Journal of Multivariate Analysis 100 (2009) 1498–1520

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

# Use of prior information in the consistent estimation of regression coefficients in measurement error models

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ARTICLE INFO

*Article history:*
Received 12 March 2007
Available online 31 December 2008

*AMS subject classifications:*
62J05
62H12

*Keywords:*
Measurement errors
Exact linear restriction
Ultrastructural model
Reliability matrix
Löwner ordering

ABSTRACT

A multivariate ultrastructural measurement error model is considered and it is assumed that some prior information is available in the form of exact linear restrictions on regression coefficients. Using the prior information along with the additional knowledge of covariance matrix of measurement errors associated with explanatory vector and reliability matrix, we have proposed three methodologies to construct the consistent estimators which also satisfy the given linear restrictions. Asymptotic distribution of these estimators is derived when measurement errors and random error component are not necessarily normally distributed. Dominance conditions for the superiority of one estimator over the other under the criterion of Löwner ordering are obtained for each case of the additional information. Some conditions are also proposed under which the use of a particular type of information will give a more efficient estimator.



## 1. Introduction

A basic assumption in most of the statistical analysis is that the observations are recorded error-free. In practice, this assumption is often violated and measurement errors creep into the observations. These measurement errors make the results invalid, which are meant for no measurement error case. For example, the ordinary least squares estimator (OLSE) has minimum variance in the class of linear and unbiased estimators in the case of linear regression analysis. The same OLSE becomes inconsistent as well as biased in the presence of measurement errors in the data. It is well known that some additional information from outside the sample is needed to obtain the consistent estimators of regression coefficients. This additional information could be available in different forms in a univariate measurement error model — e.g., either of the measurement error variances associated with explanatory variable or study variable is known, the ratio of measurement error variances or reliability ratio is known etc. See, for example, [1–3] for more details.

In many situations, some prior information on regression coefficients is available which can be used to improve upon the OLSE. When such prior information can be expressed in the form of exact linear restrictions binding the regression coefficients, the restricted least squares estimator (RLSE) is used. The RLSE is unbiased, consistent, satisfies the given linear restrictions on regression coefficients and has smaller variability around mean than the OLSE when there is no measurement error in the data — see [4,5]. However, the RLSE becomes inconsistent and biased when the observations are contaminated with measurement errors. The problem of finding the estimators which are consistent as well as which satisfy the linear restrictions in the presence of measurement errors in the data, is addressed in this paper. An iterative procedure for obtaining the estimators under restrictions using total least squares approach is discussed in [6].

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doi:10.1016/j.jmva.2008.12.014

We have considered a multivariate ultrastructural model – see [7] – for the modeling of measurement errors. The measurement errors and the random error component are assumed to be normally distributed in most of the literature. This assumption may not always hold true in many cases. So we do not assume any distributional form of the measurement errors and the random error component. Only the existence and finiteness of the first four moments of measurement errors and random error component is assumed — see, for example, [8,9] for non-normality effect in a univariate ultrastructural model.

Among the different choices available for the additional information to construct the consistent estimators, the use of covariance matrix of measurement errors and reliability matrix associated with explanatory variables, are the popular strategies in case of a multivariate measurement error model — see, for example, [10–14]. So we make use of these two types of information, along with the prior information to construct the estimators of regression coefficients that are consistent and satisfy the given linear restrictions. The large sample asymptotic approximation theory is employed to derive the asymptotic distribution of proposed estimators under the specification of an ultrastructural model. The covariance matrices of asymptotic distribution of estimators are compared under the specification of each type of additional information using the criterion of Löwner ordering. When more than one type of additional information is available and can be used to construct the consistent estimators, then the choice of estimators depends on the type of additional information used — see [15] for more details in a univariate ultrastructural model. An attempt is made in this paper to explore which of the two types of additional information yields more efficient estimators. The estimators under the two types of information are compared, and the dominance conditions for the superiority of one estimator over the other are derived under the criterion of Löwner ordering. The effect of departure from the normal distribution of measurement errors and random error component on the asymptotic distribution of the proposed estimators is also studied. A Monte-Carlo simulation experiment is conducted to study the finite sample properties of the estimators, and the effect of departure from normality of the measurement errors on them is also studied.

We would like to discuss the idea of structural equation model (SEM) of which the measurement error models are a particular case. The SEM is used for testing and estimating the causal relationships based on latent variables (indirectly measured variables). The SEM consists of several equations representing the relationships between several endogenous and exogenous variables. The area of SEM has seen an extensive growth in the last three decades and SEM has more advantages than multiple regression models — e.g. SEM has more flexible assumptions, use of confirmatory factor analysis to reduce the measurement error by having multiple indicators per latent variable, ability to test models with multiple dependents, better handling of time series data, autocorrelated data, non-normal data, missing data etc. These are a few of the advantages, among others. Lee [16] discussed the estimation of regression parameters under functional constraints on parameters in SEM by using the generalized least squares estimation — see also [17]. In order to obtain the estimators, the penalty function method [18,19 (p. 156)] is utilized and an algorithm is proposed to achieve this, followed by a numerical example. Further, Lee and Poon [20] extended the generalized least squares estimation of regression parameters in SEM under a set of inequality and equality functional restrictions, with a different approach. They proposed to fit the model by minimizing the discrepancy between the covariance matrices of regression coefficients based on the data with and without measurement errors. They have not presented any clear analytic form of the estimators but proposed an algorithm to minimize such difference. This optimization approach yields the maximum likelihood estimators when the weight matrix is changed with the parameters from iteration to iteration. This procedure yields more efficient estimators than those obtained by penalty function approach. The efficiency properties of the estimator are not derived analytically, but are analyzed numerically. Lee [21] discussed an issue of testing the validity of restrictions in SEM in the context of robust generalized least squares approach. A new test statistic is proposed which is based on minimizing the discrepancy between the covariance matrices of an identified model and sample covariance matrix. This method is claimed to be less computationally expensive than earlier available methods. The popular software for doing SEM are LISREL (LInear Structural RELations), AMOS (Analysis of MOment Structure), EQS, MPLUS etc. For example, the *CO* command in LISREL model is used to specify a parameter or an additional parameter to be a function of other parameters of the LISREL model, and the *CALIS* command in SAS does a similar job but is based on the multivariate normal distribution. The SEM is a complete path model and allows the path analysis by constructing the path diagram. It can also handle the problem of identification in the estimation of parameters, which is one of the crucial problem in measurement error models. The simultaneous equation models can be viewed as a particular case of SEM (see [1, p. 196]) and has been analyzed in the presence of measurement errors in the literature — see, for example, [22–24]. For structural equation models in measurement error case, see [25]. The LISREL can also handle the non-normality effect. The papers by Anderson [26,27] study the functional and structural linear relationships, demonstrate their relationship with SEM, and establish the asymptotic normality of the estimators. The approaches of SEM and measurement error models, in handling the identifiability problem, are not exactly the same. The SEM has a more general approach by putting various types of restrictions on the parameters to make the model identifiable. Use of different types of additional information yields different estimators in measurement error models. Goldberger [28–30] demonstrated that the errors of measurement need not destroy identifiability, provided that the model is otherwise over-identified. In fact, one can trade-off over-identifying restrictions against the under-identifiability introduced by measurement error. The multivariate measurement error model considered in this paper is a particular case of SEM, in the sense that it takes into account only one structural relationship between one endogenous and several exogenous variables under the influence of measurement errors in the observations. This has been demonstrated briefly in Section 2. A complete bibliography of SEM is out of the objectives of present work. More interested readers are referred to an excellent exposition to SEM under measurement errors in [1, chapter 8] and

references cited therein. In the works presented by, for example, Lee [16,21], Lee and Bentler [17] and Lee and Poon [20], a general approach of constructing estimators based on generalized least squares is presented, and these approaches are implemented through different approaches of numerical algorithms. No clear analytic form of the estimators is derived. Since any assumption about the distributional form of either random error, measurement error or any other type of random error is not needed for the estimation of regression parameters in the generalized least squares estimation, so the issues related to the effect of departure from the normal distribution of either the measurement errors or any other types of errors cannot be analytically ascertained. The numerical values of the estimators can only tell the values but do not give any idea about the nature of effect of departure from normality. Other issues like clear analytic form of the estimators, role and effect of choice of weight matrix for the generalized least squares estimation etc,. are not discussed in the literature. This work is an attempt, in this direction, to address these issues. The setup of ultrastructural model which can be expressed in the form of SEM is considered. Some other approaches which differ from the approaches presented in, for example, [16,17,20,21] are presented. These approaches are essentially governed by the choice of weight matrices and provide clear analytic forms of the estimators. Moreover, they do not need any algorithm to estimate the regression parameters. Since the analytical form of the estimators is clear, so the asymptotic covariance matrices clearly pronounces the effect of departure from normality of the distribution of measurement errors.

The plan of the paper is as follows. The multivariate ultrastructural model, exact linear restrictions on regression coefficients and various statistical assumptions are described in Section 2. In Section 3, the construction of consistent estimators which satisfy the given restrictions, is presented. The asymptotic distributions and dominance conditions of the estimators are given in Section 4. The findings from the Monte-Carlo simulation experiment are presented in Section 5. Section 6 contains the concluding remarks followed by the derivation of results in Appendix.

## 2. The ultrastructural model and prior information

Let a $n \times 1$ vector of observations on study variable $\eta$ and a $n \times p$ matrix $T = (\xi_{ij})$ , $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, p)$ of $n$ observations on each of the $p$ explanatory variables are related by

$$\eta = T\beta, \tag{2.1}$$

where $\beta$ is a $p \times 1$ vector of unknown regression coefficients. Suppose that the variables $\eta$ and $T$ are unobservable and can only be observed with a $n \times 1$ vector of measurement errors $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)'$ and a $n \times p$ matrix of measurement errors $\Delta = (\delta_{ij})$ as

$$y = \eta + \epsilon \quad \text{and} \quad X = T + \Delta, \tag{2.2}$$

where $y$ and $X$ are observed values of $\eta$ and $T$ respectively.

Further, assume that $\xi_{ij}$'s are randomly distributed with mean $\mu_{ij}$ and random disturbance $\phi_{ij}$, $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, p)$. Let $M = (\mu_{ij})$ and $\Phi = (\phi_{ij})$ be the $n \times p$ matrices. Then we can express

$$T = M + \Phi. \tag{2.3}$$

We assume that $\delta_{ij}$, $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, p)$ are independent and identically distributed random variables with mean 0, variance $\sigma_\delta^2$, third moment $\gamma_{1\delta}\sigma_\delta^3$ and fourth moment $(\gamma_{2\delta} + 3)\sigma_\delta^4$. Similarly, $\phi_{ij}$, $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, p)$ are assumed to be independent and identically distributed, with first four finite moments given by 0, $\sigma_\phi^2$, $\gamma_{1\phi}\sigma_\phi^3$ and $(\gamma_{2\phi} + 3)\sigma_\phi^4$ respectively. Likewise, assume that $\epsilon_i$, $(i = 1, 2, \ldots, n)$ are independent and identically distributed with first four finite moments given by 0, $\sigma_\epsilon^2$, $\gamma_{1\epsilon}\sigma_\epsilon^3$ and $(\gamma_{2\epsilon} + 3)\sigma_\epsilon^4$ respectively. Here, for a random variable $Z$, $\gamma_{1Z}$ and $\gamma_{2Z}$ denote the Pearson's coefficients of skewness and kurtosis of the random variable $Z$. Further, $\epsilon$, $\Delta$ and $\Phi$ are also assumed to be statistically independent. These specifications can be relaxed at the cost of slight algebraic complexity but without any conceptual difficulty; see, for example, [10].

