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Symmetric Gaussian mixture distributions with GGC scales

Stergios B. Fotopoulos*

Department of Finance and Management Science, Carson College of Business, Washington State University, Pullman, WA, USA

Abstract

The aim of this study is to unify and extend hyperbolic distributions when scalars are generated from the GGC family. Such distributions play an important role for modeling asset prices. Explicit expressions of multivariate densities are presented in terms of either the Laplace transform or the density of the scalar. When scalars are members of the GGC family, then the representations are articulated with respect to the Thorin measure. Several examples are provided.

Keywords: Conditional densities, Dirichlet distributions, generalized inverse Gaussian variables, Laplace transform, Spherical vectors, subordinators

1. Introduction

The main focus of this article is to develop densities from scale mixtures of Gaussian vectors. Such random variables can play an important role in understanding uncertainty driven by stock market dynamics. The inability of the normal law to explain certain empirical properties of asset returns and options prices has led to the development of a variety of models, such as stable laws, Lévy laws and various derivatives of Lévy variables that include mixtures of normal distributions; see [16, 25, 26, 31].

More specifically, the objective here is to focus on unifying various properties of multivariate elliptical contoured vectors with those of scale mixtures of Gaussian vectors and to provide a common ground where tractable density expressions can be generated. For, it is important to know the density forms when one deals with applications.

Normal mixtures have received widespread attention for over half a century. A great deal of effort has been spent on the characterization of their densities and their moments. The literature includes [1–4, 6, 10–14, 21–23], and [25]. A more complete list can be found in [16]. The laws generated from this family can possess heavy tails relative to normal laws, and yet finite moments for at least the lower power of asset prices. They also have multivariate distributions with hyperbolic form which supports the validity of the capital asset pricing model; see, e.g., [27, 28].

Adopting a few ideas from the work of Samorodnitsky and Taqqu [31], who characterized stable random vectors with respect to spectral measures of stable vectors, we extend these characterizations to scale mixtures of Gaussian vectors. The reason for this change in direction to scale mixtures is because of the one-to-one relationship between densities and characteristic functions, combined with the functional form of the scale mixture of Gaussian vectors. In particular, the characteristic function of mixtures of Gaussian vectors has a more tractable form, and thus it is used as a tool to develop densities. In the context of general scales, special cases are also examined to gain more insight into how the scales are allied with the Gaussian vectors. This permits easy computational procedures as can be seen from their expressions.

To understand further the computing context, an essential general subfamily of scales is introduced here: the generalized gamma convolution (GGC). A subfamily of GGC is the generalized inverse Gaussian (GIG) that has been of interest in many applications in applied and mathematical finance. The stability condition for the existence of the density under the scale mixture of Gaussian is stated only via the Laplace transform of the scale. However, when the

*Corresponding author.

Email address: fotopo@wsu.edu (Stergios B. Fotopoulos)

scale is a member of the generalized gamma convolutions, then the stability condition is expressed in terms of the Thorin measure of the scale. The work here is limited to symmetric vectors, e.g., when vectors $Y - \mu$ and $-(Y - \mu)$ are identically distributed. But this does not suggest that this work cannot be extended to skewed or other forms. It is also assumed that the vectors are nonsingular, i.e., their variance-covariance matrices are positive definite.

Assuming the vectors are generated from mixtures of Gaussian distributions, we next investigate their marginal and conditional distributions, which play an important role in both theory and practice. We show that under weaker stability conditions, the theory of establishing unambiguous multivariate densities is carried on to marginal and conditional mixtures of Gaussian vectors. In particular, the conditional distributions are represented as a ratio of two expected functionals of the scale or as a ratio of two Laplace transforms. To illustrate this, two examples are considered. The first one is expressed as a ratio of two independent gammas and the second as a power of a gamma. Both examples demonstrate how the method is utilized to identify identities for conditional vectors.

In summary, the goals here are twofold: First, we recap some important properties of scale mixtures taking a different approach and show their utility. Second, we show how the general hyperbolic family can be embedded into a wider general class of mixtures with the GGC family. These methods not only provide better insight, but also lead to a simple and efficient line of approach to obtain tractable identities.

The organization of this paper is as follows. Section 2 introduces general mixtures and describes symmetric elliptical contoured vectors and mixtures of normal vectors as examples. Section 3 is focused on the properties and description of the scalars and their mixtures. Finally, Section 4 continues with properties of the marginal and conditional distributions when the joint distribution is obtained from a GGC scalar mixed with normal laws.

2. Background and scale mixtures

Throughout this article we denote by F_X the distribution of the random vector $X \in \mathbb{R}^d$. If random vectors X and Y have the same distribution, we write $X =^D Y$. Let $\mathcal{P}(\mathbb{R}^d)$, $d \geq 1$, denote the set of all probability measures on $\mathbb{R}^d$ and let $\mathcal{P}[0, \infty) = \mathcal{P}_+$ be the set of probability measures on $[0, \infty)$. For each $a \in \mathbb{R}$ and $\mu \in \mathcal{P}(\mathbb{R}^d)$, the rescaling operator $T_a : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ is defined, for every Borel set $B \in \mathcal{B}(\mathbb{R}^d)$, by

$$T_a\mu(A) = \begin{cases} \mu(B/a) & \text{for } a \neq 0, \\ \delta_0(B) & \text{for } a = 0, \end{cases}$$

where δ_0 is the usual Kronecker delta function. Equivalently, $T_a\mu$ denotes the distribution of the random vector aX if μ is the distribution of the vector X.

The scale mixture $\mathrm{E}_\lambda\{\mu(B/\cdot)\} = \mu \circ \lambda(B)$, $B \in \mathcal{B}(\mathbb{R}^d)$, of a measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ with respect to the measure $\lambda \in \mathcal{P}_+$ is then defined by

$$\mu \circ \lambda(B) = \mathrm{E}_\lambda\{\mu(B/\cdot)\} = \int_{[0,\infty)} \mu(B/s)\lambda(ds) = \int_{[0,\infty)} T_s\mu(B)\lambda(ds). \tag{1}$$

When X is independent of the random variable R, $\mu \circ \lambda$ denotes the distribution of the random vector $Y =^D RX$, wherein $\mu = F_X$ and $\lambda = F_R$. In this case, the characteristic function of Y is given, for all $t \in \mathbb{R}^d$, by

$$\hat{\mathrm{E}}_\lambda(\mu)(t) = \int_{[0,\infty)} \hat{\mu}(st)\lambda(ds). \tag{2}$$

Note that when X is symmetrically distributed, the random vector $Y =^D RX \in \mathbb{R}^d$ is also symmetric. In what follows we describe three well-known multivariate families that apply to definitions (1) and/or (2).

