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The Matsumoto–Yor property on trees for matrix variates of different dimensions

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ABSTRACT

The paper is devoted to an extension of the multivariate Matsumoto–Yor (MY) independence property with respect to a tree with p vertices to the case where random variables corresponding to the vertices of the tree are replaced by random matrices. The converse of the p-variate MY property, which characterizes the product of one gamma and $p-1$ generalized inverse Gaussian distributions, is extended to characterize the product of the Wishart and $p-1$ matrix generalized inverse Gaussian distributions.

1. Introduction

Let $\mathcal{V}_n$ be the Euclidean space of $n \times n$ real symmetric matrices equipped with the inner product $\langle a, b\rangle = \mathrm{trace}(ab)$. Let dx denote the Lebesgue measure on $\mathcal{V}_n$ assigning the unit mass to the unit cube. Let $\mathcal{V}_n^+$ denote the cone of positive definite matrices in $\mathcal{V}_n$ and let $\overline{\mathcal{V}}_n^+$ denote its closure. For $x \in \mathcal{V}_n$ let $|x|$ denote the determinant of x.

Let $c \in \mathcal{V}_n^+$ and $q \in \Lambda_n = \{0, \frac{1}{2}, \frac{2}{2}, \ldots, \frac{n-1}{2}\} \cup (\frac{n-1}{2}, \infty)$. The random matrix Y taking its values in $\overline{\mathcal{V}}_n^+$ is said to follow the Wishart $W_n(q, c)$ distribution if its Laplace transform is given by

$$L_Y(\theta) = \frac{|c|^q}{|c-\theta|^q}, \quad c - \theta \in \mathcal{V}_n^+,$$

see Casalis and Letac [4] and references given therein. When $q > \frac{n-1}{2}$, that is when Y takes its values in $\mathcal{V}_n^+$, this distribution has density of the form

$$f_Y(y) = \frac{|c|^q}{\Gamma_n(q)} |y|^{q-\frac{n+1}{2}} \exp\left(-\langle c, y\rangle\right) \mathbf{1}_{\mathcal{V}_n^+}(y),$$

where Γ_n denotes the multivariate gamma function, see Muirhead [19]. When $q \in \Lambda_n$ and $q \le \frac{n-1}{2}$ the distribution is singular and is concentrated on the boundary of $\bar{\mathcal{V}}_n^+$. In the special case $q = 0$, it is the Dirac measure concentrated at the zero matrix.

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A random matrix X, taking its values in $\mathcal{V}_n^+$, is said to follow the matrix generalized inverse Gaussian distribution, $MGIG_n(-p, a, b)$, if it has density of the form

$$f_X(x) = \frac{1}{K_p^{(n)}(a, b)} |x|^{-p-\frac{n+1}{2}} \exp\left(-\langle a, x\rangle - \langle b, x^{-1}\rangle\right) \mathbf{1}_{\mathcal{V}_n^+}(x), \tag{1.1}$$

where $K_p^{(n)}$ is the matrix variate modified Bessel function of the third kind, see Herz [6].

Letac [12] has observed that the $MGIG_n(-p, a, b)$ is well defined iff p, a, b satisfy one of the following three conditions:

1. $a, b \in \mathcal{V}_n^+$ and $p \in \mathbb{R}$,
2. $a \in \bar{\mathcal{V}}_n^+$ with $\mathrm{rank}(a) = m \in \{0, 1, \ldots, n-1\}$, $b \in \mathcal{V}_n^+$ and $p > \frac{n-m-1}{2}$,
3. $a \in \mathcal{V}_n^+$, $b \in \bar{\mathcal{V}}_n^+$ with $\mathrm{rank}(b) = m \in \{0, 1, \ldots, n-1\}$ and $p < -\frac{n-m-1}{2}$.

This extends earlier definitions of the matrix variate GIG as given in Bardorff-Nielsen et al. [1] or Butler [3].

The *MGIG* distribution has the following property, which will be used later on

$$\text{if } X \sim MGIG_n(-p, a, b) \text{ then } X^{-1} \sim MGIG_n(p, b, a). \tag{1.2}$$

There are several connections between the Wishart and *MGIG* distributions considered in the literature, see e.g. Bardorff-Nielsen and Koudou [2], Butler [3], Koudou [7,8], Koudou and Ley [9], Koudou and Vallois [10,11], Seshadri and Wesołowski [20]. Here we are interested in those which are extensions of the Matsumoto–Yor (MY) property of the univariate gamma and generalized inverse Gaussian distributions. The gamma $\gamma(p, a)$ and the generalized inverse Gaussian $GIG(q, b, c)$ distributions are defined by the densities

$$f(y) \propto y^{p-1} e^{-ay} I_{(0,+\infty)}(y)$$

and

$$g(x) \propto x^{q-1} e^{-bx-c/x} I_{(0,+\infty)}(x),$$

respectively, where p, a, b, c are positive numbers and q is real.

Matsumoto and Yor [17,18] considered the transformation ψ that takes $(x, y) \in (0, +\infty)^2$ into $(0, +\infty)^2$, where

$$\psi(x, y) = \left((x+y)^{-1}, x^{-1} - (x+y)^{-1}\right).$$

They observed that if two random variables X and Y are independent and follow the $GIG(-q, a, b)$ and $\gamma(q, a)$ distributions, respectively, then the two random variables U and V defined as $(U, V) = \psi(X, Y)$ are also independent and follow the $GIG(-q, b, a)$ and $\gamma(q, b)$ distributions, respectively. Letac and Wesołowski [14] proved the converse to the MY property, that is the following characterization: if X and Y are independent and U and V are also independent, where $(U, V) = \psi(X, Y)$, then $(X, Y) \sim GIG(-q, a, b) \otimes \gamma(q, a)$. In the same paper it was shown that this result holds true also for matrix variates, namely the authors considered the transformation ψ for X and Y positive definite random matrices and proved both the direct MY property and its converse in this case (under certain smoothness conditions, weakened later on in Wesołowski [21]): if X and Y are independent $r \times r$ positive definite matrices and $U = (X+Y)^{-1}$ and $V = X^{-1} - (X+Y)^{-1}$ are also independent then X and Y follow a matrix variate GIG and Wishart distribution, respectively.

For any $s \times r$ real matrix z of full rank, denote by $\mathbf{P}(z)$ the linear mapping

$$x \in \mathcal{V}_r \mapsto \mathbf{P}(z)x = zxz^t \in \mathcal{V}_s,$$

(z^t denotes the transpose of the matrix z). Massam and Wesołowski [16] extended the MY property to more general situation, where matrix variates have different dimensions: X and Y are independent positive definite matrices of dimensions $r \times r$ and $s \times s$, respectively. They considered the transformation ψ_z defined as follows

$$\psi_z(x, y) = \left((\mathbf{P}(z)x + y)^{-1}, x^{-1} - \mathbf{P}(z^t)(\mathbf{P}(z)x + y)^{-1}\right),$$

where z is a given constant $s \times r$ matrix of full rank and obtained the following characterization (under certain smoothness conditions): if X and Y are independent and U and V are also independent, where $(U, V) = \psi_z(X, Y)$, then X and Y follow a matrix variate GIG and Wishart distribution, respectively.

On the other hand, Massam and Wesołowski [15], interpreted the original MY property as a bivariate property with respect to the simple tree with two vertices and one edge and extended it to a p-variate property with respect to any tree with p vertices. Moreover, they proved the converse of this extended version of the MY property, obtaining the characterization of the product of one gamma and $p-1$ generalized inverse Gaussian distributions. To this end they considered certain transformations induced by leaves of such a tree.

In this paper we extend the multivariate version of the MY property on trees considered in [15] to the case where the components of a random vector corresponding to the vertices of the tree are replaced by random matrices of different dimensions. We prove this generalized MY property and its counter-part being a joint characterization of one Wishart and $p-1$ matrix generalized inverse Gaussian distributions.

The proof of our characterization is given under the assumption of strict positivity and differentiability of the densities. Here we do not use Laplace transforms to identify the Wishart variables (the Laplace transform approach was used in [15] to work out the gamma distributions in the proof of the converse of the MY property on trees). Instead our approach is based on certain independence property (given in Lemma 4.1 in Section 4) and Theorem 4.1 from [16].

