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Asymptotic Distributions of Some Test Criteria for the Covariance Matrix in Elliptical Distributions under Local Alternatives

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The asymptotic distributions under local alternatives of two test criteria for testing the hypothesis that the characteristic roots of the covariance matrix of an elliptical population, assumed distinct, are equal to a set of specified numbers, are derived. The two tests are the modified likelihood ratio test and a new test criterion proposed in this context for the normal model. Similar results are given for the two tests for testing that the covariance matrix is a specified positive definite matrix, in which case the two tests are the modified likelihood ratio test and a test proposed by Rao and Nagao for the normal model, and also for a test for the covariance structure in familial data, studied by Srivastava.

1. INTRODUCTION

Hayakawa and Puri [5] and Hayakwa [4] considered the problems of testing the hypotheses that the characteristic roots of the covariance matrix of an elliptically contoured population are equal to a set of specified numbers and that the covariance matrix is a specified positive definite matrix. They assumed, however, that the population mean is known. Although, in normal models, the knowledge of the mean is not a serious restriction since the problem of the unknown mean can be reduced to one with a known mean by an orthogonal transformation of the data with a loss of one degree of freedom, in the more general context of elliptical distributions

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this is indeed a serious restriction. Moreover, the expressions obtained by them are so complicated that they lose their practical significance.

In the analysis of familial data, one usually assumes a structure on the covariance matrix of the underlying distribution of the kind described in (1) below; see, for example, Srivastava [15, 16]. Srivastava [16] has derived an asymptotic expansion of the null distribution of the likelihood ratio criterion for testing this particular structure, under the assumption of normal distribution. Deriving similar results under the general assumption of elliptical distribution would thus be an interesting problem.

Many general results on the distribution of test statistics without assuming any model, except for the existence of some moments, are available in the literature; see, for example, Fujikoshi [3] and Chan and Srivastava [1]. However, under elliptical model, a significant simplification occurs and usually it is better to treat them separately than using the results for a general model. Recently, Nagao and Srivastava [13] and Nagao [12] obtained the asymptotic distributions of several test criteria under local alternative for testing sphericity and independence without assuming that the population mean is known. The test statistics studied are the ones obtained for the normal model by the likelihood ratio method and a criterion proposed by Rao [14] and independently by Nagao [11].

The objective of this paper is to obtain under local alternatives the asymptotic expansions of the distributions of the test criteria for each of the problems, mentioned in the first two paragraphs of this section, for testing for the covariance matrix of an elliptical distribution. The organization of the paper is as follows. In Section 2, we describe the elliptical model and test statistics for each problem. Section 3 deals with the case of testing that the characteristic roots of the covariance matrix, assumed distinct, are equal to a set of specified numbers. Section 4 deals with the case of testing that the covariance matrix is equal to a specified positive definite matrix, which is assumed to be the identity matrix without loss of generality. We compare the performances of the test criteria for each of the problems in terms of the behaviour of their power functions. Section 5 deals with a modified likelihood ratio criterion obtained under normal model for testing the covariance structure in familial data, when the observations are indeed from the elliptical distribution described above. In Section 6, we compute the powers of the test statistics studied in Sections 3 and 4, by using the expansions of their distributions obtained in these sections.

2. Preliminaries

Consider a sequence of independent and identically distributed random $p \times 1$ vectors $\{X_i : i \geq 1\}$ having an elliptically contoured density

$c_p\,|V|^{-1/2}g((x-\mu)'\,V^{-1}(x-\mu))$, where c_p is a positive constant, μ is an unknown $p\times 1$ vector, V is an unknown $p\times p$ positive definite matrix, and g is a known non-negative function [7]. The characteristic function of this distribution is easily seen to be of the form $e^{it'\mu}\phi(t'Vt)$ for some function ϕ. This implies, in turn, that the mean of X_1 is μ, and the dispersion matrix of X_1 is $-2\phi'(0)\,V$, when they exist (see [2, pp. 66–67]). Moreover, all the components of X_1 have equal kurtosis, denoted by κ and given by (see [10, 5])

$$\kappa=\frac{[\phi^{(2)}(0)-\{\phi'(0)\}^2]}{\{\phi'(0)\}^2}.$$

Denote now the dispersion matrix of X_1 by W and the characteristic roots of W, taken in decreasing order of magnitude by $\lambda_1>\cdots>\lambda_p$. We consider the following testing of hypothesis problems: (i) $H_0^{(1)}:\lambda_i=\lambda_{i0}$, $i=1,...,p$, where $\lambda_{10},...,\lambda_{p0}$ are specified numbers, (ii) $H_0^{(2)}: W=I_p$, and (iii) $H_0^{(3)}: W=W_0$, where

$$W_0=\begin{pmatrix} \sigma_m^2 & \sigma_{mc} & \sigma_{mc}\cdots & \sigma_{mc} \\ \sigma_{mc} & \sigma_c^2 & \sigma_{cc} & \cdots & \sigma_{cc} \\ \sigma_{mc} & \sigma_{cc} & \sigma_c^2 & \cdots & \sigma_{cc} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{mc} & \sigma_{cc} & \sigma_{cc} & \cdots & \sigma_c^2 \end{pmatrix}, \tag{1}$$

with σ_m^2, σ_{mc}, σ_c^2, and σ_{cc} being unknown parameters. For each of these problems, we shall use the test statistics employed in the context of normal model. In what follows, we describe these statistics.

The modified log-likelihood ratio statistic for the first problem under the additional assumption of normality turns out to be

$$T_{1,n}=n\left(\sum_{i=1}^{p}\frac{d_i}{\lambda_{i0}}-p-\sum_{i=1}^{p}\log\frac{d_i}{\lambda_{i0}}\right), \tag{2}$$

where $d_1\geqslant\cdots\geqslant d_p$ are the eigenvalues of S_N, $n=N-1$, with $nS_N=(X_1-\bar{X}_N)(X_1-\bar{X}_N)'+\cdots+(X_N-\bar{X}_N)(X_N-\bar{X}_N)'$ and $N\bar{X}_N=X_1+\cdots+X_N$. An appeal to the asymptotic distribution of $(d_1,...,d_p)$ (Theorem 9.4.3 of [17], p. 285) and the delta method implies that under a general alternative, $T_{1,n}/n$ is asymptotically normal with mean

$$\sum_{i=1}^{p}\frac{\lambda_i}{\lambda_{i0}}-p-\sum_{i=1}^{p}\log\frac{\lambda_i}{\lambda_{i0}} \tag{3}$$

and variance

$$\frac{2}{n}\sum_{i=1}^{p}\left(\frac{\lambda_i}{\lambda_{i0}}-1\right)^2, \tag{4}$$

where $\lambda_1 \geqslant \cdots \geqslant \lambda_p$ denote the eigenvalues of the population dispersion matrix under the general alternative under consideration. Observe now that the quantity in (4) can be thought of as a measure of departure from the null hypothesis and that it vanishes if $\lambda_i = \lambda_{i0}$ for $i = 1, ..., p$. We, therefore, propose the following alternative test statistic for the first problem:

$$T_{2,n} = \frac{n}{2}\sum_{i=1}^{p}\left(\frac{d_i}{\lambda_{i0}}-1\right)^2. \tag{5}$$

The null hypothesis is rejected for large values of $T_{2,n}$.

Consider now the second problem. For this problem, the modified log-likelihood ratio statistic and the alternative test statistic proposed by Rao [14] and also independently by Nagao [11] are

$$T_{3,n} = n(\mathrm{tr}(S_N) - \log |S_N| - p) \tag{6}$$

and

$$T_{4,n} = \frac{n}{2}\,\mathrm{tr}(S_N - I_p)^2, \tag{7}$$

respectively.