Example 1. When $Y \in \mathbb{R}^d$ is spherically distributed, then from Theorem 2.1 in [16] and [9], the characteristic function of Y is formed as $\hat{f}_Y(t) = \varphi(\|t\|^2)$, where $\|t\|^2 = \langle t, t\rangle$, $t \in \mathbb{R}^d$, denotes the Euclidean distance and $\langle \cdot, \cdot \rangle$ is the usual inner product. Some of these results can be found in [30]. To build a similar equation for Y, as in (2), one introduces $U^{(d)}$ to be a random vector with uniform distribution $\mu = \omega_d$ on the unit sphere $S_{d-1} = \{u \in \mathbb{R}^d : \|u\| = 1\} \subset \mathbb{R}^d$. Then, following (10), the characteristic function of Y is given, for all $t \in \mathbb{R}^d$, by

$$\hat{f}_Y(t) = \int_{[0,\infty)} \hat{\omega}_d(rt)F(dr),$$

where $\hat{\omega}_d(t) = \hat{\omega}(||t||^2)$, with

$$\hat{\omega}(||t||^2) = \frac{1}{s_d} \int_{x \in S_d} e^{i\langle t,x\rangle} dx$$

and $s_d = 2\pi^{d/2}/\Gamma(d/2)$ denotes the surface of the unit sphere in $\mathbb{R}^d$. Thus, Y can be written as a scale mixture of $\mu = \omega_d$ in terms of $\lambda = F \in \mathcal{P}_+$. For instance, the random vector Y is now stated as a scale mixture of $U^{(d)}$ independently of a nonnegative random variable R by $Y =^D RU^{(d)}$. To relate the distribution of Y with the general definition (1), let Y possess a density f_Y (if it exists). As in the characteristic function, the density of Y also satisfies the form $f_Y(x) = f(\langle x, x\rangle)$, $x \in \mathbb{R}^d$, for some nonnegative function f, termed the density generator. In this case, when $F \in \mathcal{P}_+$ and $\Pr(R = 0) = 0$, it can be shown that Y is absolutely continuous. In other words, the density of Y satisfies

$$\int_{\mathbb{R}^d} f_Y(x)dx = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty r^{d/2-1} f(r)dr = 1$$

as would be expected. To further clarify the absolute continuity property of Y, it can be seen from Theorem 2.9 in [15] that Y possesses a density generator f if and only if R is absolutely continuous with density

$$f_R(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)}\, r^{d-1} f(r^2)\mathbf{1}(r > 0).$$

To avoid technical issues at zero, we further assume that $\Pr(R = 0) = 0$. Thus, as in (1),

$$\Pr(Y \in B) = \int_{[0,\infty)} \Pr(U^{(d)} \in B/s) f(\mathrm{s})ds.$$

For more information we also refer to [9, 16].

Example 2. Natural extensions to spherically distributed random vectors are the elliptically contoured vectors. In this case, $Y \in \mathbb{R}^d$ is assumed to have location and scale parameters $\mu \in \mathbb{R}^d$, and Σ, a $d \times d$ positive definite symmetric matrix (again, for convenience), respectively. Thus, definition (2) is now formed, for all $t \in \mathbb{R}^d$, as

$$\varphi_{Y-\mu}(t) = \mathrm{E}\{e^{i\langle (Y-\mu),t\rangle}\} = \int_{(0,\infty)} \hat{\omega}_{d,\Sigma}(st) f(s)ds,$$

where $\hat{\omega}_{d,\Sigma}(st) = \hat{\omega}(s||t||^2_\Sigma)$, $||t||^2_\Sigma = \langle t, \Sigma t\rangle$, $t \in \mathbb{R}^d$ and $s \in (0,\infty)$, is the characteristic function of the vector $A^\top U^{(d)}$, with A chosen such that $A^\top A = \Sigma$. For simplicity, Σ is considered to be of full rank, in which case we may write conveniently $\Sigma^{1/2}\Sigma^{1/2} = \Sigma$. Thus, the stochastic representation of $Y \in \mathbb{R}^d$ is given by

$$Y =^D \mu + RA^\top U^{(d)}. \tag{3}$$

Applying definition (1), the distribution of Y in (3) is then obtained, for all $B \in \mathcal{B}(\mathbb{R}^d)$, by

$$\Pr(Y \in B) = \int_{[0,\infty)} \Pr(A^\top U^{(d)} \in B/s - \mu) f(s)ds.$$

As in the spherical case, the density of the scalar random variable R when Y is elliptically contoured is given by

$$f_R(||x-\mu||_{\Sigma^{-1}}) = \frac{\Gamma(d/2)}{2\pi^{d/2}} \Sigma^{-1} ||x-\mu||^{d-1}_{\Sigma^{-1}} f(||x-\mu||^2_{\Sigma^{-1}}).$$

Example 3. Let $Y \in \mathbb{R}^d$ be nonsingular, which is equivalent to Σ being strictly positive definite, or for every $u \neq 0$, $\langle u, \Sigma u\rangle > 0$. Let $X \sim \mathcal{N}(0, \Sigma)$ be a d-dimensional multivariate normal random vector and R be an independent univariate strictly positive random variable. Then, the stochastic representation of Y is given, for $\mu \in \mathbb{R}^d$, by

$$Y =^D \mu + RX, \tag{4}$$

where Y has characteristic function given, for all $t \in \mathbb{R}^d$, by

$$\hat{f}_{Y-\mu}(t) = \int_{(0,\infty)} \hat{f}_X(st) f_R(s) ds = \int_{(0,\infty)} \exp(-s\|t\|_\Sigma^2/2) f_R(s) ds.$$

Applying definition (1), one has, for all $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\Pr(Y \in B) = \int_{[0,\infty)} \Pr(X \in B/s - \mu) f_R(\mathrm{s}) ds. \tag{5}$$

Note that Y is isotropic radially symmetric when Σ is a multiple of the identity matrix.

It can be seen that a normal vector $X \sim \mathcal{N}(0, \Sigma)$ can be expressed as $X =^D (\chi_d^2)^{1/2} \Sigma^{1/2} U^{(d)}$, where χ_d^2 is independent of $U^{(d)}$. In this case, the stochastic representation in (4) can now be viewed in terms of the uniform vector $U^{(d)}$ scaled by the positive random variable $R(\chi_d^2)^{1/2}$. In modeling applications, scale mixtures of univariate or multivariate normal distributions are introduced to simply alter the tail behavior of the distribution while leaving the resultant distribution symmetric. Classical examples include the multivariate t, Laplace or double exponential distribution, etc.

