



# Limit theory of quadratic forms of long-memory linear processes with heavy-tailed GARCH innovations



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## ABSTRACT

Let  $X_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$  be a moving average process with GARCH(1, 1) innovations  $\{\varepsilon_t\}$ . In this paper, the asymptotic behavior of the quadratic form  $Q_n = \sum_{j=1}^n \sum_{s=1}^n b(t-s) X_t X_s$  is derived when the innovation  $\{\varepsilon_t\}$  is a long-memory and heavy-tailed process with tail index  $\alpha$ , where  $\{b(i)\}$  is a sequence of constants. In particular, it is shown that when  $1 < \alpha < 4$  and under certain regularity conditions, the limit distribution of  $Q_n$  converges to a stable random variable with index  $\alpha/2$ . However, when  $\alpha \geq 4$ ,  $Q_n$  has an asymptotic normal distribution. These results not only shed light on the singular behavior of the quadratic forms when both long-memory and heavy-tailed properties are present, but also have applications in the inference for general linear processes driven by heavy-tailed GARCH innovations.

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## 1. Introduction

Financial data series often exhibit non-standard features such as non-Gaussianity, stochastic volatility and the long-memory feature that is reflected by the slowly decaying autocorrelation function. Amongst the various models proposed, the generalized autoregressive conditional heteroscedasticity (GARCH) model is one of the most popular ones. Specifically, consider

$$\varepsilon_t = \sigma_t \eta_t, \quad \sigma_t^2 = a_0 + \sum_{i=1}^p a_i \varepsilon_{t-i}^2 + \sum_{j=1}^s b_j \sigma_{t-j}^2, \quad (1.1)$$

where  $\{\eta_t\}$  is a sequence of i.i.d. symmetric random variables with unit variance. Under some regularity conditions  $\{\varepsilon_t\}$  has a regularly-varying tail probability, which can be used to capture the heavy-tailed properties of  $\{\varepsilon_t\}$ , see for example [21,2].

However, the GARCH process  $\{\varepsilon_t\}$  given by (1.1) is usually  $\beta$ -mixing, which is inadequate to account for the strong dependent feature of the data, see [5]. To capture the long-memory feature, Baillie, Chung and Ties [1] proposed a fractional autoregressive integrated moving average (ARFIMA)-GARCH model:

$$\phi(B)(1-B)^d X_t = \psi(B) \varepsilon_t, \quad (1.2)$$

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where  $\{\varepsilon_t\}$  is a GARCH process as defined in (1.1). This model has been extensively studied. For example, Baillie, Chung and Ties [1] used it to model the monthly post-World War II consumer price index inflation series of 10 different countries. Ling and Li [20] considered the asymptotic properties of the maximum likelihood estimate. Beran and Feng [3] considered a local polynomial estimation of semiparametric models with ARFIMA-GARCH errors. Ling [19] studied the adaptive estimation and applied this model to analyze the US consumer price index inflation series. More information about this model can be found in [19] and the references therein. However, as far as we know, all these papers only consider the long-memory feature but not the heavy-tailedness.

A more general model that captures both the long-memory and the heavy-tailed feature is a linear process with GARCH innovations given by

$$X_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \quad (1.3)$$

where  $\varepsilon_t = \sigma_t \eta_t$  is a GARCH( $p, q$ ) model given in (1.1) and  $\{c_j\}$  is a sequence of constants. To simplify the explanation, we focus on the familiar case when  $\{\varepsilon_t\}$  is a GARCH(1, 1) process given by

$$\varepsilon_t = \sigma_t \eta_t, \quad \sigma_t^2 = \omega + a\sigma_{t-1}^2 + b\varepsilon_{t-1}^2 \quad (1.4)$$

for some  $\omega > 0$ ,  $a \geq 0$  and  $b \geq 0$ . For model (1.4), it follows from Kesten [16] that under conditions H2 given in Section 2, there exists a constant  $\alpha > 0$  such that

$$E(a + b\eta_t^2)^{\alpha/2} = 1. \quad (1.5)$$

Moreover, Goldie [13] showed that there exists a positive constant  $c$  such that

$$P(\sigma_t^2 > x) = cx^{-\alpha/2}\{1 + o(1)\}, \quad \text{as } x \rightarrow \infty,$$

which gives

$$P(|\varepsilon_1| > x) = P(|\sigma_1 \eta_1| > x) = (E|\eta_1|^\alpha)P(\sigma_1^2 > x^2) = (E|\eta_1|^\alpha)cx^{-\alpha}\{1 + o(1)\} \quad (1.6)$$

as  $x \rightarrow \infty$ , provided  $E|\eta_1|^\alpha < \infty$ , see [4] for further details. Hence  $P(|\varepsilon_1| > x)$  is regularly varying with index  $\alpha$ , i.e.,  $\lim_{t \rightarrow \infty} P(|\varepsilon_1| > tx)/P(|\varepsilon_1| > t) = x^{-\alpha}$  for  $x > 0$ . When the coefficients  $c_j$  satisfy  $c_j = O(j^{-\beta}l(j))$  as  $j \rightarrow \infty$  for some slowly varying function  $l(x)$  and  $1/\alpha < \beta < 1$ , then  $\{X_t\}$  is said to be a long-memory process. This definition is similar to the one studied in [18] with i.i.d. innovations. As a result,  $\{X_t\}$  defined by (1.3) can be used to model not only long-memory and short-memory effects, but also the heavy-tailed feature.

The main concern of this paper is to develop the limiting properties of quadratic forms for the model (1.3). The asymptotic behavior of quadratic forms has long been a vibrant topic of intense research in modern probability and statistics. For example, asymptotic properties of sample autocovariance functions or periodograms of a general time series usually involve higher order moments expansion, which in turn requires sample behaviors of functions of quadratic forms. It is therefore important to pursue the study of the limiting behavior of quadratic forms of general linear processes like (1.3). Specifically, consider the following quadratic forms:

$$Q_n = \sum_{t=1}^n \sum_{s=1}^n b(t-s)X_t X_s,$$

where  $\{X_t\}$  is given by (1.3) with GARCH(1, 1) innovations  $\{\varepsilon_t\}$  and the weights  $\{b(t)\}$  are real-valued coefficients with  $b(-t) = b(t)$  for all  $t$ . The quadratic form  $Q_n$  plays many important roles, for example, in the Whittle likelihood procedure one often needs to study the order of magnitude of the dominant term governed by  $Q_n$ . This quadratic form has been studied extensively in the literature. For example, Fox and Taqqu [10] considered the asymptotic behavior of  $Q_n$  when  $\{X_t\}$  is a long-memory Gaussian process. Giraitis and Surgailis [12] extended the result of Fox and Taqqu to the cases when  $\{\varepsilon_t\}$  are i.i.d. and have finite fourth moment. Horváth and Shao [14] established the strong and weak approximation of  $Q_n$  when  $\{\varepsilon_t\}$  are i.i.d. and have finite  $4 + \iota$  ( $\iota > 0$ ) moment. Wu and Shao [24] considered the case when  $\{\varepsilon_t\}$  are martingale differences also with finite  $4 + \iota$  ( $\iota > 0$ ) moment and Kokoszka and Taqqu [17] considered the case when  $\{\varepsilon_t\}$  are i.i.d. symmetric random variables with finite variance or in the domain of normal attraction of a stable law with  $1 < \alpha < 2$ . The purpose of this paper is to consider the case when  $\{\varepsilon_t\}$  are heavy-tailed martingale innovations satisfying a GARCH(1, 1) process. When  $\{\varepsilon_t\}$  is a GARCH(1, 1) process with tail probability (1.6), it is difficult to obtain the limit distribution of the sum of the cross-product terms  $\sum_t \sum_{s \neq t} b(s-t)X_s X_t$ . In this paper, by means of a point process convergence technique for partial sum of heavy-tailed random variables (cf. [9]), we study the asymptotic behavior of  $Q_n$ . We show that under some regularity conditions and when  $1 < \alpha < 4$ , the limit distribution of  $Q_n$  converges to a stable random variable with index  $\alpha/2$ . When  $\alpha \geq 4$ ,  $Q_n$  has an asymptotic normal distribution.

The paper is organized as follows. Section 2 gives the main results. Section 3 concludes. Proofs are given in the Appendix.

## 2. Main result

In this section, we establish the limit distribution of  $Q_n$ . We impose the following assumptions:

- H1.  $E \log(a + b\eta_t^2) < 0$ ;  
 H2. There exists a  $k_0 > 0$  such that  $E(a + b\eta_t^2)^{k_0} \geq 1$  and  $E(a + b\eta_t^2)^{k_0} \log^+(a + b\eta_t^2) < \infty$ , where  $\log^+(x) = \max\{0, \log(x)\}$ ;  
 H3. The density  $g(x)$  of  $\varepsilon_1$  is positive in a neighborhood of zero;  
 H4.  $b(j) = O(j^{-\gamma} L(j))$  as  $j \rightarrow \infty$  for some slowly varying function  $L(\cdot)$ ,  $2\beta + \gamma > \max\{2 + 1/\alpha, 5/2\}$ ,  $\beta > \max\{3/(2\alpha), 3/4\}$  and  $\gamma > \max\{1/\alpha, 1/2\}$ .

**Remark 2.1.** Condition H1 is a necessary and sufficient condition for the existence of a stationary solution of  $\sigma_t^2$  (see [22]). Condition H2 is used to obtain the tail probability for  $\sigma_t^2$ , see [13]. If condition H2 holds, then condition H1 is equivalent to  $E(a + b\eta_1^2)^\mu < 1$  for some  $\mu > 0$ , see Remark 2.9 of Basrak, Davis and Mikosch [2].

**Remark 2.2.** Suppose that there exists a  $h_0 > 0$  such that  $E|\eta_t|^{h_0} = \infty$  and  $E|\eta_t|^h < \infty$  for all  $h < h_0$ , then conditions H1 and H2 are satisfied.

**Remark 2.3.** Condition H3 is a sufficient condition for the strong mixing property of a GARCH(1, 1) process. For more information, we refer to Francq and Zakoian [11].

**Remark 2.4.** The condition  $2\beta + \gamma > \max\{2 + 1/\alpha, 5/2\}$  in H4 is also required in other papers, see [17] for  $\alpha < 2$  and [12] for cases with finite variance. Assumption  $\beta > \max\{3/(2\alpha), 3/4\}$  is needed to deal with the cross-product terms, such as  $b(0) \sum_{t=1}^n \sum_{j=-\infty}^t \sum_{h=1}^{\infty} c_{t-j+h} c_{t-j} \varepsilon_j \varepsilon_{j-h}$ , where  $\varepsilon_j$  and  $\varepsilon_{j-h}$  ( $h \geq 1$ ) are no longer independent.