In the next section, we will prove some lemmas (which are just extensions of the ones given in [15] to the case of matrices) that we will use to obtain our main results. Section 3 is devoted to the statement and the proof of the matrix variate version of the MY property on trees. We also define there a matrix variate analogue of the W_G^c distribution defined in [15]. In Section 4 we formulate and prove the converse of the MY property, that is, the characterization of the product of one Wishart and $p-1$ *MGIG* distributions. The section starts with the lemma describing an independence property induced by the transformations related to the matrix variate version of the MY property on trees. This lemma enables us to use the results of [16] in our proof of the characterization.

2. Preliminaries

Let $T=(V,E)$ be an undirected tree of size p, where $V=\{1,\dots,p\}$ is the set of its vertices and E is the set of its edges (a set of unordered pairs (i,j) such that the distinct vertices i and j are linked in T). Let $L\subset V$ denote the set of leaves of T (a set of vertices with only one neighbour).

Let us choose integers $n_1,\dots,n_p$ and for $1\le i<j\le p$ let us fix arbitrary rectangular matrices k_{ij} with n_i rows and n_j columns with the only restriction that $k_{ij}=0$ when $(i,j)\notin E$. Let us call K_T this fixed set $(k_{ij})_{1\le i<j\le p}$ and define $k_{ji}=k_{ij}^t$ when $1\le i<j\le p$, where k_{ij}^t denotes the transpose of the matrix k_{ij}.

Finally, let $M(T,K_T)$ be the set of $(k_1,\dots,k_p)$ such that $k_i=k_{ii}$ is positive definite of order n_i and such that the bloc matrix $\mathbf{K}=(k_{ij})_{1\le,i,j\le p}$ is positive definite.

Thus we have

$$M(T,K_T)=\left\{k=\left(k_1,\dots,k_p\right)\in\mathcal{V}_{n_1}\times\cdots\times\mathcal{V}_{n_p}:\mathbf{K}=\begin{bmatrix}k_1 & k_{12} & \cdots & k_{1p}\\ k_{21} & k_2 & \cdots & k_{2p}\\ \cdots & \cdots & \cdots & \cdots\\ k_{p1} & k_{p2} & & k_p\end{bmatrix}\in\mathcal{V}^+_{\sum_{i=1}^p n_i},\quad k_{ij}\in K_T,\ i\neq j\right\}.$$

We can direct an undirected tree T by choosing any vertex $r\in V$ as a single root and directing all edges towards it. In a tree T directed in this way, we say that a vertex j is a child of a vertex i if there is a directed edge from i to j in T and a vertex k is a parent of a vertex i if there is a directed edge from k to i in T. Let $c_r(i)$ denote the set of children of i and let $p_r(i)$ denote the set of parents of i in the directed tree T with root r. Note that each vertex has at most one child (if i is a root then $c_i(i)=\emptyset$) and each vertex may have several parents (if i is a leaf then $p_r(i)=\emptyset$).

For a given root $r\in V$ and a leaf $l\in L$, $l\neq r$, we define $T^{-l}=(V^{-l},E^{-l})$, where $V^{-l}=V\setminus\{l\}$, $E^{-l}=E\setminus\{(l,c_r(l))\}$. Let $K_{T^{-l}}$ be a given set of off-diagonal blocks obtained from K_T by discarding $k_{c_r(l)l}\neq 0$ and $k_{il}=0,\ i\in V\setminus\{l,c_r(l)\}$.

Now we will prove some properties of the set $M(T,K_T)$. The following lemmas are the matrix variate analogues of Lemmas 2.1–2.5 obtained by [15]. Note that the second equality in Lemma 2.2 in [15] does not hold for $j=c_r(l)$ (but this small lapse does not influence the proof of the main result of [15]).

Lemma 2.1. *Let $k=\left(k_1,\dots,k_p\right)\in M(T,K_T)$. For any root $r\in V$ and leaf $l\in L$ ($l\neq r$), define $k^{-l}=\left(k_i^{-l},i\in V\setminus\{l\}\right)$ (of $p-1$ components) by*

$$\begin{aligned}&k_i^{-l}=k_i,\quad i\in V\setminus\{l,c_r(l)\},\\ &k_{c_r(l)}^{-l}=k_{c_r(l)}-\mathbf{P}(k_{c_r(l)l})k_l^{-1}.\end{aligned}\tag{2.1}$$

Then $k^{-l}\in M(T^{-l},K_{T^{-l}})$.
Furthermore, for any $k^{-l}\in M(T^{-l},K_{T^{-l}})$ and any $k_l\in\mathcal{V}^+_{n_l}$ we have

$$\left(k_1^{-l},\dots,k_{c_r(l)-1}^{-l},k_{c_r(l)}^{-l}+\mathbf{P}(k_{c_r(l)l})k_l^{-1},k_l\right)\in M(T,K_T).\tag{2.2}$$

Proof. Without loss of generality we can assume that $r=1$, $l=p$ and $c_r(l)=p-1$. Then $\mathbf{K}$ can be partitioned as

$$\mathbf{K}=\begin{bmatrix}A & B\\ C & D\end{bmatrix},$$

where

$$A=\begin{bmatrix}k_1 & k_{12} & \cdots & k_{1c_r(l)-1} & k_{1c_r(l)}\\ k_{21} & k_2 & \cdots & k_{2c_r(l)-1} & k_{2c_r(l)}\\ \cdots & \cdots & \cdots & \cdots & \\ k_{c_r(l)-11} & k_{c_r(l)-12} & \cdots & k_{c_r(l)-1} & k_{c_r(l)-1c_r(l)}\\ k_{c_r(l)1} & k_{c_r(l)2} & \cdots & k_{c_r(l)c_r(l)-1} & k_{c_r(l)}\end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ k_{c_r(l)l} \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & \cdots & 0 & k_{lc_r(l)} & k_l \end{bmatrix}, \qquad D = k_l.$$

Observing that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_1 & 0 \\ -D^{-1}C & I_2 \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & B \\ 0 & D \end{bmatrix}$$

we get

$$|\mathbf{K}| = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \left|A - BD^{-1}C\right| |D|.$$

Hence

$$|\mathbf{K}| = |k_l| \left|\mathbf{K}^{-l}\right|, \tag{2.3}$$

where

$$\mathbf{K}^{-l} = \left[A - BD^{-1}C\right] = \begin{bmatrix} k_1 & k_{12} & \cdots & k_{1c_r(l)-1} & k_{1c_r(l)} \\ k_{21} & k_2 & \cdots & k_{2c_r(l)-1} & k_{2c_r(l)} \\ \cdots & \cdots & \cdots & \cdots & \\ k_{c_r(l)-11} & k_{c_r(l)-12} & \cdots & k_{c_r(l)-1} & k_{c_r(l)-1c_r(l)} \\ k_{c_r(l)1} & k_{c_r(l)2} & \cdots & k_{c_r(l)c_r(l)-1} & k_{c_r(l)} - \mathbf{P}(k_{c_r(l)l})k_l^{-1} \end{bmatrix}. \tag{2.4}$$

Note that all principal minors of $\mathbf{K}^{-l}$ except the last one are the same as in the matrix $\mathbf{K}$. Therefore they are positive. Since $|\mathbf{K}| > 0$ and $k_l \in \mathcal{V}_{n_l}^+$, by (2.3) $\left|\mathbf{K}^{-l}\right| > 0$ and we have

$$k^{-l} = \left(k_1, \dots, k_{c_r(l)-1}, k_{c_r(l)} - \mathbf{P}(k_{c_r(l)l})k_l^{-1}\right) \in M(T^{-l}, K_{T^{-l}}).$$

To show that for any $k^{-l} \in M(T^{-l}, K_{T^{-l}})$ and any $k_l \in \mathcal{V}_{n_l}^+$ (2.2) holds true it suffices to observe that all principal minors of the matrix $\mathbf{K}$ are positive. It is obvious for the first $c_r(l) - 1$ minors. To see that this is so for the $c_r(l)$th principal minor of $\mathbf{K}$ observe that this minor is the determinant of the matrix which all entries are equal to the respective ones of $\mathbf{K}^{-l}$ except for the element with index $(c_r(l), c_r(l))$ which is equal to

$$k_{c_r(l)} = \left(\mathbf{K}^{-l}\right)_{c_r(l)\ c_r(l)} + \mathbf{P}(k_{c_r(l)l})k_l^{-1}.$$