In many situations the density of R can be unattractive (stable case) resulting in an integral equation (5) almost impossible to compute. To bypass this and other potential difficulties, one could use the inversion theorem of the characteristic function (the Laplace transform of certain positive stable variables is easier to compute, see below). This method may then provide an easier way to compute the density of $Y - \mu$ instead of computing directly from (5). In particular, when the random vector Y satisfies (4), the characteristic function of $Y - \mu$ satisfies, for all $t \in \mathbb{R}^d$,

$$\hat{f}_{Y-\mu}(t) = \mathrm{E}(e^{i\langle Y-\mu, t\rangle}) = \mathrm{E}(e^{iR\langle X, t\rangle}) = \mathrm{E}(e^{-R^2\langle t, \Sigma t\rangle/2}) = L_{R^2}(\|t\|_\Sigma^2/2). \tag{6}$$

Setting $s = \Sigma^{1/2} t$, $s, t \in \mathbb{R}^d$, and then applying polar coordinates, the following holds,

$$\begin{aligned}\int_{\mathbb{R}^d} L_{R^2}(\|t\|_\Sigma^2/2) dt = |\Sigma|^{1/2} \int_{\mathbb{R}^d} L_{R^2}(\|s\|^2/2) ds = c \int_{(0,\infty)} r^{d-1} L_{R^2}(r^2/2) dr \\ = c \int_{(0,\infty)} u^{d-2/1} L_{R^2}(u) dr < \infty,\end{aligned} \tag{7}$$

where c is a positive constant. For the existence property, we assume that (7) is bounded.

In light of (7), a general representation for the density of the random vector $Y - \mu$ with respect to the Laplace transform of R^2 can then be developed as follows.

Theorem 1. *If the Laplace transform L_{R^2} of R^2 satisfies* (7)*, then the density of $Y - \mu =^D RX \in \mathbb{R}^d$, where $X \sim \mathcal{N}(0, \Sigma)$ is independent of the scale random variable R, has the following representation*

$$\begin{aligned} f_{Y-\mu}(x) &= \frac{(\sigma^2/2)^{-1/2}}{2\pi} \int_{(0,\infty)} L_{R^2}(r^2) \cos(\sqrt{2}\, r|x-\mu|/\sigma) dr, \quad x \in \mathbb{R}, \quad d = 1 \\ f_{Y-\mu}(x) &= \frac{|\Sigma|^{-1/2}\sqrt{\pi}\Gamma\{(d-1)/2\}}{\sqrt{2}\,(2\pi)^d (\|x-\mu\|_{\Sigma^{-1}}^2/2)^{d-2/4}} \int_{(0,\infty)} r^{d/2} L_{R^2}(r^2) J_{(d-2)/2}(\sqrt{2}\|x-\mu\|_{\Sigma^{-1}}) dr, \quad x \in \mathbb{R}^d, \quad d > 1 \end{aligned}$$

where J_ν is the Bessel function of the first kind with $\nu > 0$.

Proof. The proof follows the same lines as [10]. Set $B = \Sigma^{1/2}/\sqrt{2}$ and $y = Bt$, $t \in \mathbb{R}^d$. It can be seen that $\langle t, \Sigma t\rangle/2 = \langle y, y\rangle = \|y\|^2$. From (6) and since condition (7) is satisfied, we have from the inversion of the Fourier transform that

$$f_{Y-\mu}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle t, x-\mu\rangle} L_{R^2}(\|t\|_\Sigma^2/2) dx = \frac{|B|^{-1}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle x-\mu, B^{-1}y\rangle} L_{R^2}(\|y\|^2) dx.$$

Then, using polar coordinates, set $y = rs$, $r > 0$, with $s \in S_d$. Let γ_d be a surface measure of S_d. We have

$$f_{Y-\mu}(x) = \frac{(|\Sigma|/2)^{-1/2}}{(2\pi)^d} \int_{(0,\infty)} r^{d-1} L_{R^2}(r^2) dr \int_{\mathbf{S}_d} e^{-i\langle x-\mu, B^{-1}y\rangle} \gamma_d(ds).$$

Set $y = r(\Sigma/2)^{-1/2}(x-\mu)$. Thus, $||y||^2 = 2r^2||x-\mu||^2_{\Sigma^{-1}}$. In this case, we have respectively

$$\int_{\mathbf{S}_d} e^{-i\langle x-\mu, B^{-1}y\rangle} \gamma_d(ds) = \cos(|y|), \quad d = 1$$

and

$$\int_{\mathbf{S}_d} e^{-i\langle x-\mu, B^{-1}y\rangle} \gamma_d(ds) = \int_0^\pi e^{-i||y||\cos(\theta)} \sin^{d-2}(\theta) d\theta = \frac{\pi^{1/2}\Gamma\{(d-1)/2\}}{(||y||/2)^{(d-2)/2}} J_{(d-2)/2}(||y||), \quad d > 1.$$

Finally, substituting $||y||^2 = 2r^2||x-\mu||^2_{\Sigma^{-1}}$ the result follows immediately. □

Example 4. To illustrate the practicality of Theorem 1, assume that $R^2 \sim \mathcal{G}(1, \nu+1) = \mathsf{G}_{\nu+1}$, $\nu > -1$. We then have $L_{R^2}(r^2) = (1+r^2)^{-\nu-1}$. From Eq. (6.565.4) in [19], a density representation for $Y - \mu$ is then given by

$$\begin{aligned} f_{Y-\mu}(x) &= \frac{|\Sigma|^{-1/2}\sqrt{\pi}\,\Gamma\{(d-1)/2\}}{\sqrt{2}(2\pi)^d(||x-\mu||^2_{\Sigma^{-1}}/2)^{(d-2)/4}} \int_{(0,\infty)} r^{d/2} J_{(d-2)/2}(\sqrt{2}\,r||x-\mu||_{\Sigma^{-1}}) \frac{dr}{(1+r^2)^{\nu+1}} \\ &= \frac{\Gamma\{(d-1)/2\}|\Sigma|^{-1/2}(||x-\mu||^2_{\Sigma^{-1}})^{\{\nu-(d-2)/2\}/2}}{2^{\{\nu+(3d-2)/2\}/2}\pi^{d-1/2}} K_{(d-2)/2}(\sqrt{2}\,||x-\mu||_{\Sigma^{-1}}), \quad d \le 4\nu+1 \end{aligned}$$

and $\nu > 0$, where K_ν is the Bessel function of the second kind with $\nu > 0$.