The Eqs. (2.1)–(2.3) describe the set up of an ultrastructural model – see [7,31]. The structural and functional forms of measurement error model as well as the classical regression model can be obtained as its particular cases. When all the row vectors of $M$ are assumed to be identical, implying that rows of $X$ are random and independent, having some multivariate distribution, we get the specification of a structural model. When $\Phi$ is taken identically equal to a null matrix, implying that $\sigma_\phi^2 = 0$ and consequently that the matrix $X$ is fixed but is measured with error, we obtain the specification of a functional model. When both $\Delta$ and $\Phi$ are identically equal to a null matrix, implying that $\sigma_\phi^2 = \sigma_\delta^2 = 0$ and consequently that $X$ is fixed and is measured without any measurement error, we get the classical regression model. Thus the ultrastructural model provides a general framework for the study of three interesting models in a unified manner.

Suppose some prior information about the regression coefficients is available. Such prior information can be available from different sources, for example, from some extraneous sources, similar kinds of experiments conducted in the past, long association of the experimenter with the experiment etc. Assume that such information can be expressed in the form of $J$ $(<p)$ exact linear restrictions, binding the regression coefficients as

$$r = R\beta, \tag{2.4}$$

where $r$ is a $J \times 1$ known vector and $R$ is a $J \times p$ known full row rank matrix.

We also assume that $\lim_{n\longrightarrow\infty}(\mu_{n1}, \mu_{n2}, \ldots, \mu_{np})' =: \boldsymbol{\sigma}_\mu$. This assumption implies that $\lim_{n\longrightarrow\infty} \frac{1}{n}M'M = \boldsymbol{\sigma}_\mu\boldsymbol{\sigma}_\mu' = \Sigma_\mu$(say) and $\lim_{n\longrightarrow\infty} \frac{1}{n}M'e_n = \boldsymbol{\sigma}_\mu$, where $e_n$ is a $n \times 1$ vector of elements unity. This assumption is needed to avoid the presence of any trend in the observations — see [32] and for the existence of the asymptotic distribution of the estimators.

For the sake of completeness, we present, now, how the measurement error model can be viewed in the set-up of structural equation model. We consider, here, the LISREL model in standard notations from [1, chapter 8, p. 194]. We have used asterisk sign $(*)$ additionally in the superscript in specifying the model in (2.5)–(2.7), and different variables from the standard LISREL notations just to distinguish between the variables used in this paper and variables under LISREL. The quantities inside the brackets represent the dimension of the vectors. The $i$th set of LISREL equations connecting the observations on endogenous and exogenous variables is

$$\eta_i^* = B^*\eta_i^* + \Gamma^*\xi_i^* + \zeta_i^* \quad (i = 1, 2, \ldots, n) \tag{2.5}$$

$$y_i^* = \Lambda_y^*\eta_i^* + \epsilon_i^* \tag{2.6}$$

$$x_i^* = \Lambda_x^*\xi_i^* + \delta_i^* \tag{2.7}$$

where $\eta_i^*(m^* \times 1)$ and $\xi_i^*(n^* \times 1)$ are the vectors of true but unobserved endogenous and exogenous variables, respectively, with $y_i^*(p^* \times 1)$ and $x_i^*(q^* \times 1)$ being the respective corresponding observed vector values, $\zeta_i^*(m^* \times 1)$ is the vector of usual disturbance term from regression equation, and $\epsilon_i^*(p^* \times 1)$ and $\delta_i^*(q^* \times 1)$ are the corresponding vectors of measurement errors. The matrices of regression coefficients are $B^*$ and $\Gamma^*$ whereas the matrices of factor loadings are $\Lambda_x^*$ and $\Lambda_y^*$. The random vectors $\xi_i^*, \zeta_i^*, \epsilon_i^*$ and $\delta_i^*$ are assumed to be mutually uncorrelated with mean zero and covariance matrices $\Phi^*, \Psi^*$, $\Theta_\epsilon^*$ and $\Theta_\delta^*$, respectively.

The ultrastructural model (2.1)–(2.3) becomes a particular case of LISREL in (2.5)–(2.7) by specifying $\eta_i^* = \eta_i$, $\xi_i^* = T_i$, $y_i^* = y_i$, $x_i^* = X_i$, $\zeta_i^* = 0$ (due to no error in equation model), $\epsilon_i^* = \epsilon_i$, $\delta_i^* = \delta_i$, $B^* = 0$, $\Gamma^* = \beta'$, $\Lambda_x^* = I_p$, $\Lambda_y^* = 1$, $\Phi^*\Sigma_T := \frac{1}{n}M'M + \sigma_\phi^2 I$, $\Psi^* = 0$, $\Theta_\epsilon^* = \sigma_\epsilon^2$, $\Theta_\delta^* = \sigma_\delta^2 I_p$, $p^* = 1$, $m^* = 1$, $q^* = p$ and $n^* = p$. We have additionally assumed the existence and finiteness of third and fourth order moments of $\delta_i$, $\epsilon_i$ and $\phi_i$.

## 3. Consistent estimation of parameters

First we state some basic definitions that are needed to develop the construction of consistent estimators which also satisfy the linear restrictions.

For a matrix $B$, let $(B)_{ij}$ denote the $(i, j)$th element of the matrix $B$.

**Definition 1.** Let $\{A_n : n = 1, 2, \ldots\}$ be a sequence of random matrices and let $\{b_n : n = 1, 2, \ldots\}$ be a sequence of real numbers. We say that: (i) $A_n = O_P(b_n)$ $(A_n = o_P(b_n))$ if every element of the random matrix $A_n$ is $O_P(b_n)$ $(o_P(b_n))$, and (ii) plim $A_n = A$ if plim $(A_n)_{ij} = A_{ij}\ \forall\, i, j$ where plim denotes the probability in limit.

Now we present some results in the following lemma which are used later in this paper.

**Lemma 1.** *As $n \to \infty$,*

(i) $\frac{1}{\sqrt{n}}M'\Phi = O_P(1)$, $\frac{1}{\sqrt{n}}M'\Delta = O_P(1)$, $\frac{1}{\sqrt{n}}M'\epsilon = O_P(1)$,

(ii) $\frac{1}{\sqrt{n}}\Phi'\Delta = O_P(1)$, $\frac{1}{\sqrt{n}}\Delta'\epsilon = O_P(1)$, $\frac{1}{\sqrt{n}}\Phi'\epsilon = O_P(1)$,

(iii) $\frac{1}{\sqrt{n}}\Delta'\Delta - \sqrt{n}\sigma_\delta^2 I_p = O_P(1)$, $\frac{1}{\sqrt{n}}\Phi'\Phi - \sqrt{n}\sigma_\phi^2 I_p = O_P(1)$,

(iv) $\text{plim}(\frac{1}{n}\Delta'\Delta) = \sigma_\delta^2 I_p$, $\text{plim}(\frac{1}{n}\Phi'\Phi) = \sigma_\phi^2 I_p$,

(v) $\text{plim}(\frac{1}{n}M'\Delta) = \text{plim}(\frac{1}{n}M'\Phi) = \text{plim}(\frac{1}{n}\Phi'\Delta) = \text{plim}(\frac{1}{n}M'\epsilon) = \text{plim}(\frac{1}{n}\Phi'\epsilon) = \text{plim}(\frac{1}{n}\Delta'\epsilon) = 0$.

The proof of the lemma is omitted.

The ordinary least squares estimator (OLSE) and restricted least squares estimator (RLSE) of $\beta$ under the classical regression model without measurement errors are

$$b = S^{-1}X'y$$

and

$$b_R = b + S^{-1}R'(RS^{-1}R')^{-1}(r - Rb)$$

respectively, where $S = X'X$ and $b_R$ is derived under the restrictions (2.4).

Note that $b$ and $b_R$ are also the maximum likelihood estimators of $\beta$ in the classical regression model when measurement errors are absent and disturbances follow a multivariate normal distribution.

Further, let $\Sigma = \Sigma_\mu + \sigma_\phi^2 I_p + \sigma_\delta^2 I_p$, $R_\Sigma = R\Sigma^{-1}R'$ and, assuming that as $n \to \infty$, plim $(S/n) = \Sigma$, then plim $(X'y/n) = (\Sigma - \sigma_\delta^2 I_p)\beta$. Using Lemma 1, we have

$$\text{plim}\, b = (I_p - \sigma_\delta^2\Sigma^{-1})\beta \tag{3.1}$$

and

$$\text{plim } b_R = [I_p - \sigma_\delta^2 \Sigma^{-1}(I_p - R'R_\Sigma^{-1}R\Sigma^{-1})]\beta. \tag{3.2}$$

Thus it follows from (3.1) and (3.2) that both $b$ and $b_R$ are inconsistent estimators of $\beta$. Moreover, $b$ does not satisfy the restrictions (2.4) but $b_R$ satisfies the restrictions (2.4). Note that if $\sigma_\delta^2 = 0$ (which corresponds to the classical regression model with no measurement errors on explanatory variables $T$), then both $b$ and $b_R$ are consistent estimators of $\beta$. Also note that if $\sigma_\delta^2 > 0$, then $b_R$ is consistent for estimating $\beta$ only if $J = p$, which is a trivial case where $\beta$ is completely known from the restrictions only. So we ignore this case.

In order to obtain the consistent estimators of regression coefficients in measurement error models, we need some additional information. Here we propose to utilize the following two types of additional information separately, to construct the consistent estimators which also satisfy the linear restriction:

(i) the reliability matrix associated with explanatory variables is known and

(ii) the covariance matrix of measurement errors associated with explanatory variables is known.

### 3.1. When reliability matrix of explanatory variables is known

We follow the approach proposed by Gleser [11] for the construction of consistent estimators of the regression coefficients satisfying the linear restrictions under the assumption of a known reliability matrix. The reliability matrix associated with the explanatory variables of the model is defined as $\Sigma_{\text{Obs}}^{-1}\Sigma_{\text{True}}$ where $\Sigma_{\text{Obs}}$ and $\Sigma_{\text{True}}$ are the covariance matrices of observed and true values of explanatory variables. Based on this, the reliability ratio in our context can be expressed as

$$K = \Sigma^{-1}(\Sigma^{-1} - \sigma_\delta^2 I_p). \tag{3.3}$$

This definition of the reliability matrix in (3.3) is the generalization of reliability ratio in a univariate model in psychometrics literature – see [33,3,2,34]. Gleser [11] suggests estimating the reliability matrix $K$ consistently, say by $K_X$, using some prior information and data on the explanatory variables, and then to use the obtained $K_X$ to construct an estimator of $\beta$.

We assume that

$$K_X = \Sigma_X^{-1}\Sigma_T, \tag{3.4}$$

is known with $\Sigma_X := \frac{1}{n}M'M + \sigma_\phi^2 I_p + \sigma_\delta^2 I_p$ and $\Sigma_T := \frac{1}{n}M'M + \sigma_\phi^2 I_p$.

Since, as $n \to \infty$, $\text{plim } b = (I_p - \sigma_\delta^2\Sigma^{-1})\beta$ and $\lim K_X = (I_p - \sigma_\delta^2\Sigma^{-1})$, a consistent estimator of $\beta$ is

$$b_K^{(1)} = K_X^{-1}b. \tag{3.5}$$

Gleser [11] also discusses different ways to estimate the reliability matrix under functional and structural models – see also [12]. He also showed that the knowledge of the reliability matrix makes the measurement error model identifiable and gives the consistent estimators of the parameters as well. The advantage of such an approach based on the reliability matrix is that the classical tools of regression analysis can be employed by a simple transformation on the measurement error-ridden data. We refer the readers to [11,12,2 (p. 139)] for more details on this approach. It may be noted that when $K$ is known from some outside sample information, even then a consistent estimator of $\beta$ can be obtained by replacing $K_X$ by $K$ in (3.5).

However, $b_K^{(1)}$ is consistent for $\beta$ but it does not satisfy the given linear restrictions – i.e., $Rb_K^{(1)} \neq r$.

