of Gaussian vectors) is rather exceptional among other dependence structures since it allows for very sharp conditions for CLT in terms of the decay rate of the covariance of the Gaussian process and the Hermite rank of the non-linear function. These conditions are close to being necessary and result in CLTs “in the vicinity” of non-central limit theorems; see [5,1,12,29]. The proofs of the above-mentioned results rely on specific Gaussian techniques such as the Hermite expansion and the diagram formula; however, the recent paper Nourdin et al. [20] uses a different approach based on Malliavin’s calculus and Stein’s method, yielding also convergence rates in the CLT. The main difference between our Theorem 1 and the corresponding results in [1,20] is that, contrary to these papers, we do not assume stationarity of the underlying Gaussian sequence  $(Y_n(k))$  and discuss the case of subordinated sums  $\sum_{k=1}^n f_{k,n}(Y_n(k))$  where  $f_{k,n}$  may depend on  $k$  and  $n$ . The last fact is important for statistical applications (see above). In the particular case when  $f_{k,n} = f$  do not depend on  $k$ ,  $n$  and  $(Y_n(k))$  is a stationary process independent of  $n$ , Theorem 1(iii) agrees with Arcones [1] and Nourdin et al. [20, Theorem 1.1]. The proof of Theorem 1 uses the diagram method and cumulants as in [13]. Section 4 obtains a Berry–Esseen bound in this CLT using the approach and results in [20]. Let us note that a CLT for Gaussian subordinated arrays is also proved in [25, Theorem 3.1]; however, it requires that Gaussian vectors are asymptotically independent and therefore his result is different from Theorem 1.

*Notation.* Everywhere below,  $\mathbf{X} = (X^{(1)}, \dots, X^{(v)})$  designates a standardized Gaussian vector in  $\mathbb{R}^v$ ,  $v \geq 1$ , with zero mean  $EX^{(u)} = 0$  and covariances  $EX^{(u)}X^{(v)} = \delta_{uv}$ ,  $u, v = 1, \dots, v$ . Letter  $C$  stands for a constant whose precise value is unimportant and which may change from line to line. The weak convergence of distributions is denoted by  $\xrightarrow[n \rightarrow \infty]{\mathcal{D}}$ .

## 2. A moment bound

Let  $\mathbb{L}^2(\mathbf{X})$  denote the class of all measurable functions  $f = f(\mathbf{x})$ ,  $\mathbf{x} = (x^{(1)}, \dots, x^{(v)}) \in \mathbb{R}^v$  such that  $\|f\|^2 := Ef^2(\mathbf{X}) < \infty$ . For any multiindex  $\mathbf{k} = (k^{(1)}, \dots, k^{(v)}) \in \mathbb{Z}_+^v := \{j^{(1)}, \dots, j^{(v)} \in \mathbb{Z}^v, j^{(u)} \geq 0 \ (1 \leq u \leq v)\}$ , let  $H_{\mathbf{k}}(\mathbf{x}) = H_{k^{(1)}}(x^{(1)}) \cdots H_{k^{(v)}}(x^{(v)})$  be the (product) Hermite polynomial;  $H_k(x) := (-1)^k e^{x^2/2} (e^{-x^2/2})^{(k)}$ ,  $k = 0, 1, \dots$  are standard Hermite polynomials (with  $(e^{-x^2/2})^{(k)}$  the  $k$ th derivative of the function  $x \mapsto e^{-x^2/2}$ ). Write  $|\mathbf{k}| := k^{(1)} + \dots + k^{(v)}$ ,  $\mathbf{k}! := k^{(1)}! \cdots k^{(v)}!$ ,  $\mathbf{k} = (k^{(1)}, \dots, k^{(v)}) \in \mathbb{Z}_+^v$ . A function  $f \in \mathbb{L}^2(\mathbf{X})$  is said to have a Hermite rank  $m \geq 0$  if  $J_f(\mathbf{k}) := Ef(\mathbf{X})H_{\mathbf{k}}(\mathbf{X}) = 0$  for any  $\mathbf{k} \in \mathbb{Z}_+^v$ ,  $|\mathbf{k}| < m$ , and  $J_f(\mathbf{k}) \neq 0$  for some  $\mathbf{k}$ ,  $|\mathbf{k}| = m$ . It is well-known that any  $f \in \mathbb{L}^2(\mathbf{X})$  having a Hermite rank  $m \geq 0$  admits the Hermite expansion

$$f(\mathbf{x}) = \sum_{|\mathbf{k}| \geq m} \frac{J_f(\mathbf{k})}{\mathbf{k}!} H_{\mathbf{k}}(\mathbf{x}), \quad (2.1)$$

which converges in  $\mathbb{L}^2(\mathbf{X})$ .

Let  $(\mathbf{X}_1, \dots, \mathbf{X}_n)$  be a collection of standardized Gaussian vectors  $\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(v)}) \in \mathbb{R}^v$  having a joint Gaussian distribution in  $\mathbb{R}^{vn}$ . Let  $\varepsilon \in [0, 1]$  be a fixed number. Following Taqqu [28], we call  $(\mathbf{X}_1, \dots, \mathbf{X}_n)$   $\varepsilon$ -standard if  $|EX_t^{(u)}X_s^{(v)}| \leq \varepsilon$  for any  $t \neq s$ ,  $1 \leq t, s \leq n$  and any  $1 \leq u, v \leq v$ .

As mentioned in the Introduction, Lemma 1 generalizes Taqqu [28, Lemma 4.5] to the case of a vector-valued Gaussian family  $(\mathbf{X}_1, \dots, \mathbf{X}_n)$ , taking values in  $\mathbb{R}^v$  ( $v \geq 1$ ). The lemma concerns the bound (2.4), below, where  $f_{1,t,n}, \dots, f_{p,t,n}$  are square integrable functions among which the first  $0 \leq \alpha \leq p$  functions  $f_{1,t,n}, \dots, f_{\alpha,t,n}$  for any  $1 \leq t \leq n$  have a Hermite rank at least equal to  $m \geq 1$  and where  $\sum'$  is the sum over all different indices  $1 \leq t_i \leq n$  ( $1 \leq i \leq p$ ),  $t_i \neq t_j$  ( $i \neq j$ ). In the case when  $f_{j,t,n} = f_j$  does not depend on  $t$ ,  $n$ , the bound (2.4) coincides with that of Taqqu [28, Lemma 4.5] provided  $m\alpha$  is even, but is worse than Taqqu’s bound in the more delicate case when  $m\alpha$  is odd. An advantage of our proof is its relative simplicity (we do not use the graph-theoretical argument as in [28], but rather a simple Hölder inequality). A different approach towards moment inequalities for functions in vector-valued Gaussian variables is discussed in [25], leading to a different type of moment inequalities.

**Lemma 1.** Let  $(\mathbf{X}_1, \dots, \mathbf{X}_n)$  be a  $\varepsilon$ -standard Gaussian vector,  $\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(v)}) \in \mathbb{R}^v$ ,  $v \geq 1$ , and let  $f_{j,t,n} \in \mathbb{L}^2(\mathbf{X})$ ,  $1 \leq j \leq p$ ,  $p \geq 2$ ,  $1 \leq t \leq n$  be some functions. For given integers  $m \geq 1$ ,  $0 \leq \alpha \leq p$ ,  $n \geq 1$ , define

$$Q_n := \max_{1 \leq t \leq n} \sum_{1 \leq s \leq n, s \neq t} \max_{1 \leq u, v \leq v} |EX_t^{(u)}X_s^{(v)}|^m. \quad (2.2)$$

Assume that the functions  $f_{1,t,n}, \dots, f_{\alpha,t,n}$  have a Hermite rank at least equal to  $m$  for any  $n \geq 1$ ,  $1 \leq t \leq n$ , and that

$$\varepsilon < \frac{1}{\nu p - 1}. \quad (2.3)$$

Then

$$\sum' |E[f_{1,t_1,n}(\mathbf{X}_{t_1}) \cdots f_{p,t_p,n}(\mathbf{X}_{t_p})]| \leq C(\varepsilon, p, m, \alpha, \nu) K n^{p-\frac{\alpha}{2}} Q_n^{\frac{\alpha}{2}}, \quad (2.4)$$

where the constant  $C(\varepsilon, p, m, \alpha, \nu)$  depends on  $\varepsilon, p, m, \alpha, \nu$  only, and

$$K = \prod_{j=1}^p \max_{1 \leq t \leq n} \|f_{j,t,n}\| \quad \text{with } \|f_{j,t,n}\|^2 = E[f_{j,t,n}^2(\mathbf{X})]. \quad (2.5)$$

**Proof.** Fix a collection  $(t_1, \dots, t_p)$  of disjoint indices  $t_i \neq t_j$  ( $i \neq j$ ), and write  $f_j = f_{j,t_j,n}$ ,  $1 \leq j \leq p$  for brevity. Let  $J_j(\mathbf{k}) := J_{f_j}(\mathbf{k}) = E[f_j(\mathbf{X}) H_{\mathbf{k}}(\mathbf{X})]$  be the coefficients of the Hermite expansion of  $f_j$ . Then,

$$\begin{aligned} |J_j(\mathbf{k})| &\leq \|f_j\| \prod_{i=1}^v E^{1/2} H_{k^{(i)}}^2(X) \\ &\leq \|f_j\| \prod_{i=1}^v (k^{(i)}!)^{1/2} = \|f_j\| (\mathbf{k}!)^{1/2}. \end{aligned}$$

Following Taqqu [28, p. 213, bottom, p. 214, top], we obtain

$$\begin{aligned} |Ef_1(\mathbf{X}_{t_1}) \cdots f_p(\mathbf{X}_{t_p})| &= \left| \sum_{q=0}^{\infty} \sum_{|\mathbf{k}_1|+\dots+|\mathbf{k}_p|=2q} \left\{ \prod_{j=1}^p \frac{J_j(\mathbf{k}_j)}{\mathbf{k}_j!} \right\} E[H_{\mathbf{k}_1}(\mathbf{X}_{t_1}) \cdots H_{\mathbf{k}_p}(\mathbf{X}_{t_p})] \right| \\ &\leq K_1 \sum_{q=0}^{\infty} \sum_{|\mathbf{k}_1|+\dots+|\mathbf{k}_p|=2q} \frac{|EH_{\mathbf{k}_1}(\mathbf{X}_{t_1}) \cdots H_{\mathbf{k}_p}(\mathbf{X}_{t_p})|}{(\mathbf{k}_1! \cdots \mathbf{k}_p!)^{1/2}} \\ &\leq K_1 \sum_{q=0}^{\infty} \sum_{|\mathbf{k}_1|+\dots+|\mathbf{k}_p|=2q} \frac{\varepsilon^{(|\mathbf{k}_1|+\dots+|\mathbf{k}_p|)/2} E \prod_{1 \leq u \leq \nu} \prod_{1 \leq j \leq p} H_{k_j^{(u)}}(X)}{(\mathbf{k}_1! \cdots \mathbf{k}_p!)^{1/2}} \\ &\leq K_1 \sum_{q=0}^{\infty} \sum_{|\mathbf{k}_1|+\dots+|\mathbf{k}_p|=2q} (\varepsilon(\nu p - 1))^{|k_1|+\dots+|k_p|/2} < \infty, \end{aligned}$$

where  $X \sim \mathcal{N}(0, 1)$  and

$$K_1 := \|f_{1,t_1,n}\| \cdots \|f_{p,t_p,n}\| \leq K,$$

where  $K$  is defined in (2.5) and  $K$  is independent of  $t_1, \dots, t_p$ , and where we used the assumption (2.3) to get the convergence of the last series. Therefore,

$$\sum' |E[f_{1,t_1,n}(\mathbf{X}_{t_1}) \cdots f_{p,t_p,n}(\mathbf{X}_{t_p})]| \leq K \sum_{q=0}^{\infty} \sum_{\substack{|\mathbf{k}_1|+\dots+|\mathbf{k}_p|=2q \\ |\mathbf{k}_1| \geq m, \dots, |\mathbf{k}_\alpha| \geq m}} \sum' \frac{|EH_{\mathbf{k}_1}(\mathbf{X}_{t_1}) \cdots H_{\mathbf{k}_p}(\mathbf{X}_{t_p})|}{(\mathbf{k}_1! \cdots \mathbf{k}_p!)^{1/2}}.$$

Now, the following bound remains to be proved: for any integers  $m \geq 1$ ,  $0 \leq \alpha \leq p$ ,  $n \geq 1$  and any multiindices  $\mathbf{k}_1, \dots, \mathbf{k}_p \in \mathbb{Z}_+^v$  satisfying  $|\mathbf{k}_1| + \dots + |\mathbf{k}_p| = 2q$ ,  $|\mathbf{k}_1| \geq m, \dots, |\mathbf{k}_\alpha| \geq m$ ,

$$\sum' |EH_{\mathbf{k}_1}(\mathbf{X}_{t_1}) \cdots H_{\mathbf{k}_p}(\mathbf{X}_{t_p})| \leq C_1 (\varepsilon(\nu p - 1))^{|k_1|+\dots+|k_p|/2} (\mathbf{k}_1! \cdots \mathbf{k}_p!)^{1/2} n^{p-\frac{\alpha}{2}} Q_n^{\frac{\alpha}{2}}, \quad (2.6)$$

where  $C_1$  is some constant depending only on  $p, \nu, \alpha, \varepsilon$ , and independent of  $\mathbf{k}_1, \dots, \mathbf{k}_p, n$ .