For the third problem, the modified log-likelihood ratio statistic turns out to be [16]

$$\begin{aligned} T_{5,n} = n(p-2)\log &\left\{\frac{\sum_{j=2}^{p} s_{jj}}{p-1} - \frac{2\sum_{j=2}^{p}\sum_{k=j+1}^{p} s_{jk}}{(p-1)(p-2)}\right\} \\ &- n\log |(p-1)\, S_N| + n\log\left\{s_{11}\sum_{j=2}^{p} s_{jj} - \left(\sum_{j=2}^{p} s_{1j}\right)^2\right\}, \end{aligned} \tag{8}$$

where $S_N = ((s_{ij}))$.

3. Test for Eigenvalues

In this section, we derive the asymptotic expansions of the distributions of $T_{1,n}$ and $T_{2,n}$, defined in (2) and (5), respectively. The sequence of local alternatives considered are $H_{1n}^{(1)}: \lambda_i = \lambda_{i0} + \sqrt{2}\lambda_{i0}\theta_i/\sqrt{n}$, $i = 1, ..., p$.

We begin with the observation that the distributions of $T_{1,n}$ and $T_{2,n}$ under $H_{1n}^{(1)}$ do not depend on the eigenvectors of the dispersion matrix of X_1, so that we can assume without loss of generality that under $H_{1n}^{(1)}$, $E(S_N) = \operatorname{diag}(\delta_{1,n}, ..., \delta_{p,n})$, where $\delta_{i,n} = \lambda_{i0} + \sqrt{2}\lambda_{i0}\theta_i/\sqrt{n}$, $i = 1, ..., p$. We shall see that an important consequence of this observation is that it will enable us to employ the general tool of Nagao and Srivastava [13] (henceforth abbreviated to NS) in the present situation, resulting in neat expressions for the asymptotic expansions we are looking for.

As in NS, we define the random symmetric matrix $Z = ((z_{ij}))$ by

$$S_N = D_n^{1/2} \exp(\sqrt{(2/n)}Z)\, D_n^{1/2},$$

where $D_n = \operatorname{diag}(\delta_{1,n}, ..., \delta_{p,n})$. Then the expansions of $T_{1,n}$ and $T_{2,n}$ analogous to the relation (2.3) of NS are [18]

$$\begin{aligned} T_{1,n} = {} & \sum_{i=1}^{p} (z_{ii} + \theta_i)^2 + \frac{\sqrt{2}}{3\sqrt{n}} \sum_{i=1}^{p} (z_{ii} + \theta_i)^3 \\ & + \frac{\sqrt{2}}{\sqrt{n}} \sum_{i=1}^{p} (z_{ii} + \theta_i) \left(\sum_{j=1}^{p} z_{ij}^2 - z_{ii}^2 - \theta_i^2 \right) \\ & + \frac{2\sqrt{2}}{\sqrt{n}} \sum_{i=1}^{p} \sum_{j=i+1}^{p} \frac{\lambda_{i0}(z_{ii} + \theta_i) z_{ij}^2}{\lambda_{i0} - \lambda_{j0}} + O_p\left(\frac{1}{n}\right) \end{aligned} \tag{9}$$

and

$$T_{2,n} = T_{1,n} + \frac{2\sqrt{2}}{3\sqrt{n}} \sum_{i=1}^{p} (z_{ii} + \theta_i)^3. \tag{10}$$

Moreover, the asymptotic distribution of $z = (z_{11}, ..., z_{pp}, z_{12}, ..., z_{p-1,p})$ is given by (2.14) of NS, with the quantities $g_{ab:cd}$ and $d_{ab:cd:ef}$ being given by the relations (2.4) and (2.5) of Nagao [12]. Now we make use of (9), (10), and the asymptotic distribution of z to obtain the characteristic functions of $T_{1,n}$ and $T_{2,n}$. This step consists of laborious calculation. We are not presenting the details related to this step in order to save space; they can, however, be obtained from the authors. The next step consists in inverting these characteristic functions. This leads us to the following expansions of the distributions of $T_{1,n}$ and $T_{2,n}$ under $H_{1n}^{(1)}$, denoted by P_1:

THEOREM 3.1. *$P_1(T_{k,n} \leqslant x) = P(U_{11} \leqslant x) + (1/\sqrt{n}) \sum_{i=1}^{4} \sum_{j=1}^{n_i} B_{ij}^{(k)} P(U_{ij} \leqslant x) + O(1/n)$, $k = 1, 2$, where $n_1 = 4$, $n_2 = n_3 = 2$, $n_4 = 1$, $U_{ij} = aX_{ij} + bY_{ij}$, $a = 1 + \kappa$, $b = 1 + \kappa + p\kappa/2$, for each i and j, X_{ij} has a non-central χ^2 distribution with $(p-1) + 2(i-1)$ degrees of freedom and non-centrality parameter*

δ_1, and Y_{ij} has a non-central χ^2 distribution with $2j-1$ degrees of freedom and non-centrality parameter δ_2, and X_{ij} and Y_{ij} are independent, where $\delta_1 = 1/a\{\sum_{i=1}^{p} \theta_i^2 - p\theta^2\}$, $\delta_2 = p\theta^2/b$, with $p\theta = \theta_1 + \cdots + \theta_p$, and the constants $B_{ij}^{(1)}$, $B_{ij}^{(2)}$ depend on κ, $\phi'(0)$, $\phi^{(3)}(0)$, $\lambda_{10}, \ldots, \lambda_{p0}, \theta_1, \ldots, \theta_p$. (The expressions for these constants appear in the Appendix.)

It should be mentioned that in the special case corresponding to multivariate normal distribution, the expansion of the distribution of $T_{1,n}$ obtained in the result above reduces to the one in Hayakawa and Puri [5].

We shall now see that we can use the above result to compare the performance of $T_{1,n}$ with that of $T_{2,n}$ in terms of the behaviour of their power functions. To this end, first note that the limiting null distributions of $T_{1,n}$ and $T_{2,n}$ are same, and that by Hill and Davis [6] and Withers [19] the $(1-\alpha)$th quantiles of the exact null distributions of $T_{1,n}$ and $T_{2,n}$, denoted $u_{1,n}, u_{2,n}$, satisfy $u_{1,n} - u = O(1/n)$ and $u_{2,n} - u = O(1/n)$, where u is the $(1-\alpha)$th quantile of the asymptotic null distribution of $T_{1,n}$ (and also of $T_{2,n}$). Hence, the difference of the powers of $T_{1,n}$ and $T_{2,n}$ is given by

$$\begin{aligned}
&P_1(T_{2,n} > u_{2,n}) - P_1(T_{1,n} > u_{1,n}) \\
&\quad = \frac{1}{\sqrt{n}} \Bigg[\sqrt{2}\theta\{P(U_{13} > u) - P(U_{12} > u)\} \\
&\qquad + \frac{\sqrt{2}p\theta^3}{3b}\{P(U_{14} > u) - P(U_{13} > u)\} \\
&\qquad + \sqrt{2}(p-1)\,\theta\{P(U_{22} > u) - P(U_{12} > u)\} \\
&\qquad + \frac{\sqrt{2}\theta A_5}{b}\{P(U_{32} > u) - P(U_{31} > u)\} \\
&\qquad + \frac{\sqrt{2}A_1}{3a}\{P(U_{41} > u) - P(U_{31} > u)\}\Bigg] + O\left(\frac{1}{n}\right).
\end{aligned}$$

Observe now that each of the differences of the tail probabilities in the sum above is positive, because for each i, j, k, l, X_{ij} stochastically dominates X_{kl} if $i > k$, the distribution of X_{ij} is the same as that of X_{kl} if $i = k$, a similar property is enjoyed by the random variables Y_{ij}, and X_{ij} and Y_{ij} are independent. We, therefore, have the following result.