3. GGC distributions

In this section we focus on the positive scalar random variable R by identifying specific families of distributions for R that may play or have played an important role in modeling applications. Specifically, with the aim of modeling key stylized features of observational series from finance and turbulence, [2] and [3] have considered R to possess a GIG distribution. For example, Barndorff-Nielsen [2] considered mixtures of the form

$$Y =^D \mu + mR^2 + RX, \tag{8}$$

where $X \sim \mathcal{N}(0, \Sigma)$ is independent of R, $\mu \in \mathbb{R}^d$ and $m = \beta\Sigma$ with $\beta \in \mathbb{R}^d$. This family of multivariate normal mixtures generates asymmetric shapes with the asymmetry being caused by the term $m = \beta\Sigma$. Clearly, in the absence of $m = \beta\Sigma$, the symmetry is recovered. In modeling (8) the scalar R is specified to follow a GIG distribution with density given by

$$f_R(x) = \frac{(\psi/\chi)^\lambda}{2K_\lambda(\sqrt{\chi\psi})} x^{\lambda-1} e^{-(x+\psi x)/2} \mathbf{1}(x > 0), \tag{9}$$

where K_ν is a Bessel function of the second kind with $\nu > 0$, and the range of parameters is as follows: $\chi \ge 0$, $\psi > 0$, $\lambda > 0$; $\chi > 0$, $\psi > 0$, $\lambda = 0$; $\chi > 0$, $\psi \ge 0$, $\lambda < 0$. Under these circumstances, Barndorff-Nielsen and Halgreen [3] have shown that Y has a d-dimensional generalized hyperbolic distribution with index λ denoted by $H_d(\lambda, \alpha, \beta, \delta, \mu, \Sigma)$. Here, the parameters are defined as: $\chi = \delta^2$, $\psi = \xi^2$ and $\alpha^2 = \xi^2 + \langle\beta, \Sigma\beta\rangle$. The hyperbolic distribution was also studied by [6], when $\lambda = 1$. If $\chi = 0$, $\psi = 2$ and $\alpha^2 = 2 + \langle\beta, \Sigma\beta\rangle$, one obtains the well-known asymmetric multivariate Laplace distributions.

With these areas of applications in mind, particularly those in finance, we investigate scalar distributions that may be more general than the GIG family. We are limited only to model (4) and interested in families of distributions of R that are generated from the GGC. This family does overlap with the members of the GIG but GGC is more extensive and a very general family. It can be also seen that GGC $\supset$ GIG.

To introduce the GGC family, write $R^2 = W$ with distribution function F_W. Then the random variable W is said to be a GGC, when F_W satisfies the following three properties:

(a) The probability distribution F_W is concentrated on $[0, \infty)$.

(b) The measure ν is concentrated on $(0, \infty)$ and

$$\int_{(0,1]} |\ln(x)|\nu(dx) < \infty, \quad \int_{[1,\infty)} \frac{\nu(dx)}{x} < \infty$$

(c) Let the Laplace Transform of F_W, $\mathrm{E}(e^{-sW}) = \int_{[0,\infty)} e^{-sw} F_W(dw)$, be written as

$$\mathrm{E}(e^{-sW}) = \exp\left\{-\int_{(0,\infty)} \ln(1 + s/w)\nu(dw)\right\}, \quad \mathsf{Re}(s) \geq 0. \tag{10}$$

The measure ν is known as the Thorin measure whenever the total mass $c = \nu(0, \infty) < \infty$. Then, [7] have shown in Theorems 4.1.1 and 4.1.4 that W has the representation given by

$$W \stackrel{D}{=} \mathsf{G}_c D, \tag{11}$$

where G_c is a gamma random variable independent of a positive random variable D. In this case, W is absolutely continuous with density given by

$$f_W(x) = \frac{x^{c-1}}{\Gamma(c)} \mathsf{E}\left\{\frac{\exp(-x/D)}{D^c}\right\}.$$

To explore further the GGC family, the recent survey of [18] on beta variables is of importance. To see how much further (11) can be characterized, the following remarks are relevant.

For $a, b > 0$, let $\mathsf{B}_{a,b}$ (of the first kind) represent a beta variable with parameters a, b, defined, for all $x \in (0, 1)$, by

$$\Pr(\mathsf{B}_{a,b} \in dx) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1},$$

where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$. Note that $\mathsf{B}_{a,b} \stackrel{D}{=} 1-\mathsf{B}_{b,a}$. Setting $x = u/(1 + u)$ in the above expression, the beta variable is termed the beta density of the second kind, defined as follows for all $u > 0$,

$$\Pr(\mathsf{B}^{(2)}_{a,b} \in dx) = \frac{1}{B(a,b)} u^{a-1}(1 + u)^{-(a+b)}.$$

Note that $\mathsf{B}^{(2)}_{a,b} \stackrel{D}{=} \mathsf{B}_{a,b}/(1 - \mathsf{B}_{a,b})$. Also, the following relations are repetitively used below

$$\{\mathsf{B}_{u,c-u} : 0 \leq u \leq c\} \stackrel{D}{=} \left\{\frac{\mathsf{G}_u}{\mathsf{G}_c} : 0 \leq u \leq c\right\} \stackrel{D}{=} \left\{\frac{\mathsf{G}_u}{\mathsf{G}_u + (\mathsf{G}_c - \mathsf{G}_u)} : 0 \leq u \leq c\right\}.$$

In what follows, the Thorin measure is chosen so that $\nu(dx) = cF_G(dx)$, $\nu(0, \infty) = c < \infty$, satisfying $\mathrm{E}\{\ln^+(1/G)\} < \infty$, where F_G is a distribution function of a positive random variable G. Following [18], it turns out that (11) can also be formulated as

$$W_{G,c} = \int_0^c 1/F_G^{-1}(u/c)d\mathsf{G}_u \stackrel{D}{=} \mathsf{G}_c \int_0^c 1/F^{-1}(u/c)d\mathsf{B}_{u,c-u} = \mathsf{G}_c D_{G,c}.$$

Note that GGC members with $D = \int_0^c h(u)dB_{u,c-u}$ are known to belong to the Bondesson family, $\mathbf{B}$; see, e.g., [7]. Some similar results can also be found in [32].

Throughout the following, $\mathrm{E}\{|\ln(G)|\} < \infty$ is equivalent to $\mathrm{E}\{\ln^+(G)\} < \infty$ and/or $\mathrm{E}\{\ln^+(1/G)\} < \infty$. Continuing the exploration even deeper, the duality theorem of [18] is significant. It shows relationships among the variables $W_{G,c}$, $W_{1/G,c}$, $D_{G,c}$, $D_{1/G,c}$, and Laplace exponents ψ_G and $\psi_{1/G}$. These are vital in linking algebraic equalities between functionals of beta-gamma random variables. The duality theorem in [18] is then utilized in obtaining Theorems 2 and 3 below.

It should be noted that a key sub-family of $\mathbf{B}$ that is of importance in many applications (including financial data) is the GIG with density function given in (9). For more information on the GIG family, we also refer to [12]. To stress the importance of GGC scalars, recent applications in [11] incorporate GGC as scalars that are not members of GIG.