First, we write the expectation on the left hand side of (2.6) as a sum of contributions of diagrams. Let

$$T := \begin{pmatrix} (1, 1) & (1, 2) & \cdots & (1, k_1) \\ (2, 1) & (2, 2) & \cdots & (1, k_2) \\ \cdots & & & \\ (p, 1) & (p, 2) & \cdots & (p, k_p) \end{pmatrix} \quad (2.7)$$

be a table having  $p$  rows  $\tau_1, \dots, \tau_p$  of respective lengths  $|\tau_u| = k_u = |\mathbf{k}_u| = k_u^{(1)} + \dots + k_u^{(v)}$  (we write  $T = \bigcup_{u=1}^p \tau_u$ ). A sub-table of  $T$  is a table  $T' = \bigcup_{u \in U} \tau_u$ ,  $U \subset \{1, \dots, p\}$  consisting of some rows of  $T$  written from top to bottom in the same order as rows in  $T$ ; clearly any sub-table  $T'$  of  $T$  can be identified with a (nonempty) subset  $U \subset \{1, \dots, p\}$ . A diagram is a partition  $\gamma$  of the table  $T$  by pairs (called edges of the diagram) such that no pair belongs to the same row. A diagram  $\gamma$  is called connected if the table  $T$  cannot be written as a union  $T = T' \cup T''$  of two disjoint sub-tables  $T', T''$  so that  $T'$  and  $T''$  are partitioned by  $\gamma$  separately. Write  $\Gamma(T)$ ,  $\Gamma_c(T)$  for the class of all diagrams and the class of all connected diagrams over the table  $T$ , respectively. Let

$$\rho(t, s) := \max_{1 \leq u, v \leq p} |\text{EX}_t^{(u)} X_s^{(v)}| \quad (t \neq s).$$

Note  $0 \leq \rho(t, s) \leq \varepsilon$  and  $Q_n = \max_{1 \leq t \leq n} \sum_{1 \leq s \leq n, s \neq t} \rho^m(t, s)$ . By the diagram formula for moments of Hermite (Wick) polynomials (see e.g. [26]),

$$|\text{EH}_{\mathbf{k}_1}(\mathbf{X}_{t_1}) \cdots \text{EH}_{\mathbf{k}_p}(\mathbf{X}_{t_p})| \leq \sum_{\gamma \in \Gamma(T)} \prod_{1 \leq u < v \leq p} (\rho(t_u, t_v))^{\ell_{uv}} \quad (2.8)$$

$$= \sum_{(U_1, \dots, U_h)} \prod_{r=1}^h \sum_{\gamma \in \Gamma_c(U_r)} \prod_{u, v \in U_r, u < v} (\rho(t_u, t_v))^{\ell_{uv}}, \quad (2.9)$$

where  $\ell_{uv}$  is the number of edges between rows  $\tau_u$  and  $\tau_v$  in the diagram  $\gamma$  over table  $T$ , and the sum  $\sum_{(U_1, \dots, U_h)}$  is taken over all partitions  $(U_1, \dots, U_h)$ ,  $h = 1, 2, \dots, [p/2]$  of  $\{1, \dots, p\}$  by nonempty subsets  $U_r$  of cardinality  $|U_r| \geq 2$ . (Thus, (2.9) follows from (2.8) by decomposing  $\gamma \in \Gamma(T)$  into connected components  $\gamma_r \in \Gamma_c(U_r)$ ,  $r = 1, \dots, h$ ;  $h = 1, \dots, [p/2]$ ; the restriction  $|U_r| \geq 2$  stems from the fact that any edge must necessarily connect different rows.) From (2.9) we obtain

$$\sum' |\mathbb{E}[f_1(\mathbf{X}_{t_1}) \cdots f_p(\mathbf{X}_{t_p})]| \leq \sum_{(U_1, \dots, U_h)} \prod_{r=1}^h \sum_{\gamma \in \Gamma_c(U_r)} I_{n, U_r}(\gamma), \quad (2.10)$$

where, for any sub-table  $U \subset T$  having at least two rows and for any connected diagram  $\gamma \in \Gamma_c(U)$ , the quantity  $I_{n, U}(\gamma)$  is defined by

$$I_{n, U}(\gamma) := \sum' \prod_{u, v \in U, u < v} (\rho(t_u, t_v))^{\ell_{uv}}$$

where (recall) the product is taken over all ordered pairs of rows  $(\tau_u, \tau_v)$ ,  $u < v$  of the table  $U$ , and  $\ell_{uv}$  is the number of edges in  $\gamma$  between the  $u$ th and  $v$ th rows. Below we prove the bound

$$I_{n, U}(\gamma) \leq K_3 \varepsilon^{|\mathbf{k}_U|/2} n^{|U| - \frac{\alpha(U)}{2}} (n Q_n)^{\frac{\alpha(U)}{2}}, \quad (2.11)$$

where  $|\mathbf{k}_U| := \sum_{u \in U} k_u$  is the number of points of table  $U$  and  $\alpha(U) := |\{1, \dots, \alpha\} \cap U| = \#\{u \in U : |\mathbf{k}_u| \geq m\}$  is the number of rows in  $U$  having at least  $m$  points. Clearly, it suffices to show (2.11) for  $U = T$ .

Next, let for  $1 \leq u, v \leq p$ ,  $u \neq v$ , denote

$$R_{uv} := \left( \sum_{1 \leq t \leq n} \left( \sum_{1 \leq s \leq n, s \neq t} \rho^{k_u}(s, t) \right)^{k_v/k_u} \right)^{\ell_{uv}/k_v}. \quad (2.12)$$

Let  $A := \{1, \dots, \alpha\}$ ,  $A' := \{1, \dots, p\} \setminus A = \{\alpha + 1, \dots, p\}$ . It follows immediately from the definition of  $R_{uv}$  and  $\rho(s, t)$  that

$$R_{uv} \leq \begin{cases} n^{\frac{\ell_{uv}}{k_v}} Q_n^{\frac{\ell_{uv}}{k_u}} \varepsilon^{(1 - \frac{m}{k_u}) \ell_{uv}}, & \text{if } u \in A, \\ n^{\frac{\ell_{uv}}{k_u} + \frac{\ell_{uv}}{k_v}} \varepsilon^{\ell_{uv}}, & \text{if } u \in A^c. \end{cases} \quad (2.13)$$

By the Hölder inequality (see [13, p. 202], for details),

$$I_{n, T}(\gamma) \leq \min \left( \prod_{1 \leq u < v \leq p} R_{uv}, \prod_{1 \leq u < v \leq p} R_{vu} \right). \quad (2.14)$$

For any subset  $U \subset \{1, \dots, p\}$ , let

$$L(U) := \sum_{u \in U} \sum_{u < v \leq p} \frac{\ell_{uv}}{k_u}, \quad L^*(U) := \sum_{u \in U} \sum_{1 \leq v < u} \frac{\ell_{uv}}{k_u}, \quad (2.15)$$

$L := L(T)$ ,  $L^* := L^*(T)$ . Clearly,

$$L(U) + L^*(U) = \sum_{u \in U} \frac{1}{k_u} \sum_{v=1, \dots, p, v \neq u} \ell_{uv} = |U| \quad (2.16)$$

is the number of points in  $U$ . From (2.13)–(2.14),

$$I_{n,T}(\gamma) \leq \min \left( n^{L^* + L(A^c)} Q_n^{L(A)} \varepsilon^{|T|/2 - mL(A)}, n^{L + L^*(A^c)} Q_n^{L^*(A)} \varepsilon^{|T|/2 - mL^*(A)} \right),$$

where  $|T| = \sum_{u=1}^p k_i$ . As  $0 \leq L(A)$ ,  $L^*(A) \leq p$ , see (2.16), we obtain

$$\begin{aligned} I_{n,T}(\gamma) &\leq \varepsilon^{|T|/2 - mp} \min \left( n^{L^*(A) + L^*(A^c) + L(A^c)} Q_n^{L(A)}, n^{L(A) + L(A^c) + L^*(A^c)} Q_n^{L^*(A)} \right) \\ &= \varepsilon^{|T|/2 - mp} n^{p-\alpha} \min \left( n^{L^*(A)} Q_n^{L(A)}, n^{L(A)} Q_n^{L^*(A)} \right) \\ &= \varepsilon^{|T|/2 - mp} n^{p-\alpha} (nQ_n)^{\frac{\alpha}{2}} \min \left( (n/Q_n)^{\frac{\alpha}{2} - L(A)}, (n/Q_n)^{L(A) - \frac{\alpha}{2}} \right) \\ &\leq \varepsilon^{|T|/2 - mp} n^{p-\alpha} (nQ_n)^{\frac{\alpha}{2}}, \end{aligned}$$

proving (2.11).

With (2.11)–(2.10) in mind,

$$\begin{aligned} \sum_i |EH_{k_1}(\mathbf{X}_{t_1}) \cdots H_{k_p}(\mathbf{X}_{t_p})| &\leq C_3 \varepsilon^{|T|/2} \sum_{(U_1, \dots, U_h)} \prod_{r=1}^h \sum_{\gamma \in \Gamma_c(U_r)} n^{|U_r| - \frac{\alpha(U_r)}{2}} Q_n^{\frac{\alpha(U_r)}{2}} \\ &= C_3 \varepsilon^{|T|/2} n^{p-\frac{\alpha}{2}} Q_n^{\frac{\alpha}{2}} \sum_{(U_1, \dots, U_h)} \prod_{r=1}^h \sum_{\gamma \in \Gamma_c(U_r)} 1 \\ &= C_3 \varepsilon^{|T|/2} n^{p-\frac{\alpha}{2}} Q_n^{\frac{\alpha}{2}} \sum_{\gamma \in \Gamma(T)} 1, \end{aligned}$$

where the last sum (=the number of all diagrams over the table  $T$ ) does not exceed

$$|EH_{k_1}(X) \cdots H_{k_1}(X) \cdots H_{k_p}(X) \cdots H_{k_p}(X)| \leq (p\nu - 1)^{(|k_1| + \dots + |k_p|)/2} (k_1! \cdots k_p!)^{1/2};$$

see [28, Lemma 3.1]. This proves the bound (2.6) and the lemma, too.  $\square$

**Lemma 1** can be extended to non-standardized Gaussians as follows. To this end, we introduce some definitions. Let  $\mathbf{Y} = (Y^{(1)}, \dots, Y^{(v)}) \in \mathbb{R}^v$  be a Gaussian vector with zero mean and non-degenerate covariance matrix  $\Sigma = (EY^{(u)}Y^{(v)})_{1 \leq u, v \leq v}$ . Let  $\mathbb{L}^2(\mathbf{Y})$  denote the class of all measurable functions  $f: \mathbb{R}^v \rightarrow \mathbb{R}$  with  $Ef^2(\mathbf{Y}) < \infty$ . Let  $m \geq 0$  be an integer. We say that  $f \in \mathbb{L}^2(\mathbf{Y})$  has a generalized Hermite rank not less than  $m$  if either  $m = 0$ , or  $m \geq 1$  and

$$E[P(\mathbf{Y})f(\mathbf{Y})] = 0 \quad \text{for all } P \in \mathcal{P}_{m-1}(\mathbb{R}^v) \quad (2.17)$$

hold, where  $\mathcal{P}_m(\mathbb{R}^v)$  stands for the class of all polynomials  $P$  in variables  $y^{(1)}, \dots, y^{(v)}$  of degree  $m$ , that is,  $P(\mathbf{y}) = \sum_{0 \leq |j| \leq m} c(\mathbf{j}) \mathbf{y}^{\mathbf{j}} = \sum_{j^{(1)} \geq 0, \dots, j^{(v)} \geq 0, j^{(1)} + \dots + j^{(v)} \leq m} c(j^{(1)}, \dots, j^{(v)}) (y^{(1)})^{j^{(1)}} \cdots (y^{(v)})^{j^{(v)}}$ .

Let  $\mathbf{X} := \Sigma^{-1/2} \mathbf{Y}$ ,  $\tilde{f}(\mathbf{x}) := f(\Sigma^{1/2} \mathbf{x})$ . Then  $\mathbf{X}$  has a standard Gaussian distribution in  $\mathbb{R}^v$  and  $\tilde{f} \in \mathbb{L}^2(\mathbf{X})$  with

$$\|\tilde{f}\|^2 = E|\tilde{f}(\mathbf{X})|^2 = E|f(\mathbf{Y})|^2. \quad (2.18)$$

The following proposition is known, see [20, Proposition 2.1], [25, p. 195], but we include a proof of it for completeness.

**Proposition 1.** Let  $\mathbf{Y}, \mathbf{X}$ ,  $f \in \mathbb{L}^2(\mathbf{Y})$ ,  $\tilde{f} \in \mathbb{L}^2(\mathbf{X})$  be defined as above and  $m \geq 0$  be a given integer.  $f$  has a generalized Hermite rank not less than  $m$  if and only if  $\tilde{f}$  has a Hermite rank not less than  $m$ .

**Proof.** The above proposition is true if  $\mathbf{Y} = \mathbf{X}$  has a standard Gaussian distribution; see [25, p. 194]. By definition

$$E[P(\mathbf{Y})f(\mathbf{Y})] = E[\tilde{P}(\mathbf{X})\tilde{f}(\mathbf{X})], \quad (2.19)$$

where  $\tilde{P}(\mathbf{x}) := P(\Sigma^{1/2} \mathbf{x})$ . Clearly,  $P \in \mathcal{P}_{m-1}(\mathbb{R}^v)$  implies that  $\tilde{P} \in \mathcal{P}_{m-1}(\mathbb{R}^v)$  is a polynomial of degree  $m - 1$ . Therefore  $\tilde{f}$  having a Hermite rank not less than  $m$  implies by (2.19) that  $f$  has a generalized Hermite rank not less than  $m$ . The converse statement again follows from (2.19), by taking  $P(\mathbf{y}) = \hat{P}(\Sigma^{-1/2} \mathbf{y})$ , where  $\hat{P} \in \mathcal{P}_{m-1}(\mathbb{R}^v)$  is an arbitrary polynomial of degree  $m - 1$ .  $\square$

Let  $(\mathbf{Y}_1, \dots, \mathbf{Y}_n)$  be a collection of Gaussian vectors  $\mathbf{Y}_t = (Y_t^{(1)}, \dots, Y_t^{(v)}) \in \mathbb{R}^v$  with zero mean  $E\mathbf{Y}_t = 0$  and non-degenerated covariance matrices  $\Sigma_t = (\text{Cov}(Y_t^{(u)}, Y_t^{(v)}))_{1 \leq u, v \leq v}$ , having a joint Gaussian distribution in  $\mathbb{R}^{vn}$ . Let  $\varepsilon \in [0, 1]$  be a fixed number. Call  $(\mathbf{Y}_1, \dots, \mathbf{Y}_n)$   $\varepsilon$ -correlated if  $|\text{Cor}(Y_t^{(u)}, Y_s^{(v)})| \leq \varepsilon$  for any  $t \neq s$ ,  $1 \leq t, s \leq n$  and any  $1 \leq u, v \leq v$ . Clearly, if the  $\mathbf{Y}_t$ 's are standard, this is equivalent to  $(\mathbf{Y}_1, \dots, \mathbf{Y}_n)$  being  $\varepsilon$ -standard.

