

Tests for the Mean Direction of the Langevin Distribution with Large Concentration Parameter

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In this paper we study the asymptotic behaviors of the likelihood ratio criterion ($T_L^{(s)}$), Watson statistic ($T_W^{(s)}$) and Rao statistic ($T_R^{(s)}$) for testing $H_0: \mu \in \mathbb{V}$ (a given subspace) against $H_1: \mu \notin \mathbb{V}$, based on a sample of size n from a p -variate Langevin distribution $M_p(\mu, \kappa)$ when κ is large. For the case when κ is known, asymptotic expansions of the null and nonnull distributions of these statistics are obtained. It is shown that the powers of these statistics are coincident up to the order κ^{-1} . For the case when κ is unknown, it is shown that $T_R^{(s)} \geq T_L^{(s)} \geq T_W^{(s)}$ in their powers up to the order κ^{-1} . © 1992 Academic Press, Inc.

1. INTRODUCTION

A random vector \mathbf{x} in R^p of its length $\|\mathbf{x}\|$ unity is said to have a p -variate Langevin (or von Mises–Fisher) distribution $M_p(\mu, \kappa)$ if its probability element is given by

$$\{a_p(\kappa)\}^{-1} \exp(\kappa \mu' \mathbf{x}) \quad (1.1)$$

on the surface $S_p = \{\mathbf{x} \mid \mathbf{x} \in R^p, \|\mathbf{x}\| = (\mathbf{x}'\mathbf{x})^{1/2} = 1\}$, where $\mu'\mu = 1$ and $\kappa > 0$. The normalizing constant is given by

$$a_p(\kappa) = (2\pi)^{p/2} I_{p/2-1}(\kappa) \kappa^{-p/2+1}, \quad (1.2)$$

where $I_\nu(\kappa)$ is the modified Bessel function of the first kind of order ν . The parameters μ and κ are called the mean direction vector and the concentration parameter of the Langevin distribution, respectively.

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Some testing problems for μ and κ have been studied in Mardia [4], Watson [7, 8], etc. In this paper we consider the problem of testing

$$H_0^{(s)}: \mu \in \mathbb{V} \quad \text{against} \quad H_1^{(s)}: \mu \notin \mathbb{V}, \quad (1.3)$$

based on a random sample $\mathbf{x}_1, \dots, \mathbf{x}_n$ from $M_p(\mu, \kappa)$, where \mathbb{V} is a given subspace of dimension s . Let B_0 be a $p \times s$ matrix whose columns consist of an orthonormal basis of \mathbb{V} . Then, we can write $H_0^{(s)}$ as $\mu = B_0 \theta$, $\theta' \theta = 1$. Here we note that the testing problem in a special case $s = 1$ and $B_0 = \mu_0$ is $H_0^{(1)}: \mu = \pm \mu_0$ against $H_1^{(1)}: \mu \neq \pm \mu_0$, and it is not completely coincident with the one of testing

$$H_0: \mu = \mu_0 \quad \text{against} \quad H_1: \mu \neq \mu_0. \quad (1.4)$$

When κ is known, the following two statistics have been proposed in Watson [7, 8]:

Likelihood ratio criterion,

$$T_{L0}^{(s)} = \exp\{\kappa n(\|B_0' \bar{\mathbf{x}}\| - \|\bar{\mathbf{x}}\|)\};$$

Watson statistic or Rao statistic,

$$T_{W0}^{(s)} = 2\kappa n \|(I_p - B_0 B_0') \bar{\mathbf{x}}\|^2,$$

where $\bar{\mathbf{x}} = (1/n) \sum_{j=1}^n \mathbf{x}_j$. When κ is unknown, we consider the following three statistics:

Likelihood ratio criterion,

$$T_{L1}^{(s)} = \{a_p(\hat{\kappa})/a_p(\bar{\kappa})\}^n \exp\{n(\bar{\kappa} \|B_0' \bar{\mathbf{x}}\| - \hat{\kappa} \|\bar{\mathbf{x}}\|)\};$$

Watson statistic (Watson [7, 8]),

$$T_{W1}^{(s)} = \{n\hat{\kappa}/A_p(\hat{\kappa})\} \|(I_p - B_0 B_0') \bar{\mathbf{x}}\|^2;$$

Rao statistic,

$$T_{R1}^{(s)} = \{n\bar{\kappa}/A_p(\bar{\kappa})\} \|(I_p - B_0 B_0') \bar{\mathbf{x}}\|^2.$$

Here $\bar{\kappa}$ and $\hat{\kappa}$ are the maximum likelihood estimators of κ under $H_0^{(s)}$ and $H_0^{(s)} \cup H_1^{(s)}$, respectively. Letting $A_p(\kappa) = (d/d\kappa) \log a_p(\kappa)$, $\bar{\kappa}$ and $\hat{\kappa}$ are given by the solutions of

$$A_p(\bar{\kappa}) = \|B_0' \bar{\mathbf{x}}\| \quad \text{and} \quad A_p(\hat{\kappa}) = \|\bar{\mathbf{x}}\|,$$

respectively. The statistic $T_{R1}^{(s)}$ is analogous to the one (Hayakawa [3]) for testing (1.4).

We note that the likelihood ratio criteria \tilde{T}_{Li} for (1.4) are given as the ones obtained from $T_{Li}^{(s)}$ by replacing $\|B'_0 \bar{\mathbf{x}}\|$ by $\mu'_0 \bar{\mathbf{x}}$. Watson [7, 8] obtained asymptotic null and nonnull distributions of some statistics for (1.3) and (1.4) in both the situations where n is large and κ is large. For the situation where n is large, further asymptotic results have been obtained by several authors. Chou [1] obtained asymptotic expansions of the null and nonnull distributions of $T_{Wi}^{(1)}$. Hayakawa [3] obtained similar asymptotic expansions of the null and nonnull distributions of $T_{Li}^{(1)}$ and $T_{Ri}^{(1)}$, and made numerical comparison of the powers of these statistics. Watamori [5] obtained asymptotic expansions of the null and nonnull distributions of $T_{Li}^{(s)}$ and $T_{Wi}^{(s)}$.

The purpose of this paper is to study the asymptotic behaviors of $T_{Li}^{(s)}$, $T_{Wi}^{(s)}$, and $T_{Ri}^{(s)}$ when κ is large and n is fixed. In Section 2 we give some basic results on asymptotic expansions when κ is large. In Section 3 we obtain asymptotic expansions of the null and nonnull distributions of $T_{L0}^{(s)}$ and $T_{W0}^{(s)}$. It is shown that the powers of these tests are coincident up to the order κ^{-1} . In Section 4 it is shown that $T_{R1}^{(s)} \geq T_{L1}^{(s)} \geq T_{W1}^{(s)}$ in the powers of the three criteria up to the order κ^{-1} .