THEOREM 3.2. *The power of $T_{2,n}$ is asymptotically (up to $1/\sqrt{n}$ terms) greater (smaller) than that of $T_{1,n}$ if both θ and A_1 are positive (negative). Moreover, if one of θ and A_1 is zero and the other one is positive (negative), the power of $T_{2,n}$ will be greater (smaller) than that of $T_{1,n}$.*

4. Test for a Dispersion Matrix

In this section, we derive the asymptotic expansions of the distributions of $T_{3,n}$ and $T_{4,n}$, defined in (6) and (7), respectively. The sequence of local alternatives considered are $H_{1n}^{(2)}: W_n = I_p + \sqrt{2}\Theta/\sqrt{n}$, where W_n denotes the dispersion matrix of X_1 under $H_{1n}^{(2)}$ and $\Theta = ((\theta_{ij}))$ is a $p \times p$ symmetric matrix. In this case also, we shall employ the general tool of NS.

As in NS, we define the random symmetric matrix $Z = ((z_{ij}))$ by

$$S_N = W_n^{1/2} \exp(\sqrt{(2/n)}Z)\, W_n^{1/2}. \tag{11}$$

Then the expansions of $T_{4,n}$ and $T_{3,n}$ analogous to the relation (2.3) of NS are

$$T_{4,n} = \sum_{i=1}^{p} \sum_{j=1}^{p} (z_{ij} + \theta_{ij})^2 + \frac{\sqrt{2}}{\sqrt{n}} \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{k=1}^{p} (z_{ij} + \theta_{ij})$$
$$\times [(z_{ik} + \theta_{ik})(z_{jk} + \theta_{jk}) - \theta_{ik}\theta_{jk}] + O_p\left(\frac{1}{n}\right)$$

and

$$T_{3,n} = T_{4,n} + \frac{2\sqrt{2}}{3\sqrt{n}} \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{k=1}^{p} (z_{ij} + \theta_{ij})(z_{ik} + \theta_{ik})(z_{jk} + \theta_{jk}).$$

Also, the asymptotic distribution of $z = (z_{11}, ..., z_{pp}, z_{12}, ..., z_{p-1,p})$ is the same as the one mentioned in the last section. Proceeding as in Section 3, we obtain the following expansions of the distributions of $T_{3,n}$ and $T_{4,n}$ under $H_{1n}^{(2)}$, denoted by P_1.

Theorem 4.1. *$P_1(T_{k,n} \leqslant x) = P(U_{11} \leqslant x) + (1/\sqrt{n}) \sum_{i=1}^{4} \sum_{j=1}^{n_i} B_{ij}^{(k)} P(U_{ij} \leqslant x) + O(1/n)$, $k = 3, 4$, where $n_1 = 4$, $n_2 = n_3 = 2$, $n_4 = 1$, $U_{ij} = aX_{ij} + bY_{ij}$, $a = 1 + \kappa$, $b = 1 + \kappa + p\kappa/2$, for each i and j, X_{ij} has a non-central χ^2 distribution with $p(p+1)/2 - 1 + 2(i-1)$ degrees of freedom and non-centrality parameter δ_1, and Y_{ij} has a non-central χ^2 distribution with $2j - 1$ degrees of freedom and non-centrality parameter δ_2, and X_{ij} and Y_{ij} are independent, where $\delta_1 = (1/a)\{\sum_{i=1}^{p} \sum_{j=1}^{p} \theta_{ij}^2 - p\theta^2\}$, $\delta_2 = p\theta^2/b$, with $p\theta = \theta_{11} + \cdots + \theta_{pp}$, and the constants $B_{ij}^{(3)}$, $B_{ij}^{(4)}$ depend on κ, $\phi'(0)$, $\phi^{(3)}(0)$, θ_{ij}, $i, j = 1, ..., p$.* (The expressions for these constants appear in the Appendix.)

It should be mentioned that with $\theta_{ij} = 0$ for $i, j = 1, ..., p$, the leading term of the expansion of the distribution of $T_{3,n}$ given by the above result coincides with the limiting null distribution obtained by Muirhead and Waternaux [10]. It should also be mentioned that expansion similar to the one obtained by taking $k = 3$ in the above result was obtained by Khatri and Srivastava [9] under normality; they, however, considered different kinds of local alternatives. See also Khatri and Srivastava [8].

As in Section 3, the above result can now be used to obtain the following result.

THEOREM 4.2. *The power of $T_{3,n}$ is asymptotically (up to $1/\sqrt{n}$ terms) greater (smaller) than that of $T_{4,n}$ if all of A_1, A_8, A_9, and θ* are positive *(negative). More generally, if some of these quantities are positive (negative) and the others are zero, then also the same conclusion holds.*

5. TEST FOR COVARIANCE STRUCTURE IN FAMILIAL DATA

We derive the asymptotic expansion of the distribution of $T_{5,n}$, defined in (8), in this section. The sequence of local alternatives under cosideration are $H_{1n}^{(3)}: W_n = W_0 + \sqrt{2}\Theta/\sqrt{n}$, where W_n denotes the dispersion matrix of X_1 under $H_{1n}^{(3)}$, and $\Theta = ((\theta_{ij}))$ is a $p \times p$ symmetric matrix with $\theta_{11} = 0$, $\sum_{j=2}^{p} \theta_{1j} = 0$, $\sum_{i=2}^{p} \theta_{ii} = 0$, $\sum_{i=2}^{p} \sum_{j=i+1}^{p} \theta_{ij} = 0$. As before, we shall employ the general tool of NS in this case also.

We define the random symmetric matrix $Z = ((z_{ij}))$ as in (11) and note that the asymptotic distribution of $z = (z_{11}, ..., z_{pp}, z_{12}, ..., z_{p-1,p})$ is the same as the one mentioned in Section 3. Proceeding as in previous sections, we obtain the following expansion of the distribution of $T_{5,n}$ under $H_{1n}^{(3)}$, denoted by P_1.

THEOREM 5.1. $P_1(T_{5,n} \leqslant x) = P(aX_1 \leqslant x) + (1/\sqrt{n}) \sum_{i=1}^{4} B_i P(aX_i \leqslant x) + O(1/n)$, *where $a = 1 + \kappa$, for each i, X_i has a non-central χ^2 distribution with $(p-1)(p-2)/2 + 2(p-2) - 1 + 2(i-1)$ degrees of freedom and non-centrality parameter δ, where $\delta = \operatorname{tr}(\Theta^2)/a$, and*

$$B_1 = \frac{\sqrt{2}(3A_2 + 6A_3 - 2A_1)}{a} - \frac{\sqrt{2}c_2\{A_2 + 6A_3\}}{3a^3},$$

$$B_2 = -\frac{2\sqrt{2}(3A_2 + 6A_3 - 2A_1)}{a} + \frac{\sqrt{2}c_2\{A_2 + 6A_3\}}{a^3},$$

$$B_3 = \frac{3\sqrt{2}(3A_2 + 6A_3 - 2A_1)}{2a} - \frac{\sqrt{2}c_2\{A_2 + 6A_3\}}{a^3} - \frac{\sqrt{2}(15A_2 - 10A_1 + 30A_3)}{6a},$$

$$B_4 = -\frac{\sqrt{2}(3A_2 + 6A_3 - 2A_1)}{2a} + \frac{\sqrt{2}c_2\{A_2 + 6A_3\}}{3a^3} + \frac{\sqrt{2}(15A_2 - 10A_1 + 30A_3)}{6a},$$

with

$$A_1 = \sum_{i=1}^{p} \theta_{ii}^3, \qquad A_2 = \sum_{i=1}^{p} \sum_{j=1}^{p} \theta_{ii} \theta_{ij}^2,$$

$$A_3 = \sum_{i=1}^{p} \sum_{j=i+1}^{p} \sum_{k=j+1}^{p} \theta_{ij} \theta_{ik} \theta_{jk}, \quad c_2 = -\frac{\phi^{(3)}(0)}{(\phi'(0))^3}.$$

Note that $\sum_{i=1}^{4} F_i = 0$.