Thus, it is desired to find a consistent estimator of $\beta$, which also satisfies the given restrictions (2.4). To achieve this, we propose to minimize $(b_K^{(1)} - \beta)'S(b_K^{(1)} - \beta)$ with respect to $\beta$, subject to the restrictions $R\beta = r$ and obtain the following estimator:

$$b_K^{(2)} = b_K^{(1)} + S^{-1}R'R_S^{-1}(r - Rb_K^{(1)}), \tag{3.6}$$

where $R_S = RS^{-1}R'$. Such an estimator can be viewed as arising from a two stage restricted regression estimation procedure. In the first stage, we use the method of moments (or maximum likelihood under normality) under the assumption of known reliability matrix which gives $b_K^{(1)}$. Then, in the second stage, minimize the weighted error sum of squares under the constraint $R\beta = r$ which gives $b_K^{(2)}$. Such an estimator can also be obtained by replacing $b$ in $b_R$ by $b_K^{(1)}$. Clearly $\text{plim } b_K^{(2)} = \beta$ and $Rb_K^{(2)} = r$. Thus $b_K^{(2)}$ is consistent for $\beta$ and satisfies the linear restrictions as well.

An alternative strategy to obtain the consistent estimator of $\beta$ satisfying the restrictions is to utilize the plim $b_R$. Since, as $n \to \infty$,

$$\text{plim } b_R = \left[I_p - (\sigma_\delta^2\Sigma^{-1})\left\{I_p - R'\left(R(\sigma_\delta^2\Sigma^{-1})R'\right)^{-1}R(\sigma_\delta^2\Sigma^{-1})\right\}\right]\beta$$

and

$$\lim K_X = I_p - \sigma_\delta^2\Sigma^{-1},$$

another consistent estimator of $\beta$ is obtained by adjusting plim $b_R$ as

$$b_K^{(3)} = \left[K_X + (I_p - K_X)R'\left(R(I_p - K_X)R'\right)^{-1}R(I_p - K_X)\right]^{-1} b_R$$
$$= A_{K_X}^{-1} b_R, \text{ say}, \tag{3.7}$$

where

$$A_{K_X} = K_X + (I_p - K_X)R'\left(R(I_p - K_X)R'\right)^{-1}R(I_p - K_X).$$

Clearly, plim $b_K^{(3)} = \beta$. Moreover, $RA_{K_X} = R \Rightarrow R = RA_{K_X}^{-1}$ and $Rb_K^{(3)} = RA_{K_X}^{-1}b_R = Rb_R = r$. So $b_K^{(3)}$ is consistent for $\beta$ and satisfies the linear restrictions.

Another approach to find the consistent estimator of $\beta$ satisfying the linear restrictions is to minimize $(b_K^{(1)} - \beta)'(b_K^{(1)} - \beta)$ with respect to $\beta$ such that $R\beta = r$. This yields the following estimator

$$b_K^{(4)} = b_K^{(1)} + R'(RR')^{-1}(r - Rb_K^{(1)}), \tag{3.8}$$

for which plim $b_K^{(4)} = \beta$ and $Rb_K^{(4)} = r$. Such an estimator can also be thought of as arising from a two stage least squares approach. The first stage is to use the method of moment (or maximum likelihood under normality) under the known reliability matrix and obtain $b_K^{(1)}$. The second stage involves using the least squares principle to minimize $(b_K^{(1)} - \beta)'(b_K^{(1)} - \beta)$ under constraints $R\beta = r$ which gives $b_K^{(4)}$.

### 3.2. When covariance matrix of measurement errors in explanatory variables is known

We assume that the covariance matrix of measurement errors associated with explanatory variables – i.e., $\Sigma_\delta = \sigma_\delta^2 I_p$ – is known. This essentially reduces to assuming that $\sigma_\delta^2$ is known. It may be noted that the further analysis can also be carried out with $\Sigma_\delta$ (without assuming the $\sigma_\delta^2 I_p$ structure) with suitably defining the coefficients of skewness and kurtosis for a multivariate distribution.

We have plim $b = (I_p - \sigma_\delta^2 \Sigma^{-1})\beta$ and plim$(\frac{1}{n}S) = \Sigma$. It follows that when $\sigma_\delta^2$ is known, a consistent estimator of $\beta$ is

$$b_\delta^{(1)} = (S - n\Sigma_\delta)^{-1}X'y,$$
$$= (S - n\sigma_\delta^2 I_p)^{-1}X'y, \tag{3.9}$$

see [3,14,35]. Here $(S - n\sigma_\delta^2 I_p)$ is assumed to be a positive definite matrix. Note that $b_\delta^{(1)}$ is also the maximum likelihood estimator under the assumption of normal distribution of measurement errors and known $\sigma_\delta^2$. Clearly, $b_\delta^{(1)}$ is consistent for $\beta$, but $Rb_\delta^{(1)} \neq r$ – i.e., it does not satisfy the given linear restrictions (2.4). Therefore, to obtain an estimator of $\beta$ that is consistent as well as which satisfies the given linear restrictions (2.4), Shalabh, Garg and Misra [35] discussed various approaches. Some are described here for the sake of completeness of paper. Firstly, they propose to minimize he weighted sum of squares due to errors $(b_\delta^{(1)} - \beta)'S(b_\delta^{(1)} - \beta)$ with respect to $\beta$ such that $R\beta = r$. This yields the following estimator:

$$b_\delta^{(2)} = b_\delta^{(1)} + S^{-1}R'R_S^{-1}(r - Rb_\delta^{(1)}), \tag{3.10}$$

where $R_S = RS^{-1}R'$. Clearly, plim $b_\delta^{(2)} = \beta$ and $Rb_\delta^{(2)} = r$ – i.e., $b_\delta^{(2)}$ is consistent as well as satisfying the linear restrictions (2.4). This estimator can also be thought as arising from a two stage least squares approach. Use method of moments (or maximum likelihood under normality assumption) under known $\Sigma_\delta$ in the first stage which yields $b_\delta^{(1)}$. Then minimize $(b_\delta^{(1)} - \beta)'S(b_\delta^{(1)} - \beta)$ with respect to $\beta$ under the constraints $R\beta = r$ in the second stage and then $b_\delta^{(2)}$ is obtained. The estimator $b_\delta^{(2)}$ can also be obtained by replacing $b$ in $b_R$ by $b_\delta^{(1)}$.

Next, since

$$\text{plim } b_R = [I_p - \sigma_\delta^2(I_p - \Sigma^{-1}R'R_\Sigma^{-1}R)\Sigma^{-1}]\beta$$

and plim$(\frac{1}{n}S) = \Sigma$, another consistent estimator of $\beta$ can be obtained by adjusting the plim $b_R$ as

$$b_\delta^{(3)} = [I_p - n\sigma_\delta^2(I_p - S^{-1}R'R_S^{-1}R)S^{-1}]^{-1}b_R$$
$$= A_\delta^{-1}b_R, \text{ say}, \tag{3.11}$$

where $A_\delta I_p - n\sigma_\delta^2(I_p - S^{-1}R'R_S^{-1}R)S^{-1}$. Clearly, plim $b_\delta^{(3)} = \beta$. Moreover $RA_\delta = R$ and thus $R = RA_\delta^{-1}$, so

$$Rb_\delta^{(3)} = RA_\delta^{-1}b_R = Rb_R = r.$$

Hence the estimator $b_\delta^{(3)}$ is a consistent estimator of $\beta$ and it also satisfies the linear restrictions (2.4).

Alternatively, following the two stage least squares procedure to obtain $b_\delta^{(1)}$ in the first stage and use least squares principle to minimize $(b_\delta^{(1)} - \beta)'(b_\delta^{(1)} - \beta)$ with respect to $\beta$, subject to the linear restrictions $R\beta = r$ in the second stage. This gives the following estimator:

$$b_\delta^{(4)} = b_\delta^{(1)} + R'(RR')^{-1}(r - Rb_\delta^{(1)}), \tag{3.12}$$

for which plim $b_\delta^{(4)} = \beta$ and $Rb_\delta^{(4)} = r$ – i.e., $b_\delta^{(4)}$ is also consistent for $\beta$ and satisfies the linear restrictions (2.4).

## 4. Asymptotic properties of the estimators

The exact distribution and finite sample properties of the estimators $b_K^{(l)}$ and $b_\delta^{(l)}$, $(l = 2, 3, 4)$ are difficult to derive. Even if derived, the expressions will turn out to be complicated and it may not be possible to draw any clear inference from them. Moreover, the mean of $b_\delta^{(1)}$ does not exist under the normal distribution of measurement errors – see [2 (p. 58),36]. So we propose to employ the large sample asymptotic approximation theory to study the asymptotic distribution of the estimators.

Define a function $f : \mathbb{R}^{p\times p} \times \mathbb{R}^{p\times p} \longrightarrow \mathbb{R}^{p\times p}$ as

$$f(Z_1, Z_2) = Z_1(Z_2 * I_p), \quad Z_1, Z_2 \in \mathbb{R}^{p\times p}, \tag{4.1}$$

where $*$ denotes the Hadamard product operator of matrices and $\mathbb{R}^{p\times p}$ is the collection of all $p \times p$ real matrices.

Let $U$ and $V$ be any positive definite matrices of appropriate order. Define

$$\Omega = g(U, V) = \Xi(U, V) + \Upsilon(U, V) \tag{4.2}$$

where

$$\begin{aligned}\Xi(U,V) &= (\sigma_\delta^2 + \sigma_\phi^2)[\sigma_\delta^2\{(\beta\beta' - U\beta\beta'U) + (\operatorname{tr} V\beta\beta'U)I_p\} + (\operatorname{tr} V\beta\beta'V)\Sigma] \\ &\quad + \{\sigma_\epsilon^2 + \sigma_\delta^2(\operatorname{tr}(2U - I_p)\beta\beta')\}\Sigma - \sigma_\delta^4\beta\beta' \end{aligned}\tag{4.3}$$

$$\begin{aligned}\Upsilon(U,V) &= \gamma_{1\delta}\sigma_\delta^3\left[f\left(\boldsymbol{\sigma}_\mu e_p',\ U\beta\beta'U\right) + \left\{f\left(\boldsymbol{\sigma}_\mu e_p',\ U\beta\beta'U\right)\right\}' - 2f\left(I_p,\ e_p\boldsymbol{\sigma}_\mu{}'V\beta\beta'U\right)\right] \\ &\quad + \gamma_{1\phi}\sigma_\phi^3\left[f\left(\boldsymbol{\sigma}_\mu e_p',\ V\beta\beta'V\right) + \left\{f\left(\boldsymbol{\sigma}_\mu e_p',\ V\beta\beta'V\right)\right\}' + 2f\left(I_p,\ e_p\boldsymbol{\sigma}_\mu{}'V\beta\beta'V\right)\right] \\ &\quad + \gamma_{2\delta}\sigma_\delta^4 f\left(I_p,\ U\beta\beta'U\right) + \gamma_{2\phi}\sigma_\phi^4 f\left(I_p,\ V\beta\beta'V\right). \end{aligned}\tag{4.4}$$

Now we have the following theorem.