2. BASIC RESULTS ON ASYMPTOTIC EXPANSIONS

In this section we give some basic results in obtaining asymptotic expansions of the distributions of test statistics when κ is large.

LEMMA 2.1. *Let $a_p(\kappa)$ be the normalizing constant of the density (1.1) and $A_p(\kappa) = (d/d\kappa) \log a_p(\kappa)$. Then*

$$(i) \quad A_p(\kappa) = 1 - \frac{1}{2} \kappa^{-1}(p-1) + \frac{1}{8} \kappa^{-2}(p-1)(p-3) + O(\kappa^{-3}),$$

$$(ii) \quad a_p(\kappa) = (2\pi)^{(p-1)/2} \kappa^{-(p-1)/2} e^{\kappa} \left[1 - \frac{1}{8} \kappa^{-1}(p-1)(p-3) \right. \\ \left. + \frac{1}{128} \kappa^{-2}(p+1)(p-1)(p-3)(p-5) + O(\kappa^{-3}) \right].$$

Proof. For a proof of (i), see Watson [7]. Letting $v = \frac{1}{2}(p-3)$ and $y_v(\kappa) = a_p(\kappa)$, it is known (Watson [6, p. 191] that $y_v(\kappa)$ satisfies $\kappa y_v'' + 2(v+1)y_v' - \kappa y_v = 0$. (ii) is obtained by looking for the solution of the form $y_v = (2\pi)^{v+1} \kappa^{-v+1} e^{\kappa} \{1 + c_1 \kappa^{-1} + c_2 \kappa^{-2} + \dots\}$.

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be a random sample of size n from $M_p(\boldsymbol{\mu}, \kappa)$. As a sequence of the alternatives, we consider

$$\boldsymbol{\mu} = (B_0 \boldsymbol{\theta} + \kappa^{-1/2} \boldsymbol{\delta}) \|B_0 \boldsymbol{\theta} + \kappa^{-1/2} \boldsymbol{\delta}\|^{-1} \\ = (B_0 \boldsymbol{\theta} + \kappa^{-1/2} \boldsymbol{\delta})(1 + 2\kappa^{-1} \lambda)^{-1/2}, \quad (2.1)$$

where $\boldsymbol{\theta}$ and $\boldsymbol{\delta}$ are any fixed vectors, not depending on κ such that $\boldsymbol{\theta}'\boldsymbol{\theta} = 1$ and $B'_0 \boldsymbol{\delta} = \mathbf{0}$, and $\lambda = \frac{1}{2} \boldsymbol{\delta}'\boldsymbol{\delta}$. It may be noted that (2.1) is a typical setting for

local alternatives. Let $B_1: p \times \overline{p-s}$ and $\Theta_1: s \times \overline{s-1}$ be the matrices such that $[B_0 B_1]$ and $[\Theta_0 \Theta_1]$ are orthogonal matrices. Let

$$\mathbf{z} = \sqrt{\kappa n} B_1' (\bar{\mathbf{x}} - \kappa^{-1/2} \boldsymbol{\delta}), \quad \mathbf{w} = \sqrt{\kappa n} \Theta_1' B_0' \bar{\mathbf{x}}, \quad (2.2)$$

and for $j = 1, \dots, n$, let

$$\begin{aligned} \mathbf{z}_j &= \sqrt{\kappa} B_1' (\mathbf{x}_j - \kappa^{-1/2} \boldsymbol{\delta}), & \tilde{\mathbf{z}}_j &= \mathbf{z}_j + B_1' \boldsymbol{\delta}, \\ u_j &= 2\kappa(1 - \Theta_1' B_0' \mathbf{x}_j), & \mathbf{w}_j &= \sqrt{\kappa} \Theta_1' B_0' \mathbf{x}_j. \end{aligned} \quad (2.3)$$

Note that for $s = 1$, $\mathbf{w} = \mathbf{0}$ and $\mathbf{w}_j = \mathbf{0}$ ($j = 1, \dots, n$).

The characteristic function of \mathbf{z} and \mathbf{w} can be expressed as

$$\begin{aligned} \psi_n(\mathbf{t}_1, \mathbf{t}_2) &= E[\exp(it_1' \mathbf{z} + it_2' \mathbf{w})] \\ &= \exp(-\sqrt{n} it_1' B_1' \boldsymbol{\delta}) \\ &\quad \times \{a_p(\kappa \|\boldsymbol{\mu} + i(n\kappa)^{-1/2} (B_1 \mathbf{t}_1 + B_0 \Theta_1 \mathbf{t}_2)\|) / a_p(\kappa)\}^n. \end{aligned} \quad (2.4)$$

Using Lemma 2.1(ii) we obtain an expansion of $\psi_n(\mathbf{t}_1, \mathbf{t}_2)$,

$$\begin{aligned} \psi_n(\mathbf{t}_1, \mathbf{t}_2) &= \exp\left\{\frac{1}{2} i^2 (\mathbf{t}_1' \mathbf{t}_1 + \mathbf{t}_2' \mathbf{t}_2)\right\} \left[1 - \frac{1}{8} \kappa^{-1} \{2(p-1) i^2 \mathbf{t}_1' \mathbf{t}_1 \right. \\ &\quad + 4(p-1+2\lambda) \sqrt{n} i \boldsymbol{\delta}' B_1 \mathbf{t}_1 + 4(i \boldsymbol{\delta}' B_1 \mathbf{t}_1)^2 \\ &\quad + 4i^3 n^{-1/2} \boldsymbol{\delta}' B_1 \mathbf{t}_1 \cdot \mathbf{t}_1' \mathbf{t}_1 + n^{-1} (i^2 \mathbf{t}_1' \mathbf{t}_1)^2 \\ &\quad + 4 \sqrt{n} i^3 \boldsymbol{\delta}' B_1 \mathbf{t}_1 \cdot \mathbf{t}_2' \mathbf{t}_2 + 2i^4 \mathbf{t}_1' \mathbf{t}_1 \cdot \mathbf{t}_2' \mathbf{t}_2 \\ &\quad \left. + 2(p-1) n i^2 \mathbf{t}_2' \mathbf{t}_2 + (i^2 \mathbf{t}_2' \mathbf{t}_2)^2\right\} + O(\kappa^{-2}). \end{aligned} \quad (2.5)$$

Inverting (2.5) itself and (2.5) with $\mathbf{t}_2 = \mathbf{0}$, we have the following lemma.