6. Numerical Examples

In this section we compute the powers of the test statistics $T_{1,n}$, $T_{2,n}$, $T_{3,n}$, and $T_{4,n}$ for some elliptically contoured populations. We use the expansions obtained in Theorems 3.1 and 4.1 for this purpose. The distributions considered are multivariate normal, multivariate t, and ε-contaminated normal (henceforth, abbreviated to ε-normal), all with dimension $p = 3$. The multivariate t distribution is considered with the following degrees of freedom (v): 10, 30, 50, and 100. The ε-normal distribution is considered with the following parameters: $\sigma^2 = 0.1$, $\varepsilon = 0.1$; $\sigma^2 = 0.5$, $\varepsilon = 0.1$; $\sigma^2 = 0.1$, $\varepsilon = 0.3$; $\sigma^2 = 0.5$, $\varepsilon = 0.3$; $\sigma^2 = 0.1$, $\varepsilon = 0.5$; $\sigma^2 = 0.5$, $\varepsilon = 0.5$; $\sigma^2 = 0.1$, $\varepsilon = 0.9$; $\sigma^2 = 0.5$, $\varepsilon = 0.9$. The powers are all computed for the sample sizes: $N = 50$, 80, and 100.

For the tests for eigenvalues, we choose $\lambda_{10} = \frac{1}{2}$, $\lambda_{20} = \frac{1}{6}$, $\lambda_{30} = \frac{1}{10}$. We also choose the θ_i's present in the sequence of alternative hypotheses in a way such that $\sqrt{2}\lambda_{10}\theta_1 = \sqrt{2}\lambda_{20}\theta_2 = \sqrt{2}\lambda_{30}\theta_3$. We denote this common quantity by θ_0. We consider the cases: $\theta_0 = 0.25$; 0.50. The cutoff points of both the tests are taken to be equal to the 95th percentile of the limiting null distribution under the normal model. In the present situation (i.e., when $p = 3$), this limiting null distribution is chi-square with three degrees of freedom. Hence, the cutoff point is $u = 7.81$.

For the tests for the dispersion matrix, we choose the matrix Θ to be of the form $\Theta = \alpha_0 I_3$. We consider the following cases: $\alpha_0 = -0.2/\sqrt{2}$, 0.0, and $0.2/\sqrt{2}$. As with tests for eigenvalues, in this case the cutoff point is taken to be equal to 12.59, the 95th percentile of chi-square distribution with six degrees of freedom.

The power of $T_{i,n}$ is denoted by $p_{i,n}$, $i = 1, \ldots, 4$.

For the test for eigenvalues, the computations demonstrate superiority of $T_{2,n}$ over $T_{1,n}$, for the chosen alternatives. However, for the test for dispersion matrix, itseems that $T_{4,n}$ performs better that $T_{3,n}$ when α_0 is negative, and worse when α_0 is positive. (See Tables I–XX.)

TABLE I

	Multivariate normal distribution			
	$\theta_0 = 0.25$		$\theta_0 = 0.50$	
N	$p_{1,n}$	$p_{2,n}$	$p_{1,n}$	$p_{2,n}$
50	0.328	0.481	0.844	0.883
80	0.341	0.461	0.867	0.898
100	0.346	0.454	0.876	0.904

TABLE II

	Multivariate t distribution ($\nu = 10$)					Multivariate t distribution ($\nu = 30$)			
	$\theta_0 = 0.25$		$\theta_0 = 0.50$			$\theta_0 = 0.25$		$\theta_0 = 0.50$	
N	$p_{1,n}$	$p_{2,n}$	$p_{1,n}$	$p_{2,n}$	N	$p_{1,n}$	$p_{2,n}$	$p_{1,n}$	$p_{2,n}$
50	0.398	0.514	0.782	0.829	50	0.352	0.495	0.825	0.867
80	0.413	0.504	0.811	0.848	80	0.365	0.478	0.850	0.883
100	0.419	0.501	0.822	0.856	100	0.370	0.471	0.860	0.890

TABLE III

	Multivariate t distribution ($\nu = 50$)					Multivariate t distribution ($\nu = 100$)			
	$\theta_0 = 0.25$		$\theta_0 = 0.50$			$\theta_0 = 0.25$		$\theta_0 = 0.50$	
N	$p_{1,n}$	$p_{2,n}$	$p_{1,n}$	$p_{2,n}$	N	$p_{1,n}$	$p_{2,n}$	$p_{1,n}$	$p_{2,n}$
50	0.343	0.490	0.832	0.874	50	0.335	0.485	0.838	0.878
80	0.356	0.471	0.857	0.890	80	0.348	0.466	0.862	0.894
100	0.361	0.464	0.867	0.896	100	0.353	0.459	0.872	0.900

TABLE IV

	ε-normal distribution ($\sigma^2 = 0.1$, $\varepsilon = 0.1$)					ε-normal distribution ($\sigma^2 = 0.5$, $\varepsilon = 0.1$)			
	$\theta_0 = 0.25$		$\theta_0 = 0.50$			$\theta_0 = 0.25$		$\theta_0 = 0.50$	
N	$p_{1,n}$	$p_{2,n}$	$p_{1,n}$	$p_{2,n}$	N	$p_{1,n}$	$p_{2,n}$	$p_{1,n}$	$p_{2,n}$
50	0.360	0.502	0.822	0.865	50	0.337	0.487	0.837	0.877
80	0.372	0.483	0.848	0.881	80	0.350	0.468	0.861	0.893
100	0.377	0.476	0.858	0.888	100	.0.355	0..460	0.871	0.900

TABLE V

	ε-normal distribution ($\sigma^2 = 0.1$, $\varepsilon = 0.3$)					ε-normal distribution ($\sigma^2 = 0.5$, $\varepsilon = 0.3$)			
	$\theta_0 = 0.25$		$\theta_0 = 0.50$			$\theta_0 = 0.25$		$\theta_0 = 0.50$	
N	$p_{1,n}$	$p_{2,n}$	$p_{1,n}$	$p_{2,n}$	N	$p_{1,n}$	$p_{2,n}$	$p_{1,n}$	$p_{2,n}$
50	0.425	0.542	0.787	0.834	50	0.353	0.496	0.827	0.870
80	0.434	0.526	0.815	0.852	80	0.365	0.478	0.852	0.886
100	0.438	0.520	0.827	0.860	100	0.370	0.471	0.862	0.892

TABLE VI

	ε-normal distribution ($\sigma^2 = 0.1$, $\varepsilon = 0.5$)					ε-normal distribution ($\sigma^2 = 0.5$, $\varepsilon = 0.5$)			
	$\theta_0 = 0.25$		$\theta_0 = 0.50$			$\theta_0 = 0.25$		$\theta_0 = 0.50$	
N	$p_{1,n}$	$p_{2,n}$	$p_{1,n}$	$p_{2,n}$	N	$p_{1,n}$	$p_{2,n}$	$p_{1,n}$	$p_{2,n}$
50	0.500	0.590	0.769	0.815	50	0.364	0.502	0.817	0.861
80	0.506	0.577	0.797	0.833	80	0.376	0.485	0.843	0.878
100	0.509	0.572	0.807	0.840	100	0.381	0.478	0.854	0.884

TABLE VII

	ε-normal distribution ($\sigma^2 = 0.1$, $\varepsilon = 0.9$)					ε-normal distribution ($\sigma^2 = 0.5$, $\varepsilon = 0.9$)			
	$\theta_0 = 0.25$		$\theta_0 = 0.50$			$\theta_0 = 0.25$		$\theta_0 = 0.50$	
N	$p_{1,n}$	$p_{2,n}$	$p_{1,n}$	$p_{2,n}$	N	$p_{1,n}$	$p_{2,n}$	$p_{1,n}$	$p_{2,n}$
50	0.648	0.684	0.785	0.816	50	0.350	0.493	0.825	0.868
80	0.655	0.684	0.804	0.829	80	0.363	0.476	0.851	0.884
100	0.658	0.684	0.812	0.834	100	0.368	0.469	0.861	0.892