**Theorem 1.** *The asymptotic distributions of* $\sqrt{n}(b_K^{(1)} - \beta)$, $\sqrt{n}(b_K^{(2)} - \beta)$, $\sqrt{n}(b_K^{(3)} - \beta)$ *and* $\sqrt{n}(b_K^{(4)} - \beta)$ *are normal with common mean vector* 0 *and covariance matrices given by*

$$\Omega_K^{(1)} = (\Sigma K)^{-1}\Omega_K(\Sigma K)^{-1} \tag{4.5}$$

$$\Omega_K^{(2)} = A_2\Omega_K^{(1)}A_2{}', \tag{4.6}$$

$$\Omega_K^{(3)} = A_3\Omega_{A_K}A_3{}', \tag{4.7}$$

$$\Omega_K^{(4)} = A_4\Omega_K^{(1)}A_4{}' \tag{4.8}$$

*respectively, where*

$$\bar{K} = I_p - K$$

$$\bar{K}_X = I_p - K_X$$

$$A_{K_X} = K_X + (I_p - K_X)R'\left(R(I_p - K_X)R'\right)^{-1}R(I_p - K_X)$$

$$\bar{A}_{K_X} = \bar{K}_X + (I_p - \bar{K}_X)R'\left(R(I_p - \bar{K}_X)R'\right)^{-1}R(I_p - \bar{K}_X)$$

$$A_K = \lim_{n\to\infty} A_{K_X}$$

$$\bar{A}_K = \lim_{n\to\infty} \bar{A}_{K_X}$$

$$A_2 = I_p - \bar{K}R'\{R\bar{K}R'\}^{-1}R, \tag{4.9}$$

$$A_3 = A_K^{-1}A_2\Sigma^{-1}, \tag{4.10}$$

$$A_4 = [I_p - R'(RR')^{-1}R], \tag{4.11}$$

$$\Omega_K = g(K, \bar{K}) \tag{4.12}$$

$$\Omega_{A_K} = g(A_K, \bar{A}_K) \tag{4.13}$$

$$N(K) = \Upsilon(K, \bar{K}) \tag{4.14}$$

$$N(A_K) = \Upsilon(A_K, \bar{A}_K). \tag{4.15}$$

**Proof.** See Appendix for the proof of theorem. □

The quantity $N(K)$ in (4.14) depicts the non-normality effect on the covariance matrix of the asymptotic distributions of $\sqrt{n}(b_K^{(1)} - \beta)$, $\sqrt{n}(b_K^{(2)} - \beta)$ and $\sqrt{n}(b_K^{(4)} - \beta)$ whereas the quantity $N(A_K)$ in (4.15) (which is obtained on replacing $K$ by $A_K$ in $N(K)$) shows the non-normality effect on the covariance matrix of the asymptotic distribution of $\sqrt{n}(b_K^{(3)} - \beta)$. It is clear from (4.14) and (4.15) that the skewness and kurtosis of the distributions of $\delta_{ij}$ and $\phi_{ij}$ only affect the covariance matrices in the asymptotic distributions of all the estimators. The departure from normality of the distribution of $\epsilon_i$ does not affect the asymptotic distribution of any of the estimators. The magnitude of departure from normality depends on the values and degree of skewness and kurtosis of the distributions of $\delta_{ij}$ and $\phi_{ij}$. If the distribution of measurement errors is normal, then both $N(K)$ and $N(A_K)$ become zero.

Now we find the dominance conditions of $b_K^{(l)}$, $(l = 1, 2, 3, 4)$ under Löwner ordering, based on the covariance matrices of their asymptotic distributions.

In further text, we refer Löwner ordering to the comparison of covariance matrices of the asymptotic distribution of the estimators.

It may be noted that $b_K^{(1)}$ does not satisfy the given linear restrictions (2.4), while $b_K^{(2)}$, $b_K^{(3)}$ and $b_K^{(4)}$ satisfy the given restrictions. So it would be interesting to find the conditions under which the use of prior information results gain in efficiency. So, now we compare the restricted estimators $b_K^{(l)}$, $(l = 2, 3, 4)$ with the unrestricted estimator $b_K^{(1)}$ under the criterion of Löwner ordering.

First we compare $b_K^{(1)}$ and $b_K^{(2)}$. Since $\Omega_K^{(1)}$ is a positive definite matrix, we can write

$$\Omega_K^{(1)} - \Omega_K^{(2)} = (\Omega_K^{(1)})^{\frac{1}{2}}[I_p - (\Omega_K^{(1)})^{-\frac{1}{2}} A_2 \Omega_K^{(1)} A_2' (\Omega_K^{(1)})^{-\frac{1}{2}}](\Omega_K^{(1)})^{\frac{1}{2}}.$$

Since $(\Omega_K^{(1)})^{-\frac{1}{2}} A_2 \Omega_K^{(1)} A_2' (\Omega_K^{(1)})^{-\frac{1}{2}} := W$ is a symmetric matrix, it can be written as $\Gamma \Lambda \Gamma'$, where $\Gamma$ is the diagonal matrix of the eigenvalues $\lambda_j$, $(j = 1, 2, \ldots, p)$ of $W$ and $\Lambda$ is the orthogonal matrix of the eigenvectors of $W$. Since $\Gamma\Gamma' = I_p$, we can express

$$\begin{aligned}\Omega_K^{(1)} - \Omega_K^{(2)} &= (\Omega_K^{(1)})^{\frac{1}{2}}[I_p - \Gamma\Lambda\Gamma'](\Omega_K^{(1)})^{\frac{1}{2}} \\ &= (\Omega_K^{(1)})^{\frac{1}{2}}\Gamma[I_p - \Lambda]\Gamma'(\Omega_K^{(1)})^{\frac{1}{2}}.\end{aligned}$$

Thus $\Omega_K^{(1)} - \Omega_K^{(2)}$ is positive (or negative) definite matrix when $(I_p - \Lambda)$ is positive (or negative) definite, which holds true when $\lambda_j < (\text{or } >)1;\ \forall j = 1, 2, \ldots, p$. Therefore $b_K^{(1)}$ is less efficient than $b_K^{(2)}$ under Löwner ordering when all the eigenvalues of the matrix $(\Omega_K^{(1)})^{-\frac{1}{2}} A_2 \Omega_K^{(1)} A_2' (\Omega_K^{(1)})^{-\frac{1}{2}}$ are less than unity, and vice versa.

In case $(\Omega_K^{(1)})^{-\frac{1}{2}} A_2 (\Omega_K^{(1)})^{\frac{1}{2}}$ is symmetric, then $b_K^{(2)}$ is uniformly superior to $b_K^{(1)}$ under Löwner ordering.

In a similar fashion, it can be found that $b_K^{(1)}$ is less efficient than the $b_K^{(3)}$ and $b_K^{(4)}$ when all the eigenvalues of the matrices $(\Omega_K^{(1)})^{-\frac{1}{2}} A_3 \Omega_{A_K}^{(1)} A_3' (\Omega_K^{(1)})^{-\frac{1}{2}}$ and $(\Omega_K^{(1)})^{-\frac{1}{2}} A_4 \Omega_K^{(1)} A_4' (\Omega_K^{(1)})^{-\frac{1}{2}}$, respectively are less than unity and vice versa. In case $(\Omega_K^{(1)})^{-\frac{1}{2}} A_3 (\Omega_{A_K}^{(1)})^{\frac{1}{2}}$ and $(\Omega_K^{(1)})^{-\frac{1}{2}} A_4 (\Omega_K^{(1)})^{\frac{1}{2}}$ are symmetric matrices, then $b_K^{(3)}$ and $b_K^{(4)}$, respectively, are uniformly superior to $b_K^{(1)}$ under Löwner ordering.

Now we compare the restricted estimators among themselves. First we compare $b_K^{(2)}$ and $b_K^{(3)}$ and we find that

$$\Omega_K^{(2)} - \Omega_K^{(3)} = A_2 (\Omega_K^{(1)})^{\frac{1}{2}} \left[ I_p - (\Omega_K^{(1)})^{-\frac{1}{2}} A_2^{-1} A_3 \Omega_{A_K} A_3' A_2'^{-1} (\Omega_K^{(1)})^{-\frac{1}{2}} \right] (\Omega_K^{(1)})^{\frac{1}{2}} A_2'. \tag{4.16}$$

It follows from (4.16) that $b_K^{(3)}$ is superior to $b_K^{(2)}$ under Löwner ordering when all the eigenvalues of the matrix $(\Omega_K^{(1)})^{-\frac{1}{2}} A_2^{-1} A_3 \Omega_{A_K} A_3' A_2'^{-1} (\Omega_K^{(1)})^{-\frac{1}{2}}$ are less than unity and vice versa. The uniform superiority of $b_K^{(2)}$ over $b_K^{(3)}$ holds true when $(\Omega_K^{(1)})^{-\frac{1}{2}} A_2^{-1} A_3 \Omega_{A_K}^{\frac{1}{2}}$ is symmetric.

It can also be obtained that $b_K^{(2)}$ is less efficient than $b_K^{(4)}$ under Löwner ordering when all the eigenvalues of the matrix $(\Omega_K^{(1)})^{-\frac{1}{2}} A_2^{-1} A_4 \Omega_K^{(1)} A_4' A_2'^{-1} (\Omega_K^{(1)})^{-\frac{1}{2}}$ are less than unity and vice versa. Again, in case $(\Omega_K^{(1)})^{-\frac{1}{2}} A_2^{-1} A_4 (\Omega_K^{(1)})^{\frac{1}{2}}$ is symmetric, then $b_K^{(4)}$ is uniformly superior to $b_K^{(2)}$ under Löwner ordering.

Similarly, $b_K^{(4)}$ is superior to $b_K^{(3)}$ under Löwner ordering when all the eigenvalues of the matrix $(\Omega_K^{(1)})^{-\frac{1}{2}} A_4^{-1} A_3 \Omega_{A_K} A_3' A_4'^{-1} (\Omega_K^{(1)})^{-\frac{1}{2}}$ are less than unity, and vice versa. The uniform superiority of $b_K^{(3)}$ over $b_K^{(4)}$ holds when $(\Omega_K^{(1)})^{-\frac{1}{2}} A_4^{-1} A_3 \Omega_{A_K}^{\frac{1}{2}}$ is symmetric.

Since the assumption of normality is a popular assumption in the literature, so we present the results of Theorem 1 when measurement errors and random error component follow normal distributions.

**Corollary 1.** *Assume that* $\delta_{ij} \sim N(0, \sigma_\delta^2)$, $\phi_{ij} \sim N(0, \sigma_\phi^2)$ *and* $\epsilon_i \sim N(0, \sigma_\epsilon^2)$ *for all* $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, p)$. *Then the asymptotic distributions of* $\sqrt{n}(b_K^{(1)} - \beta)$, $\sqrt{n}(b_K^{(2)} - \beta)$, $\sqrt{n}(b_K^{(3)} - \beta)$ *and* $\sqrt{n}(b_K^{(4)} - \beta)$ *are normal, with common mean vector* 0 *and covariance matrices given by*

$$\Omega_{KN}^{(1)} = (\Sigma K)^{-1} \Xi(K, \bar{K})(\Sigma K)^{-1} \tag{4.17}$$

$$\Omega_{KN}^{(2)} = A_2 \Omega_{KN}^{(1)} A_2', \tag{4.18}$$

$$\Omega_{KN}^{(3)} = A_3 \Xi(A_K, \bar{A}_K) A_3', \tag{4.19}$$

$$\Omega_{KN}^{(4)} = A_4 \Omega_{KN}^{(1)} A_4' \tag{4.20}$$

*respectively.*

The next theorem describes the asymptotic distributions of $\sqrt{n}\left(b_\delta^{(l)} - \beta\right)$, $(l = 1, 2, 3, 4)$ — see also [35].