LEMMA 2.2. Under (2.1) the probability density function of $\mathbf{z} = \sqrt{\kappa n} B_1' (\bar{\mathbf{x}} - \kappa^{-1/2} \boldsymbol{\delta})$ and $\mathbf{w} = \sqrt{\kappa n} \Theta_1' B_0' \bar{\mathbf{x}}$ can be expanded as

$$\phi_{p-s}(\mathbf{z}) \phi_{s-1}(\mathbf{w}) [1 + \kappa^{-1} \{b_{n1}(\mathbf{z}) + b_{n2}(\mathbf{w}) + b_{n3}(\mathbf{z}, \mathbf{w})\} + O(\kappa^{-2})], \quad (2.6)$$

where $\phi_{p-s}(\mathbf{z}) = (2\pi)^{-(p-s)/2} \exp(-\frac{1}{2} \mathbf{z}' \mathbf{z})$,

$$\begin{aligned} b_{n1}(\mathbf{z}) &= -\frac{1}{8} n^{-1} [-(p-s)(p+s-4) - 8\lambda \\ &\quad + 4\{n(p-1) - p + s - 2 + 2n\lambda\} \sqrt{n} \boldsymbol{\delta}' B_1' \mathbf{z} \\ &\quad + 4n(\boldsymbol{\delta}' B_1 \mathbf{z})^2 + 2\{n(p-1) - p + s - 2\} \mathbf{z}' \mathbf{z} \\ &\quad + 4 \sqrt{n} \boldsymbol{\delta}' B_1 \mathbf{z} \mathbf{z}' \mathbf{z} + (\mathbf{z}' \mathbf{z})^2], \end{aligned}$$

$$\begin{aligned} b_{n2}(\mathbf{z}) &= -\frac{1}{8} n^{-1} [-(s-1)(2(p-1)n - s - 1) \\ &\quad + 2\{(p-1)n - s - 1\} \mathbf{w}' \mathbf{w} + (\mathbf{w}' \mathbf{w})^2], \end{aligned}$$

and

$$\begin{aligned} b_{n3}(\mathbf{z}, \mathbf{w}) = & -\frac{1}{8} n^{-1} [2(p-1)(s-1) - 4\sqrt{n}(s-1)\delta' B_1' \mathbf{z} \\ & - 2(s-1)\mathbf{z}'\mathbf{z} - 2(p-s)\mathbf{w}'\mathbf{w} \\ & + 4\sqrt{n}\delta' B_1' \mathbf{z} \cdot \mathbf{w}'\mathbf{w} + 2\mathbf{z}'\mathbf{z} \cdot \mathbf{w}'\mathbf{w}]. \end{aligned}$$

Further, the marginal probability density function of \mathbf{z} can be expanded as

$$\phi_{p-s}(\mathbf{z})[1 + \kappa^{-1}b_{n1}(\mathbf{z}) + O(\kappa^{-2})]. \quad (2.7)$$

LEMMA 2.3. Under (2.1) the probability density function of $Z = [\mathbf{z}_1, \dots, \mathbf{z}_n]$ and $W = [\mathbf{w}_1, \dots, \mathbf{w}_n]$ can be expanded as

$$\phi_{(p-s) \times n}(Z) \phi_{(s-1) \times n}(W) \left[1 + \frac{1}{\kappa} b(Z, W) + O(\kappa^{-2}) \right], \quad (2.8)$$

where $\phi_{(p-s) \times n}(Z) = (2\pi)^{-n(p-s)/2} \text{etr}(-\frac{1}{2}ZZ')$, $b(Z, W) = b_1(Z) + b_2(W) + b_3(Z, W)$,

$$\begin{aligned} b_1(Z) = & \frac{1}{8} \left[n(p-s)(p+s-4) + 8n\lambda \right. \\ & - 4(s-3+2\lambda) \sum_j \text{tr } \mathbf{e}_j \delta' B_1' Z - 4 \sum_j (\text{tr } \mathbf{e}_j \delta' B_1' Z)^2 \\ & - 2(s-3) \text{tr } ZZ' - \sum_j (\text{tr } \mathbf{e}_j \mathbf{e}_j' Z' Z)^2 \\ & \left. - 4 \sum_j (\text{tr } \mathbf{e}_j \delta' B_1' Z) \text{tr } \mathbf{e}_j \mathbf{e}_j' Z' Z \right], \\ b_2(W) = & \frac{1}{8} \left[n(s-1)(2p-s-3) - 2(p-s-2) \text{tr } WW' \right. \\ & \left. - \sum_j (\text{tr } \mathbf{e}_j \mathbf{e}_j' W' W)^2 \right], \end{aligned}$$

and

$$\begin{aligned} b_3(Z, W) = & \frac{1}{8} \left[2n(p-1)(s-1) + 4 \sum_j \text{tr } \mathbf{e}_j \delta' B_1' \mathbf{z} \right. \\ & + 2(s-1) \text{tr } ZZ' + 2(p-s) \text{tr } WW' \\ & - 4 \sum_j \text{tr } \mathbf{e}_j \delta' B_1' Z \cdot \text{tr } \mathbf{e}_j \mathbf{e}_j' W' W \\ & \left. - 2 \sum_i \text{tr } \mathbf{e}_j \mathbf{e}_j' Z' Z \cdot \text{tr } \mathbf{e}_j \mathbf{e}_j' W' W \right]. \end{aligned}$$

Here \mathbf{e}_j is the column vector of dimension n whose j th element is one and others are zero.