Next, the attention of our investigation is to characterize mixtures of the multivariate normal with members of the GGC family. Let $X \sim \mathcal{N}(0, \Sigma)$ be a normally distributed random vector with mean zero and a full ranked covariance matrix $\Sigma \in \mathbb{R}^{d\times d}$. Suppose that $W_{G,c}$ and X are independent. Define the d-dimensional random vector $Y =^D \sqrt{W_{G,c}}X$. Then, applying Theorem 30.1 in [28], we have

$$\begin{aligned} \mathrm{E}\{\exp(i\langle \sqrt{W_{G,c}}X, t\rangle)\} &= \exp[-\psi[\ln\{\hat{f}_X(t)\}]] = \exp\left\{-c\int_0^\infty (1-e^{-\|t\|_\Sigma^2 x/2})\frac{\mathrm{E}(e^{-xG})}{x}dx\right\} \\ &= \exp\left[-c\mathrm{E}\left\{\int_0^\infty \ln\left(1+\frac{\|t\|_\Sigma^2/2}{G}\right)\right\}\right], \end{aligned}$$

where $\hat{f}_X(t) = \mathrm{E}(e^{i\langle X,t\rangle}) = \exp(-\|t\|_\Sigma^2/2)$, $t \in \mathbb{R}^d$, is the characteristic function of X. On the other hand, since $W_{G,c} =^D \mathsf{G}_c D_{G,c}$ we equivalently have

$$\mathrm{E}\{\exp(i\langle \sqrt{W_{G,c}}X, t\rangle)\} = \mathrm{E}\{\mathrm{E}(e^{i\langle G_c D_{G,c} X, t\rangle})|G_c D_{G,c}.\} = \mathrm{E}\{\mathrm{E}(e^{-\mathsf{G}_c D_{G,c}\|t\|_\Sigma^2/2})|D_{G,c}.\} = \mathrm{E}\{(1+D_{G,c}\|t\|_\Sigma^2/2)^{-c}\}. \tag{12}$$

Remark 1. Since $W_{G,c} =^D G_c D_{G,c} =^D G_{c_1} D_{G,c_1} + G_{c_2} D_{G,c_2}$, for $c_1 + c_2 = c$, with $\mathsf{G}_{c_1} D_{G,c_1}$ and $\mathsf{G}_{c_2} D_{G,c_2}$ being independent, we have from (12) that

$$\mathrm{E}\{\exp(i\langle \sqrt{W_{G,c}}X, t\rangle)\} = \mathrm{E}\{(1+D_{G,c}\|t\|_\Sigma^2/2)^{-c}\} = \mathrm{E}\{(1+D_{G,c_1}\|t\|_\Sigma^2/2)^{-c_1}\}\mathrm{E}\{(1+D_{G,c_2}\|t\|_\Sigma^2/2)^{-c_2}\},$$

which implies that the vector Y can be decomposed as

$$Y = \sqrt{W_{G,c}}X =^D \sqrt{W_{G,c_1}}X + \sqrt{W_{G,c_2}}X.$$

This, in turn, shows that the divisibility property is satisfied for normal GGC mixtures.

Specializing Theorem 1, in what follows below, Theorem 2 provides an equivalent formula for the density of the vector

$$Y =^D \mu + \sqrt{W_{G,c}}X, \tag{13}$$

when $X \sim \mathcal{N}(0, \Sigma)$, $\mu \in \mathbb{R}^d$, $W_{G,c} =^D \mathsf{G}_c D_{G,c}$, and $W_{G,c}$ is independent of X. Also, G_c is independent of $D_{G,c}$. The following theorem is a result of definition (1).

Theorem 2. *Let $W_{G,c}$ be such that the total mass is c, and G be a positive random variable such that* $\mathrm{E}\{\ln^+(1/G)\} < \infty$. *Then, the density of $Y =^D \mu + \sqrt{W_{G,c}}X$, for $X \sim \mathcal{N}(0, \Sigma)$ with $\mu \in \mathbb{R}^d$, $W_{G,c} = \mathsf{G}_c D_{G,c}$ and $W_{G,c}$ being independent of X, is given, for all $x \in \mathbb{R}^d$, by*

$$f_Y(x) = \frac{|\Sigma|^{-1/2} e^{c\mathrm{E}\{\ln(G)\}}}{2(2\pi)^{d/2}\Gamma(c)} \mathrm{E}\left\{\left(\frac{2D_{1/G,c}}{\|x-\mu\|_{\Sigma^{-1}}^2}\right)^{(d/2-c)/2} K_{d/2-c}(\sqrt{2}\|x-\mu\|_{\Sigma^{-1}}\sqrt{D_{1G,c}}/2)\right\}.$$

Proof. From the duality theorem in [18], the density of $W_{G,c}$ is given by

$$f_{W_{G,c}}(x) = \frac{x^{c-1}}{\Gamma(c)} e^{c\mathrm{E}\{\ln(G)\}}\mathrm{E}(e^{-xD_{1/G,c}})\mathbf{1}\{x \in (0,\infty)\}. \tag{14}$$

Applying (6) or Theorem 3.1 in [29], and from (13), we have that for $X \sim \mathcal{N}(0, \Sigma)$ the density of Y is formed as

$$f_Y(x) = \frac{|\Sigma|^{-1/2}}{(2\pi)^{d/2}\Gamma(c)} \int_0^\infty s^{c-d/2-1} e^{-\|x-\mu\|_{\Sigma^{-1}}^2/2s}\mathrm{E}(e^{-sD_{1/G,c}})$$

for all $x \in \mathbb{R}^d$. Also, from (15) below (Eq. 8.43, [19]),

$$K_\delta(xs) = \frac{x^\delta}{2}\int_0^\infty t^{-\delta-1}e^{-s(t+x^2/t)/2}dt, \tag{15}$$

and hence the representation of the Y density immediately follows. □

When $W \in \text{GGC}$ is just a gamma density, then the density can be directly computed through (1). Thus, to illustrate Theorem 2 two members of $\text{GGC} \subset \mathcal{P}_+$ are considered.