**Theorem 2.** *The asymptotic distribution of* $\sqrt{n}\left(b_\delta^{(l)} - \beta\right)$ *is normal with mean vector* 0 *and covariance matrix* $B_l \Omega_\delta B_l'$, $l = 1, 2, 3, 4$, *where*

$$\Omega_\delta = \Xi_\delta + N_\delta \tag{4.21}$$

$$\Xi_\delta = \sigma_\epsilon^2 \Sigma + \sigma_\delta^2 (\operatorname{tr} \beta\beta') \Sigma + \sigma_\delta^4 \beta\beta' \tag{4.22}$$

$$B_1 = (\Sigma - \sigma_\delta^2 I_p)^{-1} \tag{4.23}$$

$$\begin{aligned} B_2 &= \{I_p - \Sigma^{-1} R' (R\Sigma^{-1} R')^{-1} R\} (\Sigma - \sigma_\delta^2 I_p)^{-1} \\ &= A_2 (\Sigma K)^{-1}, \end{aligned} \tag{4.24}$$

$$\begin{aligned} B_3 &= [I_p - \sigma_\delta^2 \{\Sigma^{-1} - \Sigma^{-1} R' (R\Sigma^{-1} R')^{-1} R\Sigma^{-1}\}]^{-1} \{\Sigma^{-1} - \Sigma^{-1} R' (R\Sigma^{-1} R')^{-1} R\Sigma^{-1}\}, \\ &= A_3, \end{aligned} \tag{4.25}$$

$$\begin{aligned} B_4 &= \{I_p - R'(RR')^{-1} R\} (\Sigma - \sigma_\delta^2 I_p)^{-1} \\ &= A_4 (\Sigma K)^{-1}, \end{aligned} \tag{4.26}$$

*and*

$$N_\delta = \gamma_{1\delta} \sigma_\delta^3 \left\{ f(\boldsymbol{\sigma}_\mu e_p', \beta\beta') + \left( f(\boldsymbol{\sigma}_\mu e_p', \beta\beta') \right)' \right\} + \gamma_{2\delta} \sigma_\delta^4 f(I_p, \beta\beta). \tag{4.27}$$

$A_2$, $A_3$ *and* $A_4$ *are given in Theorem 1 in (4.9)–(4.11).*

The quantity $N_\delta$ is the contribution of non-normality of distribution of $\delta_{ij}$ in the covariance matrices of the asymptotic distributions of $\sqrt{n}\left(b_\delta^{(l)} - \beta\right)$ $(l = 1, 2, 3, 4)$. Clearly the effect of non-normality of distributions of $\phi_{ij}$ and $\epsilon_i$ do not affect the asymptotic distribution of the estimators. The magnitude of this effect depends on the skewness and kurtosis of the distribution of $\delta_{ij}$.

Now we find the dominance conditions for the superiority of $b_\delta^{(k)}$ over $b_\delta^{(l)}$ $(k, l = 1, 2, 3, 4, k \neq l)$ in the sense of Löwner ordering. The dominance conditions for the superiority of $b_\delta^{(1)}$ over $b_\delta^{(2)}$, $b_\delta^{(3)}$, and $b_\delta^{(4)}$ will give an idea about the role of prior information in improving the estimators.

It is observed that $b_\delta^{(k)}$ is less efficient than $b_\delta^{(l)}$ $(k, l = 1, 2, 3, 4;\ k \neq l)$ under the criterion of Löwner ordering when all the eigenvalues of the matrix $\Omega_\delta^{-\frac{1}{2}} B_k^{-1} B_l \Omega_\delta B_l' B_k'^{-1} \Omega_\delta^{-\frac{1}{2}}$ are less than unity, and vice versa. If $\Omega_\delta^{-\frac{1}{2}} B_k^{-1} B_l \Omega_\delta^{\frac{1}{2}}$ is symmetric, then the uniform superiority of $b_\delta^{(l)}$ over $b_\delta^{(k)}$ holds true under Löwner ordering.

The next corollary presents the results of Theorem 2 when measurement errors and random error component are normally distributed.

**Corollary 2.** *Assume that* $\delta_{ij} \sim N(0, \sigma_\delta^2)$, $\phi_{ij} \sim N(0, \sigma_\phi^2)$ *and* $\epsilon_i \sim N(0, \sigma_\epsilon^2)$ *for all* $i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, p$. *The asymptotic distribution of* $\sqrt{n}\left(b_\delta^{(l)} - \beta\right)$ *is normal with mean vector 0 and covariance matrix* $B_l \Xi_\delta B_l'$, $(l = 1, 2, 3, 4)$, *respectively.*

Another question arises that, given both the information viz. the covariance matrix of measurement errors associated with explanatory variables and reliability matrix of explanatory variables, which information yields more efficient estimator and under what conditions? Such guidelines will help the applied workers in choosing a good estimator, depending on their experimental conditions. To answer this, we analyze the difference of the covariance matrices of asymptotic distributions of $\sqrt{n}(b_K^{(l)} - \beta)$ and $\sqrt{n}(b_\delta^{(l)} - \beta)$ $(l = 2, 3, 4)$ and obtain the dominance conditions of one estimator over the other under the criterion of Löwner ordering.

Let $D^{(l)}$ denote the difference between covariance matrices of asymptotic distributions of $\sqrt{n}(b_K^{(l)} - \beta)$ and $\sqrt{n}(b_\delta^{(l)} - \beta)$ $(l = 2, 3, 4)$, then

$$D^{(2)} = A_2(\Sigma K)^{-1}(\Omega_K - \Omega_\delta)(\Sigma K)^{-1}A_2' \tag{4.28}$$

$$D^{(3)} = A_3(\Omega_{A_K} - \Omega_\delta)\Sigma^{-1}A_3', \tag{4.29}$$

$$D^{(4)} = A_4(\Sigma K)^{-1}(\Omega_K - \Omega_\delta)(\Sigma K)^{-1}A_4'. \tag{4.30}$$

Thus it is clear from (4.28) that $b_K^{(2)}$ is superior (or inferior) than $b_\delta^{(2)}$ under Löwner ordering if $(\Omega_K - \Omega_\delta)$ is a negative (or positive) semi-definite matrix. Moreover, it also follows from (4.28) and (4.30) that the dominance conditions of $b_K^{(2)}$ over $b_\delta^{(2)}$ and $b_K^{(4)}$ over $b_\delta^{(4)}$ are the same. Let us consider

$$\Omega_K - \Omega_\delta = \Theta + [N(K) - N_\delta] \tag{4.31}$$

where

$$\Theta = (\sigma_\delta^2 + \sigma_\phi^2)[\sigma_\delta^2\{(\operatorname{tr}\bar{K}\beta\beta'K)I_p - K\beta\beta'K\} + (\operatorname{tr}\bar{K}\beta\beta'\bar{K})\Sigma] - 2\sigma_\delta^2(\operatorname{tr}\bar{K}\beta\beta')\Sigma + \sigma_\delta^2(\sigma_\phi^2 - \sigma_\delta^2)\beta\beta'.$$

It is difficult to derive any clear conditions for the definiteness or semi-definiteness of the matrix $(\Omega_K - \Omega_\delta)$. If we assume that both $\delta_{ij}$ and $\phi_{ij}$ have a symmetric and platykurtic distribution (e.g., normal), then the quantity $(N(K) - N_\delta)$ vanishes and (semi)definiteness of the matrix $\Theta$ ensures the dominance of one estimator over the other. Using the following lemma, we check the (semi)definiteness of the matrix $\Theta$ and obtain the dominance conditions for the superiority of one estimator over the other when $[N(K) - N_\delta]$ is zero.

**Lemma 2.** *Let $A$ be a $p \times p$ non-negative definite matrix and $B$ be a $p \times p$ positive definite matrix. Then $(A - B)$ is positive semi definite matrix if and only if $\lambda_i \leq 1\ \forall\, i = 1, 2, \ldots, p$ where $\lambda_i$'s are the eigenvalues of the matrix $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$.*

**Proof.** See [5] for proof. □

If $(\operatorname{tr}\bar{K}\beta\beta'K) > 0$, then $\Theta$ is positive semi-definite if and only if

- all the eigenvalues of the matrix $\Theta_2^{-\frac{1}{2}}\Theta_1\Theta_2^{-\frac{1}{2}}$ are less than or equal to unity, provided $\sigma_\phi^2 > \sigma_\delta^2$,
- all the eigenvalues of the matrix $\Theta_4^{-\frac{1}{2}}\Theta_3\Theta_4^{-\frac{1}{2}}$ are less than or equal to unity, provided $\sigma_\phi^2 < \sigma_\delta^2$,
- all the eigenvalues of the matrix $\Theta_2^{-\frac{1}{2}}\Theta_3\Theta_2^{-\frac{1}{2}}$ are less than or equal to unity, provided $\sigma_\phi^2 = \sigma_\delta^2$,

where

$$\begin{aligned}
\Theta_1 &= (\sigma_\delta^2 + \sigma_\phi^2)\{(\operatorname{tr}\bar{K}\beta\beta'\bar{K})\Sigma + \sigma_\delta^2(\operatorname{tr}\bar{K}\beta\beta'K)I_p\} + \sigma_\delta^2(\sigma_\phi^2 - \sigma_\delta^2)\beta\beta',\\
\Theta_2 &= (\sigma_\delta^2 + \sigma_\phi^2)\sigma_\delta^2K\beta\beta'K + 2\sigma_\delta^2(\operatorname{tr}\bar{K}\beta\beta')\Sigma,\\
\Theta_3 &= (\sigma_\delta^2 + \sigma_\phi^2)\{(\operatorname{tr}\bar{K}\beta\beta'\bar{K})\Sigma + \sigma_\delta^2(\operatorname{tr}\bar{K}\beta\beta'K)I_p\},\\
\Theta_4 &= (\sigma_\delta^2 + \sigma_\phi^2)\sigma_\delta^2K\beta\beta'K + 2\sigma_\delta^2(\operatorname{tr}\bar{K}\beta\beta')\Sigma - \sigma_\delta^2(\sigma_\phi^2 - \sigma_\delta^2)\beta\beta'.
\end{aligned} \tag{4.32}$$

If $(\operatorname{tr}\bar{K}\beta\beta') < 0$, then $\Theta$ is positive semi-definite if and only if

- all the eigenvalues of the matrix $\Theta_6^{-\frac{1}{2}}\Theta_5\Theta_6^{-\frac{1}{2}}$ are less than or equal to unity, provided $\sigma_\phi^2 > \sigma_\delta^2$,
- all the eigenvalues of the matrix $\Theta_8^{-\frac{1}{2}}\Theta_7\Theta_8^{-\frac{1}{2}}$ are less than or equal to unity, provided $\sigma_\phi^2 < \sigma_\delta^2$,
- all the eigenvalues of the matrix $\Theta_6^{-\frac{1}{2}}\Theta_7\Theta_6^{-\frac{1}{2}}$ are less than or equal to unity, provided $\sigma_\phi^2 = \sigma_\delta^2$,

where

$$\begin{aligned}
\Theta_5 &= (\sigma_\delta^2 + \sigma_\phi^2)(\operatorname{tr}\bar{K}\beta\beta'\bar{K})\Sigma + \sigma_\delta^2(\sigma_\phi^2 - \sigma_\delta^2)\beta\beta' - 2\sigma_\delta^2(\operatorname{tr}\bar{K}\beta\beta')\Sigma,\\
\Theta_6 &= (\sigma_\delta^2 + \sigma_\phi^2)\sigma_\delta^2\{K\beta\beta'K - (\operatorname{tr}\bar{K}\beta\beta'K)I_p\},\\
\Theta_7 &= (\sigma_\delta^2 + \sigma_\phi^2)(\operatorname{tr}\bar{K}\beta\beta'\bar{K})\Sigma - 2\sigma_\delta^2(\operatorname{tr}\bar{K}\beta\beta')\Sigma,\\
\Theta_8 &= (\sigma_\delta^2 + \sigma_\phi^2)\sigma_\delta^2\{K\beta\beta'K - (\operatorname{tr}\bar{K}\beta\beta'K)I_p\} - \sigma_\delta^2(\sigma_\phi^2 - \sigma_\delta^2)\beta\beta'.
\end{aligned} \tag{4.33}$$

## 5. Simulation study

The asymptotic distribution of the estimators explain the behavior and properties of the distribution of estimators when sample size is large. To study the finite sample properties of the estimators, we conducted a Monte-Carlo simulation experiment. Another objective of the Monte-Carlo simulation experiment was to study the effect of departure from normality of the distribution of measurement errors and random error component, on the finite sample behavior of the estimators. We used the following distributions for this purpose:

(i) Normal distribution (having no skewness and no kurtosis),
(ii) Student's $t$ distribution (having zero skewness but non-zero kurtosis) and
(iii) Gamma Distribution (having both non-zero skewness and non-zero kurtosis).