Since $\mathbf{K}^{-l}$ and $\mathbf{P}(k_{c_r(l)l})k_l^{-1}$ are positive definite, the $c_r(l)$th principal minor of $\mathbf{K}$ is positive. From (2.3) it follows that the lth principal minor of $\mathbf{K}$ is also positive. □

For a root $r \in V$ in the directed tree T we define the mapping $\psi_r : M(T, K_T) \to \mathcal{V}_{n_1}^+ \times \cdots \times \mathcal{V}_{n_p}^+$ by

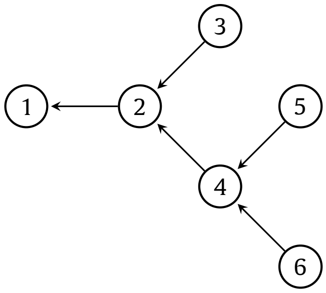
$$\psi_r\left(k_1, \dots, k_p\right) = \left(k_{1,(r)}, \dots, k_{p,(r)}\right), \tag{2.5}$$

$$k_{i,(r)} = \begin{cases} k_i & \text{if } i \in L \setminus \{r\}, \\ k_i - \sum_{j \in p_r(i)} \mathbf{P}(k_{ij})k_{j,(r)}^{-1} & \text{otherwise.} \end{cases} \tag{2.6}$$

Remark 2.2. For a root $r \in V$ in the directed tree T there exists the unique $\left(k_{1,(r)}, \dots, k_{p,(r)}\right)$ such that (2.6) holds. It can be easily proved by reversed induction on the distance $d = d(r, i)$ between the root r and a vertex $i \in V = 1, \dots, p$ of the tree $T = (V, E)$: If all the $k_{j,(r)}$ are known for $d(r, j) = d + 1$ then $k_{i,(r)}$ can be defined by (2.6).

Let us illustrate the definition of the mapping ψ_r with the following example.

Example 2.3. Let T be the tree of size $p = 6$ with $V = \{1, 2, 3, 4, 5, 6\}$, $E = \{(1, 2), (2, 3), (2, 4), (4, 5), (4, 6)\}$ and $K_T = \{k_{12}, k_{23}, k_{24}, k_{45}, k_{46}\}$. Let us choose $r = 1$ as the root of T.



The mapping ψ_1 is of the form

$$\psi_1(k_1,\ldots,k_6)=\left(k_{1,(1)},\ldots,k_{6,(1)}\right),$$

where $k_{6,(1)}=k_6$, $k_{5,(1)}=k_5$, $k_{3,(1)}=k_3$, $k_{4,(1)}=k_4-\mathbf{P}(k_{45})k_{5,(1)}^{-1}-\mathbf{P}(k_{46})k_{6,(1)}^{-1}$, $k_{2,(1)}=k_2-\mathbf{P}(k_{23})k_{3,(1)}^{-1}-\mathbf{P}(k_{24})k_{4,(1)}^{-1}$, $k_{1,(1)}=k_1-\mathbf{P}(k_{12})k_{2,(1)}^{-1}$.

Lemma 2.4. *For any root $r\in L$ and leaf $l\in L$, $l\neq r$, we have*

$$k_{j,(r)}^{-l}=k_{j,(r)}\quad \textit{for } j\in V^{-l}\quad \textit{and}\quad k_{j,(c_r(l))}^{-l}=k_{j,(l)}\quad \textit{for } j\in V^{-l}\setminus\{c_r(l)\}.$$

Proof. Let $\mathcal{P}_{lr}\subset V$ denote the unique path in T linking l and r. Since for $j\notin\mathcal{P}_{lr}$, $k_{j,(r)}$ does not depend on k_l, we have $k_{j,(r)}^{-l}=k_{j,(r)}$. On the other hand, for any $j\in\mathcal{P}_{lr}\setminus\{l\}$, $k_{j,(r)}$ depends on k_l only through $k_{c_r(l)}^{-l}=k_{c_r(l)}-\mathbf{P}(k_{c_r(l)l})k_l^{-1}$ and thus we also have $k_{j,(r)}^{-l}=k_{j,(r)}$ for such j.

To prove the second equality observe first that $k_{j,(c_r(l))}=k_{j,(l)}$ for $j\neq l,c_r(l)$. Now let us consider a tree T^{-l} directed by choosing a vertex $c_r(l)$ as a root. This gives $k_{j,(c_r(l))}^{-l}=k_{j,(l)}$ for $j\in V^{-l}\setminus\{c_r(l)\}$. $\square$

Lemma 2.5. *For any root $r\in V$, the mapping $\psi_r: M(T,K_T)\to\mathcal{V}_{n_1}^{+}\times\cdots\times\mathcal{V}_{n_p}^{+}$ defined by (2.5) and (2.6) is a bijection and its Jacobian is equal to one.*

Proof. It is obvious that the mapping ψ_r is into $\mathcal{V}_{n_1}^{+}\times\cdots\times\mathcal{V}_{n_p}^{+}$. To prove that ψ_r is onto we proceed by induction on the size p of the tree T.

For $p=2$, let $r=1$ and $l=2$. Then

$$\psi_1(k_1,k_2)=\left(k_{1,(1)},k_{2,(1)}\right)=\left(k_1-\mathbf{P}(k_{12})k_2^{-1},k_2\right)=\left(k_1^{-l},k_2\right).$$

By Lemma 2.1, $\left(k_1^{-l},k_2\right)\in M(T,K_T)$. Hence $\left(k_1-\mathbf{P}(k_{12})k_2^{-1},k_2\right)\in\mathcal{V}_{n_1}^{+}\times\mathcal{V}_{n_2}^{+}$ and we have $\mathbf{K}=\begin{bmatrix}k_1 & k_{12}\\ k_{21} & k_2\end{bmatrix}\in\mathcal{V}_{n_1+n_2}^{+}$since

$$\mathbf{K}=\begin{bmatrix}k_1 & k_{12}\\ k_{21} & k_2\end{bmatrix}=\begin{bmatrix}I_1 & k_{12}k_2^{-1}\\ 0 & I_2\end{bmatrix}\begin{bmatrix}k_1-\mathbf{P}(k_{12})k_2^{-1} & 0\\ 0 & k_2\end{bmatrix}\begin{bmatrix}I_1 & 0\\ k_2^{-1}k_{12} & I_2\end{bmatrix}$$

and thus $\mathbf{x}^t\mathbf{K}\mathbf{x}>0$ for all $\mathbf{x}\in\mathbb{R}^{n_1+n_2}$, $\mathbf{x}\neq\mathbf{0}$.

Now assume that ψ_r is onto for any tree T_{p-1} of size $p-1$, any set $K_{T_{p-1}}$ and any $k\in M(T_{p-1},K_{T_{p-1}})$. Choose an arbitrary root $r\in V$ and leaf $l\in L$, $l\neq r$. Without loss of generality we can assume that $r=1$, $l=p$ and $c_r(l)=p-1$. By Lemma 2.1, $k^{-l}=\left(k_1^{-l},\ldots,k_{p-1}^{-l}\right)=\left(k_1,\ldots,k_{c_r(l)-1},k_{c_r(l)}-\mathbf{P}(k_{c_r(l)l})k_l^{-1}\right)\in M(T^{-l},K_{T^{-l}})$. Hence, to prove that ψ_r is onto, it suffices to show that if $\mathbf{K}^{-l}\in\mathcal{V}^{+}_{\sum_{i=1}^{p-1}n_i}$ then $\mathbf{K}\in\mathcal{V}^{+}_{\sum_{i=1}^{p}n_i}$, where $\mathbf{K}^{-l}$ is defined by (2.4). But, from Lemma 2.1, $\left(k_1^{-l},\ldots,k_{c_r(l)-1}^{-l},k_{c_r(l)}^{-l}+\mathbf{P}(k_{c_r(l)l})k_l^{-1},k_l\right)\in M(T,K_T)$, so the result follows.