Proof. The probability density function of Z and W can be expressed as

$$\prod_{j=1}^n [(2\pi)^{-(p-s)/2} \exp(-\frac{1}{2} \mathbf{z}_j' \mathbf{z}_j) (2\pi)^{-(s-1)/2} \exp(-\frac{1}{2} \mathbf{w}_j' \mathbf{w}_j) \times [1 + \kappa^{-1} \{b_{11}(\mathbf{z}_j) + b_{12}(\mathbf{w}_j) + b_{13}(\mathbf{z}_j, \mathbf{w}_j)\} + O(\kappa^{-2})]. \quad (2.9)$$

Therefore, the result (2.7) is obtained by using

$$b_1(Z) = \sum_j b_{11}(\mathbf{z}_j), \quad b_2(W) = \sum_j b_{12}(\mathbf{w}_j),$$

and

$$b_3(Z, W) = \sum_j b_{13}(\mathbf{z}_j, \mathbf{w}_j).$$

LEMMA 2.4. Under (2.1) it holds that

- (i) $u_j = \tilde{\mathbf{z}}_j' \tilde{\mathbf{z}}_j + \mathbf{w}_j' \mathbf{w}_j + \frac{1}{4} \kappa^{-1} (\tilde{\mathbf{z}}_j' \tilde{\mathbf{z}}_j + \mathbf{w}_j' \mathbf{w}_j)^2 + O_p(\kappa^{-2})$,
- (ii) $\|\tilde{\mathbf{x}}\| = 1 + \kappa^{-1} \hat{d}_1 + \kappa^{-2} \hat{d}_2 + O_p(\kappa^{-2})$,
- (iii) $\|B_0' \tilde{\mathbf{x}}\| = 1 + \kappa^{-1} \bar{d}_1 + \kappa^{-2} \bar{d}_2 + O_p(\kappa^{-2})$,

where $\tilde{\mathbf{Z}} = [\tilde{\mathbf{z}}_1, \dots, \tilde{\mathbf{z}}_n]$, $P_0 = n^{-1} \mathbf{1}_n \mathbf{1}_n'$, $\mathbf{1}_n = (1, \dots, 1)'$, $P_1 = I_n - P_0$,

$$\hat{d}_1 = -\frac{1}{2} n^{-1} \{ \text{tr } \tilde{\mathbf{Z}} P_1 \tilde{\mathbf{Z}}' + \text{tr } W P_1 W' \},$$

$$\hat{d}_2 = -\frac{1}{8} n^{-1} \sum_j (\tilde{\mathbf{z}}_j' \tilde{\mathbf{z}}_j + \mathbf{w}_j' \mathbf{w}_j)^2 + \frac{1}{8} n^{-2} (\text{tr } \tilde{\mathbf{Z}} P_0 \tilde{\mathbf{Z}}' + \text{tr } W P_0 W')^2 + \frac{1}{4} n^{-2} (\text{tr } \tilde{\mathbf{Z}} P_1 \tilde{\mathbf{Z}}' + \text{tr } W P_1 W') (\text{tr } \tilde{\mathbf{Z}} P_0 \tilde{\mathbf{Z}}' + \text{tr } W P_0 W'),$$

$$\bar{d}_1 = -\frac{1}{2} n^{-1} \{ \text{tr } \tilde{\mathbf{Z}} \tilde{\mathbf{Z}}' + \text{tr } W P_1 W' \},$$

$$\bar{d}_2 = -\frac{1}{8} n^{-1} \sum_j (\tilde{\mathbf{z}}_j' \tilde{\mathbf{z}}_j + \mathbf{w}_j' \mathbf{w}_j)^2 + \frac{1}{8} n^{-2} (\text{tr } W P_0 W')^2 + \frac{1}{4} n^{-2} (\text{tr } \tilde{\mathbf{Z}} \tilde{\mathbf{Z}}' + \text{tr } W P_1 W') \text{tr } W P_0 W'.$$

Proof. Note that

$$\begin{aligned} \mathbf{x}_j &= [B_1 B_0] [B_1 B_0]' \mathbf{x}_j \\ &= B_1 B_1' \mathbf{x}_j + B_0 [\boldsymbol{\theta} \ \boldsymbol{\Theta}_1] [\boldsymbol{\theta} \ \boldsymbol{\Theta}_1]' B_0' \mathbf{x}_j \\ &= B_1 \kappa^{-1/2} \tilde{\mathbf{z}}_j + B_0 \boldsymbol{\Theta}_1 \kappa^{-1/2} \mathbf{w}_j + B_0 \boldsymbol{\theta} \boldsymbol{\theta}' B_0' \mathbf{x}_j, \end{aligned} \quad (2.10)$$

and hence $1 = \kappa^{-1}(\bar{\mathbf{z}}_j' \bar{\mathbf{z}}_j + \mathbf{w}_j' \mathbf{w}_j) + (\boldsymbol{\theta}' B_0' x_j)^2$. This implies (i). The results (ii) and (iii) are obtained by using (i) and (2.10).

Lemma 2.1(i) and Lemma 2.4(ii), (iii) imply the following stochastic expansions for $\hat{\kappa}$ and $\bar{\kappa}$.

LEMMA 2.5. *Under (2.1) it holds that*

$$(i) \quad \hat{\kappa} = \hat{c}_0 \{ \kappa + \hat{c}_1 + \kappa^{-1} \hat{c}_2 \} + O_p(\kappa^{-2}),$$

$$(ii) \quad \bar{\kappa} = \bar{c}_0 \{ \kappa + \bar{c}_1 + \kappa^{-1} \bar{c}_2 \} + O_p(\kappa^{-2}),$$

where $\hat{c}_0 = -\frac{1}{2}(p-1)\hat{d}_1^{-1}$, $\hat{c}_1 = -\hat{d}_2/\hat{d}_1 + \frac{1}{2}(p-1)^{-1}(p-3)\hat{d}_1$, and \bar{c}_j 's are the ones obtained from \hat{c}_j 's by replacing \hat{d}_j 's by \bar{d}_j 's.

3. TESTS WHEN κ IS KNOWN

First we derive asymptotic expansions of the distribution of $T_{w_0}^{(s)}$ under a sequence of the alternatives (2.1). The statistic $T_{w_0}^{(s)}$ can be expressed in terms of \mathbf{z} as

$$\begin{aligned} T_{w_0}^{(s)} &= n\kappa(B_1' \bar{\mathbf{x}})' B_1 \bar{\mathbf{x}} \\ &= (\mathbf{z} + \sqrt{n} B_1' \boldsymbol{\delta})' (\mathbf{z} + \sqrt{n} B_1 \boldsymbol{\delta}). \end{aligned} \quad (3.1)$$

Therefore, we can write the characteristic function of $T_{w_0}^{(s)}$ as

$$\begin{aligned} \Psi_{w_0}(t) &= \int \cdots \int \exp\{it(\mathbf{z} + \sqrt{n} B_1' \boldsymbol{\delta})' (\mathbf{z} + \sqrt{n} B_1 \boldsymbol{\delta})\} \\ &\quad \times \phi_{p-s}(\mathbf{z}) \{1 + \kappa^{-1} b_{n1}(\mathbf{z}) + O(\kappa^{-2})\} d\mathbf{z}. \end{aligned} \quad (3.2)$$