Before presenting the main results, first define some notations. Let

$$a_k = \sum_{j=0}^{\infty} \sum_{h=1}^{\infty} c_j c_{j+h} (b(k-h) + b(k+h)) + b(0) \sum_{j=0}^{\infty} c_j c_{j+k}; \quad (2.1)$$

$$A_t = a + b\eta_t^2;$$

$$c(\alpha) = E \left( [w + A_t \sigma_t^2]^{\alpha/2} - [A_t \sigma_t^2]^{\alpha/2} \right) / \left[ \frac{1}{2} \alpha E(A_t^{\alpha/2} \log^+ A_t) \right];$$

$$d_k(\alpha) = c(\alpha) E[\eta_1^2 A_1]^{\frac{\alpha}{4}} E|\eta_1|^{\frac{\alpha}{2}} \left[ EA_1^{\frac{\alpha}{4}} \right]^{k-1};$$

$$C_1 = \left( 4 \sum_{k=1}^{\infty} a_k^2 d_k(4) \right)^{1/2};$$

$$C_2 = \lim_{n \rightarrow \infty} n^{-1} \text{Var} \left\{ a_0^2 \sum_{i=1}^n \varepsilon_i^2 + 2 \sum_{k=1}^{n-1} \sum_{t=1}^{n-k} a_k \varepsilon_t \varepsilon_{t+k} \right\}.$$

**Theorem 2.1.** Under the conditions of H1 to H4, there exists a  $\alpha > 0$  such that

$$E(a + b\varepsilon_1^2)^\alpha = 1. \quad (2.2)$$

Further, we have the following claims.

(a) If  $1 < \alpha < 4$ , then

$$n^{-2/\alpha} (Q_n - B_n) \xrightarrow{\mathcal{D}} a_0 S_0 + 2 \sum_{k=1}^{\infty} a_k (d_k(\alpha))^{2/\alpha} S_k, \quad (2.3)$$

where  $\xrightarrow{\mathcal{D}}$  denote the weak convergence and for any  $m$ ,  $(S_1, S_2, \dots, S_m)$  is an  $m$ -dimensional stable vector with index  $\alpha/2$  and

$$B_n = \begin{cases} 0, & \text{if } 1 < \alpha < 2, \\ n \log n \left( a_0 c(2) + 2 \sum_k a_k d_k(2) \right), & \text{if } \alpha = 2, \\ EQ_n, & \text{if } 2 < \alpha < 4. \end{cases}$$

(b) If  $\alpha = 4$ , then

$$(n \log n)^{-1/2} \left( Q_n - a_0 \sum_{t=1}^n \varepsilon_t^2 \right) \xrightarrow{\mathcal{D}} C_1 N. \quad (2.4)$$

(c) If  $\alpha > 4$ , then

$$n^{-1/2} (Q_n - \mathbb{E}Q_n) \xrightarrow{\mathcal{D}} \sqrt{C_2} N, \quad (2.5)$$

where  $N$  in (b) and (c) is a standard normal random variable.

Let  $C_3 = (4 \sum_{h=1}^{\infty} \sum_{j=0}^{\infty} (c_j c_{j+h})^2 d_h(4))^{1/2}$  and  $V_n = \sum_{t=1}^n X_t^2 / n$  be the sample variance of  $X_t$ . Then  $V_n$  is a commonly used estimate of the variance of  $X_t$ . We then have the asymptotic result for the sample variance  $V_n$  as follows.

**Corollary 2.1.** Under the conditions of Theorem 2.1, we have

(a) If  $1 < \alpha < 4$ , then

$$n^{1-2/\alpha} (V_n - D_n) \xrightarrow{\mathcal{D}} \left( \sum_{j=0}^{\infty} c_j^2 \right) S_0 + 2 \sum_{h=1}^{\infty} \sum_{j=0}^{\infty} c_j c_{j+h} (d_h(\alpha))^{2/\alpha} S_h,$$

where  $D_n = 0$  when  $1 < \alpha < 2$ ,  $D_n = n \log n (c_0 + 2 \sum_{h=1}^{\infty} \sum_{j=0}^{\infty} c_j c_{j+h} d_h(2))$  when  $\alpha = 2$  and  $D_n = \mathbb{E}V_n$  when  $2 < \alpha < 4$ .

(b) If  $\alpha = 4$ , then

$$n^{1/2} (\log n)^{-1/2} \left( V_n - \frac{\nu}{n} \sum_{t=1}^n \varepsilon_t^2 \right) \xrightarrow{\mathcal{D}} C_3 N,$$

where  $\nu = \sum_{j=0}^{\infty} c_j^2$ .

(c) If  $\alpha > 4$ , then

$$(V_n - \mathbb{E}V_n) / \sqrt{\text{Var}(V_n)} \xrightarrow{\mathcal{D}} N.$$

### 3. Proofs of Theorem 2.1

Eq. (2.2) follows by virtue of Kesten's stochastic recurrence equation [16], see for example, Chan and Zhang [6]. To show Eqs. (2.3)–(2.5), we first present the main idea as follows.

First, we show that analyzing  $Q_n$  is tantamount to analyzing

$$Q_n \text{ is determined by } a_0 \sum_{t=1}^n \varepsilon_t^2 + 2 \sum_{k=1}^{n-1} a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k}, \quad (3.1)$$

where  $a_k$  is defined in (2.1) and satisfies

$$a_k = O(\max\{k^{-\gamma+\delta}, k^{-2\beta-\gamma+2+\delta}, k^{-2\beta+1+\delta}\}).$$

To this end, we apply the Fourier transform to decompose  $Q_n$ . In particular, write  $f(\lambda) = \frac{1}{2\pi} |\sum_{j=0}^{\infty} c_j e^{-i\lambda j}|^2$ ,  $\widehat{b}(\lambda) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} b(j) e^{-i\lambda j}$  and  $I_{n,Z}(\lambda) = |\sum_{t=1}^n Z_t e^{-i\lambda t}|^2$ , then

$$\begin{aligned} Q_n &= \sum_{t=1}^n \sum_{s=1}^n \int_{-\pi}^{\pi} \widehat{b}(\lambda) e^{i\lambda(t-s)} d\lambda X_t X_s \\ &= \int_{-\pi}^{\pi} \widehat{b}(\lambda) \left| \sum_{t=1}^n X_t e^{-i\lambda t} \right|^2 d\lambda = \int_{-\pi}^{\pi} \widehat{b}(\lambda) I_{n,X}(\lambda) d\lambda. \end{aligned}$$

Since

$$\begin{aligned} \sum_{t=1}^n X_t e^{-i\lambda t} &= \sum_{t=1}^n \sum_{j=0}^{\infty} c_j e^{-i\lambda j} \varepsilon_{t-j} e^{-i\lambda(t-j)} \\ &= \sum_{j=0}^{\infty} c_j e^{-i\lambda j} \sum_{t=1}^n \varepsilon_t e^{-i\lambda t} + \sum_{j=0}^{\infty} c_j e^{-i\lambda j} \left( \sum_{t=1}^n \varepsilon_{t-j} e^{-i\lambda(t-j)} - \sum_{t=1}^n \varepsilon_t e^{-i\lambda t} \right) \\ &=: \sum_{j=0}^{\infty} c_j e^{-i\lambda j} \sum_{t=1}^n \varepsilon_t e^{-i\lambda t} + Y_n(\lambda), \end{aligned}$$

define

$$R_n(\lambda) = |Y_n(\lambda)|^2 + Y_n(\lambda) \left( \sum_{j=0}^{\infty} c_j e^{i\lambda j} \sum_{t=1}^n \varepsilon_t e^{i\lambda t} \right) + Y_n(-\lambda) \left( \sum_{j=0}^{\infty} c_j e^{-i\lambda j} \sum_{t=1}^n \varepsilon_t e^{-i\lambda t} \right),$$

it follows that  $I_{n,X}(\lambda) = 2\pi f(\lambda)I_{n,\varepsilon}(\lambda) + R_n(\lambda)$  and

$$\begin{aligned} Q_n &= 2\pi \int_{-\pi}^{\pi} \widehat{b}(\lambda) f(\lambda) I_{n,\varepsilon}(\lambda) d\lambda + \int_{-\pi}^{\pi} \widehat{b}(\lambda) R_n(\lambda) d\lambda \\ &= a_0 \sum_{t=1}^n \varepsilon_t^2 + 2 \sum_{k=1}^{n-1} a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k} + \int_{-\pi}^{\pi} \widehat{b}(\lambda) R_n(\lambda) d\lambda. \end{aligned} \quad (3.2)$$

Then (3.1) follows by virtue of (3.2) and showing the fact that  $\beta_n^{-1} \int_{-\pi}^{\pi} \widehat{b}(\lambda) R_n(\lambda) d\lambda \xrightarrow{p} 0$ , where  $\beta_n = n^{2/\alpha}$  if  $1 < \alpha \leq 4$  and  $\beta_n = n^{1/2}$  if  $\alpha > 4$ . Since the proof is extremely technical, we put it in Lemmas A.2 and A.3 in the Appendix.

Second, we establish the asymptotic distribution of  $a_0 \sum_{t=1}^n \varepsilon_t^2 + 2 \sum_{k=1}^{n-1} a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k}$  via point process convergence (see Lemma A.4 for  $\alpha < 4$ ). To achieve this, we adopt the following two-step approach.

(i) First, show that  $a_0 \sum_{t=1}^n \varepsilon_t^2 + 2 \sum_{k=1}^{n-1} a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k}$  has the same asymptotic distribution as that of  $a_0 \sum_{t=1}^n \varepsilon_t^2 + 2 \sum_{k=1}^m a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k}$  for  $m$  large, independent of  $n$ .

(ii) Second, derive the asymptotic distribution of  $a_0 \sum_{t=1}^n \varepsilon_t^2 + 2 \sum_{k=1}^m a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k}$  by Lemma A.4 (which is obtained via point process convergence) and Theorem 2.1 of [25] for

$$\left( \sum_{t=1}^n \varepsilon_t^2, \dots, \sum_{t=1}^n \varepsilon_t \varepsilon_{t+m} \right)$$

and the continuous mapping theorem.