We also use some elementary facts about matrix norms. Let  $\|\mathbf{x}\| = (\sum_{i=1}^v (x^{(i)})^2)^{1/2}$  denote the Euclidean norm in  $\mathbb{R}^v$ ,  $A = (a_{ij})$  a real  $v \times v$ -matrix,  $A^t$  the transposed matrix,  $I$  the unit matrix, and  $\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$  the matrix spectral norm, respectively. Then  $\|A\|_\infty := \max_{1 \leq i, j \leq v} |a_{ij}| \leq \|A\| \leq v\|A\|_\infty$  and  $\|AB\| \leq \|A\| \|B\|$  for any such matrices  $A, B$ . An orthogonal matrix  $O = (o_{ij})$  satisfies  $OO^t = O^tO = I$  and  $\|O\| = \|O^t\| = 1$ . Any symmetric matrix  $A$  can be written as  $A = O^t \Lambda O$ , where  $O$  is an orthogonal matrix and  $\Lambda$  is a diagonal matrix. In addition, if  $A$  is positive definite, then  $\|A\| = \|\Lambda\| = \lambda_{\max}$ ,  $\|A^{-1}\| = \|\Lambda^{-1}\| = \lambda_{\min}^{-1}$ , where  $\lambda_{\max} \geq \lambda_{\min} > 0$  are the largest and smallest eigenvalues of  $A$ . We shall also use the facts that for any symmetric positive definite matrix  $A$ ,

$$\|A^{1/2}\| = \|A\|^{1/2}, \quad \|A^{-1/2}\| = \|A^{-1}\|^{1/2}, \quad (2.20)$$

since  $\|A^{1/2}\| = \|O^t \Lambda^{1/2} O\| = \|\Lambda^{1/2}\| = \lambda_{\max}^{1/2}$ ,  $\|A^{-1/2}\| = \|O^t \Lambda^{-1/2} O\| = \|\Lambda^{-1/2}\| = \lambda_{\min}^{-1/2}$ .

**Corollary 1.** Let  $(\mathbf{Y}_1, \dots, \mathbf{Y}_n)$  be an  $\varepsilon$ -correlated Gaussian vector,  $\mathbf{Y}_t = (Y_t^{(1)}, \dots, Y_t^{(v)}) \in \mathbb{R}^v$  ( $v \geq 1$ ), with zero mean  $E\mathbf{Y}_t = 0$  and non-degenerated covariance matrices  $\Sigma_t$  satisfying

$$\max_{1 \leq t \leq n} \|\Sigma_t^{-1}\| \leq c_{\max} \quad (2.21)$$

for some constant  $c_{\max} > 0$ . Let  $f_{j,t,n} \in \mathbb{L}^2(\mathbf{Y}_t)$ ,  $1 \leq j \leq p$  ( $p \geq 2$ ),  $1 \leq t \leq n$  be some functions. For given integers  $m \geq 1$ ,  $0 \leq \alpha \leq p$ ,  $n \geq 1$ , let  $Q_n$  denote the sum in (2.2) where  $X_t^{(u)}$ ,  $X_s^{(v)}$  are replaced by  $Y_t^{(u)}$ ,  $Y_s^{(v)}$ , respectively. Assume that the functions  $f_{1,t,n}, \dots, f_{\alpha,t,n}$  have a generalized Hermite rank at least equal to  $m$  for any  $n \geq 1$ ,  $1 \leq t \leq n$ , and that

$$\varepsilon < \frac{1}{(vp-1)v^2 c_{\max}}. \quad (2.22)$$

Then

$$\sum_{t=1}^n |E[f_{1,t,n}(\mathbf{Y}_{t_1}) \cdots f_{p,t,n}(\mathbf{Y}_{t_p})]| \leq C K n^{p-\frac{\alpha}{2}} Q_n^{\frac{\alpha}{2}},$$

where  $K := \prod_{j=1}^p \max_{1 \leq t \leq n} E[f_{j,t,n}^2(\mathbf{Y}_t)]$  and the constant  $C = C(\varepsilon, p, m, \alpha, v, c_{\max})$  depends on  $\varepsilon, p, m, \alpha, v, c_{\max}$  only.

**Proof.** We will reduce the above inequality to that of Lemma 1, as follows. Let  $\mathbf{X}_t := \Sigma_t^{-1/2} \mathbf{Y}_t$ ,  $\tilde{f}_{j,t,n}(\mathbf{x}) := f_{j,t,n}(\Sigma_t^{1/2} \mathbf{x})$ . The  $\mathbf{X}_t$ 's have a standard Gaussian distribution in  $\mathbb{R}^v$  and the  $\tilde{f}_{j,t,n}$ 's satisfy  $\|\tilde{f}_{j,t,n}\|^2 = E[f_{j,t,n}^2(\mathbf{Y}_t)]$ ; see (2.18). By Proposition 1,  $\tilde{f}_{j,t,n}$ ,  $j = 1, \dots, \alpha$  have a Hermite rank not less than  $m$ . Next, using (2.20), (2.21) and the fact that the  $\mathbf{Y}_t$ 's are  $\varepsilon$ -correlated, for any  $t \neq s$ ,  $1 \leq t, s \leq n$ ,  $1 \leq u, v \leq v$

$$|E X_t^{(u)} X_s^{(v)}| \leq v^2 \|\Sigma_t^{-1/2}\|_\infty \|\Sigma_s^{-1/2}\|_\infty \max_{1 \leq u, v \leq v} |E Y_t^{(u)} Y_s^{(v)}| \leq \varepsilon v^2 \|\Sigma_t^{-1/2}\| \|\Sigma_s^{-1/2}\| \leq \varepsilon v^2 c_{\max}. \quad (2.23)$$

This implies that the Gaussian vector  $(\mathbf{X}_1, \dots, \mathbf{X}_n) \in \mathbb{R}^{vn}$  is  $\tilde{\varepsilon}$ -standard, where  $\tilde{\varepsilon} := \varepsilon v^2 c_{\max}$ . Then, in view of (2.22), (2.4) of Lemma 1 applies, according to which

$$\begin{aligned} \sum_{t=1}^n |E[f_{1,t,n}(\mathbf{Y}_{t_1}) \cdots f_{p,t,n}(\mathbf{Y}_{t_p})]| &= \sum_{t=1}^n |E[\tilde{f}_{1,t,n}(\mathbf{X}_{t_1}) \cdots \tilde{f}_{p,t,n}(\mathbf{X}_{t_p})]| \\ &\leq C(\tilde{\varepsilon}, p, m, \alpha, v) \tilde{K} n^{p-\frac{\alpha}{2}} \tilde{Q}_n^{\frac{\alpha}{2}} \leq C(\varepsilon, p, m, \alpha, v, c_{\max}) K n^{p-\frac{\alpha}{2}} Q_n^{\frac{\alpha}{2}}, \end{aligned}$$

where  $\tilde{K}, \tilde{Q}_n$  are the corresponding quantities in Lemma 1 (2.4) satisfying  $\tilde{K} = K$ ,  $\tilde{Q}_n \leq (\varepsilon v^2 c_{\max})^m Q_n$  by (2.18), (2.23), respectively.  $\square$

We remark that condition (2.22) is not optimal since it does not reduce to (2.3) in the  $\varepsilon$ -standard case. This loss of optimality is due to the use of robust inequalities for matrix norms in (2.23).

**Remark 1.** As mentioned in the Introduction, Lemma 1 and Corollary 1 can be used for proving the tightness and the strong law of large numbers of various non-linear statistics from Gaussian observations. See [3,4] on application for roughness estimation and Csörgő and Mielnichuk [8], Koul and Surgailis [18] for the empirical process. The above-mentioned applications concern the fourth moment bound  $M_n := E(\sum_{t=1}^n f_{t,n}(\mathbf{Y}_n(t)))^4 = O(n^{-\kappa})$  for a suitable  $\kappa > 0$ , where  $(\mathbf{Y}_n(t))$ ,  $(f_{t,n})$  satisfy similar conditions as in Corollary 1. Clearly,  $M_n = \sum_{t_1, \dots, t_4=1}^n E[\prod_{i=1}^4 f_{t_i,n}(\mathbf{Y}_n(t_i))]$  can be decomposed into four terms according to the number of coinciding “diagonals”  $t_i = t_j$  in the last sum, where each term can be estimated with the help of Corollary 1. Let us note that condition (2.22) in the above applications is guaranteed by a preliminary “decimation” of the sum  $\sum_{t=1}^n f_{t,n}(\mathbf{Y}_n(t))$ ; see [8,4] for details.

### 3. A CLT for a triangular array of functions of Gaussian vectors

Let  $(\mathbf{X}_n(k))_{1 \leq k \leq n, n \in \mathbb{N}}$  be a triangular array of standardized Gaussian vectors with values in  $\mathbb{R}^\nu$ ,  $\mathbf{X}_n(k) = (X_n^{(1)}(k), \dots, X_n^{(\nu)}(k))$ ,  $\mathbb{E}X_n^{(p)}(k) = 0$ ,  $\mathbb{E}X_n^{(p)}(k)X_n^{(q)}(k) = \delta_{pq}$ . Now define,

$$r_n^{(p,q)}(j, k) := \mathbb{E}X_n^{(p)}(j)X_n^{(q)}(k) \quad (1 \leq j, k \leq n).$$

For a given integer  $m \geq 1$ , introduce the following assumptions: for any  $1 \leq p, q \leq \nu$ ,

$$\sup_{n \geq 1} \max_{1 \leq k \leq n} \sum_{1 \leq j \leq n} |r_n^{(p,q)}(j, k)|^m < \infty, \quad (3.1)$$

$$\sup_{n \geq 1} \frac{1}{n} \sum_{\substack{1 \leq j, k \leq n \\ |j-k| > K}} |r_n^{(p,q)}(j, k)|^m \xrightarrow{K \rightarrow \infty} 0, \quad (3.2)$$

$$\forall (j, k) \in \{1, \dots, n\}^2, \quad |r_n^{(p,q)}(j, k)| \leq |\rho(j-k)| \quad \text{with} \quad \sum_{j \in \mathbb{Z}} |\rho(j)|^m < \infty. \quad (3.3)$$

Note (3.3)  $\Rightarrow$  (3.1) and (3.3)  $\Rightarrow$  (3.2). Let  $\mathbb{L}_0^2(\mathbf{X}) := \{f \in \mathbb{L}^2(\mathbf{X}) : \mathbb{E}f(\mathbf{X}) = 0\}$ , where  $\mathbf{X} \in \mathbb{R}^\nu$  denotes a standard Gaussian vector as above.

**Theorem 1.** Let  $(\mathbf{X}_n(k))_{1 \leq k \leq n, n \in \mathbb{N}}$  be a triangular array of standardized Gaussian vectors.

(i) Assume (3.1). Let  $f_k \in \mathbb{L}_0^2(\mathbf{X})$  ( $1 \leq k \leq n$ ) have a Hermite rank at least  $m \in \mathbb{N}^*$ . Then there exists a constant  $C$  independent of  $n$  and  $f_k$ ,  $1 \leq k \leq n$  such that

$$\mathbb{E} \left( n^{-1/2} \sum_{k=1}^n f_k(\mathbf{X}_n(k)) \right)^2 \leq C \max_{1 \leq k \leq n} \|f_k\|^2. \quad (3.4)$$

(ii) Assume (3.1) and (3.2). Let  $f_{k,n} \in \mathbb{L}_0^2(\mathbf{X})$  ( $n \geq 1$ ,  $1 \leq k \leq n$ ) be a triangular array of functions all having Hermite rank at least  $m \in \mathbb{N}^*$ . Assume that there exists a  $\mathbb{L}_0^2(\mathbf{X})$ -valued continuous function  $\phi_\tau$ ,  $\tau \in [0, 1]$ , such that

$$\sup_{\tau \in (0,1)} \|f_{[\tau n],n} - \phi_\tau\|^2 = \sup_{\tau \in (0,1)} \mathbb{E}(f_{[\tau n],n}(\mathbf{X}) - \phi_\tau(\mathbf{X}))^2 \xrightarrow{n \rightarrow \infty} 0. \quad (3.5)$$

Moreover, let

$$\sigma_n^2 := \mathbb{E} \left( n^{-1/2} \sum_{k=1}^n f_{k,n}(\mathbf{X}_n(k)) \right)^2 \xrightarrow{n \rightarrow \infty} \sigma^2, \quad (3.6)$$

where  $\sigma^2 > 0$ . Then

$$n^{-1/2} \sum_{k=1}^n f_{k,n}(\mathbf{X}_n(k)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2). \quad (3.7)$$

(iii) Assume (3.3). Moreover, assume that for any  $\tau \in [0, 1]$  and any  $J \in \mathbb{N}^*$ ,

$$(\mathbf{X}_n([n\tau] + j))_{-J \leq j \leq J} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} (\mathbf{W}_\tau(j))_{-J \leq j \leq J}, \quad (3.8)$$

where  $(\mathbf{W}_\tau(j))_{j \in \mathbb{Z}}$  is a stationary Gaussian process taking values in  $\mathbb{R}^\nu$  and depending on parameter  $\tau \in (0, 1)$ . Let  $f_{k,n} \in \mathbb{L}_0^2(\mathbf{X})$  ( $n \geq 1$ ,  $1 \leq k \leq n$ ) satisfy the same conditions as in part (ii), with exception of (3.6). Then (3.6) and (3.7) hold, with

$$\sigma^2 = \int_0^1 d\tau \left( \sum_{j \in \mathbb{Z}} \mathbb{E}[\phi_\tau(\mathbf{W}_\tau(0)) \phi_\tau(\mathbf{W}_\tau(j))] \right). \quad (3.9)$$

We remark that parts (i) and (ii) of Theorem 1 are natural extensions of Theorem 2 of [1] (for instance, condition (3.1) is the same as condition (2.40) of [1] in the case of stationary sequences). We expect that parts (i) and (ii) can be also obtained following the method in [20]. Part (iii) seems more interesting. In [3], (iii) is applied when  $\mathbf{X}_n(j) = \mathbf{Z}_{j/n}$  and  $(\mathbf{Z}_t)_t$  is a vector valued continuous time process.