Considering the transformation from \mathbf{z} to

$$\mathbf{y} = (1 - 2it)^{1/2} \left\{ \mathbf{z} - \frac{2it}{1 - 2it} \sqrt{n} B_1' \boldsymbol{\delta} \right\},$$

we obtain

$$\begin{aligned} \Psi_{w_0}^{(s)}(t) &= (1 - 2it)^{-(p-s)/2} \exp\left(\frac{2itn\lambda}{1 - 2it}\right) \\ &\quad \times E_{\mathbf{y}} \{ 1 + \kappa^{-1} b_{ns}((1 - 2it)^{-1/2} \mathbf{y} \\ &\quad + ((1 - 2it)^{-1} - 1) \sqrt{n} B_1' \boldsymbol{\delta}) \} + O(\kappa^{-2}), \end{aligned} \quad (3.3)$$

where the expectation with respect to \mathbf{y} is taken under $N_{p-s}(\mathbf{0}, I_{p-s})$. After calculating the expectation in (3.3), we obtain

$$\begin{aligned} \Psi_{\mathbf{w}_1}^{(s)}(t) &= (1 - 2it)^{-(p-s)/2} \exp\left(\frac{2itn\lambda}{1 - 2it}\right) \\ &\times \left[1 + \frac{1}{4\kappa} \sum_{j=0}^4 r_j (1 - 2it)^{-j} + O(\kappa^{-2})\right], \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} r_0 &= (p-s)\{p-1 - \tfrac{1}{2}n^{-1}(p-s+2)\} \\ &\quad + 2\{n(p-1) - p+s\}\lambda + 6n\lambda^2, \\ r_1 &= -(p-s)\{p-1 - n^{-1}(p-s+2)\} + 2(p-s)\lambda - 8n\lambda^2, \\ r_2 &= -\tfrac{1}{2}n^{-1}(p-s)(p-s+2) \\ &\quad - 2\{n(p-1) - p+s-2\}\lambda + 4n\lambda^2, \\ r_3 &= -2(p-s+2)\lambda, \\ r_4 &= -2n\lambda^2. \end{aligned}$$

Inverting this characteristic function, we have the following theorem.

THEOREM 3.1. *Under a sequence of the alternatives (2.1) the distribution function of $T_{\mathbf{w}_0}^{(s)}$ can be asymptotically expanded as*

$$\begin{aligned} P(T_{\mathbf{w}_0}^{(s)} \leq x) &= P(\chi_f^2(n\lambda) \leq x) \\ &\quad + \frac{1}{4\kappa} \sum_{j=0}^4 r_j P(\chi_{f+2j}^2(n\lambda) \leq x) + O(\kappa^{-2}), \end{aligned} \quad (3.5)$$

where $f = p-s$, $\chi_f^2(\theta)$ denotes a noncentral χ^2 -variate with f degrees of freedom and noncentrality parameter θ , and r_j 's are given by (3.4).

Letting $\delta = \mathbf{0}$ in (3.5), we obtain an asymptotic expansion of the null distribution of $T_{\mathbf{w}_0}^{(s)}$ in terms of central χ^2 -variates,

$$\begin{aligned} P(T_{\mathbf{w}_0}^{(s)} \leq x) &= P(\chi_f^2 \leq x) \\ &\quad + \frac{1}{4\kappa} \sum_{j=0}^2 \tilde{r}_j P(\chi_{f+2j}^2 \leq x) + O(\kappa^{-2}), \end{aligned} \quad (3.6)$$

where $f = p - s$ and

$$\begin{aligned}\tilde{r}_0 &= (p-s)\{p-1-\tfrac{1}{2}n^{-1}(p-s+2)\}, \\ \tilde{r}_1 &= -(p-s)\{p-1-n^{-1}(p-s+2)\}, \\ \tilde{r}_2 &= -\tfrac{1}{2}n^{-1}(p-s)(p-s+2).\end{aligned}$$

Next we consider the distribution of $T_{L0}^{(s)}$ under (2.1). Using Lemma 2.4, we can expand $-2 \log T_{L0}^{(s)}$ in terms of \tilde{Z} and W as

$$-2 \log T_{L0}^{(s)} = q_0(\tilde{Z}) + \kappa^{-1} q_1(\tilde{Z}, W) + O(\kappa^{-2}), \quad (3.7)$$

where

$$\begin{aligned}q_0(\tilde{Z}) &= \text{tr } \tilde{Z} P_0 \tilde{Z}', \\ q_1(\tilde{Z}, W) &= \tfrac{1}{4} n^{-1} (\text{tr } \tilde{Z} P_0 \tilde{Z}') (2 \text{tr } W P_1 W' + 2 \text{tr } \tilde{Z} \tilde{Z}' - \text{tr } \tilde{Z} P_0 \tilde{Z}').\end{aligned}$$

By using Lemma 2.2 and Eq. (3.7) the characteristic function of $-2 \log T_{L0}^{(s)}$ can be expanded as

$$\begin{aligned}\Psi_{L0}^{(s)}(t) &= (2\pi)^{-n(p-s)/2} \int \cdots \int \text{etr}(it \tilde{Z} P_0 \tilde{Z}' - \tfrac{1}{2} Z Z') \\ &\quad \times \{1 + \kappa^{-1} (it \tilde{q}_1(\tilde{Z}) + b_s(Z))\} dz + O(\kappa^{-2}),\end{aligned}$$

where $\tilde{q}_1(\tilde{Z}) = \tfrac{1}{4} n^{-1} (\text{tr } \tilde{Z} P_0 \tilde{Z}') \{2(s-1)(n-1) + 2 \text{tr } \tilde{Z} \tilde{Z}' - \text{tr } Z P_0 Z'\}$. Consider the transformation from Z to

$$Y = [Z - ((1 - 2it)^{-1} - 1) B'_1 \delta \mathbf{1}'_n] \Gamma^{-1}, \quad (3.8)$$

where $\Gamma = (1 - 2it)^{-1/2} P_0 + P_1$. Then we can write $\Psi_{L0}^{(s)}(\mathbf{t})$ as

$$\begin{aligned}\Psi_{L0}^{(s)}(\mathbf{t}) &= (1 - 2it)^{-(p-s)/2} \exp\left(\frac{2itn\lambda}{1 - 2it}\right) \\ &\quad \times E_Y [1 + \kappa^{-1} \{it \tilde{q}_1(Y\Gamma + (1 - 2it)^{-1} B'_1 \delta \mathbf{1}'_n) \\ &\quad + b_s(Y\Gamma + ((1 - 2it)^{-1} - 1) B'_1 \delta \mathbf{1}'_n)\} + O(\kappa^{-2})]. \quad (3.9)\end{aligned}$$

The expectation in (3.9) is taken under the normal random matrix Y whose elements are independently distributed as $N(0, 1)$. Calculating the expectation in (3.9) and inverting the resultant characteristic function, we have the following theorem.