TABLE VIII

	Multivariate normal distribution					
	$\alpha_0 = -0.2/\sqrt{2}$	e	$\alpha_0 = 0.0$		$\alpha_0 = 0.2/\sqrt{2}$	
N	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$
50	0.003	0.025	0.050	0.050	0.101	0.080
80	0.014	0.031	0.050	0.050	0.090	0.074
100	0.018	0.033	0.050	0.050	0.086	0.072

TABLE IX

	Multivariate t distribution ($\nu = 10$)					
	$\alpha_0 = -0.2/\sqrt{2}$		$\alpha_0 = 0.0$		$\alpha_0 = 0.2/\sqrt{2}$	
N	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$
50	0.113	0.144	0.180	0.180	0.255	0.224
80	0.128	0.153	0.180	0.180	0.240	0.215
100	0.134	0.156	0.180	0.180	0.234	0.212

TABLE X

	Multivariate t distribution ($\nu = 30$)					
	$\alpha_0 = -0.2/\sqrt{2}$		$\alpha_0 = 0.0$		$\alpha_0 = 0.2/\sqrt{2}$	
N	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$
50	0.018	0.044	0.074	0.074	0.136	0.111
80	0.031	0.051	0.074	0.074	0.124	0.104
100	0.036	0.054	0.074	0.074	0.119	0.101

TABLE XI

	Multivariate t distribution ($\nu = 50$)					
	$\alpha_0 = -0.2/\sqrt{2}$		$\alpha_0 = 0.0$		$\alpha_0 = 0.2/\sqrt{2}$	
N	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$
50	0.011	0.035	0.063	0.063	0.121	0.097
80	0.023	0.041	0.063	0.063	0.109	0.091
100	0.027	0.044	0.063	0.063	0.104	0.088

TABLE XII

	Multivariate t distribution ($\nu = 100$)					
	$\alpha_0 = -0.2/\sqrt{2}$		$\alpha_0 = 0.0$		$\alpha_0 = 0.2/\sqrt{2}$	
N	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$
50	0.007	0.029	0.056	0.056	0.110	0.088
80	0.018	0.036	0.056	0.056	0.099	0.082
100	0.022	0.038	0.056	0.056	0.095	0.079

TABLE XIII

ε-normal distribution ($\varepsilon = 0.1$, $\sigma^2 = 0.1$)						
	$\alpha_0 = -0.2/\sqrt{2}$		$\alpha_0 = 0.0$		$\alpha_0 = 0.2/\sqrt{2}$	
N	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$
50	0.022	0.048	0.078	0.078	0.140	0.115
80	0.035	0.055	0.078	0.078	0.128	0.107
100	0.040	0.058	0.078	0.078	0.123	0.105

TABLE XIV

ε-normal distribution ($\varepsilon = 0.1$, $\sigma^2 = 0.5$)						
	$\alpha_0 = -0.2/\sqrt{2}$		$\alpha_0 = 0.0$		$\alpha_0 = 0.2/\sqrt{2}$	
N	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$
50	0.008	0.030	0.057	0.057	0.111	0.089
80	0.019	0.037	0.057	0.057	0.100	0.083
100	0.023	0.057	0.039	0.057	0.096	0.081

TABLE XV

ε-normal distribution ($\varepsilon = 0.3$, $\sigma^2 = 0.1$)						
	$\alpha_0 = -0.2/\sqrt{2}$		$\alpha_0 = 0.0$		$\alpha_0 = 0.2/\sqrt{2}$	
N	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$
50	0.108	0.139	0.174	0.174	0.247	0.216
80	0.123	0.147	0.174	0.174	0.233	0.208
100	0.128	0.150	0.174	0.174	0.227	0.205

TABLE XVI

ε-normal distribution ($\varepsilon = 0.3$, $\sigma^2 = 0.5$)						
	$\alpha_0 = -0.2/\sqrt{2}$		$\alpha_0 = 0.0$		$\alpha_0 = 0.2/\sqrt{2}$	
N	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$
50	0.023	0.046	0.073	0.073	0.129	0.106
80	0.034	0.052	0.073	0.073	0.117	0.099
100	0.039	0.055	0.073	0.073	0.113	0.097

TABLE XVII

	ε-normal distribution ($\varepsilon = 0.5$, $\sigma^2 = 0.1$)					
	$\alpha_0 = -0.2/\sqrt{2}$		$\alpha_0 = 0.0$		$\alpha_0 = 0.2/\sqrt{2}$	
N	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$
50	0.276	0.303	0.332	0.332	0.395	0.368
80	0.288	0.310	0.332	0.332	0.383	0.361
100	0.294	0.313	0.332	0.332	0.377	0.358

TABLE XVIII

	ε-normal distribution ($\varepsilon = 0.5$, $\sigma^2 = 0.5$)					
	$\alpha_0 = -0.2/\sqrt{2}$		$\alpha_0 = 0.0$		$\alpha_0 = 0.2/\sqrt{2}$	
N	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$
50	0.028	0.055	0.087	0.087	0.152	0.125
80	0.041	0.062	0.087	0.087	0.139	0.118
100	0.046	0.065	0.087	0.087	0.134	0.115

TABLE XIX

	ε-normal distribution ($\varepsilon = 0.9$, $\sigma^2 = 0.1$)					
	$\alpha_0 = -0.2/\sqrt{2}$		$\alpha_0 = 0.0$		$\alpha_0 = 0.2/\sqrt{2}$	
N	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$
50	0.715	0.724	0.723	0.723	0.732	0.723
80	0.717	0.724	0.723	0.723	0.730	0.723
100	0.718	0.724	0.723	0.723	0.729	0.723

TABLE XX

	ε-normal distribution ($\varepsilon = 0.9$, $\sigma^2 = 0.5$)					
	$\alpha_0 = -0.2/\sqrt{2}$		$\alpha_0 = 0.0$		$\alpha_0 = 0.2/\sqrt{2}$	
N	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$	$p_{3,n}$	$p_{4,n}$
50	0.017	0.043	0.074	0.074	0.136	0.110
80	0.030	0.050	0.074	0.074	0.123	0.103
100	0.035	0.053	0.074	0.074	0.118	0.100

APPENDIX

In this section, we give the expressions for the constants $B_{ij}^{(1)}$, $B_{ij}^{(2)}$, $B_{ij}^{(3)}$, $B_{ij}^{(4)}$, appearing in Theorems 3.1 and 4.1. The quantities c_1 and c_2 appearing in these expressions are

$$c_1 = 1 + \kappa, \qquad c_2 = -\frac{\phi^{(3)}(0)}{(\phi'(0))^3}.$$

The expressions for the constants $B_{ij}^{(1)}$ and $B_{ij}^{(2)}$ are given by

$$\begin{aligned}
B_{11}^{(1)} = {} & \frac{p(p-5)\theta}{2\sqrt{2}} + \frac{\sqrt{2}}{2}\left\{\frac{A_2 + A_6}{a} + \frac{4A_3\theta - \theta(p^2 - p)(p\kappa + 2a)}{4b} - \frac{\kappa p\theta A_3}{2ab}\right\} \\
& - \frac{\sqrt{2}}{6}\left\{\frac{2c_2 A_2}{a^3} + \frac{c_2\kappa^2(a+2b)\, p^3\theta^3}{2a^3b^3} - \frac{3\kappa c_2 p\theta A_3}{a^3 b}\right. \\
& + \frac{3(c_2 - c_1)\, p\theta A_3}{2a^2 b} + \frac{(c_2 - 3c_1 + 2)\, p^3\theta^3}{4b^3} - 3A_7 p\theta \\
& \left. - \frac{3(c_2 - c_1)\,\kappa(a+b)\, p^3\theta^3}{4a^2b^3}\right\} - \frac{\sqrt{2}D_3\theta}{6} - \frac{\sqrt{2}D_4}{2} - \frac{\sqrt{2}D_5 a}{2b},
\end{aligned}$$