Thus the mapping ψ_r is a bijection. From (2.5) and (2.6) it follows that the Jacobian of ψ_r is of the form

$$J=Det\begin{pmatrix}id_{\mathcal{V}_{n_1}} & * & *\\ 0 & \cdots & *\\ 0 & 0 & id_{\mathcal{V}_{n_p}}\end{pmatrix},$$

where Det denotes the determinant of operators acting on $\mathcal{V}_{n_1}\times\cdots\times\mathcal{V}_{n_p}$ and $*$ denotes the part that is not needed in calculations of the Jacobian. Hence the Jacobian of ψ_r is equal to one. $\square$

Lemma 2.6. *For any root $r\in V$, we have*

$$|\mathbf{K}|=\prod_{i\in V}|k_{i,(r)}|. \tag{2.7}$$

Proof. We proceed by induction on the size p of the tree T.

For $p = 2$ the result follows from the expression for the determinant of a partitioned matrix in terms of its submatrices:

$$\begin{aligned}\mathbf{K} &= \begin{vmatrix} k_1 & k_{12} \\ k_{21} & k_2 \end{vmatrix} \\ &= \left|k_1 - \mathbf{P}(k_{12})k_2^{-1}\right| |k_2| = |k_{1,(1)}|\,|k_{2,(1)}| \\ &= |k_1|\left|k_2 - \mathbf{P}(k_{21})k_1^{-1}\right| = |k_{1,(2)}|\,|k_{2,(2)}|.\end{aligned}$$

Now assume that (2.7) is true for any tree T_{p-1} of size $p-1$, any set $K_{T_{p-1}}$ and any $k \in M(T_{p-1}, K_{T_{p-1}})$. Choose an arbitrary root $r \in V$ and a leaf $l \in L,\ l \neq r$. Without loss of generality we can assume that $r = 1$, $l = p$ and $c_r(l) = p-1$. Then, as in the proof of Lemma 2.1, we get

$$|\mathbf{K}| = |k_l|\left|\mathbf{K}^{-l}\right| = \left|k_{l,(r)}\right|\left|\mathbf{K}^{-l}\right|,$$

where $\mathbf{K}^{-l}$ is defined by (2.4). Since, by Lemma 2.1, $k^{-l} = \left(k_1^{-l}, \ldots, k_{p-1}^{-l}\right) = \left(k_1, \ldots, k_{c_r(l)-1}, k_{c_r(l)} - \mathbf{P}(k_{c_r(l)l})k_l^{-1}\right) \in M(T^{-l}, K_{T^{-l}})$, the induction assumption implies

$$|\mathbf{K}^{-l}| = \prod_{i \in V^{-l}} |k_{i,(r)}^{-l}|.$$

Hence, by Lemma 2.4,

$$|\mathbf{K}| = \left|k_{l,(r)}\right| \prod_{i \in V^{-l}} |k_{i,(r)}^{-l}| = \left|k_{l,(r)}\right| \prod_{i \in V \setminus \{l\}} |k_{i,(r)}| = \prod_{i \in V} |k_{i,(r)}|. \quad \square$$

Lemma 2.7. *For any root $r \in V$ and for any $a = \left(a_1, \ldots, a_p\right) \in \mathcal{V}_{n_1} \times \cdots \times \mathcal{V}_{n_p}$, we have*

$$\langle a, k\rangle = \sum_{i \in V} \langle a_i, k_i\rangle = \sum_{i \in V} \left(\left\langle a_i, k_{i,(r)}\right\rangle + \left\langle \mathbf{P}(k_{ic_r(i)})a_{c_r(i)}, k_{i,(r)}^{-1}\right\rangle\right).$$

Proof. From the definition of the inner product and (2.6) we obtain

$$\begin{aligned}\langle a, k\rangle &= \sum_{i \in V} \langle a_i, k_i\rangle = \sum_{i \in V} \left\langle a_i, k_{i,(r)} + \sum_{j \in p_r(i)} \mathbf{P}(k_{ij})k_{j,(r)}^{-1}\right\rangle \\ &= \sum_{i \in V} \left\langle a_i, k_{i,(r)}\right\rangle + \sum_{i \in V} \left\langle a_i, \sum_{j \in p_r(i)} \mathbf{P}(k_{ij})k_{j,(r)}^{-1}\right\rangle \\ &= \sum_{i \in V} \left\langle a_i, k_{i,(r)}\right\rangle + \sum_{i \in V} \sum_{j \in p_r(i)} \left\langle \mathbf{P}(k_{ji})a_i, k_{j,(r)}^{-1}\right\rangle.\end{aligned}$$

Since each $i \in V \setminus \{r\}$ has only one child, changing the order of summation in the last expression yields

$$\langle a, k\rangle = \sum_{i \in V} \left(\left\langle a_i, k_{i,(r)}\right\rangle + \left\langle \mathbf{P}(k_{ic_r(i)})a_{c_r(i)}, k_{i,(r)}^{-1}\right\rangle\right). \quad \square$$

3. The matrix variate Matsumoto–Yor property

Let $T = (V, E)$ be a tree of size p and let K_T be a fixed set.

Let us define a probability distribution $W_T^c(q, K_T, a)$ on $M(T, K_T)$ by the density

$$f(k) \propto |\mathbf{K}|^q \exp\left(-\langle a, k\rangle\right) I_{M(T,K_T)}(k), \tag{3.1}$$

where $a = \left(a_1, \ldots, a_p\right) \in \mathcal{V}_{n_1}^+ \times \cdots \times \mathcal{V}_{n_p}^+$ and $q > -1$ is a scalar. It will be proved in Theorem 3.1 that the right-hand side of (3.1) is the density of a finite measure. This distribution can be viewed as a generalization of the p-variate W_G^c distribution defined in Section 3 of [15].

Note that for $p = 2$ the distribution given by the density (3.1) corresponds to the one given by Seshadri and Wesołowski [20]: they defined a distribution $W_{r,s}(q, c, a, b)$ to rephrase the MY property and related characterization for matrix variates having dimensions $r \times r$ and $s \times s$, originally stated and proved in [16]. Moreover, when $p = 2$ and $n_1 = n_2 = 1$, i.e. when $\mathbf{K}$ is a matrix valued in $\mathcal{V}_2^+$, (3.1) is the density of the conditional distribution of the diagonal elements of $\mathbf{K}$ given its off-diagonal elements in the case when the distribution of $\mathbf{K}$ is quasi-Wishart, see Geiger and Heckerman [5], Letac and Massam [13].

The direct Matsumoto–Yor property on trees for matrices of different dimensions is established by the following theorem.

Theorem 3.1. *Let $T = (V, E)$ be a tree of size p $(p \geq 2)$. Let $K = (K_1, \dots, K_p) \sim W_T^c(q, K_T, a)$. Define $X_r = \psi_r(K),\ r \in V$. Then for each $r \in V$ the components of $X_r = \left(X_{1,(r)}, \dots, X_{p,(r)}\right)$ are independent. Moreover,*

$$X_{r,(r)} \sim W_{n_r}(q_r, a_r)$$

and

$$X_{j,(r)} \sim MGIG_{n_j}(q_j, a_j, \mathbf{P}(k_{jc_r(j)})a_{c_r(j)}) \quad j \in V \setminus \{r\},$$

where $q_i - \frac{n_i+1}{2} = q,\ i \in V$.