THEOREM 3.2. *Under a sequence of the alternatives (2.1) the distribution function of $-2 \log T_{L0}^{(s)}$ can be asymptotically expanded as*

$$P(-2 \log T_{L0}^{(s)} \leq x) = P(\chi_f^2(n\lambda) \leq x) + \frac{1}{4\kappa} \sum_{j=0}^3 h_j P(\chi_{f+2j}^2(n\lambda) \leq x) + O(\kappa^{-2}), \quad (3.10)$$

where $f = p - s$ and

$$\begin{aligned} h_0 &= \frac{1}{2} n^{-1} (p-s)(p+s-4) - 2\lambda \{p-s - (p-s)n\} + 6n\lambda^2, \\ h_1 &= -\frac{1}{2} n^{-1} (p-s)(p+s-4) + 2\lambda \{2p-s-1 - (p-1)n\} - 8n\lambda^2, \\ h_2 &= -2\lambda(p-1) + 4n\lambda^2, \\ h_3 &= -2n^2\lambda^2. \end{aligned}$$

As a special case of Theorem 3.2 we obtain an asymptotic expansion of the null distribution,

$$\begin{aligned} P(-2 \log T_{L0}^{(s)} \leq x) &= P(\chi_{p-s}^2 \leq x) \\ &\quad + \frac{1}{8} (n\kappa)^{-1} (p-s)(p+s-4) \\ &\quad \times \{P(\chi_{p-s}^2 \leq x) - P(\chi_{p-s+2}^2 \leq x)\} + O(\kappa^{-2}). \end{aligned} \quad (3.11)$$

This result implies that $\bar{T}_{L0}^{(s)} = -2\rho \log T_{L0}^{(s)}$ gives a better χ^2 -approximation, since

$$P(-2\rho \log T_{L0}^{(s)} \leq x) = P(\chi_{p-s}^2 \leq x) + O(\kappa^{-2}), \quad (3.12)$$

where $\rho = 1 + \frac{1}{4} (n\kappa)^{-1} (p+s-4)$.

It may be noted that the asymptotic results for the likelihood ratio criterion $\tilde{T}_{L0}^{(1)}$ for (1.4) are the same as the ones for $T_{L0}^{(1)}$ with $B_0 = \mu_0$.

Now we compare the powers of $T_{w0}^{(s)}$ and $T_{L0}^{(s)}$ with a level of significance α . From (3.6) and (3.10) it is possible to obtain asymptotic expansions for the powers. An expansion for such powers has been obtained by Fujikoshi [2] (the coefficients c_1 and c_2 in (3.3) of his paper should be read as $c_1 = b_1 + \bar{a}_0$ and $c_2 = b_2 + 2\delta\bar{a}_0 f^{-1} - \bar{a}_2$, respectively). Applying his result to (3.6) and (3.10) we obtain the following theorem.

THEOREM 3.3. *Under a sequence of the alternatives (2.1) the powers $\beta_{w0}^{(s)}$ and $\beta_{L0}^{(s)}$ of $T_{w0}^{(s)}$ and $T_{L0}^{(s)}$, respectively, with a level of significance α are coincident up to the order κ^{-1} . Further, $\beta_{w0}^{(s)}$ (or $\beta_{L0}^{(s)}$) is given by*

$$\begin{aligned}
\beta_{w_0}^{(s)} &= P(\chi_f^2(n\lambda) \geq x_\alpha) \\
&\quad - \frac{1}{\kappa} [\{(p-1)n - (p-s) + 3n\lambda\} \lambda g_{f+2}(x_\alpha; n\lambda) \\
&\quad + \left\{ \frac{1}{2}(p-s+2) - n\lambda \right\} \lambda g_{f+4}(x_\alpha; n\lambda) \\
&\quad + n\lambda^2 g_{f+6}(x_\alpha; n\lambda)] + O(\kappa^{-2}),
\end{aligned}$$

where $f = p - s$, x_α is the upper α point of χ_f^2 , and $g_f(x; n\lambda)$ is the probability density function of $\chi_f^2(n\lambda)$.

Theorem 3.3 shows that the differences between the powers of $T_{w_0}^{(s)}$ and $T_{L_0}^{(s)}$ are small when κ is large.

4. TESTS WHEN κ IS UNKNOWN

In this section we compare the powers of $T_{L_1}^{(s)}$, $T_{R_1}^{(s)}$, and $T_{w_1}^{(s)}$ up to the order κ^{-1} under a sequence of the alternatives (2.1). Let

$$U = \text{tr } \tilde{Z}\tilde{Z}' + \text{tr } WP_1W' \quad \text{and} \quad V = \text{tr } \tilde{Z}P_0\tilde{Z}'/U. \quad (4.1)$$

Using Lemmas 2.1(ii), 2.3, and 2.4 it is shown that

$$\begin{aligned}
\tilde{T}_{L_1}^{(s)} &= 1 - \{T_{L_1}^{(s)}\}^{2/\{n(p-1)\}} \\
&= V + \frac{1}{\kappa} q(\tilde{Z}, W) + O_p(\kappa^{-2}),
\end{aligned} \quad (4.2)$$

where

$$q(\tilde{Z}, W) = \frac{\hat{d}_1}{\bar{d}_1} \left\{ \frac{p-3}{2(p-1)} (\bar{d}_1 - \hat{d}_1) + \bar{d}_2/\bar{d}_1 - \hat{d}_2/\hat{d}_1 \right\}. \quad (4.3)$$

Similarly we have

$$\begin{aligned}
\tilde{T}_{R_1}^{(s)} &= \frac{1}{n(p-1)} T_{R_1}^{(s)} \\
&= V + \frac{1}{\kappa} \left\{ q(\tilde{Z}, W) + \frac{1}{2n(p-1)} UV^2 \right\} + O_p(\kappa^{-2}),
\end{aligned} \quad (4.4)$$

and

$$\begin{aligned}\tilde{T}_{\mathbf{W}_1}^{(s)} &= \frac{1}{n(p-1)} T_{\mathbf{W}_1}^{(s)} (1 + T_{\mathbf{W}_1}^{(s)})^{-1} \\ &= V + \frac{1}{\kappa} \left\{ q(\tilde{Z}, W) + \frac{1}{2n(p-1)} UV^2(V-1) \right\} + O_0(\kappa^{-2}).\end{aligned}\quad (4.5)$$