We considered two sample sizes $n = 25$ and $n = 45$ which can be considered as small and large samples, respectively. We chose the following values of other parameters to generate the observations: $p = 5, \beta = \begin{pmatrix} 2.2 \\ 1.1 \\ 3.0 \\ 4.2 \\ 2.5 \end{pmatrix}, \begin{pmatrix} 2.0 \\ -3.2 \\ 2.5 \\ -5.1 \\ 1.5 \end{pmatrix}, R = \begin{pmatrix} 0.8 & 0.6 & 0.7 & 0.9 & 0.8 \\ 0.2 & 0.7 & 0.4 & 0.7 & 0.8 \\ 0.6 & 0.4 & 0.6 & 0.1 & 0.4 \\ 0.5 & 0 & 0.8 & 0.9 & 0.4 \end{pmatrix}$ and obtained $r = R\beta$. We fixed two matrices of orders $25 \times 5$ and $45 \times 5$ for $M$. The experiment is conducted for the following values and combinations of the variances $(\sigma_\epsilon^2, \sigma_\phi^2, \sigma_\delta^2)$: (0.4, 0.4, 0.4), (0.4, 0.4, 1), (1, 0.4, 0.4), (0.4, 1, 0.4), (1, 1, 0.4)(1, 1, 1). The observations from all the distributions were scaled suitably to have zero mean and the same variance as specified in the different combinations. The reliability matrix $K_X = (\frac{1}{n}M'M + \sigma_\phi^2 I_p + \sigma_\delta^2 I_p)^{-1}(\frac{1}{n}M'M + \sigma_\phi^2 I_p)$ can be obtained with these set of values and is used in constructing the estimators, $b_K^{(l)}$ and $b_\delta^{(l)}$, $(l = 1, 2, 3, 4)$. The bias vectors and mean squared error (MSE) matrices of $b_K^{(l)}$ and $b_\delta^{(l)}$, $(l = 1, 2, 3, 4)$ are estimated empirically based on 10 000 replications for all combinations of $(\sigma_\epsilon^2, \sigma_\phi^2, \sigma_\delta^2)$ stated above. Keeping in mind the length of the paper, only a few results are presented here in the Tables 1–6.

It is observed that the absolute values of empirical bias (EB) of $b_K^{(l)}$ and $b_\delta^{(l)}$, $(l = 1, 2, 3, 4)$ goes to a null vector as sample size increases. The magnitude of EBs and rate of such convergence depends on the values of $(\sigma_\epsilon^2, \sigma_\phi^2, \sigma_\delta^2)$. This property of the estimators is also evident from the asymptotic results. The magnitude of empirical absolute bias is smaller under the sample size $n = 43$ than $n = 25$, which shows that the estimators under consideration are asymptotically unbiased, even for this sample size. This establishes the asymptotic unbiasedness of all the estimators as suggested by the analytic results. No clear uniform superiority of any estimator under the case of known reliability ratio is clearly seen from the simulated results. In general, $b_K^{(3)}$ is found to have smaller empirical absolute bias among others, but the difference in the magnitude of bias with other estimators' bias is small. Among the estimators $b_\delta^{(l)}$, $(l = 2, 3, 4)$, $b_\delta^{(3)}$ has the smallest empirical absolute bias, while $b_\delta^{(4)}$ has the largest empirical absolute bias, in both the small and large samples.

Observing the values of the empirical mean squared error matrices (EMSEM) of $b_K^{(l)}$ and $b_\delta^{(l)}$ $(l = 2, 3, 4)$ from the Tables 1–6, it can be seen that the variability of all the estimators decreases as the sample size increases under all distributions of measurement errors under consideration, viz., normal, $t$, and gamma,. This clearly establishes the consistency property of the estimators as explained by the asymptotic theory. We observe, under all the distributions of measurement errors, that

$$\text{EMSEM}(b_K^{(2)}) \leq_{\text{L}} \text{EMSEM}(b_K^{(3)}) \leq_{\text{L}} \text{EMSEM}(b_K^{(4)}), \tag{5.1}$$

$$\text{EMSEM}(b_\delta^{(3)}) \leq_{\text{L}} \text{EMSEM}(b_\delta^{(2)}) \leq_{\text{L}} \text{EMSEM}(b_\delta^{(4)}), \tag{5.2}$$

where $\leq_{\text{L}}$ implies Löwner ordering. For the two square matrices $G_1$ and $G_2$, $G_1 \leq_{\text{L}} G_2$ implies that $G_1 - G_2$ is negative semi definite. Thus it is clear from (5.1) that $b_K^{(2)}$ is better than $b_K^{(3)}$ and $b_K^{(4)}$ both $b_\delta^{(3)}$ is better then $b_\delta^{(2)}$ and $b_\delta^{(4)}$ both under the criterion of Löwner ordering of EMSEM. Since the dominance conditions stated in the earlier Section depend on the unknown parameters, so we also verified them under the given parametric set up. The dominance under simulated results goes as per the dominance conditions based on asymptotic theory in most of the cases.

In order to have an idea about the effect of prior information on the efficiency of the estimators, we compared the EMSEM of unrestricted estimators $b_K^{(1)}$ and $b_\delta^{(1)}$ with the restricted estimators $b_K^{(l)}$ and $b_\delta^{(l)}$ $(l = 2, 3, 4)$, respectively. It is observed that under all the distributions, $b_K^{(1)}$ and $b_\delta^{(1)}$ are less efficient than $b_K^{(l)}$ and $b_\delta^{(1)}$ $(l = 2, 3, 4)$, respectively in the sense of mean squared errors of the estimators.

The effect of variances $\sigma_\epsilon^2$, $\sigma_\phi^2$ and $\sigma_\delta^2$ on the variability of the estimators is observed by increasing one of these variances while keeping other two fixed, and also by increasing two of them while keeping the remaining one fixed. It is noticed that $\sigma_\epsilon^2$ has no significant effect, but $\sigma_\delta^2$ plays a dominating role in affecting the variability of any of the estimators $b_K^{(l)}$ and $b_\delta^{(l)}$ $(l = 2, 3, 4)$ under all the distributions of measurement errors, viz., normal, $t$ and gamma. As $\sigma_\delta^2$ is increased, the

**Table 1**

Empirical bias vectors and empirical mean squared error matrices of $b_K^{(2)}$, $b_K^{(3)}$ and $b_K^{(4)}$ when measurement errors and random error components follow normal distribution.

| | $n = 25$ | | | | | $n = 45$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma_\epsilon^2 = 0.4,\ \sigma_\phi^2 = 0.4,\ \sigma_\delta^2 = 0.4$ | | | | | | | | | | |
| $\mathrm{EB}(b_K^{(2)})'$ | 0.0006 | −0.0051 | −0.0014 | −0.0021 | 0.0069 | 0.0003 | −0.0023 | −0.0006 | −0.0010 | 0.0032 |
| $\mathrm{EB}(b_K^{(3)})'$ | 0.0005 | −0.0048 | −0.0013 | −0.0020 | 0.0064 | 0.0003 | −0.0031 | −0.0008 | −0.0013 | 0.0042 |
| $\mathrm{EB}(b_K^{(4)})'$ | 0.0004 | −0.0038 | −0.0010 | −0.0016 | 0.0052 | 0.0004 | −0.0040 | −0.0011 | −0.0017 | 0.0054 |
| | 0.0011 | −0.0098 | −0.0027 | −0.0041 | 0.0133 | 0.0006 | −0.0054 | −0.0015 | −0.0023 | 0.0073 |
| | −0.0098 | 0.0900 | 0.0246 | 0.0377 | −0.1216 | −0.0054 | 0.0494 | 0.0135 | 0.0207 | −0.0667 |
| $\mathrm{EMSEM}(b_K^{(2)})$ | −0.0027 | 0.0246 | 0.0067 | 0.0103 | −0.0332 | −0.0015 | 0.0135 | 0.0037 | 0.0056 | −0.0182 |
| | −0.0041 | 0.0377 | 0.0103 | 0.0157 | −0.0508 | −0.0023 | 0.0207 | 0.0056 | 0.0086 | −0.0279 |
| | 0.0133 | −0.1216 | −0.0332 | −0.0508 | 0.1641 | 0.0073 | −0.0667 | −0.0182 | −0.0279 | 0.0900 |
| | 0.0011 | −0.0104 | −0.0028 | −0.0044 | 0.0141 | 0.0006 | −0.0056 | −0.0015 | −0.0024 | 0.0076 |
| | −0.0104 | 0.0955 | 0.0260 | 0.0399 | −0.1289 | −0.0056 | 0.0516 | 0.0141 | 0.0216 | −0.0697 |
| $\mathrm{EMSEM}(b_K^{(3)})$ | −0.0028 | 0.0260 | 0.0071 | 0.0109 | −0.0352 | −0.0015 | 0.0141 | 0.0038 | 0.0059 | −0.0190 |
| | −0.0044 | 0.0399 | 0.0109 | 0.0167 | −0.0539 | −0.0024 | 0.0216 | 0.0059 | 0.0090 | −0.0291 |
| | 0.0141 | −0.1289 | −0.0352 | −0.0539 | 0.1740 | 0.0076 | −0.0697 | −0.0190 | −0.0291 | 0.0941 |
| | 0.0013 | −0.0117 | −0.0032 | −0.0049 | 0.0157 | 0.0007 | −0.0061 | −0.0017 | −0.0026 | 0.0083 |
| | −0.0117 | 0.1069 | 0.0291 | 0.0447 | −0.1443 | −0.0061 | 0.0562 | 0.0153 | 0.0235 | −0.0759 |
| $\mathrm{EMSEM}(b_K^{(4)})$ | −0.0032 | 0.0291 | 0.0079 | 0.0122 | −0.0393 | −0.0017 | 0.0153 | 0.0042 | 0.0064 | −0.0207 |
| | −0.0049 | 0.0447 | 0.0122 | 0.0187 | −0.0603 | −0.0026 | 0.0235 | 0.0064 | 0.0098 | −0.0317 |
| | 0.0157 | −0.1443 | −0.0393 | −0.0603 | 0.1947 | 0.0083 | −0.0759 | −0.0207 | −0.0317 | 0.1024 |
| $\sigma_\epsilon^2 = 1.0,\ \sigma_\phi^2 = 1.0,\ \sigma_\delta^2 = 0.4$ | | | | | | | | | | |
| $\mathrm{EB}(b_K^{(2)})'$ | −0.0001 | 0.0011 | 0.0003 | 0.0004 | −0.0014 | 0 | −0.0003 | −0.0001 | −0.0001 | 0.0005 |
| $\mathrm{EB}(b_K^{(3)})'$ | −0.0001 | 0.0013 | 0.0004 | 0.0005 | −0.0017 | 0.0001 | −0.0011 | −0.0003 | −0.0005 | 0.0015 |
| $\mathrm{EB}(b_K^{(4)})'$ | −0.0002 | 0.0018 | 0.0005 | 0.0008 | −0.0025 | 0 | 0 | 0 | 0 | 0 |
| | 0.0009 | −0.0083 | −0.0023 | −0.0035 | 0.0112 | 0.0005 | −0.0045 | −0.0012 | −0.0019 | 0.0061 |
| | −0.0083 | 0.0759 | 0.0207 | 0.0317 | −0.1024 | −0.0045 | 0.0412 | 0.0112 | 0.0172 | −0.0556 |
| $\mathrm{EMSEM}(b_K^{(2)})$ | −0.0023 | 0.0207 | 0.0056 | 0.0087 | −0.0279 | −0.0012 | 0.0112 | 0.0031 | 0.0047 | −0.0152 |
| | −0.0035 | 0.0317 | 0.0087 | 0.0133 | −0.0428 | −0.0019 | 0.0172 | 0.0047 | 0.0072 | −0.0232 |
| | 0.0112 | −0.1024 | −0.0279 | −0.0428 | 0.1383 | 0.0061 | −0.0556 | −0.0152 | −0.0232 | 0.0750 |
| | 0.0009 | −0.0087 | −0.0024 | −0.0036 | 0.0117 | 0.0005 | −0.0047 | −0.0013 | −0.0019 | 0.0063 |
| | −0.0087 | 0.0794 | 0.0217 | 0.0332 | −0.1072 | −0.0047 | 0.0427 | 0.0116 | 0.0179 | −0.0576 |
| $\mathrm{EMSEM}(b_K^{(3)})$ | −0.0024 | 0.0217 | 0.0059 | 0.0091 | −0.0292 | −0.0013 | 0.0116 | 0.0032 | 0.0049 | −0.0157 |
| | −0.0036 | 0.0332 | 0.0091 | 0.0139 | −0.0448 | −0.0019 | 0.0179 | 0.0049 | 0.0075 | −0.0241 |
| | 0.0117 | −0.1072 | −0.0292 | −0.0448 | 0.1448 | 0.0063 | −0.0576 | −0.0157 | −0.0241 | 0.0778 |
| | 0.0011 | −0.0098 | −0.0027 | −0.0041 | 0.0132 | 0.0005 | −0.0050 | −0.0014 | −0.0021 | 0.0067 |
| | −0.0098 | 0.0894 | 0.0244 | 0.0374 | −0.1207 | −0.0050 | 0.0458 | 0.0125 | 0.0192 | −0.0618 |
| $\mathrm{EMSEM}(b_K^{(4)})$ | −0.0027 | 0.0244 | 0.0066 | 0.0102 | −0.0329 | −0.0014 | 0.0125 | 0.0034 | 0.0052 | −0.0169 |
| | −0.0041 | 0.0374 | 0.0102 | 0.0156 | −0.0505 | −0.0021 | 0.0192 | 0.0052 | 0.0080 | −0.0259 |
| | 0.0132 | −0.1207 | −0.0329 | −0.0505 | 0.1629 | 0.0067 | −0.0618 | −0.0169 | −0.0259 | 0.0835 |
| $\sigma_\epsilon^2 = 1.0,\ \sigma_\phi^2 = 1.0,\ \sigma_\delta^2 = 1.0$ | | | | | | | | | | |
| $\mathrm{EB}(b_K^{(2)})'$ | −0.0002 | 0.0022 | 0.0006 | 0.0009 | −0.0030 | 0.0003 | −0.0025 | −0.0007 | −0.0011 | 0.0034 |
| $\mathrm{EB}(b_K^{(3)})'$ | 0.0002 | −0.0014 | −0.0004 | −0.0006 | 0.0019 | 0.0002 | −0.0017 | −0.0005 | −0.0007 | 0.0024 |
| $\mathrm{EB}(b_K^{(4)})'$ | −0.0008 | 0.0075 | 0.0020 | 0.0031 | −0.0101 | 0.0004 | −0.0038 | −0.001 | −0.0016 | 0.0051 |
| | 0.0024 | −0.0223 | −0.0061 | −0.0093 | 0.0301 | 0.0013 | −0.0122 | −0.0033 | −0.0051 | 0.0165 |
| | −0.0223 | 0.2047 | 0.0558 | 0.0856 | −0.2763 | −0.0122 | 0.1120 | 0.0305 | 0.0468 | −0.1512 |
| $\mathrm{EMSEM}(b_K^{(2)})$ | −0.0061 | 0.0558 | 0.0152 | 0.0233 | −0.0754 | −0.0033 | 0.0305 | 0.0083 | 0.0128 | −0.0412 |
| | −0.0093 | 0.0856 | 0.0233 | 0.0358 | −0.1156 | −0.0051 | 0.0468 | 0.0128 | 0.0196 | −0.0632 |
| | 0.0301 | −0.2763 | −0.0754 | −0.1156 | 0.3730 | 0.0165 | −0.1512 | −0.0412 | −0.0632 | 0.2041 |
| | 0.0014 | −0.0131 | −0.0036 | −0.0055 | 0.0177 | 0.0005 | −0.0047 | −0.0013 | −0.0019 | 0.0063 |
| | −0.0131 | 0.1204 | 0.0328 | 0.0503 | −0.1625 | −0.0047 | 0.0427 | 0.0116 | 0.0179 | −0.0576 |
| $\mathrm{EMSEM}(b_K^{(3)})$ | −0.0036 | 0.0328 | 0.0090 | 0.0137 | −0.0443 | −0.0013 | 0.0116 | 0.0032 | 0.0049 | −0.0157 |
| | −0.0055 | 0.0503 | 0.0137 | 0.0210 | −0.0679 | −0.0019 | 0.0179 | 0.0049 | 0.0075 | −0.0241 |
| | 0.0177 | −0.1625 | −0.0443 | −0.0679 | 0.2194 | 0.0063 | −0.0576 | −0.0157 | −0.0241 | 0.0778 |