Proof. Choose any $r \in V$. We know from Lemma 2.5 that the mapping $\psi_r : M(T, K_T) \to \mathcal{V}_{n_1}^+ \times \cdots \times \mathcal{V}_{n_p}^+$ is a bijection and its Jacobian is equal to one. Hence, by Lemmas 2.6 and 2.7, we can rewrite the density (3.1) of K as

$$\begin{aligned} f(k) &\propto |\mathbf{K}|^q \exp\left(-\langle a, k\rangle\right) I_{M(T,K_T)}(k) \\ &= \prod_{i\in V} |k_{i,(r)}|^{q_i - \frac{n_i+1}{2}} \exp\left\{-\sum_{i\in V}\left(\left\langle a_i, k_{i,(r)}\right\rangle + \left\langle \mathbf{P}(k_{ic_r(i)})a_{c_r(i)}, k_{i,(r)}^{-1}\right\rangle\right)\right\} I_{\mathcal{V}_{n_i}^+}(k_{i,(r)}) \\ &= |k_{r,(r)}|^{q_r - \frac{n_r+1}{2}} \exp\left\{-\left\langle a_r, k_{r,(r)}\right\rangle\right\} I_{\mathcal{V}_{n_r}^+}(k_{r,(r)}) \\ &\quad \times \prod_{i\in V\setminus\{r\}} |k_{i,(r)}|^{q_i - \frac{n_i+1}{2}} \exp\left\{-\sum_{i\in V}\left(\left\langle a_i, k_{i,(r)}\right\rangle + \left\langle \mathbf{P}(k_{ic_r(i)})a_{c_r(i)}, k_{i,(r)}^{-1}\right\rangle\right)\right\} I_{\mathcal{V}_{n_i}^+}(k_{i,(r)}). \end{aligned}$$

Therefore we obtain the product of one Wishart and $p - 1$ matrix GIG distributions. □

4. The characterization of the Wishart and MGIG distributions

In this section we will prove a characterization of one Wishart and $p - 1$ matrix GIG distributions, that is the converse to the Matsumoto–Yor property on trees for matrix variates of different dimensions. The essential part of the proof of this characterization is the following observation.

Lemma 4.1. *Let $T = (V, E)$ be a tree of size p $(p \geq 2)$. Let $L \subset V$ be its set of leaves. Let the set K_T be given and let $K = (K_1, \dots, K_p)$ be a vector of block random matrices taking its values in $M(T, K_T)$. Let $X_r = \psi_r(K),\ r = l_1, l_2 \in L$. If the components of $X_{l_1} = (X_{1,(l_1)}, \dots, X_{p,(l_1)})$ are jointly independent, then the random matrices $X_{l_2,(l_1)}$ and $X_{s_2,(l_1)} - \mathbf{P}(k_{s_2c_{l_1}(s_2)})X^{-1}_{c_{l_1}(s_2),(l_2)}$ are independent, where $s_2 = c_{l_1}(l_2)$.*

Proof. We have

$$\psi_r(K) = \left(X_{1,(r)}, \dots, X_{p,(r)}\right), \tag{4.1}$$

where

$$X_{i,(r)} = \begin{cases} K_i & \text{if } i \in L \setminus \{r\}, \\ K_i - \sum_{j\in p_r(i)} \mathbf{P}(k_{ij})X_{j,(r)}^{-1} & \text{if } i \in V \setminus L, \end{cases} \tag{4.2}$$

$r = l_1, l_2$. Since we assume that the components of $X_{l_1} = (X_{1,(l_1)}, \dots, X_{p,(l_1)})$ are jointly independent, it suffices to show that $X_{c_{l_1}(s_2),(l_2)}$ can be written in terms of some $X_{j,(l_1)}$'s, except for $X_{l_2,(l_1)}$.

Obviously, each $X_{i,(r)}$ is a function of K. From (4.1) and (4.2) it follows that $X_{c_{l_1}(s_2),(l_2)}$ depends on K only through the components K_j such that $j \in V \setminus \{s_2, l_2\}$. It is therefore of interest to look how each K_j can be written in terms of $X_{j,(l_1)}$'s. Observe that for $j \in V \setminus \{s_2, l_2\}$, each K_j can be written in terms of some $X_{j,(l_1)}$'s, except for $X_{l_2,(l_1)}$. Thus $X_{c_{l_1}(s_2),(l_2)}$ is a function of some $X_{j,(l_1)}$'s where $j \neq l_2$. Hence $X_{l_2,(l_1)}$ and $X_{s_2,(l_1)} - \mathbf{P}(k_{s_2c_{l_1}(s_2)})X^{-1}_{c_{l_1}(s_2),(l_2)}$ are independent. □

Our characterization of one Wishart and $p - 1$ matrix GIG distributions will be given under the assumption of strict positivity and differentiability of the densities. These technical conditions are due to the fact that we will use the following result (Theorem 4.1) from the paper of Massam and Wesołowski [16]:

Theorem 4.2. *Let X and Y be two independent random matrices taking their values in $\mathcal{V}_r^+$ and $\mathcal{V}_s^+$ respectively. Assume that X and Y have strictly positive differentiable densities with respect to the Lebesgue measure. Let*

$$\psi_z(x, y) = \left((\mathbf{P}(z)x + y)^{-1}, x^{-1} - \mathbf{P}(z^t)(\mathbf{P}(z)x + y)^{-1}\right),$$

where z is a given constant $s \times r$ matrix of full rank. Let $(U, V) = \psi_z(X, Y)$.

If U and V are independent, then there exist $(a, b) \in \mathcal{V}_r^+ \times \mathcal{V}_s^+$ and scalars p and q satisfying $p - q = \frac{r-s}{2}$, $p > \frac{r-1}{2}$, such that X and Y are independent GIG and Wishart with

$$(X, Y) \sim MGIG_r(-p, \mathbf{P}(z^t)a, b) \otimes W_s(q, a).$$

It also follows immediately that U and V are independent GIG and Wishart with

$$(U, V) \sim MGIG_s(-q, \mathbf{P}(z)b, a) \otimes W_s(p, b).$$

We will need Lemma 4.1 and Theorem 4.2 to identify the matrices with Wishart distributions. To obtain further the matrix GIG distributions, we will use the induction argument, similar to the one used in [15] in the proof of the characterization of the product of one gamma and $p - 1$ GIG laws (Theorem 4.1). It is worth noting that the proof in [15] was split into two parts. In the first part by applying Laplace transforms (which was an adaptation of the method used in Theorem 4.1 in Letac and Wesołowski [14]) the gamma distributions were identified. Then the induction argument gave GIG's variables.

Here we do not use Laplace transforms to work out the Wishart variables. Instead our approach is based on the independence property observed in Lemma 4.1 and on Theorem 4.2. To get the matrix GIG variables we adopt the induction approach from [15] to the matrix variate situation.

Theorem 4.3. *Let $T = (V, E)$ be a tree of size p $(p \geq 2)$. Let $L \subset V$ be its set of leaves. Let the set K_T be given and let $K = (K_1, \ldots, K_p)$ be a vector of p random matrices $n_i \times n_i$ $(i = 1, \ldots, p)$ taking its values in $M(T, K_T)$. Let $X_r = \psi_r(K)$, $r \in V$. If, for any root $r \in L$, the components of $X_r = (X_{1,(r)}, \ldots, X_{p,(r)})$ are mutually independent and have strictly positive differentiable densities with respect to the Lebesgue measure then there exist $a_j \in \mathcal{V}_{n_j}^+$, $j \in V$ and scalars q_j satisfying*

$$q_j - \frac{n_j}{2} = q_i - \frac{n_i}{2} > -\frac{1}{2}, \quad i, j \in V,$$

such that

$$X_{r,(r)} \sim W_{n_r}(q_r, a_r)$$

and

$$X_{j,(r)} \sim MGIG_{n_j}(q_j, a_j, \mathbf{P}(k_{jc_r(j)})a_{c_r(j)}) \quad j \in V \setminus \{r\}.$$

Proof. Let us consider two leaves $l_1, l_2 \in L$. Let $\mathcal{P}_{l_1 l_2} \subset V$ denote a unique path in T linking l_1 and l_2. By choosing l_1 as a root and then by choosing l_2 as a root we obtain two different directions of T. Consider two mappings ψ_{l_1} and ψ_{l_2} defined by (2.5) and (2.6), corresponding to these two directions of T and let $X_r = \psi_r(K)$, $r = l_1, l_2$. We have

$$\psi_r\left(K_1, \ldots, K_p\right) = \left(X_{1,(r)}, \ldots, X_{p,(r)}\right), \tag{4.3}$$

where

$$X_{i,(r)} = \begin{cases} K_i & \text{if } i \in L \setminus \{r\}, \\ K_i - \sum_{j \in p_r(i)} \mathbf{P}(k_{ij}) X_{j,(r)}^{-1} & \text{if } i \in V \setminus L, \end{cases} \tag{4.4}$$

$r = l_1, l_2$ and we know that for $r = l_1, l_2$ the components of $(X_{1,(r)}, \ldots, X_{p,(r)})$ are mutually independent.