Now, we need to show that for  $1 < \alpha < 4$ ,

$$n^{-2/\alpha} \left( a_0 \sum_{t=1}^n \varepsilon_t^2 + 2 \sum_{k=1}^{n-1} a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k} - B_n \right) \xrightarrow{\mathcal{D}} a_0 S_0 + 2 \sum_{k=1}^{\infty} a_k (d_k(\alpha))^{2/\alpha} S_k. \quad (3.3)$$

Let

$$B^*(n, m) = \begin{cases} 0, & \text{if } 1 < \alpha < 2, \\ B(n, m), & \text{if } 2 \leq \alpha < 4. \end{cases}$$

From Lemma A.4, it follows that when  $1 < \alpha < 4$ ,

$$\frac{1}{n^{2/\alpha}} \left( \sum_{t=1}^n \varepsilon_t^2, \sum_{t=1}^n \varepsilon_t \varepsilon_{t+1}, \dots, \sum_{t=1}^n \varepsilon_t \varepsilon_{t+m} \right) - B^*(n, m) \xrightarrow{\mathcal{D}} (S^0, S^1, \dots, S^m). \quad (3.4)$$

By (3.4) and the continuous mapping theorem, it follows that

$$\begin{aligned} S_{n,m} &=: \frac{1}{n^{2/\alpha}} \left( a_0 \sum_{t=1}^n \varepsilon_t^2 + 2 \sum_{k=1}^m a_k \sum_{t=1}^n \varepsilon_t \varepsilon_{t+k} \right) \\ &\xrightarrow{\mathcal{D}} a_0 S^0 + 2 \sum_{k=1}^m a_k S^k =: S_m. \end{aligned} \quad (3.5)$$

For any  $p < \min\{\alpha/2, 1\}$ , we have

$$\begin{aligned} P \left\{ \left| \frac{1}{n^{2/\alpha}} \left( a_0 \sum_{t=1}^n \varepsilon_t^2 + 2 \sum_{k=1}^m a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k} \right) - S_{n,m} \right| > \delta \right\} &= P \left\{ \left| \frac{1}{n^{2/\alpha}} \left| \sum_{k=1}^m a_k \sum_{t=n-k+1}^n \varepsilon_t \varepsilon_{t+k} \right| \right| > \delta \right\} \\ &\leq \frac{1}{n^{2p/\alpha} \delta^p} \sum_{k=1}^m k |a_k|^p E |\varepsilon_1 \varepsilon_{1+k}|^p \\ &\leq \frac{C}{n^{2p/\alpha} \delta^p} \sum_{k=1}^m k |a_k|^p \rightarrow 0, \end{aligned} \quad (3.6)$$

holds uniformly for all fixed integer  $m$ . On the other hand, note that

$$\begin{aligned} \frac{1}{n^{2/\alpha}} \left( \sum_{k=1}^{n-1} a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k} - \sum_{k=1}^m a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k} \right) &= \frac{1}{n^{2/\alpha}} \sum_{i=m+2}^n \sum_{j=m+1}^{i-1} a_j \varepsilon_{i-j} \varepsilon_i \\ &= \frac{1}{n^{2/\alpha}} \sum_{i=m+2}^n \sum_{j=m+1}^{i-1} a_j \varepsilon_{i-j} \varepsilon_i \{I(|\varepsilon_{i-j} \sigma_i| \leq n^{2/\alpha}) + I(|\varepsilon_{i-j} \sigma_i| > n^{2/\alpha})\} \\ &=: H_1 + H_2. \end{aligned}$$

By condition H4 and taking  $\delta$  sufficiently small, we have

$$\begin{aligned} EH_1^2 &= \frac{1}{n^{4/\alpha}} \sum_{i=m+2}^n \sum_{j=m+1}^{i-1} a_j^2 E\{(\varepsilon_{i-j} \varepsilon_i)^2 I(|\varepsilon_{i-j} \sigma_i| \leq n^{2/\alpha})\} \\ &\leq C' n^{-1} \sum_{i=m+2}^n \sum_{j=m+1}^{i-1} a_j^2 \\ &\leq C'' \max\{m^{-2\gamma+1+2\delta}, m^{-4\beta-2\gamma+5+2\delta}, m^{-4\beta+3+2\delta}\} \rightarrow 0 \end{aligned} \quad (3.7)$$

as  $m \rightarrow \infty$ . Further, similar to (A.28) and (A.29) (see Appendix), there exist two constants  $c_1$  and  $c_2$  such that when  $x$  is large enough,

$$\begin{aligned} P(|\varepsilon_{t-h} \sigma_t| > x) &= P(\varepsilon_{t-h}^2 \sigma_t^2 > x^2) \\ &\leq P\left\{\omega \left[1 + \sum_{k=t-h+1}^t \prod_{m=k+1}^t A_m\right] \eta_{t-h}^2 \sigma_{t-h}^2 > x^2/2\right\} + P\left\{\prod_{k=t-h+1}^t A_k \eta_{t-h}^2 \sigma_{t-h}^4 > x^2/2\right\} \\ &\leq c_1 x^{-\alpha} + c_2 \rho^h x^{-\alpha/2}. \end{aligned}$$

This gives that for any  $p < \min\{\alpha/2, 1\}$ ,

$$\begin{aligned} E|H_2|^p &\leq n^{-2p/\alpha} \sum_{i=m+2}^n \sum_{j=m+1}^{i-1} |a_j|^p E\{|\varepsilon_{i-j} \varepsilon_i|^p I(|\varepsilon_{i-j} \sigma_i| > n^{2/\alpha})\} \\ &\leq C n^{-2p/\alpha} \sum_{i=m+2}^n \sum_{j=m+1}^{i-1} |a_j|^p \{n^{2(p-\alpha)/\alpha} + \rho^j n^{2(p-\alpha/2)/\alpha}\} \\ &\leq C \max\{m^{p(-\gamma+\delta)}, m^{p(-2\beta+1+\delta)}, m^{p(-2\beta-\gamma+2+\delta)}\} + n^{-1} \sum_{i=m+2}^n \sum_{j=m+1}^{i-1} |a_j|^p \rho^j \\ &\rightarrow 0, \end{aligned} \quad (3.8)$$

by condition H4 and taking  $m$  large and  $\delta$  small. Thus, when  $1 < \alpha < 4$  and  $m$  is large,

$$\frac{1}{n^{2/\alpha}} \left( \sum_{k=1}^{n-1} a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k} - \sum_{k=1}^m a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k} \right) \xrightarrow{p} 0.$$

Combining this with (3.5) and (3.6) gives that when  $m$  is large,  $n^{-2/\alpha} (a_0 \sum_{t=1}^n \varepsilon_t^2 + 2 \sum_{k=1}^{n-1} a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k})$  has the same distribution as  $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} S_{n,m}$  if

$$\lim_{m \rightarrow \infty} S_m = a_0 S^0 + 2 \sum_{h=1}^{\infty} a_h S^h = a_0 \sum_{i,j \geq 1} P_i^2 (Q_{ij}^0)^2 + 2 \sum_{h=1}^{\infty} a_h \sum_{i,j \geq 1} P_i^2 (Q_{ij}^0 Q_{ij}^h) =: Z_{\frac{\alpha}{2}} \quad (3.9)$$

is well defined. By (A.30) and the fact that  $\sum_{t=1}^n (\varepsilon_t \varepsilon_{t-h})/n^{2/\alpha} \Rightarrow S^h$ , we know that  $S^h$  have the same distribution of  $c \rho^h S^0$  for some constant  $c$ . Thus, for any  $\delta > 0$  and  $\gamma < \alpha/2 < 1$ ,

$$\begin{aligned} P\left(\left|\sum_{h=m}^{\infty} a_h \sum_{i,j \geq 1} P_i^2 (Q_{ij}^0 Q_{ij}^h)\right| > \delta\right) &\leq \delta^{-\gamma} \sum_{h=m}^{\infty} |a_h|^\gamma E\left|\sum_{i,j \geq 1} P_i^2 (Q_{ij}^0 Q_{ij}^h)\right|^\gamma \\ &\leq C \delta^{-\gamma} \sum_{h=m}^{\infty} \left(\rho^{\frac{2h}{\alpha}}\right)^\gamma \xrightarrow{m \rightarrow \infty} 0, \end{aligned}$$

by Property 1.2.17 of [23]. This completes the proof of (3.3). By (3.1), we have part (a) of Theorem 2.1.

We now turn to part (b) to show that when  $\alpha = 4$ ,

$$(n \log n)^{-1/2} 2 \sum_{k=1}^{n-1} a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k} \xrightarrow{\mathcal{D}} C_1 N. \quad (3.10)$$

Set  $d_0(2) = c(2)$  and  $d_h(2)$  defined as in Section 2. For  $\alpha = 4$ , we have for  $h \geq 0$ ,

$$\lim_{x \rightarrow \infty} x^2 P(|\varepsilon_1 \varepsilon_{1+h}| > x) = d_h(2). \quad (3.11)$$

For any  $m$ , let  $\mathbf{Z}_{t,m} = (\varepsilon_t \varepsilon_{t+1}, \varepsilon_t \varepsilon_{t+2}, \dots, \varepsilon_t \varepsilon_{t+m})$ , then the random vectors  $\mathbf{Z}_{t,m}$ ,  $t = 1, 2, \dots$  are regularly varying with index 2. By Theorem 2.1 of [25], we have that for all  $h \geq 1$ ,

$$\frac{1}{\sqrt{d_h(2)n \log n}} \sum_{t=1}^n (\varepsilon_t \varepsilon_{t+h} - E \varepsilon_t \varepsilon_{t+h}) \xrightarrow{\mathcal{D}} N.$$

This implies that

$$\frac{1}{\sqrt{n \log n}} \sum_{t=1}^n (\mathbf{Z}_{t,m} - E \mathbf{Z}_{t,m}) \xrightarrow{\mathcal{D}} (d_1(2)N_1, \dots, d_m(2)N_m), \quad (3.12)$$

where  $N_i$ ,  $i \geq 1$  are standard normal variables. Using (3.11), (3.12) and the argument for the case  $1 < \alpha < 4$ , we have that

$$\frac{2}{\sqrt{n \log n}} \sum_{k=1}^{n-1} a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k} \xrightarrow{\mathcal{D}} C_1 N.$$

This gives (3.10) as desired and completes the proof of part (b) of Theorem 2.1 by (3.1).