Similarly to Lemma 1, Theorem 1 can be extended to nonstandardized Gaussian vectors. Corollary 2 refers to the most interesting part (iii) of Theorem 1.



**Corollary 2.** Let  $\mathbf{Y}_n(k) = (Y_n^{(1)}(k), \dots, Y_n^{(v)}(k)) \in \mathbb{R}^v$ ,  $1 \leq k \leq n$ ,  $n \in \mathbb{N}$  be a triangular array of jointly Gaussian vectors, with zero mean  $E\mathbf{Y}_n(k) = 0$  and non-degenerate covariance matrices  $\Sigma_{k,n} = E\mathbf{Y}_n(k)\mathbf{Y}_n(k)^\top$ . Assume that covariances  $r_n^{(p,q)}(j, k) := \text{Cov}(Y_n^{(p)}(j), Y_n^{(q)}(k))$  satisfy (3.3), for some  $m \geq 1$ . Moreover, assume that (3.8) holds with  $\mathbf{X}_n(\cdot)$  replaced by  $\mathbf{Y}_n(\cdot)$ , where  $(\mathbf{W}_\tau(j))_{j \in \mathbb{Z}}$  is a stationary Gaussian  $\mathbb{R}^v$ -valued process with non-degenerate covariance matrix  $\Sigma_\tau := E\mathbf{W}_\tau(0)\mathbf{W}_\tau(0)^\top$  such that

$$\sup_{\tau \in (0,1]} \|\Sigma_\tau^{-1}\| < \infty \quad (3.10)$$

and

$$\sup_{\tau \in (0,1]} \|\Sigma_{[n\tau],n} - \Sigma_\tau\| \xrightarrow{n \rightarrow \infty} 0. \quad (3.11)$$

Let  $f_{k,n} \in \mathbb{L}_0^2(\mathbf{Y}_n(k))$ ,  $1 \leq k \leq n$ ,  $n \in \mathbb{N}$  be a triangular array of functions all having a generalized Hermite rank not less than  $m$  and such that

$$\sup_{\tau \in (0,1]} E \left( \tilde{f}_{[n\tau],n}(\mathbf{X}) - \tilde{\phi}_\tau(\mathbf{X}) \right)^2 \xrightarrow{n \rightarrow \infty} 0, \quad (3.12)$$

where  $\tilde{f}_{k,n}(\mathbf{x}) := f_{k,n}(\Sigma_{k,n}^{1/2}\mathbf{x})$  and where  $\tilde{\phi}_\tau$ ,  $\tau \in [0, 1]$  is a  $\mathbb{L}_0^2(\mathbf{X})$ -valued continuous function, with  $\mathbf{X}$  a standard Gaussian vector in  $\mathbb{R}^v$  as usual. Then

$$n^{-1/2} \sum_{k=1}^n f_{k,n}(\mathbf{Y}_n(k)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2) \quad (3.13)$$

where  $\sigma^2$  is defined in (3.9), with  $\phi_\tau(\mathbf{x}) := \tilde{\phi}_\tau(\Sigma_\tau^{-1/2}\mathbf{x})$ .

**Proof of Corollary 2.** Similarly as in the proof of Corollary 1, let  $\mathbf{X}_n(k) := \Sigma_{k,n}^{-1/2}\mathbf{Y}_n(k)$ . The  $\mathbf{X}_n(k)$ 's are standardized Gaussian vectors in  $\mathbb{R}^v$  and the  $\tilde{f}_{k,n}$ 's belong to  $\mathbb{L}^2(\mathbf{X})$  and have a Hermite rank not less than  $m$ . Assumptions (3.8) and (3.11) entail for any  $\tau \in (0, 1)$ ,  $(\mathbf{X}_n(j + [n\tau]))_{-J \leq j \leq J} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} (\tilde{\mathbf{W}}_\tau(j))_{-J \leq j \leq J}$ , where  $\tilde{\mathbf{W}}_\tau(j) := \Sigma_\tau^{-1/2}\mathbf{W}_\tau(j)$ ,  $j \in \mathbb{Z}$  is a stationary Gaussian process having a unit covariance matrix  $E\tilde{\mathbf{W}}_\tau(0)\tilde{\mathbf{W}}_\tau(0)^\top = I$ . Conditions (3.10) and (3.11) imply that  $\max_{1 \leq k \leq n} \|\Sigma_{k,n}^{-1/2}\| \leq C$ . The last fact together with condition (3.3) for covariances  $r_n^{(p,q)}(j, k) := \text{Cov}(Y_n^{(p)}(j), Y_n^{(q)}(k))$  implies a similar condition for  $\text{Cov}(\mathbf{X}_n^{(p)}(j), \mathbf{X}_n^{(q)}(k))$ : for all  $(j, k) \in \{1, \dots, n\}^2$  we have that  $\max_{1 \leq u, v \leq v} |E\mathbf{X}_n^{(u)}(j)\mathbf{X}_n^{(v)}(k)| \leq C|\rho(j-k)|$ ; see (2.23). This way we see that the conditions of Theorem 1(iii) including (3.5) are satisfied and can be applied to the families of Gaussian vectors  $(\mathbf{X}_n(k))$  and functions  $(\tilde{f}_{k,n})$ , yielding (3.13).  $\square$

**Proof of Theorem 1.** (i) Using Arcones' inequality (see [1, (2.44)] or [25, (2.4)]), one obtains

$$\begin{aligned} E \left( n^{-1/2} \sum_{k=1}^n f_k(\mathbf{X}_n(k)) \right)^2 &= \frac{1}{n} \sum_{k=1}^n \|f_k\|^2 + \frac{1}{n} \sum_{k=1}^n E f_k(\mathbf{X}_n(k)) f_\ell(\mathbf{X}_n(\ell)) \\ &\leq \max_{1 \leq k \leq n} \|f_k\|^2 + C \left( \max_{1 \leq k \leq n} \|f_k\| \right)^2 \max_{1 \leq k \leq n} \sum_{1 \leq \ell \leq n, \ell \neq k} \max_{1 \leq p, q \leq v} |r_n^{(p,q)}(k, \ell)|^m, \end{aligned}$$

where  $C$  is a positive real number not depending on  $n$  or  $f_k$ . Now, using assumption (3.1), (i) is proved.

(ii) We use the following well-known fact. Let  $(Z_n)_{n \geq 1}$  be a sequence of r.v.'s with zero mean and finite variance. Then  $Z_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2)$  if and only if for any  $\epsilon > 0$  one can find an integer  $n_0(\epsilon) \geq 1$  and a sequence  $(Z_{n,\epsilon})_{n \geq 1}$  satisfying  $Z_{n,\epsilon} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \sigma_\epsilon^2)$  and  $\forall n > n_0(\epsilon)$ ,  $E(Z_n - Z_{n,\epsilon})^2 < \epsilon$ .

Let  $Z_n := n^{-1/2} \sum_{k=1}^n f_{k,n}(\mathbf{X}_n(k))$ . We shall construct an approximating sequence  $Z_{n,\epsilon}$  with the above properties in two steps.

First, by condition (3.5) and continuity of  $\phi_\tau$ , for a given  $\epsilon > 0$  one can find integers  $M$ ,  $n_0(\epsilon)$  and a partition  $0 =: \tau_0 < \tau_1 < \dots < \tau_M < \tau_{M+1} := 1$  such that  $\forall n > n_0(\epsilon)$ ,

$$\max_{0 \leq i \leq M} \max_{k/n \in (\tau_i, \tau_{i+1}]} \|f_{k,n} - \phi_{\tau_i}\| = \max_{0 \leq i \leq M} \max_{k/n \in (\tau_i, \tau_{i+1}]} (E(f_{k,n}(\mathbf{X}) - \phi_{\tau_i}(\mathbf{X}))^2)^{1/2} < \epsilon. \quad (3.14)$$

Put

$$\tilde{Z}_{n,\epsilon} := n^{-1/2} \sum_{i=0}^M \sum_{k/n \in (\tau_i, \tau_{i+1}]} \phi_{\tau_i}(\mathbf{X}_n(k)).$$



Note for any  $\tau \in (0, 1]$ , the function  $\psi_\tau$  has Hermite rank not less than  $m$ , being the limit of a sequence of  $\mathbb{L}_0^2(\mathbf{X})$ -valued functions of Hermite rank  $\geq m$ . Therefore for the difference  $Z_n - \tilde{Z}_{n,\epsilon}$  the inequality (3.4) applies, yielding  $\forall n > n_0(\epsilon)$

$$\mathbb{E}(Z_n - \tilde{Z}_{n,\epsilon})^2 \leq C \max_{0 \leq i \leq M} \max_{k/n \in (\tau_i, \tau_{i+1}]} \|f_{k,n} - \phi_{\tau_i}\|^2 \leq C\epsilon^2 \quad (3.15)$$

in view of (3.14), with a constant  $C$  independent of  $n, \epsilon$ .

Second, we expand each  $\phi_{\tau_i}$  in Hermite polynomials:

$$\phi_{\tau_i}(\mathbf{x}) = \sum_{m \leq |\mathbf{k}|} \frac{J_i(\mathbf{k})}{\mathbf{k}!} H_{\mathbf{k}}(\mathbf{x}), \quad (i = 0, 1, \dots, M) \quad (3.16)$$

where

$$J_i(\mathbf{k}) := J_{\phi_{\tau_i}}(\mathbf{k}) = \mathbb{E}\phi_{\tau_i}(\mathbf{X})H_{\mathbf{k}}(\mathbf{X}), \quad |J_i(\mathbf{k})| \leq \|\phi_{\tau_i}\|(\mathbf{k}!)^{1/2}.$$

We can choose  $t(\epsilon) \in \mathbb{N}$  large enough so that

$$\|\phi_{\tau_i} - \phi_{\tau_i,\epsilon}\| \leq \epsilon, \quad (i = 0, 1, \dots, M), \quad (3.17)$$

where  $\phi_{\tau_i,\epsilon}$  is a finite sum of Hermite polynomials:

$$\phi_{\tau_i,\epsilon}(\mathbf{x}) := \sum_{m \leq |\mathbf{k}| \leq t(\epsilon)} \frac{J_i(\mathbf{k})}{\mathbf{k}!} H_{\mathbf{k}}(\mathbf{x}), \quad (i = 0, 1, \dots, M). \quad (3.18)$$

Note  $t(\epsilon)$  does not depend on  $i = 0, 1, \dots, M$ , and  $\epsilon > 0$  is the same as in (3.14). Put

$$Z_{n,\epsilon} := n^{-1/2} \sum_{i=0}^M \sum_{k/n \in (\tau_i, \tau_{i+1}]} \phi_{\tau_i,\epsilon}(\mathbf{X}_n(k)). \quad (3.19)$$

Applying (3.4) to the difference  $\tilde{Z}_{n,\epsilon} - Z_{n,\epsilon}$  and using (3.17) and (3.15), we obtain  $\forall n > n_0(\epsilon)$ ,

$$\mathbb{E}(Z_n - Z_{n,\epsilon})^2 \leq C\epsilon^2 \quad (3.20)$$

where the constant  $C$  is independent of  $n, \epsilon$ . Let  $\sigma_{n,\epsilon}^2 := \mathbb{E}Z_{n,\epsilon}^2$ . From (3.20) and condition (3.6) it follows that  $\forall n > n_0(\epsilon)$ ,

$$\sigma^2 - C\epsilon \leq \sigma_{n,\epsilon}^2 \leq \sigma^2 + C\epsilon, \quad (3.21)$$

with some  $C$  independent of  $n, \epsilon$ . In particular, by choosing  $\epsilon > 0$  small enough, it follows that  $\liminf_{n \rightarrow \infty} \sigma_{n,\epsilon}^2 > 0$ . We shall prove below that for any fixed  $\epsilon > 0$ ,

$$U_n := \frac{Z_{n,\epsilon}}{\sigma_{n,\epsilon}} = \frac{1}{\sigma_{n,\epsilon} n^{1/2}} \sum_{i=1}^M \sum_{k/n \in (\tau_i, \tau_{i+1}]} \phi_{\tau_i,\epsilon}(\mathbf{X}_n(k)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, 1). \quad (3.22)$$

As noted in the beginning of the proof of the theorem, the CLT in (3.7) follows from (3.22), (3.20), (3.21). Indeed, write

$$\mathbb{E}e^{iaZ_n} - e^{-a^2\sigma^2/2} = (\mathbb{E}e^{iaZ_n} - \mathbb{E}e^{iaZ_{n,\epsilon}}) + (\mathbb{E}e^{ia\sigma_{n,\epsilon}U_n} - e^{-a^2\sigma_{n,\epsilon}^2/2}) + (e^{-a^2\sigma_{n,\epsilon}^2/2} - e^{-a^2\sigma^2/2}) := \sum_{i=1}^3 \ell_i(n).$$

Here, for some constant  $C$  independent of  $n, a, \epsilon$ ,

$$|\ell_1(n)| \leq \mathbb{E}^{1/2} |e^{ia(Z_n - Z_{n,\epsilon})} - 1|^2 \leq |a| \mathbb{E}^{1/2} |Z_n - Z_{n,\epsilon}|^2 \leq C|a|\epsilon,$$

$$|\ell_3(n)| \leq Ca^2 |\sigma_{n,\epsilon}^2 - \sigma^2| \leq Ca^2\epsilon,$$

and therefore  $\ell_i(n)$ ,  $i = 1, 3$  can be made arbitrarily small by choosing  $\epsilon > 0$  small enough; see (3.20), (3.21). On the other hand, the convergence in (3.22) implies uniform convergence of characteristic functions on compact intervals and therefore  $\sup_{|a| \leq A} |\ell_2(n)| \leq \sup_{|a| \leq 2A} |\mathbb{E}e^{iaU_n} - e^{-a^2/2}| \xrightarrow[n \rightarrow \infty]{} 0$  for any  $A > 0$ . This proves (3.7).