We begin by proving that the distributions of $X_{r,(r)}$, $r = l_1, l_2$, are Wisharts.

Note that for $i \notin \mathcal{P}_{l_1 l_2}$, the ith component of $\psi_{l_1}(K)$ is equal to the ith component of $\psi_{l_2}(K)$:

$$X_{i,(l_1)} = X_{i,(l_2)}.$$

Let s_i denote the only neighbour of l_i, $i = 1, 2$. From (4.3) and (4.4) it follows that

$$X_{l_2,(l_1)} = K_{l_2}, \tag{4.5}$$

$$X_{s_2,(l_1)} = K_{s_2} - \mathbf{P}(k_{s_2 l_2}) K_{l_2}^{-1} - \sum_{j \in p_{l_1}(s_2) \setminus \mathcal{P}_{l_1 l_2}} \mathbf{P}(k_{s_2 j}) X_{j,(l_1)}^{-1}, \tag{4.6}$$

and since $\{c_{l_1}(s_2)\} = p_{l_2}(s_2) \cap \mathcal{P}_{l_1 l_2}$ we obtain

$$X_{s_2,(l_2)} = K_{s_2} - \mathbf{P}(k_{s_2 c_{l_1}(s_2)}) X_{c_{l_1}(s_2),(l_2)}^{-1} - \sum_{j \in p_{l_2}(s_2) \setminus \mathcal{P}_{l_1 l_2}} \mathbf{P}(k_{s_2 j}) X_{j,(l_2)}^{-1}, \tag{4.7}$$

$$X_{l_2,(l_2)} = K_{l_2} - \mathbf{P}(k_{l_2 s_2}) X_{s_2,(l_2)}^{-1}. \tag{4.8}$$

From (4.5) and (4.6) we have

$$K_{l_2} = X_{l_2,(l_1)}, \tag{4.9}$$

$$K_{s_2} = X_{s_2,(l_1)} + \mathbf{P}(k_{s_2l_2})X^{-1}_{l_2,(l_1)} + \sum_{j\in p_{l_1}(s_2)\setminus\mathcal{P}_{l_1l_2}} \mathbf{P}(k_{s_2j})X^{-1}_{j,(l_1)}. \tag{4.10}$$

Plugging (4.10) into (4.7) and using the fact that $p_{l_1}(s_2)\setminus\mathcal{P}_{l_1l_2} = p_{l_2}(s_2)\setminus\mathcal{P}_{l_1l_2}$ and for $j\notin\mathcal{P}_{l_1l_2}$ we have $X_{j,(l_1)} = X_{j,(l_2)}$, we obtain

$$X_{s_2,(l_2)} = X_{s_2,(l_1)} + \mathbf{P}(k_{s_2l_2})X^{-1}_{l_2,(l_1)} - \mathbf{P}(k_{s_2c_{l_1}(s_2)})X^{-1}_{c_{l_1}(s_2),(l_2)}. \tag{4.11}$$

Inserting (4.9) and (4.11) into (4.8) we get

$$X_{l_2,(l_2)} = X_{l_2,(l_1)} - \mathbf{P}(k_{l_2s_2})\left[X_{s_2,(l_1)} + \mathbf{P}(k_{s_2l_2})X^{-1}_{l_2,(l_1)} - \mathbf{P}(k_{s_2c_{l_1}(s_2)})X^{-1}_{c_{l_1}(s_2),(l_2)}\right]^{-1}. \tag{4.12}$$

Let

$$X = X^{-1}_{l_2,(l_1)}$$

and

$$Y = X_{s_2,(l_1)} - \mathbf{P}(k_{s_2c_{l_1}(s_2)})X^{-1}_{c_{l_1}(s_2),(l_2)}.$$

By Lemma 4.1, X and Y are independent.

Now define

$$U = X^{-1}_{s_2,(l_2)}$$

and

$$V = X_{l_2,(l_2)}.$$

From the assumption that the components of $(X_{1,(l_2)},\dots,X_{p,(l_2)})$ are mutually independent it follows that U and V are independent. Moreover, using (4.11) and (4.12), we can rewrite U and V in terms of X and Y as follows

$$U = \left[Y + \mathbf{P}(k_{s_2l_2})X\right]^{-1}$$

and

$$V = X^{-1} - \mathbf{P}(k_{l_2s_2})\left[Y + \mathbf{P}(k_{s_2l_2})X\right]^{-1}.$$

Thus, from Theorem 4.2 and from (1.2), we get that there exist $a_{l_2}\in\mathcal{V}^+_{n_{l_2}}$, $a_{s_2}\in\mathcal{V}^+_{n_{s_2}}$ and scalars q_{l_2}, q_{s_2} satisfying

$$q_{l_2} - \frac{n_{l_2}}{2} = q_{s_2} - \frac{n_{s_2}}{2} > -\frac{1}{2}$$

such that

$$\begin{aligned} X_{l_2,(l_2)} &= V \sim W_{n_{l_2}}(q_{l_2}, a_{l_2}), \\ X_{l_2,(l_1)} &= X^{-1} \sim MGIG_{n_{l_2}}(q_{l_2}, a_{l_2}, \mathbf{P}(k_{l_2s_2})a_{s_2}), \\ X_{s_2,(l_2)} &= U^{-1} \sim MGIG_{n_{s_2}}(q_{s_2}, a_{s_2}, \mathbf{P}(k_{s_2l_2})a_{l_2}) \end{aligned} \tag{4.13}$$

and

$$Y \sim W_{n_{s_2}}(q_{s_2}, a_{s_2}).$$

Similarly, swapping the roles of l_1 and l_2 we have that there exist $a_{l_1}\in\mathcal{V}^+_{n_{l_1}}$, $a_{s_1}\in\mathcal{V}^+_{n_{s_1}}$ and scalars q_{l_1}, q_{s_1} satisfying

$$q_{l_1} - \frac{n_{l_1}}{2} = q_{s_1} - \frac{n_{s_1}}{2} > -\frac{1}{2}$$

such that

$$\begin{aligned} X_{l_1,(l_1)} &= \tilde{V} \sim W_{n_{l_1}}(q_{l_1}, a_{l_1}), \\ X_{l_1,(l_2)} &= \tilde{X}^{-1} \sim MGIG_{n_{l_1}}(q_{l_1}, a_{l_1}, \mathbf{P}(k_{l_1s_1})a_{s_1}), \\ X_{s_1,(l_1)} &= \tilde{U}^{-1} \sim MGIG_{n_{s_1}}(q_{s_1}, a_{s_1}, \mathbf{P}(k_{s_1l_1})a_{l_1}) \end{aligned} \tag{4.14}$$

and

$$\tilde{Y} \sim W_{n_{s_1}}(q_{s_1}, a_{s_1}).$$

Now we will prove that all $X_{j,(r)}$, where $j \in V \setminus \{r\}$, $r = l_1, l_2$, have matrix GIG distributions. We will apply induction with respect to the size p of the tree G.

Observe first that the condition that for any root $r \in L$, the components of $X_r = (X_{1,(r)}, \dots, X_{p,(r)})$ are mutually independent, can be rewritten as

$$f_{r,(r)}(k_{r,(r)}) \prod_{i \in V \setminus \{r\}} f_{i,(r)}(k_{i,(r)}) = h(k) \tag{4.15}$$

a.s. with respect to the Lebesgue measure for $k = (k_1, \dots, k_p) \in M(T, K_T)$, where $f_{i,(r)}$ is the density of $X_{i,(r)}$, $i \in V$, $r \in L$, $k_{i,(r)}$ is the ith component of $\psi_r(k)$ and h is a function independent of r. Moreover, $f_{r,(r)}$ is the density of the Wishart distribution $W_{n_r}(q_r, a_r)$.