From these expressions it is easy to see that the asymptotic distributions of $\tilde{T}_{\mathbf{L}_1}^{(s)}$, $\tilde{T}_{\mathbf{R}_1}^{(s)}$, and $\tilde{T}_{\mathbf{W}_1}^{(s)}$ are the same as the one of a noncentral beta variate $\beta_{1/2f_1, 1/2f_2}(\theta)$ with f_1, f_2 degrees of freedom and noncentrality parameter θ , where

$$f_1 = p - s, \quad f_2 = (p-1)(n-1), \quad \theta = n\lambda. \quad (4.6)$$

Now we consider asymptotic expansions of the distributions of these statistics up to the order κ^{-1} . The characteristic function of $\tilde{T}_{\mathbf{L}_1}^{(s)}$ can be expressed as

$$\tilde{\Psi}_{\mathbf{L}_1}^{(s)}(t) = E \left[e^{itV} \left\{ 1 + \frac{1}{\kappa} [itq(\tilde{Z}, W) + b(Z, W)] \right\} \right] + O(\kappa^{-2}). \quad (4.7)$$

Here the expectation in (4.7) is taken under the normal random matrices Z and W whose elements are independently distributed as $N(0, 1)$. Evaluating (4.7) and inverting the resultant one, we can obtain the form,

$$\begin{aligned}P(\tilde{T}_{\mathbf{L}_1}^{(s)} \leq x) &= P(\beta_{1/2f_1, 1/2f_2}(\theta) \leq x) \\ &\quad + \frac{1}{\kappa} b_{1/2f_1, 1/2f_2}(x; \theta) \gamma_{\mathbf{L}}(x, \theta) + O(\kappa^{-1}),\end{aligned}\quad (4.8)$$

where $b_{1/2f_1, 1/2f_2}(x; \theta)$ is the probability density function of $\beta_{1/2f_1, 1/2f_2}(\theta)$. Since our interest is the comparison of powers, we will not try to obtain an explicit expression for $\gamma_{\mathbf{L}}(x, \theta)$. The characteristic functions of $\tilde{T}_{\mathbf{R}_1}^{(s)}$ and $\tilde{T}_{\mathbf{W}_1}^{(s)}$ can be expressed as

$$\tilde{\Psi}_{\mathbf{R}_1}^{(s)}(t) = \tilde{\Psi}_{\mathbf{L}_1}^{(s)}(t) + \frac{1}{\kappa} \Psi_{\mathbf{R}}^*(t) + O(\kappa^{-2}) \quad (4.9)$$

and

$$\tilde{\Psi}_{\mathbf{W}_1}^{(s)}(t) = \tilde{\Psi}_{\mathbf{L}_1}^{(s)}(t) + \frac{1}{\kappa} \Psi_{\mathbf{W}}^*(t) + O(\kappa^{-2}). \quad (4.10)$$

Here

$$\begin{aligned}\Psi_{\mathbf{R}}^*(t) &= \frac{it}{2n(p-1)} E[e^{it\nu} UV^2], \\ \Psi_{\mathbf{W}}^*(t) &= \frac{it}{2n(p-1)} E[e^{it\nu} UV^2(V-1)].\end{aligned}\quad (4.11)$$

Therefore, we obtain the distribution functions of $\tilde{T}_{\mathbf{R}1}^{(s)}$ and $\tilde{T}_{\mathbf{W}1}^{(s)}$ as the (4.8) replaced $\gamma_{\mathbf{L}}(x, \theta)$ by

$$\gamma_{\mathbf{R}}(x, \theta) = \gamma_{\mathbf{L}}(x, \theta) + \gamma_{\mathbf{R}}^*(x, \theta) \quad (4.12)$$

and

$$\gamma_{\mathbf{W}}(x, \theta) = \gamma_{\mathbf{L}}(x, \theta) + \gamma_{\mathbf{W}}^*(x, \theta), \quad (4.13)$$

respectively. Here $\gamma_{\mathbf{R}}^*(x, \theta) = \gamma_{\mathbf{R}}^*(x, \theta)$ or $\gamma_{\mathbf{W}}^*(x, \theta)$ is defined by

$$\gamma_{\mathbf{R}}^*(x, \theta) b_{1/2f_1, 1/2f_2}(x; \theta) = \int_0^x \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \Psi_{\mathbf{R}}^*(t) dt \right\} dx. \quad (4.14)$$

In the evaluation of (4.12), we may assume, without loss of generality, that the probability density function of (U, V) is

$$\sum_{k=0}^{\infty} e^{-\theta} \frac{\theta^k}{k!} g_{f_1+f_2+2k}(u) b_{1/2f_1+k, 1/2f_2}(v), \quad (4.15)$$

where g_f and $b_{1/2f_1, 1/2f_2}$ are the probability density functions of χ_f^2 and $\beta_{1/2f_1, 1/2f_2} = \beta_{1/2f_1, 1/2f_2}(0)$, respectively.

LEMMA 4.1. Suppose that (U, V) has the density function (4.15). Then, for any negative integer m

$$\begin{aligned}E[e^{it\nu} UV^m] &= \sum_{k=0}^{\infty} e^{\theta} \frac{\theta^k}{k!} (f_1 + f_2 + 2k) \frac{(\frac{1}{2}f_1 + k)_m}{(\frac{1}{2}f_1 + \frac{1}{2}f_2 + k)_m} \\ &\quad \times {}_1F_1\left(\frac{1}{2}f_1 + k + m; \frac{1}{2}f_1 + \frac{1}{2}f_2 + k + m; it\right),\end{aligned}$$

where $(a)_m = a(a+1)\cdots(a+m-1)$ and $(a)_0 = 1$.

Proof. Using the expression (4.15) we have

$$\begin{aligned} E[e^{itV}U] &= \sum_{k=0}^{\infty} e^{-\theta} \frac{\theta^k}{k!} (f_1 + f_2 + 2k) \\ &\quad \times \int_0^1 e^{itv} b_{1/2f_1+k, 1/2f_2}(v) dv \\ &= \sum_{k=0}^{\infty} e^{-\theta} \frac{\theta^k}{k!} (f_1 + f_2 + 2k) \\ &\quad \times {}_1F_1\left(\frac{1}{2}f_1 + k; \frac{1}{2}f_1 + \frac{1}{2}f_2 + k; it\right). \end{aligned}$$

Differentiating the both sides of this result with respect to t , we obtain the required result.