(*continued on next page*)

Table 1 (*continued*)

| | $n = 25$ | | | | | $n = 45$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.0029 | −0.0270 | −0.0074 | −0.0113 | 0.0365 | 0.0015 | −0.0139 | −0.0038 | −0.0058 | 0.0187 |
| | −0.0270 | 0.2478 | 0.0676 | 0.1036 | −0.3345 | −0.0139 | 0.1272 | 0.0347 | 0.0532 | −0.1717 |
| EMSEM($b_K^{(4)}$) | −0.0074 | 0.0676 | 0.0184 | 0.0283 | −0.0912 | −0.0038 | 0.0347 | 0.0095 | 0.0145 | −0.0468 |
| | −0.0113 | 0.1036 | 0.0283 | 0.0433 | −0.1399 | −0.0058 | 0.0532 | 0.0145 | 0.0222 | −0.0718 |
| | 0.0365 | −0.3345 | −0.0912 | −0.1399 | 0.4516 | 0.0187 | −0.1717 | −0.0468 | −0.0718 | 0.2318 |

**Table 2**

Empirical bias vectors and empirical mean squared error matrices of $b_K^{(2)}$, $b_K^{(3)}$ and $b_K^{(4)}$ when measurement errors and random error component follow $t$-distribution with 12 degrees of freedom.

| | $n = 25$ | | | | | $n = 45$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma_\epsilon^2 = 0.4,\ \sigma_\phi^2 = 0.4,\ \sigma_\delta^2 = 0.4$ | | | | | | | | | | |
| EB($b_K^{(2)}$)′ | 0.0006 | −0.0054 | −0.0015 | −0.0022 | 0.0072 | 0.0002 | −0.0015 | −0.0004 | −0.0006 | 0.0021 |
| EB($b_K^{(3)}$)′ | 0.0006 | −0.0055 | −0.0015 | −0.0023 | 0.0074 | 0.0002 | −0.0018 | −0.0005 | −0.0007 | 0.0024 |
| EB($b_K^{(4)}$)′ | 0.0007 | −0.0066 | −0.0018 | −0.0028 | 0.0089 | 0.0003 | −0.0027 | −0.0007 | −0.0011 | 0.0036 |
| | 0.0011 | −0.0100 | −0.0027 | −0.0042 | 0.0134 | 0.0006 | −0.0055 | −0.0015 | −0.0023 | 0.0074 |
| | −0.0100 | 0.0913 | 0.0249 | 0.0382 | −0.1232 | −0.0055 | 0.0500 | 0.0136 | 0.0209 | −0.0675 |
| EMSEM($b_K^{(2)}$) | −0.0027 | 0.0249 | 0.0068 | 0.0104 | −0.0336 | −0.0015 | 0.0136 | 0.0037 | 0.0057 | −0.0184 |
| | −0.0042 | 0.0382 | 0.0104 | 0.0160 | −0.0515 | −0.0023 | 0.0209 | 0.0057 | 0.0087 | −0.0282 |
| | 0.0134 | −0.1232 | −0.0336 | −0.0515 | 0.1664 | 0.0074 | −0.0675 | −0.0184 | −0.0282 | 0.0911 |
| | 0.0011 | −0.0105 | −0.0029 | −0.0044 | 0.0142 | 0.0006 | −0.0057 | −0.0016 | −0.0024 | 0.0077 |
| | −0.0105 | 0.0966 | 0.0263 | 0.0404 | −0.1304 | −0.0057 | 0.0522 | 0.0142 | 0.0218 | −0.0704 |
| EMSEM($b_K^{(3)}$) | −0.0029 | 0.0263 | 0.0072 | 0.0110 | −0.0356 | −0.0016 | 0.0142 | 0.0039 | 0.0059 | −0.0192 |
| | −0.0044 | 0.0404 | 0.0110 | 0.0169 | −0.0545 | −0.0024 | 0.0218 | 0.0059 | 0.0091 | −0.0294 |
| | 0.0142 | −0.1304 | −0.0356 | −0.0545 | 0.1761 | 0.0077 | −0.0704 | −0.0192 | −0.0294 | 0.0951 |
| | 0.0013 | −0.0117 | −0.0032 | −0.0049 | 0.0159 | 0.0007 | −0.0061 | −0.0017 | −0.0026 | 0.0083 |
| | −0.0117 | 0.1077 | 0.0294 | 0.0450 | −0.1454 | −0.0061 | 0.0562 | 0.0153 | 0.0235 | −0.0758 |
| EMSEM($b_K^{(4)}$) | −0.0032 | 0.0294 | 0.0080 | 0.0123 | −0.0397 | −0.0017 | 0.0153 | 0.0042 | 0.0064 | −0.0207 |
| | −0.0049 | 0.0450 | 0.0123 | 0.0188 | −0.0608 | −0.0026 | 0.0235 | 0.0064 | 0.0098 | −0.0317 |
| | 0.0159 | −0.1454 | −0.0397 | −0.0608 | 0.1963 | 0.0083 | −0.0758 | −0.0207 | −0.0317 | 0.1024 |
| $\sigma_\epsilon^2 = 1.0,\ \sigma_\phi^2 = 1.0,\ \sigma_\delta^2 = 0.4$ | | | | | | | | | | |
| EB($b_K^{(2)}$)′ | −0.0007 | 0.0066 | 0.0018 | 0.0028 | −0.0089 | 0.0001 | −0.0008 | −0.0002 | −0.0003 | 0.0010 |
| EB($b_K^{(3)}$)′ | −0.0007 | 0.0067 | 0.0018 | 0.0028 | −0.0091 | 0.0001 | −0.0012 | −0.0003 | −0.0005 | 0.0016 |
| EB($b_K^{(4)}$)′ | −0.0007 | 0.0060 | 0.0016 | 0.0025 | −0.0081 | 0.0002 | −0.0021 | −0.0006 | −0.0009 | 0.0029 |
| | 0.0009 | −0.0085 | −0.0023 | −0.0035 | 0.0114 | 0.0005 | −0.0044 | −0.0012 | −0.0019 | 0.0060 |
| | −0.0085 | 0.0776 | 0.0212 | 0.0325 | −0.1048 | −0.0044 | 0.0407 | 0.0111 | 0.0170 | −0.0550 |
| EMSEM($b_K^{(2)}$) | −0.0023 | 0.0212 | 0.0058 | 0.0089 | −0.0286 | −0.0012 | 0.0111 | 0.0030 | 0.0046 | −0.0150 |
| | −0.0035 | 0.0325 | 0.0089 | 0.0136 | −0.0438 | −0.0019 | 0.0170 | 0.0046 | 0.0071 | −0.0230 |
| | 0.0114 | −0.1048 | −0.0286 | −0.0438 | 0.1414 | 0.0060 | −0.0550 | −0.0150 | −0.0230 | 0.0742 |
| | 0.0010 | −0.0089 | −0.0024 | −0.0037 | 0.0120 | 0.0005 | −0.0046 | −0.0012 | −0.0019 | 0.0062 |
| | −0.0089 | 0.0813 | 0.0222 | 0.0340 | −0.1098 | −0.0046 | 0.0420 | 0.0115 | 0.0176 | −0.0567 |
| EMSEM($b_K^{(3)}$) | −0.0024 | 0.0222 | 0.0061 | 0.0093 | −0.0299 | −0.0012 | 0.0115 | 0.0031 | 0.0048 | −0.0155 |
| | −0.0037 | 0.0340 | 0.0093 | 0.0142 | −0.0459 | −0.0019 | 0.0176 | 0.0048 | 0.0073 | −0.0237 |
| | 0.0120 | −0.1098 | −0.0299 | −0.0459 | 0.1482 | 0.0062 | −0.0567 | −0.0155 | −0.0237 | 0.0766 |
| | 0.0011 | −0.0100 | −0.0027 | −0.0042 | 0.0134 | 0.0005 | −0.0050 | −0.0014 | −0.0021 | 0.0068 |
| | −0.0100 | 0.0912 | 0.0249 | 0.0381 | −0.1231 | −0.0050 | 0.0459 | 0.0125 | 0.0192 | −0.0620 |
| EMSEM($b_K^{(4)}$) | −0.0027 | 0.0249 | 0.0068 | 0.0104 | −0.0336 | −0.0014 | 0.0125 | 0.0034 | 0.0052 | −0.0169 |
| | −0.0042 | 0.0381 | 0.0104 | 0.0160 | −0.0515 | −0.0021 | 0.0192 | 0.0052 | 0.0080 | −0.0259 |
| | 0.0134 | −0.1231 | −0.0336 | −0.0515 | 0.1662 | 0.0068 | −0.0620 | −0.0169 | −0.0259 | 0.0837 |
| $\sigma_\epsilon^2 = 1.0,\ \sigma_\phi^2 = 1.0,\ \sigma_\delta^2 = 1.0$ | | | | | | | | | | |
| EB($b_K^{(2)}$)′ | 0.0005 | −0.0046 | −0.0013 | −0.0019 | 0.0063 | 0.0001 | −0.0005 | −0.0001 | −0.0002 | 0.0007 |
| EB($b_K^{(3)}$)′ | 0.0005 | −0.0043 | −0.0012 | −0.0018 | 0.0058 | −0.0001 | 0.0005 | 0.0001 | 0.0002 | −0.0007 |
| EB($b_K^{(4)}$)′ | 0.0002 | −0.0017 | −0.0005 | −0.0007 | 0.0023 | 0 | −0.0003 | −0.0001 | −0.0001 | 0.0004 |