Finally, we consider part (c) for the case  $\alpha > 4$  to establish that

$$n^{-1/2} \left( a_0 \sum_{t=1}^n \varepsilon_t^2 + 2 \sum_{k=1}^{n-1} a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k} - EQ_n \right) \xrightarrow{\mathcal{D}} C_2 N. \quad (3.13)$$

Let  $\mathbf{Y}_{t,m} = (\varepsilon_t^2, \varepsilon_t \varepsilon_{t+1}, \dots, \varepsilon_t \varepsilon_{t+m})$ . When  $\alpha > 4$ , then  $\mathbf{Y}_{t,m}$ ,  $t = 1, 2, \dots$  are regularly varying index with index  $\alpha/2 > 2$  and

$$\lim_{x \rightarrow \infty} x^{\alpha/2} P(|\varepsilon_1 \varepsilon_{1+h}| > x) = d_h(\alpha). \quad (3.14)$$

From (3.14), it follows that there exists a  $\delta > 0$  such that for any  $h \geq 0$ ,  $E|\varepsilon_1 \varepsilon_{1+h}|^{2+\delta} < \infty$ . By Lemma A.1, we see that  $\{\varepsilon_t \varepsilon_{t+h}\}$  is a stationary  $\beta$ -mixing process with exponential decay. It follows from the CLT of stationary process (cf, Theorem 1.7 of [15]) that

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n (\varepsilon_t \varepsilon_{t+h} - E \varepsilon_1 \varepsilon_{1+h}) \xrightarrow{\mathcal{D}} \sigma_h N, \quad (3.15)$$

where  $\sigma_h^2 = E(\varepsilon_t \varepsilon_{t+h})^2$ . This give that

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n (\mathbf{Y}_{t,m} - E \mathbf{Y}_{t,m}) \xrightarrow{\mathcal{D}} (\sigma_0 N_0, \sigma_1 N_1, \dots, \sigma_m N_m).$$

Combining this with the continuous mapping theorem gives that

$$\begin{aligned} S_{n,m} &:= \frac{1}{n^{1/2}} \left( a_0 \sum_{t=1}^n (\varepsilon_t^2 - E \varepsilon_1^2) + 2 \sum_{k=1}^m a_k \sum_{t=1}^n (\varepsilon_t \varepsilon_{t+k} - E \varepsilon_1 \varepsilon_{1+k}) \right) \\ &\xrightarrow{\mathcal{D}} a_0 N_0 + 2 \sum_{k=1}^m a_k N_k =: S_m \quad (\text{a normal random variable}). \end{aligned} \quad (3.16)$$

Using (3.14) and (3.16), along the same argument as in the case  $1 < \alpha < 4$ , we obtain (3.13) and part (c) of Theorem 2.1 by (3.1). This completes the proof of Theorem 2.1.  $\square$

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## Appendix. Technical lemmas

In this section, we furnish the technical lemmas to complete the proof of [Theorem 2.1](#). We first show that  $\{\varepsilon_t\}$  of model (1.4) is a strong mixing stationary process with regularly varying tails.

**Lemma A.1.** Under conditions H1, H2 and H3, we have

(a)  $\{\varepsilon_t\}$  is a  $\beta$ -mixing stationary sequence with

$$\beta_k = \mathbb{E} \left[ \sup_{B \in \sigma(\varepsilon_t, t \geq k)} |P(B|\sigma(\varepsilon_s, s \leq 0)) - P(B)| \right] = O(\theta^k)$$

for some  $0 < \theta < 1$ .

(b)  $\sigma_t^2$  has a unique strictly stationary solution and  $\sigma_t^2$  satisfies the following regularly varying condition

$$\lim_{x \rightarrow \infty} x^{\alpha/2} P\{\sigma_t^2 > x\} = c(\alpha), \quad (\text{A.1})$$

where  $c(\alpha)$  is defined in Section 2.

**Proof.** Observe that  $\sigma_t^2 = \omega + a\sigma_{t-1}^2 + b\varepsilon_{t-1}^2 = \omega + (a + b\eta_{t-1}^2)\sigma_{t-1}^2$ . Conclusion (a) follows from Theorem 3 of [11] and conclusion (b) follows from Theorem 4.1 of [13] (see also [16]).  $\square$

**Lemma A.2.** Under the conditions of [Theorem 2.1](#), as  $n \rightarrow \infty$ ,

$$\beta_n^{-1} \int_{-\pi}^{\pi} \widehat{b}(\lambda) |Y_n(\lambda)|^2 d\lambda \xrightarrow{p} 0, \quad (\text{A.2})$$

where  $\xrightarrow{p}$  denotes convergence in probability.

**Proof.** Split  $Y_n(\lambda)$  as in [17]:  $Y_n(\lambda) = \sum_{u=1}^4 \Gamma_{un}$ , where

$$\begin{aligned} \Gamma_{1n} &= \sum_{k=0}^{n-2} \left( \sum_{j=k+1}^{n+k} c_j e^{i(k-j)\lambda} \right) \varepsilon_{-k}, \\ \Gamma_{2n} &= \sum_{k=n-1}^{\infty} \left( \sum_{j=k+1}^{n+k} c_j e^{i(k-j)\lambda} \right) \varepsilon_{-k}, \\ \Gamma_{3n} &= -e^{-in\lambda} \sum_{k=0}^{n-2} \left( \sum_{j=k+1}^{n+k} c_j e^{i(k-j)\lambda} \right) \varepsilon_{-k}, \\ \Gamma_{4n} &= - \left( \sum_{j=n}^{\infty} c_j e^{i\lambda j} \right) \sum_{k=1}^n \varepsilon_k e^{-i\lambda k}. \end{aligned}$$

To prove (A.2), it suffices to show that when  $\alpha > 1$ ,

$$\beta_n^{-1} \int_{-\pi}^{\pi} |\widehat{b}(\lambda)| |\Gamma_{un}(\lambda)|^2 d\lambda \xrightarrow{p} 0, \quad \text{for } u = 1, 2, 3, 4. \quad (\text{A.3})$$

First consider the case  $u = 1$ . Note that

$$\int_{-\pi}^{\pi} |\widehat{b}(\lambda)| |\Gamma_{1n}(\lambda)|^2 d\lambda = \sum_{k=0}^{n-2} v_n(k, k) \varepsilon_{-k}^2 + \sum_{k, t=0, k \neq t} v_n(k, t) \varepsilon_{-k} \varepsilon_{-t}, \quad (\text{A.4})$$

where  $v_n(k, t) = \int_{-\pi}^{\pi} \left( \sum_{j=k+1}^{n+k} c_j e^{i\lambda(k-j)} \right) \left( \sum_{j=t+1}^{n+t} c_j e^{i\lambda(j-t)} \right) |\widehat{b}(\lambda)| d\lambda$ . Since as  $j \rightarrow \infty$ ,  $b(j) = O(j^{-\gamma} L(j))$ , it follows that  $|\widehat{b}(\lambda)| = |\sum_j b(j) e^{-i\lambda j}| = O(\lambda^{-1+\gamma} L(1/\lambda))$  as  $\lambda \rightarrow 0$ . Along the line of proof of Proposition 2.1 in [17], we have for any  $\delta > 0$ ,

$$|v_n(k, t)| \leq C \begin{cases} (k \vee t)^{-\beta+\delta} n^{2-\beta-\gamma+\delta}, & \text{if } \beta + \gamma \leq 2. \\ (k \vee t)^{-\beta+\delta}, & \text{if } \beta + \gamma \geq 2. \end{cases}$$

We now show that (A.4) holds for two situations: (a)  $\alpha \leq 4$  and (b)  $\alpha > 4$ .



(a) When  $\alpha \leq 4$ , we first consider the bound of the first term  $\sum_{k=0}^{n-2} v_n(k, k) \varepsilon_{-k}^2$ . It follows from Lemma A.1 that  $P(|\varepsilon_1| > x) \sim c(\alpha) E|\eta|^\alpha x^{-\alpha}$  (see (1.6)). Thus, for all  $\mu < \alpha$ ,  $E|\varepsilon_1|^\mu < \infty$  and therefore for any  $0 < p < \min\{1, \alpha/2\}$ ,

$$\begin{aligned} E \left| n^{-2/\alpha} \sum_{k=0}^{n-2} v_n(k, k) \varepsilon_{-k}^2 \right|^p &\leq n^{-2p/\alpha} \sum_{k=0}^{n-2} |v_n(k, k)|^p E|\varepsilon_{-k}|^{2p} \\ &\leq C n^{-2p/\alpha} \max\{n^{p(2-\beta-\gamma+\delta)}, 1\} \sum_{k=0}^{n-2} k^{p(-\beta+\delta)} \\ &\leq C' \max\{n^{p(2-2\beta-\gamma-2/\alpha+2\delta)+1}, n^{p(-\beta-2/\alpha+\delta)+1}, n^{p(-2/\alpha-\beta-\gamma+2)} I(\beta > 1)\} \rightarrow 0, \end{aligned} \quad (\text{A.5})$$

by virtue of the conditions that  $2\beta + \gamma > \max\{2 + 1/\alpha, 5/2\}$  and  $\gamma > \max\{1/2, 1/\alpha\}$ .

Next, we turn to estimate the bound of the cross-product term  $\sum_{k,t=0, k \neq t} v_n(k, t) \varepsilon_{-k} \varepsilon_{-t}$  in (A.4) for  $\alpha \leq 4$ . We split the proof into two cases: (i)  $\alpha \leq 2$  and (ii)  $2 < \alpha \leq 4$ .

(i) When  $\alpha \leq 2$ , since  $\{\varepsilon_{-t}\}$  is a martingale difference sequence, it follows from Lemma 1 of [24] that there exist constants  $C, C', C''$  such that for any  $p < \alpha/2 \leq 1$ ,

$$\begin{aligned} E \left| n^{-2/\alpha} \sum_{k,t=0, k \neq t} v_n(k, t) \varepsilon_{-k} \varepsilon_{-t} \right|^p &= 2^p n^{-2p/\alpha} E \left| \sum_{t=1}^{n-2} \sum_{h=1}^{n-2-t} v_n(t+h, t) \varepsilon_{-t-h} \varepsilon_{-t} \right|^p \\ &\leq 2^p n^{-2p/\alpha} \sum_{t=1}^{n-2} E \left| \sum_{h=1}^{n-2-t} v_n(t+h, t) \varepsilon_{-t-h} \varepsilon_{-t} \right|^p \\ &\leq 2^p n^{-2p/\alpha} \sum_{t=1}^{n-2} \left[ E \left| \sum_{h=1}^{n-2-t} v_n(t+h, t) \varepsilon_{-t-h} \right|^{2p} \right]^{1/2} [E|\varepsilon_{-t}|^{2p}]^{1/2} \\ &\leq C n^{-2p/\alpha} \sum_{t=1}^{n-2} \left[ \sum_{h=1}^{n-2-t} E|v_n(t+h, t) \varepsilon_{-t-h}|^{2p} \right]^{1/2} \\ &\leq C' n^{-2p/\alpha} \max\{n^{p(2-\beta-\gamma+\delta)}, 1\} \sum_{t=1}^{n-2} \left[ \sum_{h=1}^{n-2-t} (t+h)^{2p(-\beta+\delta)} \right]^{1/2} \\ &\leq C'' \max\{n^{p(2-2\beta-\gamma-2/\alpha+2\delta)+3/2}, n^{p(-\beta-2/\alpha+\delta)+3/2}\} \rightarrow 0 \end{aligned} \quad (\text{A.6})$$

by virtue of the conditions that  $2\beta + \gamma > \max\{2 + 1/\alpha, 5/2\}$  and  $\beta > 1/\alpha$ . Thus, when  $\alpha \leq 2$ ,

$$n^{-2/\alpha} \int_{-\pi}^{\pi} |\widehat{b}(\lambda)| |\Gamma_{1n}(\lambda)|^2 d\lambda \xrightarrow{p} 0.$$