LEMMA 4.2. Under the assumption of Lemma 4.1 it holds that

$$\begin{aligned} \int_0^x \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}(it) E[e^{itV}UV^m] dt dx \\ = -x^m \{ (f_1 + f_2) b_{1/2f_1, 1/2f_2}(x; \theta) + 2\theta b_{1/2f_1+1, 1/2f_2}(x; \theta) \}. \quad (4.16) \end{aligned}$$

Proof. It is well known that ${}_1F_1(\alpha + 1; \beta + 1; z) = \beta [{}_1F_1(\alpha + 1; \beta; z) - {}_1F_1(\alpha; \beta; z)]$, and the characteristic function of a central beta variate $\beta_{1/2f_1, 1/2f_2}$ is ${}_1F_1(\frac{1}{2}f_1; \frac{1}{2}f_1 + \frac{1}{2}f_2; it)$. Hence, from Lemma 4.1 we have
The left-hand side of (4.16)

$$\begin{aligned} &= \sum_{k=0}^{\infty} e^{-\theta} \frac{\theta^k}{k!} (f_1 + f_2 + 2k) \frac{(\frac{1}{2}f_1 + k)_m}{(\frac{1}{2}f_1 + \frac{1}{2}f_2 + k)_m} \\ &\quad \times \left(\frac{1}{2}f_1 + \frac{1}{2}f_2 + k + m - 1 \right) \\ &\quad \times \{ P(\beta_{1/2f_1+k+m, 1/2f_2-1} \leq x) - P(\beta_{1/2f_1+k+m-1, 1/2f_2} \leq x) \}. \end{aligned}$$

Further, using a partial integration method, we have

$$\begin{aligned} &P(\beta_{1/2f_1+k+m, 1/2f_2-1} \leq x) - P(\beta_{1/2f_1+k+m-1, 1/2f_2} \leq x) \\ &= (\frac{1}{2}f_1 + \frac{1}{2}f_2 + k)_m / \{ (\frac{1}{2}f_1 + k)_m (\frac{1}{2}f_1 + \frac{1}{2}f_2 + k + m - 1) \}, \end{aligned}$$

which implies (4.16).

THEOREM 4.1. *Let $\beta_{L1}^{(s)}$, $\beta_{R1}^{(s)}$, and $\beta_{W1}^{(s)}$ be the powers of test statistics $T_{L1}^{(s)}$, $T_{R1}^{(s)}$, and $T_{W1}^{(s)}$ with a level of significance α under a sequence of the alternatives (2.1), respectively. Then*

$$\beta_{R1}^{(s)} - \beta_{L1}^{(s)} = \frac{1}{\kappa} \cdot \frac{\theta v_\alpha^2}{n(p-1)} \cdot \frac{b_{1/2f_1+1, 1/2f_2}(v_\alpha; \theta)}{b_{1/2f_1, 1/2f_2}(v_\alpha; \theta)} + O(\kappa^{-2}),$$

and

$$\beta_{L1}^{(s)} - \beta_{W1}^{(s)} = \frac{1}{\kappa} \cdot \frac{\theta v_\alpha^2(1-v_\alpha)}{n(p-1)} \cdot \frac{b_{1/2f_1+1, 1/2f_2}(v_\alpha; \theta)}{b_{1/2f_1, 1/2f_2}(v_\alpha; \theta)} + O(\kappa^{-2}),$$

where $f_1 = p - s$, $f_2 = (p-1)(n-1)$, $\theta = n\lambda$, v_α is the upper α point of $\beta_{1/2f_1, 1/2f_2}$, and $b_{1/2f_1, 1/2f_2}(v; \theta)$ is the probability density function of $\beta_{1/2f_1, 1/2f_2}(\theta)$.

Proof. From (4.8), we can write the null distribution of $\tilde{T}_{L1}^{(s)}$ as

$$P(\tilde{T}_{L1}^{(s)} \leq x) = P(\beta_{1/2f_1, 1/2f_2} \leq x) + \frac{1}{\kappa} b_{1/2f_1, 1/2f_2}(x) \gamma_L(x) + O(\kappa^{-2}),$$

where $\gamma_L(x) = \gamma_L(x; 0)$. This implies that the upper α point of $\tilde{T}_{L1}^{(s)}$ is expressed as

$$v_{L\alpha} = v_\alpha - \kappa^{-1} \gamma_L(v_\alpha) + O(\kappa^{-2})$$

in the term of v_α . Using this expansion of $v_{L\alpha}$ we obtain

$$\begin{aligned} \beta_{L1}^{(s)} &= 1 - P(\tilde{T}_{L1}^{(s)} \leq v_{L1}) = P(\beta_{1/2f_1, 1/2f_2}(\theta) \geq v_\alpha) \\ &\quad + \frac{1}{\kappa} b_{1/2f_1, 1/2f_2}(v_\alpha; \theta) \{ \gamma_L(v_\alpha) - \gamma_L(v_\alpha; \theta) \} + O(\kappa^{-2}). \end{aligned}$$

Similarly we obtain the expansions of $\beta_{R1}^{(s)}$ and $\beta_{W1}^{(s)}$, which are given from the one of $\beta_{L1}^{(s)}$ by replacing γ_L by γ_R and γ_W , respectively. Therefore, using (4.12) we have

$$\begin{aligned} \beta_{R1}^{(s)} - \beta_{L1}^{(s)} &= \frac{1}{\kappa} b_{1/2f_1, 1/2f_2}(v_\alpha; \theta) \\ &\quad \times \{ \gamma_R^*(v_\alpha) - \gamma_R^*(v_\alpha; \theta) \} + O(\kappa^{-2}), \end{aligned}$$

where $\gamma_R^*(v_\alpha) = \gamma_R^*(v_\alpha; 0)$. Further, from Lemma 4.2 it follows that

$$\gamma_R^*(v_\alpha; \theta) = -\frac{x^2}{2n(p-1)} \\ \times \{f_1 + f_2 + 2\theta b_{1/2f_1+1, 1/2f_2}(v_\alpha; \theta)/b_{1/2f_1, 1/2f_2}(v_\alpha; \theta)\},$$

which proves the first result of Theorem 4.1. Similarly we can prove the second result of Theorem 4.1.

Theorem 4.1 shows that, neglecting the term of the order κ^{-2} , the power of the Rao statistic is greater than that of the likelihood ratio criterion, which is also greater than that of the Watson statistic.

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