$$\begin{aligned}
B_{12}^{(1)} = {} & \frac{\sqrt{2}(6 + 3p - p^2)\theta}{4} \\
& - \frac{\sqrt{2}}{2}\left\{\frac{3\theta A_3}{a} + \frac{4A_3\theta - \theta(p^2 - p)(p\kappa + 2a)}{4b} - \frac{p\theta\kappa A_3 + 2\kappa p^2\theta^3}{2ab}\right\} \\
& + \frac{\sqrt{2}}{6}\left\{\frac{6c_2\theta A_3}{a^3} - \frac{3\kappa c_2 p\theta(A_3 + 2p\theta^2)}{a^3 b}\right. \\
& + \frac{3c_2\kappa^2(a+2b)\, p^3\theta^3}{2a^3b^3} + \frac{3(c_2 - 3c_1 + 2)\, p^3\theta^3}{4b^3} \\
& \left. + \frac{3(c_2 - c_1)\, p\theta(A_3 + 2p\theta^2)}{2a^2 b} - 3A_7 p\theta\right\} \\
& - \frac{\sqrt{2}D_2\theta}{6} + \frac{\sqrt{2}D_3\theta}{6} + \frac{\sqrt{2}D_5 a}{2b},
\end{aligned}$$

$$B_{13}^{(1)} = -\sqrt{2}\theta + \frac{\sqrt{2}}{2}\left\{\frac{3p\theta^3}{a} - \frac{3p^2\theta^3\kappa}{2ab} - \frac{p\theta^3}{3b}\right\} - \frac{\sqrt{2}}{6}\left\{\frac{6c_2 p\theta^3}{a^3} - \frac{9\kappa c_2 p^2\theta^3}{a^3 b}\right.$$

$$+\frac{3c_2\kappa^2(a+2b)\,p^3\theta^3}{2a^3b^3} + \frac{9(c_2-c_1)\,p^2\theta^3}{2a^2b}$$

$$\left.+\frac{3(c_2-3c_1+2)\,p^3\theta^3}{4b^3}\right\} + \frac{\sqrt{2}D_2\theta}{6},$$

$$B_{14}^{(1)} = \frac{\sqrt{2}}{2}\left\{\frac{\kappa p^2\theta^3}{2ab} - \frac{p\theta^3}{a} + \frac{p\theta^3}{3b}\right\}$$

$$+\frac{\sqrt{2}}{6}\left\{\frac{2c_2 p\theta^3}{a^3} - \frac{3\kappa c_2 p^2\theta^3}{a^3b} + \frac{c_2\kappa^2(a+2b)\,p^3\theta^3}{2a^3b^3}\right.$$

$$\left.+\frac{3(c_2-c_1)\,p^2\theta^3}{2a^2b} + \frac{(c_2-3c_1+2)\,p^3\theta^3}{4b^3}\right\},$$

$$B_{21}^{(1)} = \frac{3\sqrt{2}(p-1)\theta}{2} - \frac{\sqrt{2}}{2}\left\{\frac{4A_6}{a} + \frac{\kappa(p^2-p)\,a\theta - 2\kappa p\theta A_5}{2ab}\right\}$$

$$+\frac{\sqrt{2}}{6}\left\{\frac{6c_2A_6}{a^3} - \frac{6\kappa c_2 p\theta A_5}{a^3b} + \frac{3(c_2-c_1)\,p\theta A_5}{a^2b}\right.$$

$$\left.-\frac{9(c_2-c_1)\,\kappa(a+b)\,p^3\theta^3}{4a^2b^3}\right\} + \frac{\sqrt{2}D_4}{2} - \frac{\sqrt{2}D_1\theta}{6},$$

$$B_{22}^{(1)} = -\sqrt{2}(p-1)\theta + \frac{\sqrt{2}}{2}\left\{\frac{6\theta A_5}{a} + \frac{(p^2-p)\theta\kappa}{2b} - \frac{p\theta\kappa A_5}{ab}\right\}$$

$$-\frac{\sqrt{2}}{6}\left\{\frac{12c_2\theta A_5}{a^3} - \frac{6\kappa c_2 p\theta A_5}{a^3b} + \frac{3(c_2-c_1)\,p\theta A_5}{a^2b}\right\} + \frac{\sqrt{2}D_1\theta}{6},$$

$$B_{31}^{(1)} = \sqrt{2}\left\{\frac{3A_4}{2a} - \frac{c_2A_4}{a^3} + \frac{\kappa c_2\theta pA_5}{2a^3b} - \frac{(c_2-c_1)\,p\theta A_5}{4a^2b}\right.$$

$$\left.-\frac{A_1}{6a} - \frac{\theta A_5}{2b} - \frac{p\kappa\theta A_5}{4ab}\right\} + \frac{3\sqrt{2}(c_2-c_1)\,\kappa(a+b)\,p^3\theta^3}{8a^2b^3},$$

$$B_{32}^{(1)} = \frac{\sqrt{2}\theta A_5}{4a^2b}\{4c_2 + c_2p - c_1p - 4a^2 - 2ap\kappa\},$$

$$B_{41}^{(1)} = \frac{\sqrt{2}A_1}{3}\left\{\frac{c_2}{a^3} - \frac{1}{a}\right\} - \frac{\sqrt{2}(c_2 - c_1)\,\kappa(a+b)\,p^3\theta^3}{8a^2b^3},$$

$$B_{11}^{(2)} = B_{11}^{(1)}, \qquad B_{12}^{(2)} = B_{12}^{(1)} - \sqrt{2}p\theta, \qquad B_{13}^{(2)} = B_{13}^{(1)} + \sqrt{2}\theta - \frac{\sqrt{2}p\theta^3}{3b},$$

$$B_{14}^{(2)} = B_{14}^{(1)} + \frac{\sqrt{2}p\theta^3}{3b}, \qquad B_{21}^{(2)} = B_{21}^{(1)}, \qquad B_{22}^{(2)} = B_{22}^{(1)} + \sqrt{2}(p-1)\theta,$$

$$B_{31}^{(2)} = B_{31}^{(1)} - \frac{\sqrt{2}A_1}{3a} - \frac{\sqrt{2}\theta A_5}{b},$$

$$B_{32}^{(2)} = B_{32}^{(1)} + \frac{\sqrt{2}\theta A_5}{b}, \qquad B_{41}^{(2)} = B_{41}^{(1)} + \frac{\sqrt{2}A_1}{3a},$$

where

$$A_1 = \sum_{i=1}^{p} (\theta_i - \theta)^3, \qquad A_2 = \sum_{i=1}^{p} \theta_i^3, \qquad A_3 = \sum_{i=1}^{p} \theta_i^2,$$

$$A_4 = \sum_{i=1}^{p} (\theta_i - \theta)^2\,\theta_i, \qquad A_5 = \sum_{i=1}^{p} (\theta_i - \theta)^2, \qquad A_6 = \sum_{i=1}^{p} (\theta_i - \theta)\,\theta_i^2,$$

$$A_7 = \frac{\{(8c_2 - 4c_1) + p(6c_2 - 4c_1 + 2) + p^2(c_2 - c_1)\}}{4ab}$$

$$- \frac{\kappa\{8c_2 + 6p(c_2 - c_1) + p^2(c_2 - 3c_1 + 2)\}}{8ab^2},$$

$$D_1 = \frac{3\{p^2(c_2 - c_1) + p(3c_2 + c_1) - 4c_2\}}{2ab},$$

$$D_2 = \frac{3\{p^2(c_2 - 3c_1 + 1) + p(6c_2 - 6c_1 + 2) + (9c_2 - 1)\}}{4b^2},$$

$$D_3 = \frac{3p\{p^2(c_2 - c_1) + p(3c_2 + c_1) - 4c_2\}}{4ab},$$

$$D_4 = \sum_{i=1}^{p} \sum_{j=i+1}^{p} \frac{\lambda_{j0}(\theta_i - \theta)}{\lambda_{i0} - \lambda_{j0}}, \qquad D_5 = \theta \sum_{i=1}^{p} \sum_{j=i+1}^{p} \frac{\lambda_{j0}}{\lambda_{i0} - \lambda_{j0}}.$$