It remains to prove (3.22). The proof of the corresponding CLTs for sums of Hermite polynomials in [1,5] refers to stationary processes and uses Fourier methods. Therefore we present an independent proof of (3.22) based on cumulants and the Hölder inequality in (2.14). Again, our proof appears to be much simpler than computations in the above mentioned papers.

Accordingly, it suffices to show that cumulants of order  $p \geq 3$  of  $U_n$  asymptotically vanish. In view of (3.21) and linearity of cumulants, this follows from the fact that for any  $p \geq 3$  and any multiindices  $\mathbf{k}_u = (k_u^{(1)}, \dots, k_u^{(v)}) \in \mathbb{Z}_+^v$ ,  $u = 1, \dots, p$  with  $k_u = |\mathbf{k}_u| = k_u^{(1)} + \dots + k_u^{(v)} \geq m$  ( $1 \leq u \leq p$ ),

$$\Sigma_n := \sum_{t_1, \dots, t_p=1}^n |\text{cum}(t_1, \dots, t_p)| = o(n^{p/2}), \quad (3.23)$$

where  $\text{cum}(t_1, \dots, t_p)$  stands for the joint cumulant:

$$\text{cum}(t_1, \dots, t_p) := \text{cum}(H_{\mathbf{k}_1}(\mathbf{X}_n(t_1)), \dots, H_{\mathbf{k}_p}(\mathbf{X}_n(t_p))). \quad (3.24)$$

Split  $\Sigma_n = \Sigma'_n(K) + \Sigma''_n(K)$ , where

$$\Sigma'_n(K) := \sum_{t_1, \dots, t_p=1}^n |\text{cum}(t_1, \dots, t_p)| \mathbf{1}(|t_i - t_j| \leq K \forall i \neq j)$$

and where  $K$  will be chosen large enough. Then for any fixed  $K$ , we have  $\Sigma'_n(K) = O(n) = o(n^{p/2})$  as  $p \geq 3$ . The remaining sum  $\Sigma''_n(K)$  does not exceed  $\sum_{1 \leq i \neq j \leq p} \Sigma''_{n,i,j}(K)$ , where

$$\Sigma''_{n,i,j}(K) := \sum_{t_1, \dots, t_p=1}^n |\text{cum}(t_1, \dots, t_p)| \mathbf{1}(|t_i - t_j| > K).$$

Therefore, relation (3.23) follows if we show that there exist  $\delta(K) \xrightarrow{K \rightarrow \infty} 0$  and  $\tilde{n}_0$  such that for any  $1 \leq i \neq j \leq p$  and any  $n > \tilde{n}_0$

$$\limsup_{n \rightarrow \infty} \Sigma''_{n,i,j}(K) < \delta(K)n^{p/2}. \quad (3.25)$$

The proof below is limited to  $(i, j) = (1, 2)$  as the general case is analogous. It is well-known that the joint cumulant in (3.24), similarly to the joint moment in (2.6), can be expressed as a sum over all *connected* diagrams  $\gamma \in \Gamma_c(T)$  over the table  $T$  in (2.7). By introducing  $\bar{\rho}(s, t) := \max_{1 \leq p, q \leq v} |r_n^{(p,q)}(s, t)|$ , we obtain

$$|\text{cum}(t_1, \dots, t_p)| \leq \sum_{\gamma \in \Gamma_c(T)} \prod_{1 \leq u < v \leq p} (\bar{\rho}(t_u, t_v))^{\ell_{uv}}, \quad (3.26)$$

where we use the notation in (2.6). Therefore,

$$\Sigma''_{n,1,2}(K) \leq \sum_{\gamma \in \Gamma_c(T)} \sum_{t_1, \dots, t_p=1}^n \prod_{1 \leq u < v \leq p} (\bar{\rho}(t_u, t_v))^{\ell_{uv}} \mathbf{1}(|t_1 - t_2| > K) := \sum_{\gamma \in \Gamma_c(T)} \bar{I}_{n,T}(\gamma).$$

Next, by applying the Hölder inequality as in (2.14),

$$\bar{I}_{n,T}(\gamma) \leq \min \left( \prod_{1 \leq u < v \leq p} \bar{R}_{uv}, \prod_{1 \leq u < v \leq p} \bar{R}_{vu} \right) \quad (3.27)$$

where (cf. (2.12))

$$\bar{R}_{uv} := \begin{cases} \left( \sum_{1 \leq t \leq n} \left( \sum_{1 \leq s \leq n} \bar{\rho}^{k_u}(s, t) \right)^{k_v/k_u} \right)^{\ell_{uv}/k_v}, & (u, v) \neq (1, 2), (2, 1), \\ \left( \sum_{1 \leq t \leq n} \left( \sum_{1 \leq s \leq n} \bar{\rho}^{k_1}(s, t) \mathbf{1}(|t - s| > K) \right)^{k_2/k_1} \right)^{\ell_{12}/k_2}, & (u, v) = (1, 2), \\ \left( \sum_{1 \leq t \leq n} \left( \sum_{1 \leq s \leq n} \bar{\rho}^{k_2}(t, s) \mathbf{1}(|t - s| > K) \right)^{k_1/k_2} \right)^{\ell_{12}/k_1}, & (u, v) = (2, 1). \end{cases}$$

From assumptions (3.1), (3.2), there exists a constant  $C$  and  $\delta(K) \xrightarrow{K \rightarrow \infty} 0$  independent of  $n$  such that for any  $k \geq m$  and any  $n \geq 1$

$$\sup_{1 \leq t \leq n} \sum_{s=1}^n \bar{\rho}^k(s, t) \leq Cn,$$

$$\sup_{1 \leq t \leq n} \sum_{s=1}^n \bar{\rho}^k(s, t) \mathbf{1}(|t - s| > K) \leq \delta(K)n.$$

Therefore

$$\bar{R}_{uv} \leq \begin{cases} Cn^{\ell_{uv}/k_v}, & (u, v) \neq (1, 2), (2, 1), \\ \delta(K)n^{\ell_{12}/k_2}, & (u, v) = (1, 2), \\ \delta(K)n^{\ell_{12}/k_1}, & (u, v) = (2, 1), \end{cases}$$

with some  $\tilde{\delta}(K) \xrightarrow{K \rightarrow \infty} 0$  independent of  $n$ . Consequently, the minimum on the right-hand side of (3.27) does not exceed

$$C\tilde{\delta}(K) \min \left( n^{\sum_{1 \leq u < v \leq p} \ell_{uv}/k_v}, n^{\sum_{1 \leq u < v \leq p} \ell_{uv}/k_u} \right) = C\tilde{\delta}(K)n^{\min(L(T), L^*(T))}$$

where the quantities  $L(T)$ ,  $L^*(T)$  introduced in (2.15) satisfy  $L(T) + L^*(T) = p$ , see (2.16), and therefore  $\min(L(T), L^*(T)) \leq p/2$ . This proves (3.25) and the CLT in (3.22), thereby completing the proof of part (ii).

(iii) Let us first prove (3.6) with  $\sigma^2$  given in (3.9) in the case when  $f_{k,n} \equiv f$  do not depend on  $k, n$  (in such case, one has  $\phi_\tau \equiv f$ , too). We have

$$\sigma_n^2 = n^{-1} \sum_{k, k'=1}^n \mathbb{E}[f(\mathbf{X}_n(k))f(\mathbf{X}_n(k'))] = \int_0^1 F_n(\tau) d\tau, \quad (3.28)$$

where

$$F_n(\tau) := \sum_{j=1-\lfloor n\tau \rfloor}^{n-\lfloor n\tau \rfloor} \mathbb{E}[f(\mathbf{X}_n(\lfloor n\tau \rfloor))f(\mathbf{X}_n(\lfloor n\tau \rfloor + j))]. \quad (3.29)$$

Condition (3.8) implies that

$$\mathbb{E}[f(\mathbf{X}_n(\lfloor n\tau \rfloor))f(\mathbf{X}_n(\lfloor n\tau \rfloor + j))] \rightarrow \mathbb{E}[f(\mathbf{W}_\tau(0))f(\mathbf{W}_\tau(j))]$$

for each  $j \in \mathbb{Z}$  as  $n \rightarrow \infty$ . From (3.3) and with the inequality of previous part (i), there exists  $C > 0$  such that

$$|\mathbb{E}[f(\mathbf{X}_n(\lfloor n\tau \rfloor))f(\mathbf{X}_n(\lfloor n\tau \rfloor + j))]| \leq C|\rho(j)|^m, \quad (3.30)$$

and  $\sum_{j \in \mathbb{Z}} |\rho(j)|^m < \infty$ . Hence, from the Lebesgue theorem,

$$F_n(\tau) = \sum_{j \in \mathbb{Z}} \mathbf{1}_{j \in \{1-\lfloor n\tau \rfloor, \dots, n-\lfloor n\tau \rfloor\}} \mathbb{E}[f(\mathbf{X}_n(\lfloor n\tau \rfloor))f(\mathbf{X}_n(\lfloor n\tau \rfloor + j))] \xrightarrow{n \rightarrow \infty} \sum_{j \in \mathbb{Z}} \mathbb{E}[f(\mathbf{W}_\tau(0))f(\mathbf{W}_\tau(j))].$$

The dominated convergence theorem allows one to pass to the limit under the integral, thereby proving (3.6) with  $\sigma^2$  given in (3.9) in the case  $f_{k,n} \equiv f$ .

To end the proof, consider the general case of  $f_{k,n}$  as in (iii). Let  $Z_{n,\epsilon}$  be defined as in (3.19). Note relation (3.20) holds as its proof does not use (3.6). In part (ii), we used (3.6) to prove (3.21). Now we want to prove (3.21) using (3.8) instead of (3.6). This will suffice for the proof of (iii), as the remaining argument is the same as in part (ii).

Consider the variance  $\sigma_{n,\epsilon}^2 = \mathbb{E}Z_{n,\epsilon}^2$  of  $Z_{n,\epsilon}$  defined in (3.19):

$$\sigma_{n,\epsilon}^2 = n^{-1} \left( \sum_{0 \leq i \leq M} \mathbb{E}D_i^2 + 2 \sum_{0 \leq i < j \leq M} \mathbb{E}D_i D_j \right),$$

where

$$D_i := \sum_{k/n \in [\tau_i, \tau_{i+1})} \phi_{\tau_i, \epsilon}(\mathbf{X}_n(k)).$$

Let us show that for  $\epsilon, M$  fixed, and as  $n \rightarrow \infty$ ,

$$\mathbb{E}D_i D_j = o(n) \quad (i \neq j), \quad (3.31)$$

$$n^{-1} \mathbb{E}D_i^2 \xrightarrow{n \rightarrow \infty} \int_{\tau_i}^{\tau_{i+1}} \sum_{j \in \mathbb{Z}} \mathbb{E}[\phi_{\tau_i, \epsilon}(\mathbf{W}_\tau(0))\phi_{\tau_i, \epsilon}(\mathbf{W}_\tau(j))] d\tau. \quad (3.32)$$

Here, (3.32) follows from the argument in the beginning of the proof of (iii), as  $\phi_{\tau_i, \epsilon}$  does not depend on  $k, n$ . Relation (3.31) is implied by the following computations. Using the Hermitian rank of functions  $\phi_{\tau_i, \epsilon}$ , for  $i < j$  one obtains

$$\begin{aligned} |\mathbb{E} \phi_{\tau_i, \epsilon}(\mathbf{X}_n([n\tau_i] + k)) \phi_{\tau_j, \epsilon}(\mathbf{X}_n([n\tau_j] + \ell))| &\leq C \|\phi_{\tau_i, \epsilon}\| \cdot \|\phi_{\tau_j, \epsilon}\| \max_{1 \leq p, q \leq v} |r_n^{(p, q)}([n\tau_i] + k, [n\tau_j] + \ell)|^m \\ &\leq C \|\phi_{\tau_i, \epsilon}\| \cdot \|\phi_{\tau_j, \epsilon}\| |\rho([n\tau_j] - [n\tau_i] + \ell - k)|^m. \end{aligned}$$

Therefore, for  $i < j$ , and  $\epsilon$  small enough,

$$\begin{aligned} |\mathbb{E} D_i D_j| &\leq C \max_{\tau \in [0, 1]} \|\phi_\tau\|^2 \sum_{k=0}^{[\tau_{i+1}n] - [\tau_i n]} \sum_{\ell=0}^{[\tau_{j+1}n] - [\tau_j n]} |\rho([n\tau_j] - [n\tau_i] + \ell - k)|^m \\ &\leq C \max_{\tau \in [0, 1]} \|\phi_\tau\|^2 \sum_{k=1}^n k |\rho(k)|^m = o(n) \end{aligned}$$

since  $\sum_{k=1}^n k |\rho(k)|^m \leq \sqrt{n} \sum_{1 \leq k \leq \sqrt{n}} |\rho(k)|^m + n \sum_{k > \sqrt{n}} |\rho(k)|^m = o(n)$ . Thus, (3.31) is proved. From (3.31), (3.32) it follows that for any  $\epsilon > 0$

$$\lim_{n \rightarrow \infty} \sigma_{n, \epsilon}^2 = \bar{\sigma}_\epsilon^2 := \sum_{i=0}^M \int_{\tau_i}^{\tau_{i+1}} \sum_{j \in \mathbb{Z}} \mathbb{E} [\phi_{\tau_i, \epsilon}(\mathbf{W}_\tau(0)) \phi_{\tau_i, \epsilon}(\mathbf{W}_\tau(j))] d\tau.$$