(ii) When  $2 < \alpha \leq 4$ , we split  $\{\varepsilon_{-t} \varepsilon_{-t-h}\}$  as follows.

$$\begin{aligned} \sum_{k,t=0, k \neq t} v_n(k, t) \varepsilon_{-k} \varepsilon_{-t} &= 2 \sum_{t=1}^{n-2} \sum_{h=1}^{n-2-t} v_n(t+h, t) \varepsilon_{-t-h} \varepsilon_{-t} \left[ I\left(\sigma_{-t-h} > n^{\frac{1}{\alpha}} \text{ or } \sigma_{-t} > n^{\frac{1}{\alpha}}\right) \right. \\ &\quad \left. + I\left(\sigma_{-t-h} \leq n^{\frac{1}{\alpha}}\right) I\left(\sigma_{-t} \leq n^{\frac{1}{\alpha}}\right) \right] \\ &=: 2(L_{1n} + L_{2n}). \end{aligned}$$

Since  $\{\sum_{h=1}^{n-2-t} \varepsilon_{-t-h} I(\sigma_{-t-h} > n^{1/\alpha}) \varepsilon_{-t}\}$  and  $\{\varepsilon_{-t} I(\sigma_{-t} > n^{1/\alpha})\}$  are two martingale difference sequences with respect to the  $\sigma$ -fields  $\sigma\{\eta_s, s \leq -t\}$ , by Lemma 1 of [24] again, we have that for any  $1 < q < \alpha/2$ ,

$$\begin{aligned} E \left| n^{-\frac{2}{\alpha}} L_{1n} \right|^q &\leq E \left| n^{-\frac{2}{\alpha}} \sum_{t=1}^{n-2} \sum_{h=1}^{n-2-t} v_n(t+h, t) \varepsilon_{-t} \varepsilon_{-t-h} I(\sigma_{-t-h} > n^{1/\alpha}) \right|^q \\ &\quad + E \left| n^{-\frac{2}{\alpha}} \sum_{t=1}^{n-2} \sum_{h=1}^{n-2-t} v_n(t+h, t) \varepsilon_{-t} \varepsilon_{-t-h} I(\sigma_{-t} > n^{1/\alpha}) \right|^q \\ &\leq C (E|\varepsilon_1|^{2q})^{1/2} n^{-2q/\alpha} \sum_{t=1}^{n-2} \left[ E \left| \sum_{h=1}^{n-2-t} v_n(t+h, t) \varepsilon_{-t-h} I(\sigma_{-t-h} > n^{1/\alpha}) \right|^{2q} \right]^{1/2} \end{aligned}$$

$$\begin{aligned}
& + Cn^{-2q/\alpha} \sum_{t=1}^{n-2} \left[ \mathbb{E} \left| \sum_{h=1}^{n-2-t} v_n(t+h, t) \varepsilon_{-t-h} \right|^{2q} \right]^{1/2} \left( \mathbb{E} |\varepsilon_1|^{2q} I(\sigma_{-t} > n^{1/\alpha}) \right)^{1/2} \\
& \leq C'n^{-2q/\alpha} \sum_{t=1}^{n-2} \left[ \sum_{h=1}^{n-2-t} |v_n(t+h, t)|^2 [\mathbb{E} |\varepsilon_{-t-h}|^{2q} I(\sigma_{-t-h} > n^{1/\alpha})]^{1/q} \right]^{q/2} \\
& \quad + C'n^{-2q/\alpha} \sum_{t=1}^{n-2} \left[ \sum_{h=1}^{n-2-t} |v_n(t+h, t)|^2 [\mathbb{E} |\varepsilon_{-t-h}|^{2q}]^{1/q} \right]^{q/2} n^{(2q-\alpha)/(2\alpha)} \\
& \leq C'n^{-2q/\alpha} \max\{n^{q(2-\beta-\gamma+\delta)}, 1\} \sum_{t=1}^{n-2} \left[ \sum_{h=1}^{n-2-t} (t+h)^{2(-\beta+\delta)} n^{(2q-\alpha)/(q\alpha)} \right]^{q/2} \\
& \leq C'' \max\{n^{1/2-q(1/\alpha+\beta-1/2-\delta)}, n^{1/2-q(1/\alpha+2\beta+\gamma-5/2-2\delta)}\} \rightarrow 0,
\end{aligned} \tag{A.7}$$

by conditions  $2\beta + \gamma > 5/2$  and  $\beta > 1/2$  and taking  $\delta \rightarrow 0$  and  $q$  near enough to  $\alpha/2$ .

Since  $\mathbb{E}[\varepsilon_{-t-h} I(\sigma_{-t-h} \leq n^{1/\alpha}) \varepsilon_{-s} I(\sigma_{-s} \leq n^{1/\alpha}) \varepsilon_{-s-j} I(\sigma_{-s-j} \leq n^{1/\alpha}) \varepsilon_{-s} I(\sigma_{-s} \leq n^{1/\alpha})] = 0$  if  $s \neq t$  and  $\mathbb{E}[\varepsilon_{-t-h} I(\sigma_{-t-h} \leq n^{1/\alpha}) \varepsilon_{-t-j} I(\sigma_{-t-j} \leq n^{1/\alpha})] = 0$  if  $h \neq j$ , and  $\{\varepsilon_t\}$  is a  $\beta$ -mixing process with exponential decay rate, it follows that for any  $b > 0$ ,

$$\begin{aligned}
& \mathbb{E}[n^{-2/\alpha} L_{2n}]^2 \\
& = \frac{4}{n^{4/\alpha}} \sum_{t=1}^{n-2} \sum_{h=1}^{n-2-t} v_n^2(t+h, t) \mathbb{E} \left[ \varepsilon_{-t-h} I(\sigma_{-t-h} \leq n^{1/\alpha}) \varepsilon_{-t} I(\sigma_{-t} \leq n^{1/\alpha}) \right]^2 \\
& \quad + \frac{8}{n^{4/\alpha}} \sum_{t=1}^{n-2} \sum_{h=1}^{n-2-t} \sum_{l=1}^{n-2-t-h} v_n(t+h, t) v_n(t+h+l, t) \mathbb{E} \left[ \varepsilon_{-t}^2 I(\sigma_{-t} \leq n^{1/\alpha}) \varepsilon_{-t-h} I(\sigma_{-t-h} \leq n^{1/\alpha}) \varepsilon_{-t-h-l} I(\sigma_{-t-h-l} \leq n^{1/\alpha}) \right] \\
& \leq Cn^{-4/\alpha} \max\{n^{2(2-\beta-\gamma+\delta)}, 1\} \left[ n^{(4-\alpha)/\alpha} \sum_{t=1}^{n-2} \sum_{h=1}^{n-2-t} (t+h)^{2(-\beta+\delta)} \right. \\
& \quad + \sum_{t=1}^{n-2} \sum_{h=1}^{n-2-t} \sum_{l=1}^{n-2-t-h} [(t+h)(t+h+l)]^{-\beta+\delta} \theta^{\frac{bh}{2+b}} [\mathbb{E} \sigma_t^{2(2+b)} I(\sigma_t \leq n^{1/\alpha})]^{1/(2+b)} \\
& \quad \left. \times \{\mathbb{E}[\sigma_{-t-h}^{2+b} I(\sigma_{-t-h} \leq n^{1/\alpha}) |\varepsilon_{-t-h-l}|^{2+b} I(\sigma_{-t-h-l} \leq n^{1/\alpha})] \}^{1/(2+b)} \right] \\
& \leq C' \max\{n^{-2\beta+1+2\delta+b/(2+b)}, n^{5-2(2\beta+\gamma)+4\delta+b/(2+b)}\} \rightarrow 0
\end{aligned} \tag{A.8}$$

by conditions  $2\beta + \gamma > 5/2$ ,  $\beta > 1/2$  and taking  $b = \min\{\beta - 1/2, 2\beta + \gamma - 5/2\}$  and  $\delta$  small enough. Thus the bound for the cross-product term is established for  $\alpha \leq 4$ .

(b) For  $\alpha > 4$ , the proof is easier than  $\alpha \leq 4$ . For the first term  $\sum_{k=0}^{n-2} v_n(k, k) \varepsilon_{-k}^2$ , we have

$$\begin{aligned}
\mathbb{E} \left[ n^{-\frac{1}{2}} \sum_{k=0}^{n-2} v_n(k, k) \varepsilon_{-k}^2 \right] & \leq n^{-\frac{1}{2}} \sum_{k=0}^{n-2} |v_n(k, k)| \mathbb{E} |\varepsilon_{-k}|^2 \\
& \leq Cn^{-\frac{1}{2}} \max\{n^{2-\beta-\gamma+\delta}, 1\} \sum_{k=0}^{n-2} k^{-\beta+\delta} \\
& \leq C' \max \left\{ n^{\frac{5}{2}-2\beta-\gamma+2\delta}, n^{\frac{1}{2}-\beta+\delta}, n^{\frac{3}{2}-\beta-\gamma} I(\beta > 1) \right\} \rightarrow 0,
\end{aligned} \tag{A.9}$$

by assumptions  $2\beta + \gamma > 5/2$  and  $\beta, \gamma > 1/2$ .