Example 5. We assume model (13), where $X \sim \mathcal{N}(0, \Sigma)$, $\mu \in \mathbb{R}^d$, and $W_{c,m} = \mathsf{G}_c/\mathsf{G}_m$, $m > 0$, $c \in (0,1)$ has a beta density of second kind, independent of X. Let G_c and G_m be independent. Simple algebra then shows that

$$\Pr(W_{c,m} \in dx) = \frac{1}{B(c,m)} x^{c-1}(1+x)^{-(c+m)}\mathbf{1}\{x \in (0,\infty)\}dx = \frac{1}{B(c,m)} x^{c-1}\mathrm{E}(e^{-xG_{c+m}})\mathbf{1}\{x \in (0,\infty)\}dx. \tag{16}$$

It can be seen that $\mathrm{E}\{\ln^+(1/\mathsf{G}_c)\} < \infty$, $c \in (0,1)$. Also, because $W_{c,m} =^D \mathsf{G}_c D_{c,m} =^D \mathsf{G}_c/\mathsf{G}_m$, with $D_{c,m} = 1/\mathsf{G}_m$, it follows from the duality theorem in [18] that

$$\mathrm{E}(D_{c,m}^{-c}) = \exp[c\mathrm{E}\{\ln(1/\mathsf{G}_m)\}] = \Gamma(c+m)/\Gamma(m) \quad \text{and} \quad D_{1/B_{c,m}} = \mathsf{G}_{c+m}. \tag{17}$$

Applying Theorem 2, (14), (16) and (17), we have, for all $x \in \mathbb{R}^d$,

$$f_Y(x) = \frac{|\Sigma|^{-1/2}}{2(2\pi)^{d/2}B(c,m)} \mathrm{E}\left\{\left(\frac{2\mathsf{G}_{c+m}}{\|x-\mu\|^2_{\Sigma^{-1}}}\right)^{(d/2-c)/2} K_{d/2-c}(\sqrt{2}\,\|x-\mu\|_{\Sigma^{-1}}\sqrt{\mathsf{G}_{c+m}}/2)\right\}.$$

Remark 2. Note that when $c \in (0,1)$, $W_{c,1-c} = \mathsf{G}_c/\mathsf{G}_{1-c} =^D |\mathsf{C}_a|^a$, $a = 1/c$, the Cauchy density

$$\Pr(\mathsf{C}_a \in dx) = \frac{\sin(\pi/a)}{2\pi/a}\frac{1}{1+|x|^a}\mathbf{1}(x \in \mathbb{R}),$$

can be found in [32]. When $a = 2$, it is of course the standard Cauchy density.

Example 6. Using model (14) we again assume $X \sim \mathcal{N}(0,\Sigma)$, $\mu \in \mathbb{R}^d$, and $W_c = \mathsf{G}_{1/c}$, $c \in (0,1)$, to be a power of an exponential independent of X. Thus, the density of W_c can be computed by

$$\Pr(W_c \in dx) = cx^{c-1}e^{-x^c}\mathbf{1}\{x \in (0,\infty)\}dx = cx^{c-1}\mathrm{E}(e^{-x\mathsf{C}_c})\mathbf{1}\{x \in (0,\infty)\}dx,$$

where C_c represents a positive stable random variable with index c. Note that $\mathrm{E}(e^{-\lambda W_c}) = e^{-\lambda^c}$. Thus, as in Example 5, we can compute the density of Y to be, for all $x \in \mathbb{R}^d$,

$$f_Y(x) = \frac{c|\Sigma|^{-1/2}}{2(2\pi)^{d/2}} \mathrm{E}\left\{\left(\frac{2\mathsf{C}_c}{\|x-\mu\|^2_{\Sigma^{-1}}}\right)^{(d/2-c)/2} K_{d/2-c}(\sqrt{2}\,\|x-\mu\|_{\Sigma^{-1}}\sqrt{\mathsf{C}_c}/2)\right\}.$$

4. Marginal and conditional distributions of Gaussian GGC mixtures

To obtain marginals and conditional distributions of $Y \in \mathbb{R}^d$, where Y might be elliptical contoured or a scale mixture of a Gaussian vector, we first partition the vector $Y = (Y_1, Y_2)^\top$ with Y_1 to be a d_1−dimensional sub-vector of Y for $1 \le d_1 < d$. Note that when $X \sim \mathcal{N}(0,\Sigma) \in \mathbb{R}^d$ is a multivariate normal vector, we have

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad X \sim \mathcal{N}(\mu,\Sigma) = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}. \tag{18}$$

When $Y = (Y_1, Y_2)^\top \sim EC_d(0, I, \phi)$, we have that $Y =^D RX$, where $X \sim \mathcal{N}(0, I) \in \mathbb{R}^d$. Then, following [10], it can be seen that Y has the same law as

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} =^D R \times \begin{pmatrix} \sqrt{\mathsf{B}_{d_1/2,(d-d_1)/2}}\,U^{(d_1)} \\ \sqrt{1-\mathsf{B}_{d_1/2,(d-d_1)/2}}\,U^{(d-d_1)} \end{pmatrix}.$$

In this case, R is independent of $\mathsf{B}_{d_1/2,(d-d_1)/2}$, $U^{(d_1)}$ and $U^{(d-d_1)}$. Further, upon R being absolutely continuous, the density of $R_{d_1} =^D \sqrt{\mathsf{B}_{d_1/2,(d-d_1)/2}}R$ exists and can be represented by

$$f_{d_1}(r) = \frac{2r^{d_1-1}g_{d_1}(r^2)}{\Gamma(d_1/2)}\mathbf{1}\{r \in (0,\infty)\}.$$

As in [9], we have that

$$g_{d_1}(u) = s_d^{-1}s_{d_1}\int_r^\infty s^{-(d-2)}(r^2-s^2)^{d_1/2-1}f_d(r)dr\,\mathbf{1}\{s \in (0,\infty)\} = s_{d_1}\int_r^\infty r^{d_1-1}g_d(u+r^2)dr\,\mathbf{1}\{u \in (0,\infty)\}.$$

When $d = 1$, $g_1(u) = \Pr(|Y_1| \le r) = \Pr(R_1 \le r) = F_X(r) - F(-r)$, in which case, $f_1(r) = 2f_{X_1}(x)$. Thus, Y is any continuous variable that has symmetric distribution. In this case, the density of the sub-vector Y_1 is $g_{d_1}(\|x\|)$, $x \in \mathbb{R}^{d_1}$. Further, when $Y = (Y_1, Y_2)^\top \sim EC_d(0, I, \phi) \subset \mathcal{P}(\mathbb{R}^d)$, the conditional density of Y_1 given Y_2, $Y_2|Y_1.$ say, can be computed, for all $x_2 \in R^{d-d_1}$, through

$$f_{Y_2|Y_1.}(x_2 : x_1) = \frac{g_d(\|x_2\| + \|x_1\|)}{g_{d_1}(\|x_1\|)}. \tag{19}$$

When $Y \sim EC_d(0, I, \phi)$, provided the $Y_2|Y_1. = x_1$ exists, $Y_2|Y_1. = x_1$ is spherically distributed with stochastic representation as

$$Y_2|Y_1. = x_1 =^D R_{\|x_1\|}U^{(d-d_1)},$$

where $U^{(d-d_1)}$ is uniformly distributed on S_{d-d_1}, i.e., independent of $R_{\|x_1\|}$. The distribution of $R_{\|x_1\|}$ is computed using

$$R_{\|x_1\|} =^D \sqrt{R^2 - \|x_1\|^2}|(R^2\mathsf{B}_{d_1/2,(d-d_1)/2} = \|x_1\|^2).$$