Consider the difference  $\bar{\sigma}_\epsilon^2 - \sigma^2 = \sum_{i=0}^M \int_{\tau_i}^{\tau_{i+1}} \sum_{j \in \mathbb{Z}} \Theta_{M, \epsilon}(\tau, j) d\tau$ , where

$$\begin{aligned} |\Theta_{M, \epsilon}(\tau, j)| &= |\mathbb{E} \phi_{\tau_i, \epsilon}(\mathbf{W}_\tau(0)) \phi_{\tau_i, \epsilon}(\mathbf{W}_\tau(j)) - \mathbb{E} \phi_\tau(\mathbf{W}_\tau(0)) \phi_\tau(\mathbf{W}_\tau(j))| \\ &\leq |\mathbb{E} (\phi_{\tau_i, \epsilon}(\mathbf{W}_\tau(0)) - \phi_\tau(\mathbf{W}_\tau(0))) \phi_{\tau_i, \epsilon}(\mathbf{W}_\tau(j))| + |\mathbb{E} (\phi_{\tau_i, \epsilon}(\mathbf{W}_\tau(j)) - \phi_\tau(\mathbf{W}_\tau(j))) \phi_\tau(\mathbf{W}_\tau(0))| \\ &\leq \|\phi_{\tau_i, \epsilon} - \phi_\tau\| (\|\phi_{\tau_i, \epsilon}\| + \|\phi_\tau\|). \end{aligned} \quad (3.33)$$

Using uniform continuity of  $\phi_\tau$ ,  $\tau \in [0, 1]$  (in the sense of  $\mathbb{L}^2$ -norm continuity), we obtain that the right-hand side of (3.33) can be made arbitrarily small by choosing  $M$  (= the number of partition intervals of  $[0, 1]$ ) and  $t(\epsilon)$  (= the truncation level of Hermite expansion) sufficiently large, uniformly in  $\tau \in [0, 1]$  and  $j \in \mathbb{Z}$ . On the other hand,  $|\Theta_{M, \epsilon}(\tau, j)| \leq C \sup_{\tau \in [0, 1]} \|\phi_\tau\|^2 |\rho(j)|^m$  by Arcones' inequality; c.f. (3.30). Therefore  $|\Theta_{M, \epsilon}(\tau, j)|$  is dominated by a summable function uniformly in  $M, \epsilon$ . Now, (3.21) follows by an application of the Lebesgue theorem. This proves part (iii) and Theorem 1 too.  $\square$

#### 4. A Berry–Esseen-type bound for nonstationary Gaussian subordinated triangular arrays

This section obtains a Berry–Esseen-type upper bound in the CLT (3.7) for non-stationary Gaussian subordinated triangular arrays following the method and results presented in [20]. We will refer NPP to the last paper in the rest of this section. To simplify the discussion, we restrict our task to the case when the functions  $f_{k, n} = f$  in Theorem 1(iii) do not depend on  $k, n$ . As in NPP, our starting point is the Hermite expansion (2.1) written as

$$f = \sum_{\ell=m}^{\infty} f_{(\ell)}, \quad f_{(\ell)} := \sum_{|\mathbf{k}|=\ell} J_f(\mathbf{k}) H_{\mathbf{k}} / \mathbf{k}! \quad (4.1)$$

Following NPP and using the Hermite expansion in (4.1), we first define the following quantities: for  $j \in \mathbb{Z}$ ,  $\ell \geq m$ ,  $N \geq m$ ,  $n \in \mathbb{N}^*$  and  $J \in \{1, \dots, n\}$ :

$$\theta(j) := |\rho(j)|, \quad K := \inf \left\{ k \in \mathbb{N} : \theta(j) \leq \frac{1}{v}, \forall |j| \geq k \right\}, \quad \theta := \sum_{j \in \mathbb{Z}} \theta(j)^m, \quad (4.2)$$

$$\sigma_{\ell, n}^2 := n^{-1} \sum_{t, t'=-n}^n \text{Cov}(f_{(\ell)}(\mathbf{X}_n(t)), f_{(\ell)}(\mathbf{X}_n(t'))), \quad (4.3)$$

$$\gamma_{n, \ell, e} := \frac{1}{n^{1/2}} \left( 2\theta \sum_{|j| \leq n} \theta(j)^e \sum_{|j'| \leq n} \theta(j')^{\ell-e} \right)^{1/2} \quad (\text{for } 1 \leq e \leq \ell - 1), \quad (4.4)$$

$$A_{2, N} := 2(2K + v^m \theta) \left( \mathbb{E}[f^2(\mathbf{X})] \sum_{\ell=N+1}^{\infty} \mathbb{E}[f_{(\ell)}^2(\mathbf{X})] \right)^{1/2}, \quad (4.5)$$

$$A_{3,n,N} := \frac{1}{2} \mathbb{E}[f^2(\mathbf{X})] \sum_{\ell=m}^N \left( \frac{\nu^\ell}{\ell!} \sum_{j=1}^{\ell-1} j! \binom{\ell}{j}^2 \sqrt{(2\ell-2j)!} \gamma_{n,\ell,j} \right), \quad (4.6)$$

$$A_{4,n,N} := \frac{1}{2} \mathbb{E}[f^2(\mathbf{X})] \sum_{m \leq \ell < \ell' \leq N} \nu^{\ell'/2} \sqrt{\frac{\ell'!}{\ell!} \frac{\ell + \ell'}{\ell}} \binom{\ell' - 1}{\ell - 1} ((\ell' - \ell)! \gamma_{n,\ell',\ell' - \ell})^{1/2}, \quad (4.7)$$

$$A_{5,n,N} := \frac{\mathbb{E}[f^2(\mathbf{X})]}{2\sqrt{2}} \sum_{m \leq \ell < \ell' \leq N} (\ell + \ell') \sum_{j=1}^{\ell-1} (j-1)! \binom{\ell-1}{j-1} \binom{\ell'-1}{j-1} \\ \times \sqrt{(\ell + \ell' - 2j)!} \left( \frac{\nu^\ell}{\ell!} \gamma_{n,\ell,\ell-j} + \frac{\nu^{\ell'}}{\ell'!} \gamma_{n,\ell',\ell'-j} \right), \quad (4.8)$$

$$A_{6,n,J} := \frac{1}{2} |\partial f|_\infty^2 \sup_{0 \leq \tau \leq 1} \sum_{|j| \leq J} \left\| \mathbb{E}[\mathbf{X}_n([n\tau]) \mathbf{X}_n^\top([n\tau] + j)] - \mathbb{E}[\mathbf{W}_\tau(0) \mathbf{W}_\tau^\top(j)] \right\|, \quad (4.9)$$

$$A_{7,J} := \frac{1}{2} \mathbb{E}[f^2(\mathbf{X})] \nu^m \sum_{|k| > J} \theta^m(k). \quad (4.10)$$

Note that terms  $A_{2,n}$ ,  $A_{3,n,N}$  and  $A_{5,n,N}$  are the same as in NPP,  $A_{4,n,N}$  is a minor improvement of the corresponding term in NPP, and  $A_{6,n,J}$  reflects the “convergence rate” in (3.8). Term  $A_{1,n}$  of NPP (which does not appear in our bounds) is “absorbed” in the term  $\inf_{1 \leq j \leq n} A_{7,j}$  in the bounds (i)–(iii), below, due to a somewhat different approximation (see (4.15)).

**Proposition 2.** Let the assumptions of Theorem 1(iii) prevail, with  $f_{k,n} \equiv f$  for all  $1 \leq k \leq n$ ,  $n \in \mathbb{N}^*$  where  $f : \mathbb{R}^\nu \rightarrow \mathbb{R}$  is a Lipschitz function with  $|f(x) - f(y)| \leq |\partial f|_\infty |x - y|$  for all  $x \neq y \in \mathbb{R}^\nu$ . Define  $S_n := n^{-1/2} \sum_{t=1}^n f(\mathbf{X}_n(t))$  and let  $S$  be a zero-mean Gaussian random variable with a variance  $\sigma_S^2 := \int_0^1 \sum_{j \in \mathbb{Z}} \text{Cov}(f(\mathbf{W}_\tau(0)), f(\mathbf{W}_\tau(j))) d\tau < \infty$ . Then, we have the following.

(i) For any function  $h$  twice continuously differentiable with bounded second derivative and for every  $n > K$ ,

$$|\mathbb{E}[h(S_n)] - \mathbb{E}[h(S)]| \leq |h''|_\infty \left( \inf_{N \geq m} \{A_{2,N} + A_{3,n,N} + A_{4,n,N} + A_{5,n,N}\} + \inf_{1 \leq j \leq n} \{A_{6,n,J} + A_{7,j}\} \right). \quad (4.11)$$

(ii) For any Lipschitz function  $h$ , and for every  $n > K$ ,

$$|\mathbb{E}[h(S_n)] - \mathbb{E}[h(S)]| \leq |h'|_\infty \left( \frac{2}{\sigma_S} \inf_{1 \leq j \leq n} \{A_{6,n,J} + A_{7,j}\} \right. \\ \left. + \inf_{N \geq m} \left\{ \left( \frac{1}{2\sigma_S} + \frac{1}{((2K + \nu^m) \mathbb{E}[f^2(\mathbf{X})])^{1/2}} \right) A_{2,N} + \frac{A_{3,n,N} + A_{4,n,N} + A_{5,n,N}}{\left( \sum_{\ell=m}^N \sigma_{\ell,n}^2 \right)^{1/2}} \right\} \right). \quad (4.12)$$

(iii) For any  $z \in \mathbb{R}$ , and for every  $n > K$ ,

$$|\mathbb{P}(S_n \leq z) - \mathbb{P}(S \leq z)| \leq \frac{2}{\sigma_S} \left( \frac{2}{\sigma_S} \inf_{1 \leq j \leq n} \{A_{6,n,J} + A_{7,j}\} \right. \\ \left. + \inf_{N \geq m} \left\{ \left( \frac{1}{2\sigma_S} + \frac{1}{((2K + \nu^m) \mathbb{E}[f^2(\mathbf{X})])^{1/2}} \right) A_{2,N} + \frac{A_{3,n,N} + A_{4,n,N} + A_{5,n,N}}{\left( \sum_{\ell=m}^N \sigma_{\ell,n}^2 \right)^{1/2}} \right\} \right)^{1/2}. \quad (4.13)$$

**Proof of Proposition 2.** Let us introduce a similar notation to NPP. Consider the Hilbert space  $\mathfrak{H} = \mathbb{R}^{\nu\nu}$  with elements  $u = (u_{t,l}, 1 \leq t \leq n, 1 \leq l \leq \nu) \in \mathfrak{H}$  and the scalar product  $\langle u_{t,j}, u_{t',j'} \rangle_{\mathfrak{H}} := \mathbb{E} X_n^{(j)}(t) X_n^{(j')}(t') = r_n^{(j,j')}(t, t')$ . The

$\ell$ -fold tensor product and the symmetrized tensor product of  $\mathfrak{H}$  are denoted by  $\mathfrak{H}^{\otimes \ell}$  and  $\mathfrak{S}^{\otimes \ell}$ , respectively. Let  $\mathbb{L}^2(\mathfrak{X}_n)$  denote the space of r.v.'s subordinated to the Gaussian vector  $\mathfrak{X}_n := (\mathbf{X}_n(t))_{1 \leq t \leq n}$ . Any element  $\xi \in \mathbb{L}^2(\mathfrak{X}_n)$  admits a chaotic expansion  $\xi = \sum_{\ell=0}^{\infty} I_{(\ell)}(g_{(\ell)})$ , where  $g_{(\ell)} \in \mathfrak{H}^{\otimes \ell}$  and the linear mapping  $I_{(\ell)} : \mathfrak{H}^{\otimes \ell} \rightarrow \mathbb{L}^2(\mathfrak{X}_n)$  satisfies  $I_{(\ell)}(g) = I_{(\ell)}(\text{sym}(g))$ ,  $\text{El}_{(\ell)}^2(g) = \ell! \|\text{sym}(g)\|_{\mathfrak{H}^{\otimes \ell}}^2$ , and  $\mathbb{E}[I_{(\ell)}(g)I_{(\ell')}(g')] = 0$ ,  $\ell \neq \ell'$ ,  $g_{(\ell)} \in \mathfrak{H}^{\otimes \ell}$ ,  $g_{(\ell')} \in \mathfrak{H}^{\otimes \ell'}$ , where  $\text{sym}$  denotes the symmetrization operator. In particular, for any  $t = 1, \dots, n$ ,  $\mathbf{k} \in \mathbb{Z}_+^v$ ,  $|\mathbf{k}| =: \ell$  we have  $H_{\mathbf{k}}(\mathbf{X}_n(t)) = I_{(\ell)}(g_{\ell}(\mathbf{k}))$ , where

$$g_{\ell}(\mathbf{k}) := \text{sym}(u_{t,1}^{\otimes k(1)} \otimes \dots \otimes u_{t,v}^{\otimes k(v)}) = \sum_{\mathbf{v} \in \{1, \dots, v\}^{\ell}} b(\mathbf{v}; \mathbf{k}) u_{t,v_1} \otimes \dots \otimes u_{t,v_{\ell}}$$

and where  $b(\mathbf{v}; \mathbf{k}) = \text{sym}[\tilde{b}(\mathbf{v}; \mathbf{k})]$  is the symmetrization of the function  $\{1, \dots, v\}^{\ell} \ni \mathbf{v} = (v_1, \dots, v_{\ell}) \mapsto \tilde{b}(\mathbf{v}; \mathbf{k}) := \prod_{r=1}^v \mathbf{1}(v_i = r, k_1 + \dots + k_{r-1} < i \leq k_1 + \dots + k_r)$ . Thus,  $S_n = n^{-1/2} \sum_{t=1}^n f(\mathbf{X}_n(t))$  admits the chaotic expansion