For the cross-product term  $\sum_{k,t=0, k \neq t} v_n(k, t) \varepsilon_{-k} \varepsilon_{-t}$ , since  $\alpha > 4$ ,  $\mathbb{E} \varepsilon_1^{2(1+\alpha/4)} < \infty$ , it follows that

$$\begin{aligned}
\mathbb{E} \left[ n^{-1/2} \sum_{k,t=0, k \neq t} v_n(k, t) \varepsilon_{-k} \varepsilon_{-t} \right]^2 & = \frac{4}{n} \sum_{t=1}^{n-2} \sum_{h=1}^{n-2-t} v_n^2(t+h, t) \mathbb{E} [\varepsilon_{-t-h} \varepsilon_{-t}]^2 \\
& \quad + \frac{8}{n} \sum_{t=1}^{n-2} \sum_{h=1}^{n-2-t} \sum_{l=1}^{n-2-t-h} v_n(t+h, t) v_n(t+h+l, t) \mathbb{E} [\varepsilon_{-t}^2 \varepsilon_{-t-h} \varepsilon_{-t-h-l}]
\end{aligned}$$

$$\begin{aligned}
&\leq Cn^{-1} \max\{n^{2(2-\beta-\gamma+\delta)}, 1\} \sum_{t=1}^{n-2} \sum_{h=1}^{n-2-t} (t+h)^{2(-\beta+\delta)} \\
&\quad + \sum_{t=1}^{n-2} \sum_{h=1}^{n-2-t} \sum_{l=1}^{n-2-t-h} [(t+h)(t+h+l)]^{-\beta+\delta} \theta^{\frac{h(\frac{\alpha}{4}-1)}{1+\frac{\alpha}{4}}} \\
&\quad \times [E|\varepsilon_t|^{2(1+\frac{\alpha}{4})}]^{\frac{1}{1+\frac{\alpha}{4}}} \left\{ E|\varepsilon_{-t-h}\varepsilon_{-t-h-l}|^{1+\frac{\alpha}{4}} \right\}^{\frac{1}{1+\frac{\alpha}{4}}} \\
&\leq C' \max\{n^{-2\beta+2-1+2\delta}, n^{5-2(2\beta+\gamma)+4\delta}, n^{2(2-\beta-\gamma+\delta)-1} I(\beta > 1)\} \rightarrow 0 \quad (\text{A.10})
\end{aligned}$$

by conditions  $2\beta + \gamma > 5/2$  and  $\beta, \gamma > 1/2$  and taking  $\delta$  small enough. Thus, from (A.9) and (A.10), we have for  $\alpha > 4$ ,

$$n^{-1/2} \int_{-\pi}^{\pi} |\widehat{b}(\lambda)| |\Gamma_{1n}(\lambda)|^2 d\lambda \xrightarrow{p} 0.$$

Second, consider  $u = 2$ . Observe that

$$\int_{-\pi}^{\pi} |\widehat{b}(\lambda)| |\Gamma_{2n}(\lambda)|^2 d\lambda = \sum_{k=n-1}^{\infty} v_n(k, k) \varepsilon_{-k}^2 + \sum_{n-1 \leq k \neq t < \infty} v_n(k, t) \varepsilon_{-k} \varepsilon_{-t}, \quad (\text{A.11})$$

where  $v_n(k, t)$  is given as in (A.4). Similar to Kokoszka and Taqqu [17], we have

$$|v_n(k, t)| \leq C \max\{n^{2-\gamma+\delta}, 1\} k^{-\beta+\delta} t^{-\beta+\delta}. \quad (\text{A.12})$$

To obtain the bound of (A.11), we still need to consider the two cases: (a)  $\alpha \leq 4$  and (b)  $\alpha > 4$ .

(a) When  $\alpha \leq 4$ , for the first term  $\sum_{k=n-1}^{\infty} v_n(k, k) \varepsilon_{-k}^2$  of (A.11), by (A.12), we have that for any  $0 < p < \min\{1, \alpha/2\}$

$$E \left| n^{-2/\alpha} \sum_{k=n-1}^{\infty} v_n(k, k) \varepsilon_{-k}^2 \right|^p \leq Cn^{-2p/\alpha} \max\{n^{p(2-\gamma+\delta)}, 1\} \sum_{k=n-1}^{\infty} k^{p(-2\beta+2\delta)}. \quad (\text{A.13})$$

Since  $\beta > \max\{1/\alpha, 1/2\}$ , we can select a small  $\delta > 0$  and a  $p$  close to  $\min\{1, \alpha/2\}$  such that  $p(-2\beta + \delta) < -1$  and  $p(-2/\alpha - 2\beta - \gamma + 2 + 2\delta) + 1 < 0$  by virtue of the condition  $2\beta + \gamma > 2 + 1/\alpha$ . This yields that the right-hand side of (A.13) is no greater than  $C\{n^{p(-2/\alpha-2\beta-\gamma+2+2\delta)+1} + n^{-2p/\alpha} + n^{p(-2/\alpha-2\beta+\delta)+1}\}$  and converges to 0.

For the cross-product term  $\sum_{n-1 \leq k \neq t < \infty} v_n(k, t) \varepsilon_{-k} \varepsilon_{-t}$  of (A.11), we still split the proof for  $\alpha \leq 4$  into two cases:  $\alpha \leq 2$  and  $2 < \alpha \leq 4$ .

(i) When  $\alpha \leq 2$  and  $\beta > 3/(2\alpha)$ , similar to (A.6), we have for any  $p < \alpha/2$ ,

$$\begin{aligned}
E \left| n^{-2/\alpha} \sum_{n-1 \leq k \neq t < \infty} v_n(k, t) \varepsilon_{-k} \varepsilon_{-t} \right|^p &\leq 2^p n^{-2p/\alpha} \sum_{t=n-1}^{\infty} E \left| \sum_{h=1}^{\infty} v_n(t+h, t) \varepsilon_{-t-h} \varepsilon_{-t} \right|^p \\
&\leq Cn^{-2p/\alpha} \sum_{t=n-1}^{\infty} \left[ \sum_{h=1}^{\infty} |v_n(t+h, t)|^{2p} E|\varepsilon_{-t-h}|^{2p} \right]^{1/2} \\
&= C' \max\{n^{p(2-\gamma-2\beta-2/\alpha+3\delta)+3/2}, n^{p(-2\beta-2/\alpha+2\delta)+3/2}\} \rightarrow 0 \quad (\text{A.14})
\end{aligned}$$

by assumption  $2\beta + \gamma > 2 + 1/\alpha$  and  $\beta > 3/(2\alpha)$ .

(ii) When  $2 < \alpha \leq 4$  and  $\beta > 3/4$ , similar to (A.7), we have for any  $p < \alpha/2$ ,

$$\begin{aligned}
E \left| n^{-2/\alpha} \sum_{n-1 \leq k \neq t < \infty} v_n(k, t) \varepsilon_{-k} \varepsilon_{-t} \right|^p &\leq 2^p n^{-2p/\alpha} \sum_{t=n-1}^{\infty} E \left[ \sum_{h=1}^{\infty} v_n(t+h, t) \varepsilon_{-t-h} \varepsilon_{-t} \right]^p \\
&\leq Cn^{-2p/\alpha} \sum_{t=n-1}^{\infty} \left[ \sum_{h=1}^{\infty} |v_n(t+h, t)|^2 [E|\varepsilon_{-t-h}|^{2p}]^{1/p} \right]^{p/2} \\
&= C' \max\{n^{p(5/2-\gamma-2\beta-2/\alpha+3\delta)+1}, n^{p(-2\beta-2/\alpha+1/2+2\delta)+1}\} \rightarrow 0 \quad (\text{A.15})
\end{aligned}$$

by assumption  $2\beta + \gamma > 5/2$  and  $\beta > 3/4$ . Thus, by (A.13)–(A.15), we have when  $1 < \alpha \leq 4$ ,

$$\int_{-\pi}^{\pi} |\widehat{b}(\lambda)| |\Gamma_{2n}(\lambda)|^2 d\lambda \xrightarrow{p} 0.$$

(b) When  $\alpha > 4$ ,  $\int_{-\pi}^{\pi} |\widehat{b}(\lambda)| |\Gamma_{2n}(\lambda)|^2 d\lambda \xrightarrow{p} 0$  can be shown as that in  $u = 1$ .

Third, consider  $u = 3$ . Similar to the case of  $u = 1$ , it can be shown that (A.3) holds, i.e.,

$$n^{-2/\alpha} \int_{-\pi}^{\pi} |\widehat{b}(\lambda)| |\Gamma_{3n}(\lambda)|^2 d\lambda \xrightarrow{p} 0.$$

Finally, consider  $u = 4$ . In this case, we have

$$\int_{-\pi}^{\pi} |\widehat{b}(\lambda)| |\Gamma_{4n}(\lambda)|^2 d\lambda = \sum_{k=1}^n \kappa_n(k, k) \varepsilon_k^2 + \sum_{1 \leq k \neq t \leq n} \kappa_n(k, t) \varepsilon_k \varepsilon_t, \quad (\text{A.16})$$

where  $\kappa_n(k, t) = \int_{-\pi}^{\pi} |\sum_{j=n}^{\infty} c_j e^{-ij}|^2 |\widehat{b}(\lambda)| e^{i\lambda(k-t)} d\lambda$ . By (3.27) of [17], we have

$$|\kappa_n(k, t)| \leq \begin{cases} n^{-\beta+\delta}, & \text{if } \beta + \gamma > 2, \\ n^{2-\beta-\gamma} n^{-\beta+\delta}, & \text{if } \beta + \gamma \leq 2. \end{cases}$$

Note that for  $1 \leq k, t \leq n$ ,  $n^{-\beta+\delta} \leq (k \vee t)^{-\beta+\delta}$ , by the proof in the case of  $u = 1$ , we have

$$n^{-2/\alpha} \int_{-\pi}^{\pi} |\widehat{b}(\lambda)| |\Gamma_{4n}(\lambda)|^2 d\lambda \xrightarrow{p} 0.$$

The proof of (A.3) is complete.  $\square$

**Lemma A.3.** Under the condition of Theorem 2.1,

$$\beta_n^{-1} \int_{-\pi}^{\pi} \widehat{b}(\lambda) \left( \sum_{j=0}^{\infty} c_j e^{-ij} \right) Y_n(\lambda) \left( \sum_{t=1}^n \varepsilon_t e^{i\lambda t} \right) d\lambda \xrightarrow{p} 0,$$

where  $\beta_n$  is given as in Lemma A.2.