Note that $\sum_{i=1}^{4} \sum_{j=1}^{n_i} B_{ij}^{(1)} = \sum_{i=1}^{4} \sum_{j=1}^{n_i} B_{ij}^{(2)} = 0.$

The expressions for the constants $B_{ij}^{(3)}$ and $B_{ij}^{(4)}$ are given by

$$B_{11}^{(3)} = -\frac{\sqrt{2}(p+1)p\theta}{2} + \frac{\sqrt{2}}{2}\left\{\frac{A_2 + A_6 + 5A_7 + 6A_8 + 12A_9}{a}\right.$$

$$\left. + \frac{A_7 + \theta A_3}{b} - \frac{p\kappa(A_7 + \theta A_3)}{2ab}\right\} - \frac{\sqrt{2}}{6}\left\{\frac{c_2\kappa^2(a+2b)p^3\theta^3}{2a^3b^3}\right.$$

$$+ \frac{4c_2(A_2 + 3A_7 + 3A_8 + 6A_9) + 3(c_2 - c_1)pA_7}{2a^3}$$

$$+ \frac{3(c_2 - c_1)p\theta A_3}{2a^2b} - \frac{12\kappa c_2 p(\theta A_3 + A_7) + 3(c_2 - c_1)p^2\kappa A_7}{4a^3b}$$

$$\left. + \frac{(c_2 - 3c_1 + 2)p^3\theta^3}{4b^3} - \frac{3(c_2 - c_1)\kappa(a+b)p^3\theta^3}{4a^2b^3} - 3A_{10}p\theta\right\},$$

$$B_{12}^{(3)} = \frac{\sqrt{2}(p^2 - 7p + 6)\theta}{4}$$

$$- \frac{\sqrt{2}}{2}\left\{\frac{3\theta A_3 + 3A_7}{a} - \frac{p\kappa(A_7 + \theta A_3 + 2p\theta^3)}{2ab} + \frac{A_7 + \theta A_3}{b}\right\}$$

$$+ \frac{\sqrt{2}}{6}\left\{\frac{12c_2(\theta A_3 + A_7) + 3(c_2 - c_1)pA_7}{2a^3}\right.$$

$$+ \frac{3(c_2 - c_1)p\theta(A_3 + 2p\theta^2)}{2a^2b} + \frac{3c_2\kappa^2(a+2b)p^3\theta^3}{2a^3b^3}$$

$$- \frac{12c_2 p\kappa(\theta A_3 + A_7 + 2p\theta^3) + 3(c_2 - c_1)p^2\kappa A_7}{4a^3b}$$

$$\left. + \frac{3(c_2 - 3c_1 + 2)p^3\theta^3}{4b^3} - D_2\theta - 3A_{10}p\theta\right\},$$

$$B_{13}^{(3)} = \sqrt{2}\theta + \frac{\sqrt{2}}{2}\left\{\frac{3p\theta^3}{a} - \frac{3\kappa p^2\theta^3}{2ab} - \frac{5p\theta^3}{3b}\right\}$$

$$- \frac{\sqrt{2}}{6}\left\{\frac{6c_2p\theta^3}{a^3} - \frac{9\kappa c_2 p^2\theta^3}{a^3b} + \frac{3c_2\kappa^2(a+2b)p^3\theta^3}{2a^3b^3}\right.$$

$$\left. + \frac{9(c_2 - c_1)p^2\theta^3}{2a^2b} + \frac{3(c_2 - 3c_1 + 2)p^3\theta^3}{4b^3} - D_2\theta\right\},$$

$$B_{14}^{(3)} = -\frac{\sqrt{2}}{2}\left\{\frac{p\theta^3}{a} - \frac{\kappa p^2\theta^3}{2ab} - \frac{5p\theta^3}{3b}\right\}$$

$$+\frac{\sqrt{2}}{6}\left\{\frac{2c_2 p\theta^3}{a^3} - \frac{3\kappa c_2 p^2\theta^3}{a^3 b} + \frac{c_2\kappa^2(a+2b)\,p^3\theta^3}{2a^3b^3}\right.$$

$$\left. + \frac{3(c_2-c_1)\,p^2\theta^3}{2a^2b} + \frac{(c_2-3c_1+2)\,p^3\theta^3}{4b^3}\right\},$$

$$B_{21}^{(3)} = \frac{\sqrt{2}(p^2+5p-6)\theta}{4}$$

$$-\frac{\sqrt{2}}{2}\left\{\frac{4(A_6+2A_7+3A_8+6A_9)}{a} + \frac{\theta\kappa(p^3+p^2-2p)}{4b}\right.$$

$$\left. - \frac{\kappa p(\theta A_5 + A_7)}{ab} - \frac{a(p^2-p)\theta}{2}\right\}$$

$$-\frac{\sqrt{2}}{6}\left\{\frac{9(c_2-c_1)\,\kappa(a+b)\,p^3\theta^3}{4a^2b^3}\right.$$

$$- \frac{6c_2(A_6+2A_7+3A_8+6A_9) + 3(c_2-c_1)\,pA_7}{a^3} - \frac{3(c_2-c_1)\,p\theta A_5}{a^2 b}$$

$$\left. + \frac{12\kappa c_2\, p(\theta A_5 + A_7) + 3(c_2-c_1)\,p^2\kappa A_7}{2a^3 b} + D_1\theta + D_3\theta\right\}$$

$$-\frac{\sqrt{2}\theta a p(p-1)}{b},$$

$$B_{22}^{(3)} = \sqrt{2}(p-1)\theta + \frac{\sqrt{2}}{2}\left\{\frac{6(\theta A_5 + A_7)}{a} + \frac{\theta\kappa(p^3+p^2-2p)}{4b}\right.$$

$$\left. - \frac{p\kappa(A_7+\theta A_5)}{ab} - \frac{a(p^2-p)\theta}{2}\right\}$$

$$-\frac{\sqrt{2}}{6}\left\{\frac{12c_2(\theta A_5 + A_7) + 3(c_2-c_1)\,pA_7}{a^3} + \frac{3(c_2-c_1)\,p\theta A_5}{a^2 b}\right.$$

$$\left. - \frac{12p\kappa c_2(\theta A_5 + A_7) + 3(c_2-c_1)\,p^2\kappa A_7}{2a^3 b} - D_1\theta - D_3\theta\right\}$$

$$+\frac{\sqrt{2}\theta a p(p-1)}{b},$$

$$B_{31}^{(3)} = \frac{\sqrt{2}}{2}\left\{\frac{9A_4+9A_7+12A_8+24A_9-5A_1}{3a}\right.$$

$$\left. -\frac{5(\theta A_5+A_7)}{b}-\frac{p\kappa(\theta A_5+A_7)}{2ab}\right\}$$

$$+\frac{\sqrt{2}}{6}\left\{\frac{9(c_2-c_1)\,\kappa(a+b)\,p^3\theta^3}{4a^2b^3}\right.$$

$$+\frac{12\kappa c_2\,p(\theta A_5+A_7)+3(c_2-c_1)\,p^2\kappa A_7}{4a^3b}$$

$$-\frac{12c_2(A_4+A_7+3A_8+6A_9)+3(c_2-c_1)\,pA_7}{2a^3}$$

$$\left.-\frac{3(c_2-c_1)\,p\theta A_5}{2a^2b}\right\},$$

$$B_{32}^{(3)} = -\frac{\sqrt{2}}{2}\left\{\frac{3(\theta A_5+A_7)}{a}-\frac{5(\theta A_5+A_7)}{b}-\frac{p\kappa(\theta A_5+A_7)}{2ab}\right\}$$