We then extend the spherical vectors to elliptical by letting $Y \sim EC_d(\mu, \Sigma, \phi)$. We next define the conditional location parameter, conditional scale matrix and quadratic form above as follows,

$$\mu_{2.1} = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1), \quad \Sigma_{22.1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} = \Sigma_{22.1}^{1/2}\Sigma_{22.1}^{1/2}$$

and

$$q_1(x) = \|x_1 - \mu_1\|^2_{\Sigma_{11}^{-1}} = \langle x_1 - \mu_1, \Sigma_{11}^{-1}(x_1 - \mu_1)\rangle, \quad x_1 - \mu_1 \in \mathbb{R}^{d_1}.$$

In this case, the marginal and conditional random vectors have the following stochastic representations

$$Y_1 =^D \mu_1 + R\mathsf{B}_{d_1/2,(d-d_1)/2}U^{(d_1)}\Sigma_{11}^{1/2}, \quad Y_2|(Y_1 = x_1). =^D \mu_{2.1} + R_{\|x_1\|}U^{(d-d_1)}\Sigma_{22.1}^{1/2}.$$

Since the norm expressions are unusually large, set

$$\|x_2 : x_1\|^2_{(2)} = \|x_1 - \mu_1\|^2_{\Sigma_{11}^{-1}} + \|x_2 - \mu_{2.1}\|^2_{\Sigma_{22.1}^{-1}}$$

and $\|x_1\|^2_{(1)} = \|x_1 - \mu_1\|^2_{\Sigma_{11}^{-1}}$. Then, applying (30), the conditional density can be expressed as

$$f_{Y_2|Y_1.}(x_2 : x_1) = \frac{|\Sigma_{22.1}|^{-1/2}}{(\sqrt{2\pi})^{d-d_1}}\frac{\int_{(0,\infty)} s^{-d/2}\exp(-\|x_2 : x_1\|^2_{(2)}/2s)F_{R^2}(ds)}{\int_{(0,\infty)} s^{-d_1/2}\exp(-\|x_1\|^2_{(1)}/2s)F_{R^2}(ds)}. \tag{20}$$

When the Laplace transform of the amplitude R^2 is available instead of the density of R^2, one can incorporate (20) and Theorem 1 to obtain the following result.

Theorem 3. *Let the Laplace transform of R^2, L_{R^2}, satisfy* (7)*. Assume that $Y - \mu = RX$, where $X \sim \mathcal{N}(0, \Sigma)$ and the scale variable R and the vector X are independent. Then, the conditional distribution of $Y_2|Y_1. = x_1$ has the following representation in terms of the Bessel function J_ν of the first kind with $\nu > 0$:*

$$f_{Y_2|Y_1.}(x_2 : x_1) = \frac{|\Sigma_{22.1}|^{-1/2}\Gamma\{(d-1)/2\}}{(2\pi)^{d-d_1}\Gamma\{(d_1-1)/2\}}\frac{\int_{(0,\infty)} r^{d/2}L_{R^2_\infty}(r^2)J^{(2)}_{(d-2)/2}(\sqrt{2r}\|x_2 : x_1\|_{(2)})dr}{\int_{(0,\infty)} r^{d_1/2}L_{R^2_\infty}(r^2)J^{(1)}_{(d_1-2)/2}(\sqrt{2r}\|x_1\|_{(1)})dr}.$$

Proof. We have that the conditional density of $Y_2|Y_1. = x_1$ is given by

$$f_{Y_2|Y_1.}(x_2 : x_1) = f_{Y_1,Y_2}(x_1, x_2)/f_{Y_1}(x_1) \ \ a.e.$$

Now, since condition (7) is satisfied, one applies the inversion theorem of the Fourier transform for both numerator and denominator to obtain the following

$$f_{Y_2|Y_1.}(x_2 : x_1) = \frac{\frac{|B|^{-1}}{(2\pi)^d}\int_{R^d} e^{-i\langle x-\mu, B^{-1}y\rangle} L_{R^2}(\|y\|^2)dx}{\frac{|B_1|^{-1}}{(2\pi)^{d_1}}\int_{R^{d_1}} e^{-i\langle x_1-\mu_1, B_1^{-1}y_1\rangle} L_{R^2}(\|y_1\|^2)dx} \ \ a.e.,$$

where $B = \Sigma^{1/2}/\sqrt{2}$, $y = Bt$, $B_1 = \Sigma_{11}^{1/2}/\sqrt{2}$ and $y_1 = B_1 t_1$. Then as in Theorem 1, using polar coordinates, set $y = rs$, $r > 0$, where $s \in S_d$ and $y_1 = rs_1$, $r > 0$, where $s_1 \in S_{d_1}$ and let γ_d and γ_{d_1} be surface measures of S_d and S_{d_1}, respectively. From (20) and applying Theorem 1, the result immediately follows. □

If R^2 is a member of the GGC family, then the above results can be further specialized as follows.

Corollary 1. *Let $Y \in \mathbb{R}^d$ be such that $Y = \mu + RX$, where $X \sim \mathcal{N}(0, \Sigma)$ with Σ being positive definite and $R^2 =^D W_{G,c}$ as defined in* (13). *Let Y, μ and Σ be partitioned as in* (18), *where $Y_1 \in \mathbb{R}^{d_1}$, $1 \le d_1 < d$. Then the conditional distribution of $Y_2|Y_1. = x_1$ is computed from*

$$f_{Y_2|Y_1.}(x_2 : x_1) = \frac{|\Sigma_{22.1}|^{-1/2} e^{c\mathrm{E}\{\ln(G)\}}}{2(\sqrt{2\pi})^{d-d_1}\Gamma(c)} \frac{\mathrm{E}\left\{\left(\frac{2D_{1/G,c}}{\|x_2:x_1\|^2_{(2)}}\right)^{(d/2-c)/2} K_{d/2-c}(\sqrt{2}\,\|x_1\|^2_{(1)}\sqrt{D_{1/G,c}}/2) K_{d/2-c}(\sqrt{2}\,\|x_2 : x_1\|^2_{(2)}\sqrt{D_{1/G,c}}/2)\right\}}{\mathrm{E}\left\{\left(\frac{2D_{1/G,c}}{\|x_1\|^2_{(1)}}\right)^{(d/2-c)/2} K_{d/2-c}(\sqrt{2}\,\|x_1\|^2_{(1)}\sqrt{D_{1/G,c}}/2)\right\}}.$$

To illustrate the applicability of Corollary 1, we set $R^2_\infty =^D W_{G,c} =^D \mathsf{G}_c/\mathsf{G}_m$. Then the conditional density for $x_2 \in \mathbb{R}^{d-d_1}$ and $x_1 \in \mathbb{R}^{d_1}$ can be obtained a.e., viz.