We will prove that if for any tree T_p of size $p \geq 2$, for any set of off-diagonal elements K_{T_p} and for any $k \in M(T_p, K_T)$, Eq. (4.15) holds under the assumption that $f_{r,(r)}$ is the density of the Wishart distribution $W_{n_r}(q_r, a_r)$, then there exist $a_i \in \mathcal{V}^+_{n_i}$ and scalars q_i $i \in V \setminus \{r\}$, satisfying

$$q_j - \frac{n_j}{2} = q_i - \frac{n_i}{2} > -\frac{1}{2}, \quad i, j \in V$$

such that the densities $f_{i,(r)}$, where $i \neq r$, are the densities of the matrix GIG distributions $MGIG_{n_i}(q_i, a_i, \mathbf{P}(k_{ic_r(i)})a_{c_r(i)})$.

For $p = 2$ the result follows directly from Theorem 4.2, since in this case $T = \{l_1, l_2\}$ and we have that there exist $a_{l_1} \in \mathcal{V}^+_{n_{l_1}}$, $a_{l_2} \in \mathcal{V}^+_{n_{l_2}}$ and scalars q_{l_1}, q_{l_2} satisfying

$$q_{l_1} - \frac{n_{l_1}}{2} = q_{l_2} - \frac{n_{l_2}}{2} > -\frac{1}{2}$$

such that

$$\begin{aligned}
&X_{l_1,(l_1)} \sim W_{n_{l_1}}(q_{l_1}, a_{l_1}),\\
&X_{l_2,(l_1)} \sim MGIG_{n_{l_2}}(q_{l_2}, a_{l_2}, \mathbf{P}(k_{l_2 l_1})a_{l_1}) = MGIG_{n_{l_2}}(q_{l_2}, a_{l_2}, \mathbf{P}(k_{l_2 c_{l_1}(l_2)})a_{c_{l_1}(l_2)}),\\
&X_{l_2,(l_2)} \sim W_{n_{l_2}}(q_{l_2}, a_{l_2}),\\
&X_{l_1,(l_2)} \sim MGIG_{n_{s_2}}(q_{l_1}, a_{l_1}, \mathbf{P}(k_{l_1 l_2})a_{l_2}) = MGIG_{n_{s_2}}(q_{l_1}, a_{l_1}, \mathbf{P}(k_{l_1 c_{l_2}(l_1)})a_{c_{l_2}(l_1)}).
\end{aligned} \tag{4.16}$$

Now we assume that the result holds for any tree T_{p-1} of size $p-1$, for any set of off-diagonal elements $K_{T_{p-1}}$ and for any $k \in M(T_{p-1}, K_{T_{p-1}})$.

From (4.15) it follows that

$$f_{l_2,(l_1)}(k_{l_2,(l_1)}) f_{l_1,(l_1)}(k_{l_1,(l_1)}) \prod_{i \in V \setminus \{l_1, l_2\}} f_{i,(l_1)}(k_{i,(l_1)}) = f_{l_2,(l_2)}(k_{l_2,(l_2)}) f_{s_2,(l_2)}(k_{s_2,(l_2)}) \prod_{i \in V \setminus \{s_2, l_2\}} f_{i,(l_2)}(k_{i,(l_2)}).$$

Using (4.7) and (4.8) and noting that $k_{l_2,(l_1)} = k_{l_2}$, we can rewrite the above equation as

$$\begin{aligned}
f_{l_2,(l_1)}(k_{l_2}) f_{l_1,(l_1)}(k_{l_1,(l_1)}) \prod_{i \in V \setminus \{l_1, l_2\}} f_{i,(l_1)}(k_{i,(l_1)}) &= f_{l_2,(l_2)}\left(k_{l_2} - \mathbf{P}(k_{l_2 s_2})\left(k_{s_2} - \sum_{j \in p_{l_2}(s_2)} \mathbf{P}(k_{s_2 j}) k^{-1}_{j,(l_2)}\right)^{-1}\right)\\
&\quad \times f_{s_2,(l_2)}\left(k_{s_2} - \sum_{j \in p_{l_2}(s_2)} \mathbf{P}(k_{s_2 j}) k^{-1}_{j,(l_2)}\right) \prod_{i \in V \setminus \{s_2, l_2\}} f_{i,(l_2)}(k_{i,(l_2)}).
\end{aligned} \tag{4.17}$$

Let now fix k_{l_2} and consider $k^{-l_2} \in M(T^{-l_2}, K_{T^{-l_2}})$, defined as in Lemma 2.1. By Lemma 2.4, we can rewrite (4.17) as

$$\begin{aligned}
&f_{l_2,(l_1)}(k_{l_2}) f_{l_1,(l_1)}(k^{-l_2}_{l_1,(l_1)}) \prod_{i \in V^{-l_2} \setminus \{l_1\}} f_{i,(l_1)}(k^{-l_2}_{i,(l_1)})\\
&\quad = f_{l_2,(l_2)}\left(k_{l_2} - \mathbf{P}(k_{l_2 s_2})\left(k^{-l_2}_{s_2} + \mathbf{P}(k_{s_2 l_2}) k^{-1}_{l_2} - \sum_{j \in p_{l_2}(s_2)} \mathbf{P}(k_{s_2 j}) \left[k^{-l_2}_{j,(l_2)}\right]^{-1}\right)^{-1}\right)\\
&\quad\quad \times f_{s_2,(l_2)}\left(k^{-l_2}_{s_2} + \mathbf{P}(k_{s_2 l_2}) k^{-1}_{l_2} - \sum_{j \in p_{l_2}(s_2)} \mathbf{P}(k_{s_2 j}) \left[k^{-l_2}_{j,(l_2)}\right]^{-1}\right) \prod_{i \in V^{-l_2} \setminus \{s_2\}} f_{i,(l_2)}(k^{-l_2}_{i,(s_2)})
\end{aligned} \tag{4.18}$$

and obtain the following equation

$$f_{l_1,(l_1)}(k^{-l_2}_{l_1,(l_1)}) \prod_{i\in V^{-l_2}\setminus\{l_1\}} f_{i,(l_1)}(k^{-l_2}_{i,(l_1)}) = g\,(x) \prod_{i\in V^{-l_2}\setminus\{s_2\}} f_{i,(l_2)}(k^{-l_2}_{i,(s_2)}),$$

where

$$g\,(x) = \frac{f_{l_2,(l_2)}\left(k_{l_2} - \mathbf{P}(k_{l_2 s_2})\left(x+\mathbf{P}(k_{s_2 l_2})k_{l_2}^{-1}\right)^{-1}\right) f_{s_2,(l_2)}\left(x+\mathbf{P}(k_{s_2 l_2})k_{l_2}^{-1}\right)}{f_{l_2,(l_1)}(k_{l_2})}$$

and

$$x = k^{-l_2}_{s_2} - \sum_{j\in p_{l_2}(s_2)} \mathbf{P}(k_{s_2 j})\left[k^{-l_2}_{j,(l_2)}\right]^{-1}.$$

Since the set $p_{l_2}(s_2)$ in T is equal to the set $p_{s_2}(s_2)$ in T^{-l_2} we get

$$x = k^{-l_2}_{s_2,(s_2)}$$

and our equation takes the form

$$f_{l_1,(l_1)}(k^{-l_2}_{l_1,(l_1)}) \prod_{i\in V^{-l_2}\setminus\{l_1\}} f_{i,(l_1)}(k^{-l_2}_{i,(l_1)}) = g\left(k^{-l_2}_{s_2,(s_2)}\right) \prod_{i\in V^{-l_2}\setminus\{s_2\}} f_{i,(l_2)}(k^{-l_2}_{i,(s_2)}). \tag{4.19}$$

Note that we can always choose k_{l_2} in such a way that (4.19) holds a.s. with respect to the Lebesgue measure for $k^{-l_2} \in M(T^{-l_2}, K_{T^{-l_2}})$ and that $L^{-l_2} \subset (L \setminus \{l_2\}) \cup \{s_2\}$. Moreover, (4.19) implies that g is also a density.