**Proof.** Similar to Kokoszka and Taqqu [17], split  $Y_n(\lambda)$  as  $Y_n(\lambda) = \Delta_{1n}(\lambda) + \Delta_{2n}(\lambda) + \Delta_{3n}(\lambda)$ , where  $\Delta_{1n}(\lambda) = \Gamma_{1n} + \Gamma_{2n} = \sum_{t=1}^n e^{-i\lambda t} (\sum_{j=t}^{\infty} c_j \varepsilon_{t-j})$ ,  $\Delta_{2n}(\lambda) = \Gamma_{3n}$  and  $\Delta_{3n}(\lambda) = \Gamma_{4n}$ . Write  $h_k = \sum_{j=0}^{\infty} |k-j|^{-\gamma} L(|k-j|) j^{-\beta} l(j)$ . Then, for any  $\delta > 0$ ,

$$\begin{aligned} h_k &= O \left( \left( \sum_{j=0}^{\lfloor k/2 \rfloor} + \sum_{\lfloor k/2 \rfloor+1}^{k-1} \right) (k-j)^{-\gamma} L(k-j) j^{-\beta} l(j) + \left( \sum_{j=k+1}^{k+\lfloor k/2 \rfloor} + \sum_{k+\lfloor k/2 \rfloor+1}^{\infty} \right) (j-k)^{-\gamma} L(j-k) j^{-\beta} l(j) \right) \\ &= O(k^{-\beta+\delta} \max\{k^{-\gamma+1}, 1\} + k^{-\gamma+\delta} \max\{k^{-\beta+1}, 1\}) \end{aligned} \quad (\text{A.17})$$

and

$$\begin{aligned} &\int_{-\pi}^{\pi} \widehat{b}(\lambda) \left( \sum_{j=0}^{\infty} c_j e^{-ij} \right) \Delta_{1n}(\lambda) \left( \sum_{t=1}^n \varepsilon_t e^{i\lambda t} \right) d\lambda \\ &= \int_{-\pi}^{\pi} \left( \sum_{k=-\infty}^{\infty} h_k e^{-i\lambda k} \right) \left[ \sum_{s=1}^{n-1} \left( \sum_{t=1}^{n-s} \sum_{j=t}^{\infty} c_j \varepsilon_{t-j} \varepsilon_{t+s} \right) e^{i\lambda s} + \sum_{s=0}^{n-2} \left( \sum_{t=1}^{n-1-s} \varepsilon_t \sum_{j=t+s}^{\infty} c_j \varepsilon_{t+s-j} \right) e^{-i\lambda s} \right] d\lambda \\ &= \sum_{s=1}^{n-1} h_s \left( \sum_{t=1}^{n-s} \sum_{j=t}^{\infty} c_j \varepsilon_{t-j} \varepsilon_{t+s} \right) + \sum_{s=0}^{n-2} h_{-s} \left( \sum_{t=1}^{n-1-s} \varepsilon_t \sum_{j=t+s}^{\infty} c_j \varepsilon_{t+s-j} \right) \\ &= I_{1n} + I_{2n}. \end{aligned} \quad (\text{A.18})$$

When  $1 < \alpha \leq 2$ , we have that for any  $0 < p < \alpha/2$ ,

$$\begin{aligned} \mathbb{E} |n^{-2/\alpha} I_{1n}|^p &= n^{-2p/\alpha} \mathbb{E} \left| \sum_{k=1}^{n-1} \sum_{t=1}^{k-1} \sum_{j=-\infty}^0 c_{t-j} h_{k-t} \varepsilon_j \varepsilon_k \right|^p \\ &\leq n^{-2p/\alpha} \sum_{k=1}^{n-1} \mathbb{E} \left| \sum_{t=1}^{k-1} \sum_{j=-\infty}^0 c_{t-j} h_{k-t} \varepsilon_j \varepsilon_k \right|^p \quad (\text{by Bahr-Esseen's inequality}) \\ &\leq n^{-2p/\alpha} \sum_{k=1}^{n-1} [\mathbb{E} |\varepsilon_k|^{2p}]^{1/2} \left[ \mathbb{E} \left| \sum_{j=-\infty}^0 \sum_{t=1}^{k-1} c_{t-j} h_{k-t} \varepsilon_j \right|^{2p} \right]^{1/2} \quad (\text{by Hölder's inequality}) \end{aligned}$$

$$\begin{aligned}
&\leq n^{-2p/\alpha} \sum_{k=1}^{n-1} [E|\varepsilon_k|^{2p}]^{1/2} \left[ \sum_{j=-\infty}^0 E \left| \sum_{t=1}^{k-1} c_{t-j} h_{k-t} \varepsilon_j \right|^{2p} \right]^{1/2} \quad (\text{by Bahr-Esseen's inequality}) \\
&\leq C n^{-2p/\alpha} \sum_{k=1}^{n-1} \sum_{j=-\infty}^0 \left[ E \left| \sum_{t=1}^{k-1} c_{t-j} h_{k-t} \varepsilon_j \right|^{2p} \right]^{1/2} \quad (\text{by Minkowski's inequality}) \\
&\leq C' n^{-2p/\alpha} \sum_{k=1}^{n-1} \left[ \sum_{j=0}^{\infty} \left( \sum_{t=1}^{k-1} |(t+j)^{-\beta+\delta} [(k-t)^{-\beta+\delta} \max\{(k-t)^{-\gamma+1}, 1\} \right. \right. \\
&\quad \left. \left. + (k-t)^{-\gamma+\delta} \max\{(k-t)^{-\beta+1}, 1\}] \right| \right)^{2p} \right]^{1/2} \\
&\leq C'' \{n^{p(1-2/\alpha-2\beta+2\delta)+3/2} I(\gamma > 1) + n^{p(1-\beta-\gamma-2/\alpha+2\delta)+3/2} I(\beta > 1) \\
&\quad + n^{p(2-2\beta-\gamma-2/\alpha+2\delta)+3/2}\} \rightarrow 0, \tag{A.19}
\end{aligned}$$

by noting that  $\beta > 3/(2\alpha)$ ,  $2\beta + \gamma > 2 + 1/\alpha$  and  $\gamma > 1/\alpha$  and taking  $p$  close to  $\alpha/2$  and  $\delta$  small enough.

When  $2 < \alpha \leq 4$ , by Lemma 1 of [24], it follows that any  $1 < q < \alpha/2$

$$\begin{aligned}
E|n^{-2/\alpha} I_{1n}|^q &= n^{-2q/\alpha} E \left| \sum_{k=1}^{n-1} \sum_{t=1}^{k-1} \sum_{j=-\infty}^0 c_{t-j} h_{k-t} \varepsilon_j \varepsilon_k \right|^q \\
&\leq n^{-2q/\alpha} \sum_{k=1}^{n-1} E \left| \sum_{t=1}^{k-1} \sum_{j=-\infty}^0 c_{t-j} h_{k-t} \varepsilon_j \varepsilon_k \right|^q \\
&\leq n^{-2q/\alpha} \sum_{k=1}^{n-1} [E|\varepsilon_k|^{2q}]^{1/2} \left[ E \left| \sum_{j=-\infty}^0 \sum_{t=1}^{k-1} c_{t-j} h_{k-t} \varepsilon_j \right|^{2q} \right]^{1/2} \\
&\leq C n^{-2q/\alpha} \sum_{k=1}^{n-1} \left[ \sum_{j=-\infty}^0 \left( \sum_{t=1}^{k-1} c_{t-j} h_{k-t} \right)^2 [E|\varepsilon_j|^{2q}]^{1/q} \right]^{q/2} \\
&\leq C' n^{-2q/\alpha} \sum_{k=1}^{n-1} \left[ \sum_{j=0}^{\infty} \left( \sum_{t=1}^{k-1} |(t+j)^{-\beta+\delta} [(k-t)^{-\beta+\delta} \max\{(k-t)^{-\gamma+1}, 1\} \right. \right. \\
&\quad \left. \left. + (k-t)^{-\gamma+\delta} \max\{(k-t)^{-\beta+1}, 1\}] \right| \right)^2 \right]^{q/2} \\
&\leq C'' \{n^{q(3/2-2/\alpha-2\beta+2\delta)+1} I(\gamma > 1) + n^{q(3/2-\beta-\gamma-2/\alpha+2\delta)+1} I(\beta > 1) \\
&\quad + n^{q(5/2-2\beta-\gamma-2/\alpha+2\delta)+1}\} \rightarrow 0, \tag{A.20}
\end{aligned}$$

by assumption  $2\beta + \gamma > 5/2$  and  $\beta > 3/4$ ,  $\gamma > 1/2$ . Thus, for  $\alpha \leq 4$ ,  $n^{-2/\alpha} I_{1n} \xrightarrow{p} 0$ . Similarly, we can show that  $n^{-2/\alpha} I_{2n} \xrightarrow{p} 0$ . As a result,

$$n^{-2/\alpha} \int_{-\pi}^{\pi} \widehat{b}(\lambda) \left( \sum_{j=0}^{\infty} c_j e^{-i\lambda j} \right) \Delta_{1n}(\lambda) \left( \sum_{t=1}^n \varepsilon_t e^{i\lambda t} \right) d\lambda \xrightarrow{p} 0. \tag{A.21}$$

For  $\Delta_{2n}$ , we have

$$\begin{aligned}
\int_{-\pi}^{\pi} \widehat{b}(\lambda) \left( \sum_{j=0}^{\infty} c_j e^{-i\lambda j} \right) \Delta_{2n}(\lambda) \left( \sum_{t=1}^n \varepsilon_t e^{i\lambda t} \right) d\lambda &= \sum_{s=1}^{n-1} h_{n-s} \left( \sum_{t=1}^{n-s} \sum_{j=t}^{n-1} c_j \varepsilon_{n+t-j} \varepsilon_{t+s} \right) \\
&\quad + \sum_{s=0}^{n-2} h_{n+s} \left( \sum_{t=1}^{n-1-s} \varepsilon_t \sum_{j=t+s}^{n-1} c_j \varepsilon_{t+s-j} \right) \\
&= H_{1n} + H_{2n}.
\end{aligned}$$

Similar to (A.19) and (A.20), it can be shown that  $n^{-2/\alpha} H_{1n} \xrightarrow{p} 0$  and  $n^{-2/\alpha} H_{2n} \xrightarrow{p} 0$ . As a result, we have

$$n^{-2/\alpha} \int_{-\pi}^{\pi} \widehat{b}(\lambda) \left( \sum_{j=0}^{\infty} c_j e^{-i\lambda j} \right) \Delta_{2n}(\lambda) \left( \sum_{t=1}^n \varepsilon_t e^{i\lambda t} \right) d\lambda \xrightarrow{p} 0. \quad (\text{A.22})$$

By (A.21) and (A.22), we see that Lemma A.3 holds for  $1 < \alpha \leq 4$  if we can show

$$n^{-2/\alpha} \int_{-\pi}^{\pi} \widehat{b}(\lambda) \left( \sum_{j=0}^{\infty} c_j e^{-i\lambda j} \right) \Delta_{3n}(\lambda) \left( \sum_{t=1}^n \varepsilon_t e^{i\lambda t} \right) d\lambda \xrightarrow{p} 0,$$

which can be proved in the same way as (A.3) with  $u = 4$ .