$$S_n = \sum_{\ell=m}^{\infty} I_{(\ell)}(g_{\ell}^n) \quad \text{with } g_{\ell}^n := \frac{1}{\sqrt{n}} \sum_{t=1}^n \sum_{\mathbf{v} \in \{1, \dots, v\}^{\ell}} b_{\ell}(\mathbf{v}) u_{t,v_1} \otimes \dots \otimes u_{t,v_{\ell}},$$

where  $b_{\ell}(\mathbf{v}) := \sum_{|\mathbf{k}|=\ell} (J_f(\mathbf{k})/|\mathbf{k}|!) b(\mathbf{v}; \mathbf{k})$  depend only on  $f \in \mathbb{L}^2(\mathbf{X}_n(t)) = \mathbb{L}^2(\mathbf{X})$  and satisfy  $\text{El}_{(\ell)}^2(\mathbf{X}) = \ell! \sum_{\mathbf{v} \in \{1, \dots, v\}^{\ell}} b_{\ell}^2(\mathbf{v})$ , as in NPP. It is important that here the  $g_{\ell}^n$ 's are symmetric since the  $b_{\ell}(\mathbf{v})$ 's are symmetric. Therefore  $\text{El}_{(\ell)}^2(g_{\ell}^n) = \ell! \|g_{\ell}^n\|_{\mathfrak{H}^{\otimes \ell}}^2$ . Next, for  $N \geq m$  consider the truncated expansion

$$S_{n,N} := \sum_{\ell=m}^N I_{(\ell)}(g_{\ell}^n).$$

Note that

$$\begin{aligned} \text{ES}_{n,N}^2 &= \sum_{\ell=m}^N \text{El}_{(\ell)}^2(g_{\ell}^n) = \sum_{\ell=m}^N \ell! \|g_{\ell}^n\|_{\mathfrak{H}^{\otimes \ell}}^2 \\ &= \frac{1}{n} \sum_{\ell=m}^N \ell! \sum_{t,t'=1}^n \sum_{\mathbf{v}, \mathbf{v}' \in \{1, \dots, v\}^{\ell}} b_{\ell}(\mathbf{v}) b_{\ell}(\mathbf{v}') \langle u_{t,v_1} \otimes \dots \otimes u_{t,v_{\ell}}, u_{t',v'_1} \otimes \dots \otimes u_{t',v'_{\ell}} \rangle_{\mathfrak{H}^{\otimes \ell}} \\ &= \frac{1}{n} \sum_{\ell=m}^N \ell! \sum_{t,t'=1}^n \sum_{\mathbf{v}, \mathbf{v}' \in \{1, \dots, v\}^{\ell}} b_{\ell}(\mathbf{v}) b_{\ell}(\mathbf{v}') \prod_{i=1}^{\ell} r_n^{(v_i, v'_i)}(t, t'). \end{aligned}$$

Using  $|r_n^{(j,j')}(t, t')| \leq \theta(t - t')$  similarly as in NPP we obtain

$$|\mathbb{E}[h(S_n)] - \mathbb{E}[h(S_{n,N})]| \leq \frac{3}{2} (2K + v^m \theta) |h''|_{\infty} \left( \mathbb{E}[f^2(\mathbf{X})] \sum_{\ell=N+1}^{\infty} \text{El}_{(\ell)}^2(\mathbf{X}) \right)^{1/2} \leq \frac{3}{4} |h''|_{\infty} A_{2,N}. \quad (4.14)$$

For  $N \geq m$ , let  $Z_{n,N}$  be a centered Gaussian random variable with variance  $\text{ES}_{n,N}^2 = \sum_{\ell=m}^N \sigma_{\ell,n}^2$ , with  $\sigma_{\ell,n}^2$  defined in (4.3). (Note that the last variance is slightly different from the variance of  $Z_N$  in (NPP, Section 4.2).) Let  $D$  denote the Malliavin derivative in  $\mathbb{L}^2(\mathfrak{X}_n)$ ; see NPP. Using  $\ell^{-1} \mathbb{E} \|DI_{(\ell)}(g_{\ell}^n)\|_{\mathfrak{H}}^2 = \ell! \|g_{\ell}^n\|_{\mathfrak{H}^{\otimes \ell}}^2 = \sigma_{\ell,n}^2$ , see (4.14), as in (NPP, (4.46)) we obtain

$$\begin{aligned} |\mathbb{E}[h(Z_{n,N})] - \mathbb{E}[h(S_{n,N})]| &\leq \frac{1}{2} |h''|_{\infty} \sum_{\ell, \ell'=m}^N \|\delta_{\ell \ell'} \sigma_{\ell,n}^2 - \ell^{-1} \langle DI_{(\ell)}(g_{\ell}^n), DI_{(\ell')}(g_{\ell'}^n) \rangle_{\mathfrak{H}}\|_{\mathbb{L}^2(\mathbb{P})} \\ &\leq |h''|_{\infty} (A_{3,n,N} + A_{4,n,N} + A_{5,n,N}). \end{aligned} \quad (4.15)$$

Next, using (NPP, (3.39))

$$|\mathbb{E}[h(Z_{n,N})] - \mathbb{E}[h(S)]| \leq \frac{1}{2} |h''|_{\infty} \left| \sum_{\ell=m}^N \sigma_{\ell,n}^2 - \sigma_S^2 \right| \leq \frac{1}{2} |h''|_{\infty} \left( |\sigma_n^2 - \sigma_S^2| + \left| \sigma_n^2 - \sum_{\ell=m}^N \sigma_{\ell,n}^2 \right| \right).$$

To estimate the difference  $\sigma_n^2 - \sigma_S^2$ , we use an interpolation identity from [16]. Let  $(\mathbf{X}_1, \mathbf{X}_2)$ ,  $(\mathbf{W}_1, \mathbf{W}_2)$  be two  $(2v)$ -dimensional Gaussian vectors with zero means and respective covariance matrices  $\mathbb{E}[\mathbf{X}_i \mathbf{X}_i^T] = \mathbb{E}[\mathbf{W}_i \mathbf{W}_i^T] = I$ ,  $i = 1, 2$ ,  $\mathbb{E}[\mathbf{X}_1 \mathbf{X}_2^T] = \Sigma_1$ ,  $\mathbb{E}[\mathbf{W}_1 \mathbf{W}_2^T] = \Sigma_0$ . For  $\alpha \in [0, 1]$  let  $(\mathbf{X}_{1\alpha}, \mathbf{X}_{2\alpha})$  denote the “interpolated” Gaussian vector with zero mean and  $\mathbb{E}[\mathbf{X}_{i\alpha} \mathbf{X}_{i\alpha}^T] = I$ ,  $i = 1, 2$ ,  $\mathbb{E}[\mathbf{X}_{1\alpha} \mathbf{X}_{2\alpha}^T] = (1 - \alpha) \Sigma_0 + \alpha \Sigma_1$ . Let  $f \in \mathbb{L}^2(\mathbf{X})$  be a real function satisfying the conditions

of Proposition 2. Then from [16, (1.1), (1.3)] we obtain

$$\begin{aligned} |\text{Cov}(f(\mathbf{X}_1), f(\mathbf{X}_2)) - \text{Cov}(f(\mathbf{W}_1), f(\mathbf{W}_2))| &= \left| \int_0^1 \mathbb{E}[\partial f(\mathbf{X}_{1\alpha})^\top (\Sigma_1 - \Sigma_0) \partial f(\mathbf{X}_{2\alpha})] d\alpha \right| \\ &\leq |\partial f|_\infty^2 \|\Sigma_1 - \Sigma_0\|, \end{aligned} \quad (4.16)$$

where  $\partial f = (\partial f / \partial x^{(1)}, \dots, \partial f / \partial x^{(v)})^\top \in \mathbb{R}^v$ . Let  $F_n(\tau) := \sum_{t'=1}^n \text{Cov}(f(\mathbf{X}_n([n\tau])), f(\mathbf{X}_n(t')))$ ,  $\tau \in [0, 1]$  so that  $\sigma_n^2 = \int_0^1 F_n(\tau) d\tau$ . Using (4.16), for  $1 \leq J \leq n$  we can write  $|\sigma_n^2 - \sigma_S^2| \leq R_1(n, J) + R_2(n, J)$ , where

$$\begin{aligned} R_1(n, J) &:= \int_0^1 \sum_{|j| \leq J} |\text{Cov}(f(\mathbf{X}_n([n\tau])), f(\mathbf{X}_n([n\tau] + j)) - \text{Cov}(f(\mathbf{W}_\tau(0)), f(\mathbf{W}_\tau(j)))| d\tau \leq 2A_{6,n,J}, \\ R_2(n, J) &\leq 2 \mathbb{E}[f^2(\mathbf{X})] v^m \sum_{|k| > J} \theta^m(k) = 2A_{7,J}. \end{aligned}$$

We also have  $|\sigma_n^2 - \sum_{\ell=m}^N \sigma_{\ell,n}^2| = \sum_{\ell=N+1}^\infty \sigma_{\ell,n}^2 \leq \frac{1}{2} A_{2,N}$ , as in (4.14). Therefore,  $|\sum_{\ell=m}^N \sigma_{\ell,n}^2 - \sigma_S^2| \leq 2A_{6,n,J} + 2A_{7,J} + \frac{1}{2} A_{2,N}$ , implying

$$|\mathbb{E}[h(Z_{n,N})] - \mathbb{E}[h(S)]| \leq |h''|_\infty \left( A_{6,n,J} + A_{7,J} + \frac{1}{4} A_{2,N} \right) \quad \text{for } 1 \leq J \leq n. \quad (4.17)$$

Finally combining (4.14), (4.15) and (4.17) results in

$$\begin{aligned} |\mathbb{E}[h(S_n)] - \mathbb{E}[h(S)]| &\leq |h''|_\infty \left( A_{2,N} + A_{3,n,N} + A_{4,n,N} + A_{5,n,N} + \inf_{1 \leq J \leq n} (A_{6,n,J} + A_{7,J}) \right) \\ &\leq |h''|_\infty \left( \inf_{N \geq m} \{A_{2,N} + A_{3,n,N} + A_{4,n,N} + A_{5,n,N}\} + \inf_{1 \leq J \leq n} \{A_{6,n,J} + A_{7,J}\} \right), \end{aligned}$$

proving the bound in (4.11).

(ii) Following (NPP, Proof of Theorem 2.2-(2)) and the previous results, for a Lipschitz function  $h$  we obtain:

$$\begin{aligned} |\mathbb{E}[h(S_n)] - \mathbb{E}[h(S_{n,N})]| &\leq |h'|_\infty ((2K + v^m) \mathbb{E}[f^2(\mathbf{X})])^{-1/2} A_{2,N}, \\ |\mathbb{E}[h(Z_{n,N})] - \mathbb{E}[h(S_{n,N})]| &\leq 2 |h'|_\infty \left( \sum_{\ell=m}^N \sigma_{\ell,n}^2 \right)^{-1/2} (A_{3,n,N} + A_{4,n,N} + A_{5,n,N}) \\ \text{and } |\mathbb{E}[h(Z_{n,N})] - \mathbb{E}[h(S)]| &\leq \frac{|h'|_\infty}{\sigma_S} \left( \frac{1}{2} A_{2,N} + \inf_{1 \leq J \leq n} (A_{6,n,J} + A_{7,J}) \right) \end{aligned}$$

and therefore (4.12) is established.

(iii) Bound (4.13) is obtained exactly as in (NPP, Proof of Theorem 2.2-(3)).  $\square$

## 5. Applications of Lemma 1 and Theorem 1

### 5.1. Application to the IR statistic

This application was developed in [3,4]. Let  $(X_t)_{t \in [0,1]}$  be a continuous time Gaussian process with zero mean and generally nonstationary increments locally resembling a fractional Brownian motion with Hurst parameter  $H(t) \in (0, 1)$ . Consider the Increment Ratio (IR) statistic

$$R^{2,n}(X) := \frac{1}{n-2} \sum_{k=0}^{n-3} \frac{|\Delta_k^{2,n} X + \Delta_{k+1}^{2,n} X|}{|\Delta_k^{2,n} X| + |\Delta_{k+1}^{2,n} X|},$$

with  $\Delta_k^{2,n} X = X_{(k+2)/n} - 2X_{(k+1)/n} + X_{k/n}$  and the convention  $\frac{0}{0} := 1$ . Let  $\sigma_{2,n}^2(k) := \mathbb{E}[(\Delta_k^{2,n} X)^2]$  and

$$Y_n^{(1)}(k) := \frac{\Delta_k^{2,n} X}{\sigma_{2,n}(k)}, \quad Y_n^{(2)}(k) := \frac{\Delta_{k+1}^{2,n} X}{\sigma_{2,n}(k)}.$$

Then  $R^{2,n}(X) = \frac{1}{n-2} \sum_{k=0}^{n-3} f(\mathbf{Y}_n(k))$ ,  $f(x^{(1)}, x^{(2)}) := |x^{(1)} + x^{(2)}| / (|x^{(1)}| + |x^{(2)}|)$  can be written as the sum of nonlinear function  $f$  of Gaussian vectors  $\mathbf{Y}_n(k) = (Y_n^{(1)}(k), Y_n^{(2)}(k)) \in \mathbb{R}^2$ ,  $0 \leq k \leq n-3$ . These Gaussian vectors can be standardized, leading to the expression  $R^{2,n}(X) = \frac{1}{n-2} \sum_{k=0}^{n-3} f_{n,k}(\mathbf{X}_n(k))$  of the IR statistics as the sum of some functions  $f_{n,k}$  of standardized



Gaussian vectors  $\mathbf{X}_n(k)$ ,  $0 \leq k \leq n-3$ . (It is easy to check that the centered functions  $f_{n,k} - \mathbb{E}[f_{n,k}(\mathbf{X})]$  have the Hermite rank 2.) If  $(X_t)$  satisfies some additional conditions (specifying the decay rate of correlations of increments and the convergence rate to the tangent process), [Theorem 1](#) can be applied to establish that  $\sqrt{n}(R^{2,n}(X) - \int_0^1 \Lambda(H(t)) dt) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2)$  with an explicit function  $\Lambda$  and a variance  $\sigma^2$ . An application of [Lemma 1](#) to bound the fourth moment  $\mathbb{E}(R^{2,n}(X) - \mathbb{E}R^{2,n}(X))^4$  provides a crucial step in the proof of the almost sure consistency of the IR statistic, i.e.  $R^{2,n}(X) \xrightarrow[n \rightarrow \infty]{a.s.} \int_0^1 \Lambda(H(t)) dt$ . See [\[3\]](#) for details. Local versions of the IR statistic for point-wise estimation of  $H(t)$  are developed in [\[4\]](#). The study of the asymptotic properties of these estimators in the last paper is also based on [Theorem 1](#) and [Lemma 1](#).