$$f_{Y_2|Y_1.}(x_2 : x_1) = \frac{|\Sigma_{22.1}|^{-1/2}}{2(\sqrt{2\pi})^{d-d_1}B(c, m)} \frac{\mathrm{E}\left\{\left(\frac{2G_{c+m}}{\|x_2:x_1\|^2_{(2)}}\right)^{(d/2-c)/2} K_{d/2-c}(\sqrt{2}\,\|x_2 : x_1\|^2_{(2)}\sqrt{G_{c+m}}/2)\right\}}{\mathrm{E}\left\{\left(\frac{2G_{c+m}}{\|x_1\|^2_{(1)}}\right)^{(d/2-c)/2} K_{d/2-c}(\sqrt{2}\,\|x_1\|^2_{(1)}\sqrt{G_{c+m}}/2)\right\}}.$$

Finally, when $R^2_\infty =^D W_{G,c} =^D \mathsf{G}_1^{1/c}$, $c \in (0, 1)$, then again the conditional density is computed a.e. through

$$f_{Y_2|Y_1.}(x_2 : x_1) = \frac{c|\Sigma_{22.1}|^{-1/2}}{2(\sqrt{2\pi})^{d-d_1}} \frac{\mathrm{E}\left\{\left(\frac{2C_c}{\|x_2:x_1\|^2_{(2)}}\right)^{(d/2-c)/2} K_{d/2-c}(\sqrt{2}\,\|x_2 : x_1\|^2_{(2)}\sqrt{C_2}/2)\right\}}{\mathrm{E}\left\{\left(\frac{2C_c}{\|x_1\|^2_{(1)}}\right)^{(d/2-c)/2} K_{d/2-c}(\sqrt{2}\,\|x_1\|^2_{(1)}\sqrt{C_c}/2)\right\}},$$

where C_c is a positive stable random variable with index $c \in (0, 1)$.

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References

[1] D.F. Andrews, C.L. Mallows, Scale mixtures of normal distributions, J. Royal Stat. Soc. Ser. B, 36 (1974) 99–102.

[2] O.E. Barndorff-Nielsen, Process of normal inverse Gaussian type, Finance and Stochastics 2 (1998) 41–68.

[3] O.E. Barndorff-Nielsen, C. Halgreen, Infinite divisibility of the hyperbolic and generalized inverse Gaussian distributions, Z. Wahrschein. Verw. Geb. 38 (1977) 41–68.

[4] B.M. Bennett, On a certain multivariate non-normal distribution, Math. Proc. Camb. Phil. Soc. 57 (1961) 434–436.

[5] J. Bertoin, Lévy Processes, Cambridge University Press, 1996.

[6] P. Blæsild, The two-dimensional hyperbolic distribution and related distributions, with an application to Johannsen's bean data, Biometrika 68 (1981) 251–263.

[7] L. Bondesson, A remarkable property of generalized gamma convolutions, Prob. Theor. Rel. Fields 78 (1988) 321–333.

[8] L. Bondesson, Generalized Gamma Convolutions and Related Classes of Distributions and Densities, Springer, New York, 1992.

[9] S. Cambanis, S.B. Fotopoulos, L. He, On the conditional variance for scale mixtures of normal distributions, J. Multivariate Anal. 74 (2000) 163–192.

[10] S. Cambanis, S. Huang, G. Simons, On the theory of elliptically contoured distributions, J. Multivariate Anal. 11 (1981) 388–385.

[11] R. Cont, P. Tankov, Financial Modelling with Jump Processes, Chapman & Hall/CRC, London, 2004.

[12] E. Eberlein, E.A. Hammerstein, Generalized hyperbolic and inverse Gaussian distributions: Limiting cases and approximations of processes, In: Seminar on Stochastic Analysis, Random Fields and Applications IV, Birkhäuser, New York, 2002.

[13] E. Eberlein, U. Keller, Hyperbolic distributions in finance, Bernoulli 1 (1998) 281–299.

[14] E. Eberlein, U. Keller, K. Prause, New insights into smile, mispricing and value at risk: the hyperbolic model, J. Business 71 (1998) 371–405.

[15] E. Eberlein, K. Prause, The generalized hyperbolic model: financial derivatives and risk measures, In: H. Geman, D. Madan, S. Pliska, T. Vorst, Eds., Mathematical Finance-Bachelier Congress 2000 (Springer, Berlin), 2001, pp. 245–267.

[16] K.T. Fang, S. Kotz, W.K. Ng, Symmetric Multivariate and Related Distributions, Chapman &Hall, New York, 1990.

[17] G. Frahm, Generalized Elliptical Distributions: Theory and Applications, Ph.D. Thesis, Universität zu Köln, Germany, 2004.

[18] I.S. Gradshteyn, I.M. Ryzhik, Tables of Integrals, Series and Products, 6th Ed., Academic Press New York, 2000.

[19] F.A. Graybill, Theory and Applications in Linear Models, Duxbury, 1976.

[20] L.F. James, B. Roynette, M. Yor, Generalized gamma convolutions, Dirichlet means, Thorin measures, with explicit examples, Probability Surveys 5 (2008) 346–415.

[21] Y. Kano, Consistency property of elliptical probability density functions, J. Multivariate Anal. 51 (1994) 139–147.

[22] D. Kelker, Distribution theory of spherical distributions and a location scale parameter, Sankhyā Ser. A 32 (1970) 419–430.

[23] D. Kelker, Infinite divisibility and variance mixtures of the normal distribution, Ann. Math. Statist. 42 (1971) 802–808.

[24] J.F.C. Kingman, On random sequences with spherical symmetry, Biometrika 59 (1972) 492–498.

[25] S.G. Kou, A jump diffusion model for option pricing with three properties: leptokurtic feature, volatility smile, and analytical tractability, Manag. Sci. 48 (2002) 1087–1101.

[26] K.L. Lange, J.S. Sinsheimer, Normal/independent distributions and their applications in robust regression, J. Comput. Graph. Stat. 17 (1993) 81–92.

[27] D.B. Madan, E. Seneta, The variance gamma (V.G.) model for share market returns, J. Business 63 (1990) 511–524.

[28] J. Owen, R. Rabinovitch, On the class of elliptical distributions and their applications to the theory of portfolio choice, J. Finance 38 (1983) 745–752.

[29] K. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, New York, 1999.

[30] I.J. Schoenberg, Metric spaces and completely monotone functions, Ann. Math. 24 (1938) 811–841.

[31] G. Samorodnitsky, M. Taqqu, Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance, Chapman & Hall, New York, 1994.

[32] K. Yano, Y. Yano, M. Yor, On the laws of the first hitting of points for one-dimensional symmetric stable Lévy processes, In: Séminaire de Probabilités XLII, 1979, Springer, 2008.