Since (4.19) holds for any $l_1 \in L^{-l_2}$, we get an analogue of (4.15) for a tree of size $p-1$: for any root $r \in L^{-l_2}$

$$f^{-l_2}_{r,(r)}(k^{-l_2}_{r,(r)}) \prod_{i\in V\setminus\{r\}} f^{-l_2}_{i,(r)}(k^{-l_2}_{i,(r)}) = \tilde{h}(k^{-l_2}) \tag{4.20}$$

a.s. with respect to the Lebesgue measure for $k^{-l_2} \in M(T^{-l_2}, K_{T^{-l_2}})$, where either $s_2 \notin L^{-l_2}$ and then $f^{-l_2}_{i,(r)} = f_{i,(r)}$ is the density of $X^{-l_2}_{i,(r)} = X_{i,(r)}$ for $i \in V^{-l_2}$ and $r \in L^{-l_2}$ or $s_2 \in L^{-l_2}$ and then $f^{-l_2}_{i,(r)} = f_{i,(r)}$ is the density of $X^{-l_2}_{i,(r)} = X_{i,(r)}$ for $i \in V^{-l_2}$ and $r \in L^{-l_2} \setminus \{s_2\}$, $f^{-l_2}_{i,(s_2)} = f_{i,(l_2)}$ is the density of $X_{i,(l_2)}$ for $i \in V^{-l_2} \setminus \{s_2\}$, $f^{-l_2}_{s_2,(s_2)} = g$ is the density of $X^{-l_2}_{s_2,(s_2)}$ and $\tilde{h}$ is a function independent of r.

Moreover, $f^{-l_2}_{r,(r)}$ is the density of the Wishart distribution $W_{n_r}(q_r, a_r)$: either $s_2 \notin L^{-l_2}$ and then it follows from the fact that $f_{r,(r)}$ in (4.15) is Wishart or $s_2 \in L^{-l_2}$ and then (4.20) corresponds to the independence conditions analogous to the ones defined by (4.3) and (4.4) but taken with respect to the tree with leaves in the set $(L \setminus \{l_2\}) \cup \{s_2\}$. This, due to the first part of the proof in which Wishart matrices were identified, implies that $f^{-l_2}_{s_2,(s_2)}$ is also the density of the Wishart distribution.

Thus from our induction assumption it follows that the functions $f_{i,(l_1)}$, $i \in V^{-l_2} \setminus \{l_1\}$, on the left hand side of (4.19) are densities of the matrix GIG distributions. More precisely, there exist $\tilde{a}_j \in \mathcal{V}^+_{n_j}$, $j \in V$, and scalars $\tilde{q}_j$ satisfying

$$\tilde{q}_i - \frac{n_i}{2} = \tilde{q}_j - \frac{n_j}{2} > -\frac{1}{2}$$

such that

$$X^{-l_2}_{l_1,(l_1)} = X_{l_1,(l_1)} \sim W_{n_{l_1}}(\tilde{q}_{l_1}, \tilde{a}_{l_1})$$

and

$$X^{-l_2}_{j,(l_1)} = X_{j,(l_1)} \sim MGIG_{n_j}(\tilde{q}_j, \tilde{a}_j, \mathbf{P}(k_{jc_{l_1}(j)})\tilde{a}_{c_{l_1}(j)}) \quad j \in V^{-l_2} \setminus \{l_1\},$$
$$X^{-l_2}_{j,(s_2)} = X_{j,(s_2)} \sim MGIG_{n_j}(\tilde{q}_j, \tilde{a}_j, \mathbf{P}(k_{jc_{s_2}(j)})\tilde{a}_{c_{s_2}(j)}) \quad j \in V^{-l_2} \setminus \{s_2\}.$$

Similarly, swapping the roles of l_1 and l_2 we have that there exist $a_j \in \mathcal{V}^+_{n_j}$, $j \in V$, and scalars q_j satisfying

$$q_i - \frac{n_i}{2} = q_j - \frac{n_j}{2} > -\frac{1}{2}$$

such that

$$X^{-l_1}_{l_2,(l_2)} = X_{l_2,(l_2)} \sim W_{n_{l_2}}(q_{l_2}, a_{l_2})$$

and

$$X^{-l_1}_{j,(l_2)} = X_{j,(l_2)} \sim MGIG_{n_j}(q_j, a_j, \mathbf{P}(k_{jc_{l_2}(j)})a_{c_{l_2}(j)}) \quad j \in V^{-l_1} \setminus \{l_2\},$$
$$X^{-l_1}_{j,(s_1)} = X_{j,(s_1)} \sim MGIG_{n_j}(q_j, a_j, \mathbf{P}(k_{jc_{s_1}(j)})a_{c_{s_1}(j)}) \quad j \in V^{-l_1} \setminus \{s_1\}.$$

Hence we know the distributions of $X_{j,(r)}$ for any $j \in V$ and $r = l_1, l_2 \in L$ ($X_{l_1,(l_2)}$ and $X_{l_2,(l_1)}$ were found earlier, see (4.13) and (4.14)).

It remains to prove that the two sets of parameters, for $r = l_1$ and $r = l_2$, are identical, that is $\tilde{q}_j = q_j$ and $\tilde{a}_j = a_j$ for any $j \in V$. Adopting the notation from (4.16) we can define

$$\tilde{q}_{l_1} = q_{l_1}, \qquad \tilde{q}_{l_2} = q_{l_2}, \qquad \tilde{a}_{l_1} = a_{l_1}, \qquad \tilde{a}_{l_2} = a_{l_2}, \qquad \tilde{a}_{c_{l_1}(l_2)} = a_{c_{l_1}(l_2)} \quad \text{and} \quad \tilde{a}_{c_{l_2}(l_1)} = \tilde{a}_{c_{l_1}(l_2)}.$$

Using our independence equation

$$\prod_{i\in V} f_{i,(l_1)}(k_{i,(l_1)}) = \prod_{i\in V} f_{i,(l_2)}(k_{i,(l_2)}), \tag{4.21}$$

and knowing that $f_{i,(l_j)},\ j = l_1, l_2,\ i \in V$, are densities of Wisharts and matrix GIG distributions identified above, we obtain that the left-hand side of (4.21) is proportional to

$$\prod_{i\in V} |k_{i,(l_1)}|^{\tilde{q}_i - \frac{n_i+1}{2}} \exp\left(-\left\langle \tilde{a}_{l_1}, k_{l_1,(l_1)}\right\rangle\right) \prod_{i\in V\setminus\{l_1\}} \exp\left(-\left\langle \tilde{a}_i, k_{i,(l_1)}\right\rangle - \left\langle \mathbf{P}(k_{ic_{l_1}(i)})\tilde{a}_{c_{l_1}(i)}, k^{-1}_{i,(l_1)}\right\rangle\right).$$

Since $\tilde{q}_i - \frac{n_i}{2} = \tilde{q}_{l_1} - \frac{n_{l_1}}{2}$ for any $i \in V$, it follows from Lemmas 2.6 and 2.7 that the above expression may be rewritten in the form

$$|\mathbf{K}|^{\tilde{q}_{l_1} - \frac{n_{l_1}+1}{2}} \exp\left(\sum_{i\in V} \left\langle \tilde{a}_i, k_i\right\rangle\right). \tag{4.22}$$

Similarly, on the right-hand side of (4.21) we have

$$|\mathbf{K}|^{q_{l_2} - \frac{n_{l_2}+1}{2}} \exp\left(\sum_{i\in V} \langle a_i, k_i\rangle\right). \tag{4.23}$$

Taking logarithms of (4.22) and (4.23) we get, respectively,

$$\left(\tilde{q}_{l_1} - \frac{n_{l_1}+1}{2}\right) \log(|\mathbf{K}|) + \sum_{i\in V} \left\langle \tilde{a}_i, k_i\right\rangle \tag{4.24}$$

and

$$\left(q_{l_2} - \frac{n_{l_2}+1}{2}\right) \log(|\mathbf{K}|) + \sum_{i\in V} \langle a_i, k_i\rangle. \tag{4.25}$$

Comparing (4.24) and (4.25) and using linear independence of the functions $\log(|\mathbf{K}|)$ and $\langle a_i, k_i\rangle$ as functions of $k_i,\ i \in V$, we get

$$\tilde{q}_i = q_i \quad \text{and} \quad a_i = \tilde{a}_i \quad \text{for any } i \in V.$$

This completes the proof. □

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