## 5.2. A central limit theorem for functions of locally stationary Gaussian processes

Using an adaptation of Dahlhaus and Polonik [\[9,10\]](#), we will say that  $(X_{t,n})_{1 \leq t \leq n, n \in \mathbb{N}^*}$  is a locally stationary Gaussian process if

$$X_{t,n} := \sum_{j \in \mathbb{Z}} a_{t,n}(j) \varepsilon_{t-j}, \quad \text{for all } 1 \leq t \leq n, n \in \mathbb{N}^*, \quad (5.1)$$

where  $(\varepsilon_k)_{k \in \mathbb{Z}}$  is a sequence of independent standardized Gaussian variables and for  $1 \leq t \leq n, n \in \mathbb{N}^*$  the sequences  $(a_{t,n}(j))_{j \in \mathbb{Z}}$  are such that there exist  $K \geq 0$  and  $\alpha < 1/2$  satisfying for all  $n \in \mathbb{N}^*$  and  $j \in \mathbb{Z}$ ,

$$\max_{1 \leq t \leq n} |a_{t,n}(j)| \leq \frac{K}{u_j}, \quad \text{with } u_j := \max(1, |j|^{\alpha-1}) \text{ for } j \in \mathbb{Z} \quad (5.2)$$

and such that there exist functions  $\tau \in (0, 1] \mapsto a(\tau, j) \in \mathbb{R}$  satisfying the following conditions:

$$\sup_{\tau \in (0, 1]} |a(\tau, j)| \leq \frac{K}{u_j}, \quad \forall j \in \mathbb{Z}, \quad (5.3)$$

$$\text{and } \sup_{\tau \in (0, 1]} \max_{|[n\tau] - k| \leq L} |(a)_{k,n}(j) - a(\tau, j)| \rightarrow 0, \quad \forall j \in \mathbb{Z}, \forall L > 0. \quad (5.4)$$

For  $\tau \in (0, 1]$  introduce a stationary Gaussian process

$$W_\tau(t) := \sum_{j \in \mathbb{Z}} a(\tau, j) \varepsilon_{t-j}, \quad t \in \mathbb{Z}$$

with spectral density  $g_\tau(v) = |\hat{a}(\tau, v)|^2$ ,  $\hat{a}(\tau, v) := (2\pi)^{-1/2} \sum_{j \in \mathbb{Z}} e^{-ijv} a(\tau, j)$ ,  $v \in [-\pi, \pi]$ . Let

$$\mathbf{Y}_n(k) := (X_{k+1,n}, \dots, X_{k+v,n})^\top, \quad \mathbf{W}_\tau(j) := (W_\tau(j+1), \dots, W_\tau(j+v))^\top.$$

Note  $(\mathbf{W}_\tau(j))_{j \in \mathbb{Z}}$  is a  $\mathbb{R}^v$ -valued stationary Gaussian process. Let

$$\Sigma_{k,n} := \mathbb{E}[\mathbf{Y}_n(k) \mathbf{Y}_n(k)^\top], \quad \Sigma_\tau := \mathbb{E}[\mathbf{W}_\tau(0) \mathbf{W}_\tau(0)^\top].$$

**Proposition 3.** In addition to (5.1)–(5.4), assume that

$$\sup_{\tau \in (0, 1]} \|\Sigma_\tau^{-1}\| < \infty. \quad (5.5)$$

Let  $f_{k,n} \in \mathbb{L}_0^2(\mathbf{Y}_n(k))$ ,  $1 \leq k \leq n$ ,  $n \geq 1$  be a triangular array of functions all having a generalized Hermite rank at least  $m > 1/(1 - 2\alpha)$ . Let there exists a  $\mathbb{L}_0^2(\mathbf{X})$ -valued continuous function  $\tilde{\phi}_\tau$ ,  $\tau \in (0, 1]$  such that relation (3.12) holds, with  $\tilde{f}_{k,n}(\mathbf{x}) := f_{k,n}(\Sigma_{k,n}^{1/2} \mathbf{x})$ . Then the CLT of (3.13) holds, with

$$\sigma^2 := \int_0^1 d\tau \sum_{j \in \mathbb{Z}} \mathbb{E}[\phi_\tau(\mathbf{W}_\tau(0)) \phi_\tau(\mathbf{W}_\tau(j))] \quad (5.6)$$

and  $\phi_\tau(\mathbf{x}) := \tilde{\phi}_\tau(\Sigma_\tau^{-1/2} \mathbf{x})$  defined as in [Corollary 2](#).

**Proof.** We apply [Corollary 2](#). Let us first check

$$\sup_{\tau \in (0, 1]} \|\Sigma_{[n\tau],n} - \Sigma_\tau\| \xrightarrow[n \rightarrow \infty]{} 0. \quad (5.7)$$

We have

$$|\sigma_{[n\tau],n}(p, q) - \sigma_\tau(p, q)| = \left| \sum_{j \in \mathbb{Z}} (a_{[n\tau]+p,n}(p+j) a_{[n\tau]+q,n}(q+j) - a(\tau, p+j) a(\tau, q+j)) \right| \leq T_{nJ} + T''_{nJ},$$

where

$$T'_{n,j} := 2K^2 \sum_{|j|>J} u_{p+j} u_{q+j}, \quad T''_{n,j} := \sum_{|j|\leq J} |a_{[n\tau]+p,n}(p+j) a_{[n\tau]+q,n}(q+j) - a(\tau, p+j) a(\tau, q+j)|$$

according to (5.2) and (5.3). Clearly,  $T'_{n,j}$  can be made arbitrarily small by choosing  $J$  large enough. Then for any  $J < \infty$  fixed, we have that  $\sup_{\tau \in (0,1]} T'_{n,j} \rightarrow 0$  according to assumption (5.4). This proves (5.7). In a similar way, one verify that for any  $\tau \in (0, 1]$ ,  $j, j' \in \mathbb{Z}$ ,  $\|E[\mathbf{Y}_n([n\tau] + j)\mathbf{Y}_n([n\tau] + j')^\top] - E[\mathbf{W}_\tau(j)\mathbf{W}_\tau(j')^\top]\| \xrightarrow{n \rightarrow \infty} 0$  implying condition (3.8). The dominating condition (3.3) on cross-covariances is ensured by (5.2) and the fact that  $(1 - 2\alpha)m > 1$ . The remaining conditions of Corollary 2 are trivially satisfied.  $\square$

**Remark 2.** Dahlhaus and Polonik [9,10] discussed the short-memory case  $(a_{t,n}(j))_{j \in \mathbb{Z}} \in \ell^1$ ,  $1 \leq t \leq n$  only. On the other hand, condition (5.2) allows for the long-memory case  $(a_{t,n}(j))_{j \in \mathbb{Z}} \in \ell^2$ ,  $\sum_{j \in \mathbb{Z}} |a_{t,n}(j)| = \infty$ . The last case is also discussed in [23], where similar conditions as (5.2) and (5.3) are provided in spectral terms. It is not clear whether condition (5.4) allows for jumps of the parameter curves  $\tau \mapsto a(\tau, \cdot)$  as in [9,10], in particular, for abrupt changes of the memory intensity of Gaussian process (5.1). See also [19] for a related class of nonstationary moving average processes with long memory.

**Remark 3.** Note that  $\mathbf{x}^\top \Sigma_\tau \mathbf{x} = \int_{-\pi}^{\pi} g_\tau(v) \left| \sum_{j=1}^v e^{ijv} x^{(j)} \right|^2 dv$  for any  $\mathbf{x} = (x^{(1)}, \dots, x^{(v)})^\top \in \mathbb{R}^v$ . Therefore condition  $\inf_{v \in [-\pi, \pi], \tau \in (0,1]} g_\tau(v) \geq \gamma > 0$  on the spectral density of  $(W_\tau(t))$  implies condition (5.5), since  $\mathbf{x}^\top \Sigma_\tau \mathbf{x} \geq c|\mathbf{x}|^2$ ,  $c := 2\pi v \gamma > 0$ .

**Remark 4.** For stationary Gaussian long memory processes, condition  $m(1-2\alpha) > 1$  was first obtained in [27]. Proposition 3 can be applied to prove the asymptotic normality of various statistics of locally stationary processes; see, e.g., [23].

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## References

- [1] M.A. Arcones, Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors, *Ann. Probab.* 22 (1994) 2242–2274.
- [2] J.-M. Bardet, P. Doukhan, G. Lang, N. Ragache, The standard Lindeberg method applied to weakly dependent processes, *ESAIM Probab. Stat.* 12 (2008) 154–172.
- [3] J.-M. Bardet, D. Surgailis, Measuring the roughness of random paths by increment ratios, *Bernoulli* 17 (2011) 749–780.
- [4] J.-M. Bardet, D. Surgailis, A new nonparametric estimator of the local Hurst function of multifractional processes, preprint, 2012.
- [5] P. Breuer, P. Major, Central limit theorems for nonlinear functionals of Gaussian fields, *J. Multivariate Anal.* 13 (1983) 425–441.
- [6] D. Chambers, E. Slud, Central limit theorems for nonlinear functional of stationary Gaussian process, *Probab. Theory Related Fields* 80 (1989) 323–349.
- [7] C. Coulon-Prieur, P. Doukhan, A triangular central limit theorem under a new weak dependence condition, *Statist. Probab. Lett.* 47 (2000) 61–68.
- [8] M. Csörgő, J. Mielniczuk, The empirical process of a short-range dependent stationary sequence under Gaussian subordination, *Probab. Theory Related Fields* 104 (1996) 15–25.
- [9] R. Dahlhaus, W. Polonik, Nonparametric quasi-maximum likelihood estimation for Gaussian locally stationary processes, *Ann. Statist.* 34 (2006) 2790–2824.
- [10] R. Dahlhaus, W. Polonik, Empirical spectral processes for locally stationary time series, *Bernoulli* 15 (2009) 1–39.
- [11] J. Dedecker, F. Merlevède, Necessary and sufficient conditions for the conditional central limit theorem, *Ann. Probab.* 30 (2002) 1044–1081.
- [12] R.L. Dobrushin, P. Major, Non-central limit theorems for nonlinear functionals of Gaussian fields, *Z. Wahrscheinlichkeitstheor. Verwandte Geb.* 50 (1979) 27–52.
- [13] L. Giraitis, D. Surgailis, CLT and other limit theorems for functionals of Gaussian processes, *Z. Wahrscheinlichkeitstheor. Verwandte Geb.* 70 (1985) 191–212.
- [14] H. Guo, H.L. Koul, Asymptotic inference in some heteroscedastic regression models with long memory design and errors, *Ann. Statist.* 36 (2008) 458–487.
- [15] X. Guyon, J. Leõn, Convergence en loi des H-variations d'un processus gaussien stationnaire, *Ann. Inst. H. Poincaré* 25 (1989) 265–282.
- [16] C. Houdré, V. Pérez-Abreu, D. Surgailis, Interpolation, correlation identities, and inequalities for infinitely divisible variables, *J. Fourier Anal. Appl.* 4 (1998) 651–668.
- [17] J. Jacod, A.B. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin, 1987.
- [18] H.L. Koul, D. Surgailis, Asymptotic expansion of the empirical process of long memory moving averages, in: H. Dehling, Th. Mikosch, M. Sørensen (Eds.), *Empirical Process Techniques for Dependent Data*, Birkhäuser, Boston, 2002, pp. 213–239.
- [19] F. Lavancier, R. Leipus, A. Philippe, D. Surgailis, Detection of non-constant long memory parameter, preprint, 2011.
- [20] I. Nourdin, G. Peccati, M. Podolskij, Quantitative Breuer–Major theorems, *Stochastic Process. Appl.* 121 (2011) 793–812.
- [21] M. Peligrad, S. Utev, Central limit theorem for linear processes, *Ann. Probab.* 25 (1997) 443–456.
- [22] E. Rio, About the Lindeberg method for strongly mixing sequences, *ESAIM Probab. Stat.* 1 (1995) 35–61.
- [23] F. Roueff, R. von Sachs, Locally stationary long memory estimation, *Stochastic Process. Appl.* 121 (2010) 813–844.
- [24] M.V. Sanchez de Naranjo, Non-central limit theorems for nonlinear functionals of  $k$  Gaussian fields, *J. Multivariate Anal.* 44 (1993) 227–255.
- [25] Ph. Soulier, Moment bounds and central limit theorem for functions of Gaussian vectors, *Statist. Probab. Lett.* 54 (2001) 193–203.
- [26] D. Surgailis, Long-range dependence and Appell rank, *Ann. Probab.* 28 (2000) 478–497.
- [27] M.S. Taqqu, Weak convergence to the fractional Brownian motion and to the Rosenblatt process, *Z. Wahrscheinlichkeitstheor. Verwandte Geb.* 31 (1975) 287–302.
- [28] M.S. Taqqu, Law of the iterated logarithm for sums of non-linear functions of Gaussian variables that exhibit a long range dependence, *Z. Wahrscheinlichkeitstheor. Verwandte Geb.* 40 (1977) 203–238.
- [29] M.S. Taqqu, Convergence of integrated processes of arbitrary Hermite rank, *Z. Wahrscheinlichkeitstheor. Verwandte Geb.* 50 (1979) 53–83.