When  $\alpha > 4$ , since  $E|\varepsilon_1|^2 < \infty$  and  $\beta > 3/4$ , similar to (A.20) (by taking  $p = 2$  and  $\alpha = 4$ ), we can show that

$$n^{-1/2} (|I_{1n}| + |I_{2n}| + H_{1n} + H_{2n}) \xrightarrow{p} 0.$$

Further, by a similar argument of the case  $\alpha > 4$  in Lemma A.2, it can be shown that

$$n^{-1/2} \int_{-\pi}^{\pi} \widehat{b}(\lambda) \left( \sum_{j=0}^{\infty} c_j e^{-i\lambda j} \right) \Delta_{3n} \left( \sum_{t=1}^n \varepsilon_t e^{i\lambda t} \right) d\lambda \xrightarrow{p} 0.$$

Thus, Lemma A.3 holds also for  $\alpha > 4$  and the proof of Lemma A.3 is complete.  $\square$

For any given integers  $j, m$ , denote

$$\mathbf{Z}_{t,j,m} = (\varepsilon_{t+j}^2, \varepsilon_{t+j}\varepsilon_{t+j+1}, \dots, \varepsilon_{t+j}\varepsilon_{t+j+m}) =: (Z_{t,j}^0, Z_{t,j}^1, \dots, Z_{t,j}^m), \quad t \in \mathbb{Z}$$

and

$$\mathbf{B}_{n,m} = (E\varepsilon_0^2 I(\varepsilon_0^2 \leq n^{2/\alpha}), E\varepsilon_0\varepsilon_1 I(|\varepsilon_0\varepsilon_1| \leq n^{2/\alpha}), \dots, E\varepsilon_0\varepsilon_m I(|\varepsilon_0\varepsilon_m| \leq n^{2/\alpha})).$$

In the following, we establish a general useful lemma, which was used in establishing the second part of Theorem 2.1.

**Lemma A.4.** Under the conditions of Theorem 2.1, for any positive integer  $K$  and  $m$ , we have the following conclusions.

(a) For  $1 \leq \alpha < 2$ ,

$$\frac{1}{n^{2/\alpha}} \left( \sum_{t=1}^n \mathbf{Z}_{t,0,m}, \sum_{t=1}^n \mathbf{Z}_{t,1,m}, \dots, \sum_{t=1}^n \mathbf{Z}_{t,K,m} \right) \xrightarrow{\mathcal{D}} (\mathbf{S}_m, \mathbf{S}_m, \dots, \mathbf{S}_m)_{1 \times K}; \quad (\text{A.23})$$

(b) For  $2 \leq \alpha < 4$ ,

$$\begin{aligned} & \frac{1}{n^{2/\alpha}} \left( \sum_{t=1}^n (\mathbf{Z}_{t,0,m} - B(n, m)), \sum_{t=1}^n (\mathbf{Z}_{t,1,m} - B(n, m)), \dots, \sum_{t=1}^n (\mathbf{Z}_{t,K,m} - B(n, m)) \right) \\ & \xrightarrow{\mathcal{D}} (\mathbf{S}_m, \mathbf{S}_m, \dots, \mathbf{S}_m)_{1 \times K} \end{aligned} \quad (\text{A.24})$$

where  $\mathbf{S}_m = (S^0, S^1, \dots, S^m)$  is an  $m$ -dimension stable random vector with index  $\alpha/2$ . The specific form of  $\mathbf{S}_m$  will be given in (A.26) for  $1 < \alpha < 2$  and (A.27) for  $2 \leq \alpha < 4$  respectively.

**Proof.** For any given integers  $j, m$ , define ( $m$ -dimensional random vector)

$$\mathbf{X}_{t,j,m} = (\varepsilon_{t+j}, \varepsilon_{t+j+1}, \dots, \varepsilon_{t+j+m}) =: (X_{t,j}^0, X_{t,j}^1, \dots, X_{t,j}^m).$$

Then,  $\{\mathbf{X}_{t,j,m}, t \geq 1\}$  is a regularly varying random vector sequence with index  $\alpha$ . By Theorem 2.8 of [7] (see also Theorem 3.1 of [21]), there exists a Poisson process  $\sum_{i=1}^{\infty} \delta_{P_i}$  defined on  $\mathbb{R}_+$  with intensity measure  $\nu(dy) = \gamma \alpha y^{\alpha-1} dy$  and a sequence of i.i.d. point processes  $\{\sum_{j=1}^{\infty} \delta_{\mathbf{Q}_{ij,m}}\}$  with distribution  $Q_m$  such that

$$\sum_{t=1}^n \delta_{\mathbf{X}_{t,j,m}/n^{1/\alpha}} := N_n \xrightarrow{\mathcal{D}} N =: \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta_{P_i \mathbf{Q}_{ij,m}}, \quad (\text{A.25})$$

where  $\mathbf{Q}_{ij,m} = (Q_{ij}^0, Q_{ij}^1, \dots, Q_{ij}^m)$ ,  $\{\sum_{j=1}^{\infty} \delta_{\mathbf{Q}_{ij,m}}\}$  is independent of process  $\{P_i\}$ ,  $\gamma, Q_m$  is given by  $\gamma = \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} P\left\{\sum_{t=1}^k \|\mathbf{X}_{t,1,m}\| \leq n^{1/\alpha} \mid \|\mathbf{X}_{0,1,m}\| > n^{1/\alpha}\right\}$  and

$$Q_m(\cdot) = \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} P\left(\sum_{|t| \leq k} \delta_{\mathbf{X}_{t,1,m}/(\sup_{|t| \leq k} \|\mathbf{X}_{t,1,m}\|)} \in \cdot \mid \sup_{1 \leq t \leq k} \|\mathbf{X}_{t,1,m}\| \leq n^{1/\alpha} \leq \|\mathbf{X}_{0,1,m}\|\right)$$

and  $\|\mathbf{X}_{t,j,m}\| = \max_{1 \leq l \leq m} |X_{t,j}^l|$ . Thus, by virtue of Theorem 3.1 of [21], we can show that for  $1 < \alpha < 2$ ,

$$\begin{aligned} \frac{1}{n^{2/\alpha}} \sum_{t=1}^n \mathbf{Z}_{t,j,m} &\xrightarrow{\mathcal{D}} \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P_i^2 (Q_{ij}^0)^2, \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P_i^2 Q_{ij}^0 Q_{ij}^1, \dots, \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P_i^2 Q_{ij}^0 Q_{ij}^m \right) \\ &=: (S^0, S^1, \dots, S^m) = \mathbf{S}_m, \end{aligned} \quad (\text{A.26})$$

and for  $2 \leq \alpha < 4$ ,

$$\begin{aligned} \frac{1}{n^{2/\alpha}} \sum_{t=1}^n (\mathbf{Z}_{t,j,m} - B(n, m)) &\xrightarrow{\mathcal{D}} \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P_i^2 Q_{ij}^0 Q_{ij}^h I(|P_i^2 Q_{ij}^0 Q_{ij}^h| > 0) - \int_{(\mathbf{X} \in \mathbb{R}^{m+1}, |x_0 x_h| > 0)} x_0 x_h d\mu(\mathbf{X}) \right)_{h=0, \dots, m} \\ &=: (S^0, S^1, \dots, S^m) = \mathbf{S}_m, \end{aligned} \quad (\text{A.27})$$

where  $\mu(\cdot)$  is a measure on  $\mathbb{R}^m$  given by  $\mu(\cdot) = \lim_{n \rightarrow \infty} nP(\mathbf{Z}_{t,j,m}/n^{2/\alpha} \in \cdot)$  and  $\mathbf{X} = (x_0, x_1, \dots, x^m)$ .

Since  $\sigma_t^2 = \prod_{k=t-h+1}^t A_k \sigma_{t-h}^2 + \omega \left[ 1 + \sum_{k=t-h+1}^t \prod_{m=k+1}^t A_m \right]$ , it follows that

$$\varepsilon_t^2 \varepsilon_{t-h}^2 = \prod_{k=t-h+1}^t A_k \eta_{t-h}^2 \sigma_{t-h}^4 \eta_t^2 + \omega \left[ 1 + \sum_{k=t-h+1}^t \prod_{m=k+1}^t A_m \right] \eta_{t-h}^2 \sigma_{t-h}^2 \eta_t^2.$$

By Lemma 4.1 and Proposition 3 of [4], we have that

$$\begin{aligned} \lim_{y \rightarrow \infty} P \left\{ \omega \left[ 1 + \sum_{k=t-h+1}^t \prod_{m=k+1}^t A_m \right] \eta_{t-h}^2 \sigma_{t-h}^2 \eta_t^2 > y \right\} &= c(\alpha) E \left\{ \omega \left[ 1 + \sum_{k=t-h+1}^t \prod_{m=k+1}^t A_m \right] \eta_{t-h}^2 \eta_t^2 \right\}^{\alpha/2} y^{-\alpha/2} \\ &=: c_0(\alpha) y^{-\alpha/2} \end{aligned} \quad (\text{A.28})$$

and

$$\begin{aligned} \lim_{y \rightarrow \infty} P \left\{ \prod_{k=t-h+1}^t A_k \eta_{t-h}^2 \sigma_{t-h}^4 \eta_t^2 > y \right\} &= c(\alpha) E[\eta_0^2 A_1]^{\alpha/4} [EA_1^{\alpha/4}]^{h-1} [E|\eta_0|^{\alpha/2}] y^{-\alpha/4}, \\ &:= c_1(\alpha) \rho^h y^{-\frac{\alpha}{4}}. \end{aligned} \quad (\text{A.29})$$

Further, by the Cauchy inequality and the fact that  $\eta_0$  is non-degenerate,  $0 < \rho = EA_1^{\alpha/4} \leq [EA_1^{\alpha/2}]^{1/2} = 1$ . By Theorem 4 of [16] (see also Lemma A.1 of [6]),  $\alpha/2$  is the unique solution to the equation  $EA_1^x = 1$ ,  $x > 0$  and it follows that  $0 < \rho < 1$ . Combining (A.28) with (A.29) yields that for any given  $h$ , as  $x$  is large enough,

$$P(|\varepsilon_t \varepsilon_{t-h}| > x) = P(\varepsilon_t^2 \varepsilon_{t-h}^2 > x^2) \simeq c_1(\alpha) \rho^h x^{-\alpha/2}. \quad (\text{A.30})$$

Using (A.30), along the proof line of Theorem 4.1 of [8], it can be shown that for any positive integer  $K$  and  $m$ ,

$$\frac{1}{a_n^2} \left( \sum_{t=1}^n \mathbf{Z}_{t,0,m}, \sum_{t=1}^n \mathbf{Z}_{t,1,m}, \dots, \sum_{t=1}^n \mathbf{Z}_{t,K,m} \right) \xrightarrow{\mathcal{D}} (\mathbf{S}_m, \mathbf{S}_m, \dots, \mathbf{S}_m)_{1 \times K}. \quad (\text{A.31})$$

This completes the proof of Lemma A.4.  $\square$

## References

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