$$+\frac{\sqrt{2}}{6}\left\{\frac{3(c_2-c_1)\,p\theta A_5}{2a^2b}+\frac{12c_2(\theta A_5+A_7)+3(c_2-c_1)\,pA_7}{2a^3}\right.$$

$$\left.-\frac{12\kappa c_2\,p(\theta A_5+A_7)+3(c_2-c_1)\,p^2\kappa A_7}{4a^3b}\right\},$$

$$B_{41}^{(3)} = \frac{\sqrt{2}(A_1+3A_8+6A_9)}{3a}+\frac{\sqrt{2}c_2(A_1+3A_8+6A_9)}{3a^3}$$

$$-\frac{\sqrt{2}(c_2-c_1)\,\kappa(a+b)\,p^3\theta^3}{8a^2b^3},$$

$$B_{11}^{(4)} = B_{11}^{(3)}, \qquad B_{12}^{(4)} = B_{12}^{(3)}+\sqrt{2}p\theta, \qquad B_{13}^{(4)} = B_{13}^{(3)}-\sqrt{2}\theta+\frac{\sqrt{2}p\theta^3}{3b},$$

$$B_{14}^{(4)} = B_{14}^{(3)}-\frac{\sqrt{2}p\theta^3}{3b}, \qquad B_{21}^{(4)} = B_{21}^{(3)}+\frac{\sqrt{2}\theta ap(p-1)}{2b},$$

$$B_{22}^{(4)} = B_{22}^{(3)}-\frac{\sqrt{2}\theta(p-1)(pa+2b)}{2b},$$

$$B_{31}^{(4)} = B_{31}^{(3)}+\sqrt{2}\left\{\frac{A_8+2A_9}{a}+\frac{A_5\theta+A_7}{b}+\frac{A_1}{3a}\right\},$$

$$B_{32}^{(4)} = B_{32}^{(3)}-\frac{\sqrt{2}(A_5\theta+A_7)}{b}, \qquad B_{41}^{(4)} = B_{41}^{(3)}-\frac{\sqrt{2}(A_1+3A_8+6A_9)}{3a},$$

where

$$A_1=\sum_{i=1}^{p}(\theta_{ii}-\theta)^3,\qquad A_2=\sum_{i=1}^{p}\theta_{ii}^3,$$

$$A_3=\sum_{i=1}^{p}\theta_{ii}^2,\qquad A_4=\sum_{i=1}^{p}(\theta_{ii}-\theta)^2\theta_{ii},$$

$$A_5=\sum_{i=1}^{p}(\theta_{ii}-\theta)^2,\qquad A_6=\sum_{i=1}^{p}(\theta_{ii}-\theta)\,\theta_{ii}^2,\qquad A_7=2\theta\sum_{i=1}^{p}\sum_{j=i+1}^{p}\theta_{ij}^2,$$

$$A_8=\sum_{i=1}^{p}\sum_{j=i+1}^{p}\theta_{ij}^2(\theta_{ii}+\theta_{jj}-2\theta),\qquad A_9=\sum_{i=1}^{p}\sum_{j=i+1}^{p}\sum_{k=j+1}^{p}\theta_{ij}\,\theta_{ik}\theta_{jk},$$

$$A_{10}=\frac{\{(8c_2-4c_1)+p(6c_2-4c_1+2)+p^2(c_2-c_1)\}}{4ab}$$

$$-\frac{\kappa\{8c_2+6p(c_2-c_1)+p^2(c_2-3c_1+2)\}}{8ab^2},$$

$$D_1=\frac{3\{p^2(c_2-c_1)+p(3c_2+c_1)-4c_2\}}{2ab},$$

$$D_2=\frac{3\{p^2(c_2-3c_1+2)+p(6c_2-6c_1+1)+(9c_2-1)\}}{4b^2},$$

$$D_3=\frac{3p\{p^2(c_2-c_1)+p(3c_2+c_1)-4c_2\}}{4ab}.$$

Note that $\sum_{i=1}^{4}\sum_{j=1}^{n_i}B_{ij}^{(3)}=\sum_{i=1}^{4}\sum_{j=1}^{n_i}B_{ij}^{(4)}=0$.

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REFERENCES

[1] CHAN, Y. M., AND SRIVASTAVA, M. S. (1988). Comparison of powers for the sphericity tests using both the asymptotic distribution and the bootstrap method. *Comm. Statist.* **17** 671–690.

[2] FANG, K. T., AND ZHANG, Y. T. (1990). *Generalized Multivariate Analysis.* Springer-Verlag, Berlin.

[3] FUJIKOSHI, Y. (1980). Asymptotic expansions for the distributions of the sample roots under nonnormality. *Biometrika* **67** 45–51.

[4] HAYAKAWA, T. (1986). On testing hypotheses of covariance matrices under an elliptical population. *J. Statist. Plann. Inference* **13** 193–202.

[5] HAYAKAWA, T., AND PURI, M. L. (1985). Asymptotic distributions of likelihood ratio criterion for testing latent roots and latent vectors of a covariance matrix under an elliptical population. *Biometrika* **75** 331–338.

[6] HILL, G. W., AND DAVIS, A. W. (1968). Generalized asymptotic expansions of Cornish Fisher type. *Ann. Math. Statist.* **39** 1264–1273.

[7] KELKER, D. (1970). Distribution theory of spherical distributions and a location-scale parameter generalization. *Sankhyā A* **32** 419–430.

[8] KHATRI, C. G., AND SRIVASTAVA, M. S. (1974). Asymptotic expansions of the non-null distributions of likelihood ratio criteria for covariance matrices. *Ann. Statist.* **2** 109–117.

[9] KHATRI, C. G., AND SRIVASTAVA, M. S. (1976). Asymptotic expansions of the non-null distributions of the likelihood ratio criteria for covariances II. *Metron* **36** 55–71.

[10] MUIRHEAD, R. J., AND WATERNAUX, C. M. (1980) Asymptotic distributions in canonical correlation analysis and other multivariate procedures for nonnormal populations. *Biometrika* **67** 31–43.

[11] NAGAO, H. (1973). On some test criteria for covariance matrix. *Ann. Statist.* **1** 700–709.

[12] NAGAO, H. (1993). Asymptotic expansions of some test criteria for sphericity and independence under local alternatives from an elliptical population. *Math. Japonica* **38** 165–170.

[13] NAGAO, H., AND SRIVASTAVA, M. S. (1992). On the distributions of some test criteria for a covariance matrix under local alternatives and bootstrap approximations. *J. Multivariate Anal.* **43** 331–350.

[14] RAO, C. R. (1948). Large sample tests of statistical hypotheses concerning several parameters with applications to problems of estimation. *Proc. Cambridge Philos. Soc.* **44** 50–57.

[15] SRIVASTAVA, M. S. (1984). Estimation of interclass correlations in familial data. *Biometrika* **71** 177–185.

[16] SRIVASTAVA, M. S. (1987). Tests for covariance structure in familial data and principal component. In *Advances in Multivariate Statistical Analysis* (A. K. Gupta, Ed.), pp. 341–352. Reidel, Dordrecht.

[17] SRIVASTAVA, M. S., AND KHATRI, C. G. (1979). *An Introduction to Multivariate Statistics*. North Holland, New York.

[18] SUGIURA, N. (1973). Derivatives of the characteristic root of a symmetric or a Hermitian matrix with two applications in multivariate analysis. *Comm. Statist.* **1** 393–417.

[19] WITHERS, C. S. (1984) Asymptotic expansions for distributions and quantiles with power series cumulants. *J. Roy. Statist. Soc. Ser. B* **46** 389–396.