Table 2 (*continued*)

| | $n = 25$ | | | | | $n = 45$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.0025 | −0.0233 | −0.0063 | −0.0097 | 0.0314 | 0.0013 | −0.0123 | −0.0034 | −0.0051 | 0.0166 |
| | −0.0233 | 0.2133 | 0.0582 | 0.0892 | −0.2880 | −0.0123 | 0.1127 | 0.0307 | 0.0471 | −0.1522 |
| EMSEM($b_K^{(2)}$) | −0.0063 | 0.0582 | 0.0159 | 0.0243 | −0.0785 | −0.0034 | 0.0307 | 0.0084 | 0.0129 | −0.0415 |
| | −0.0097 | 0.0892 | 0.0243 | 0.0373 | −0.1204 | −0.0051 | 0.0471 | 0.0129 | 0.0197 | −0.0636 |
| | 0.0314 | −0.2880 | −0.0785 | −0.1204 | 0.3888 | 0.0166 | −0.1522 | −0.0415 | −0.0636 | 0.2054 |
| | 0.0028 | −0.0255 | −0.0070 | −0.0107 | 0.0344 | 0.0014 | −0.0132 | −0.0036 | −0.0055 | 0.0178 |
| | −0.0255 | 0.2339 | 0.0638 | 0.0978 | −0.3157 | −0.0132 | 0.1209 | 0.0330 | 0.0506 | −0.1632 |
| EMSEM($b_K^{(3)}$) | −0.0070 | 0.0638 | 0.0174 | 0.0267 | −0.0861 | −0.0036 | 0.0330 | 0.0090 | 0.0138 | −0.0445 |
| | −0.0107 | 0.0978 | 0.0267 | 0.0409 | −0.1320 | −0.0055 | 0.0506 | 0.0138 | 0.0211 | −0.0683 |
| | 0.0344 | −0.3157 | −0.0861 | −0.1320 | 0.4263 | 0.0178 | −0.1632 | −0.0445 | −0.0683 | 0.2204 |
| | 0.0030 | −0.0278 | −0.0076 | −0.0116 | 0.0375 | 0.0015 | −0.0141 | −0.0038 | −0.0059 | 0.0190 |
| | −0.0278 | 0.2545 | 0.0694 | 0.1064 | −0.3436 | −0.0141 | 0.1293 | 0.0353 | 0.0541 | −0.1746 |
| EMSEM($b_K^{(4)}$) | −0.0076 | 0.0694 | 0.0189 | 0.0290 | −0.0937 | −0.0038 | 0.0353 | 0.0096 | 0.0147 | −0.0476 |
| | −0.0116 | 0.1064 | 0.0290 | 0.0445 | −0.1437 | −0.0059 | 0.0541 | 0.0147 | 0.0226 | −0.0730 |
| | 0.0375 | −0.3436 | −0.0937 | −0.1437 | 0.4638 | 0.0190 | −0.1746 | −0.0476 | −0.0730 | 0.2357 |

**Table 3**

Empirical bias vectors and empirical mean squared error matrices of $b_K^{(2)}$, $b_K^{(3)}$ and $b_K^{(4)}$ when measurement errors and random error component follow gamma distribution.

| | $n = 25$ | | | | | $n = 45$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma_\epsilon^2 = 0.4,\ \sigma_\phi^2 = 0.4,\ \sigma_\delta^2 = 0.4$ | | | | | | | | | | |
| EB($b_K^{(2)}$)′ | 0.0001 | −0.0012 | −0.0003 | −0.0005 | 0.0016 | 0.0003 | −0.0025 | −0.0007 | −0.0010 | 0.0034 |
| EB($b_K^{(3)}$)′ | 0.0002 | −0.0018 | −0.0005 | −0.0007 | 0.0024 | 0.0003 | −0.0027 | −0.0007 | −0.0011 | 0.0037 |
| EB($b_K^{(4)}$)′ | 0 | 0.0002 | 0.0001 | 0.0001 | −0.0003 | 0.0005 | −0.0048 | −0.0013 | −0.002 | 0.0065 |
| | 0.0011 | −0.0100 | −0.0027 | −0.0042 | 0.0135 | 0.0006 | −0.0055 | −0.0015 | −0.0023 | 0.0074 |
| | −0.0100 | 0.0916 | 0.0250 | 0.0383 | −0.1237 | −0.0055 | 0.0502 | 0.0137 | 0.0210 | −0.0677 |
| EMSEM($b_K^{(2)}$) | −0.0027 | 0.0250 | 0.0068 | 0.0105 | −0.0337 | −0.0015 | 0.0137 | 0.0037 | 0.0057 | −0.0185 |
| | −0.0042 | 0.0383 | 0.0105 | 0.0160 | −0.0517 | −0.0023 | 0.0210 | 0.0057 | 0.0088 | −0.0283 |
| | 0.0135 | −0.1237 | −0.0337 | −0.0517 | 0.1670 | 0.0074 | −0.0677 | −0.0185 | −0.0283 | 0.0914 |
| | 0.0012 | −0.0107 | −0.0029 | −0.0045 | 0.0144 | 0.0006 | −0.0058 | −0.0016 | −0.0024 | 0.0078 |
| | −0.0107 | 0.0981 | 0.0267 | 0.0410 | −0.1324 | −0.0058 | 0.0527 | 0.0144 | 0.0221 | −0.0712 |
| EMSEM($b_K^{(3)}$) | −0.0029 | 0.0267 | 0.0073 | 0.0112 | −0.0361 | −0.0016 | 0.0144 | 0.0039 | 0.0060 | −0.0194 |
| | −0.0045 | 0.0410 | 0.0112 | 0.0171 | −0.0554 | −0.0024 | 0.0221 | 0.0060 | 0.0092 | −0.0298 |
| | 0.0144 | −0.1324 | −0.0361 | −0.0554 | 0.1787 | 0.0078 | −0.0712 | −0.0194 | −0.0298 | 0.0961 |
| | 0.0013 | −0.0116 | −0.0032 | −0.0049 | 0.0157 | 0.0007 | −0.0061 | −0.0017 | −0.0026 | 0.0083 |
| | −0.0116 | 0.1064 | 0.0290 | 0.0445 | −0.1436 | −0.0061 | 0.0563 | 0.0154 | 0.0236 | −0.0760 |
| EMSEM($b_K^{(4)}$) | −0.0032 | 0.0290 | 0.0079 | 0.0121 | −0.0392 | −0.0017 | 0.0154 | 0.0042 | 0.0064 | −0.0207 |
| | −0.0049 | 0.0445 | 0.0121 | 0.0186 | −0.0601 | −0.0026 | 0.0236 | 0.0064 | 0.0098 | −0.0318 |
| | 0.0157 | −0.1436 | −0.0392 | −0.0601 | 0.1939 | 0.0083 | −0.0760 | −0.0207 | −0.0318 | 0.1026 |
| $\sigma_\epsilon^2 = 1.0,\ \sigma_\phi^2 = 1.0,\ \sigma_\delta^2 = 0.4$ | | | | | | | | | | |
| EB($b_K^{(2)}$)′ | 0.0002 | −0.0021 | −0.0006 | −0.0009 | 0.0028 | 0 | −0.0001 | 0 | 0 | 0.0001 |
| EB($b_K^{(3)}$)′ | 0.0003 | −0.0025 | −0.0007 | −0.0011 | 0.0034 | 0 | 0 | 0 | 0 | 0 |
| EB($b_K^{(4)}$)′ | 0.0003 | −0.0024 | −0.0007 | −0.001 | 0.0033 | 0 | 0.0002 | 0.0001 | 0.0001 | −0.0003 |
| | 0.0009 | −0.0085 | −0.0023 | −0.0035 | 0.0115 | 0.0005 | −0.0045 | −0.0012 | −0.0019 | 0.0060 |
| | −0.0085 | 0.0778 | 0.0212 | 0.0325 | −0.1050 | −0.0045 | 0.0410 | 0.0112 | 0.0171 | −0.0553 |
| EMSEM($b_K^{(2)}$) | −0.0023 | 0.0212 | 0.0058 | 0.0089 | −0.0286 | −0.0012 | 0.0112 | 0.0030 | 0.0047 | −0.0151 |
| | −0.0035 | 0.0325 | 0.0089 | 0.0136 | −0.0439 | −0.0019 | 0.0171 | 0.0047 | 0.0072 | −0.0231 |
| | 0.0115 | −0.1050 | −0.0286 | −0.0439 | 0.1417 | 0.0060 | −0.0553 | −0.0151 | −0.0231 | 0.0747 |
| | 0.0010 | −0.0089 | −0.0024 | −0.0037 | 0.0120 | 0.0005 | −0.0047 | −0.0013 | −0.0020 | 0.0063 |
| | −0.0089 | 0.0816 | 0.0223 | 0.0341 | −0.1102 | −0.0047 | 0.0428 | 0.0117 | 0.0179 | −0.0577 |
| EMSEM($b_K^{(3)}$) | −0.0024 | 0.0223 | 0.0061 | 0.0093 | −0.0301 | −0.0013 | 0.0117 | 0.0032 | 0.0049 | −0.0157 |
| | −0.0037 | 0.0341 | 0.0093 | 0.0143 | −0.0461 | −0.0020 | 0.0179 | 0.0049 | 0.0075 | −0.0241 |
| | 0.0120 | −0.1102 | −0.0301 | −0.0461 | 0.1488 | 0.0063 | −0.0577 | −0.0157 | −0.0241 | 0.0779 |
| | 0.0011 | −0.0099 | −0.0027 | −0.0041 | 0.0133 | 0.0005 | −0.0050 | −0.0014 | −0.0021 | 0.0067 |
| | −0.0099 | 0.0904 | 0.0247 | 0.0378 | −0.1221 | −0.0050 | 0.0456 | 0.0124 | 0.0191 | −0.0615 |
| EMSEM($b_K^{(4)}$) | −0.0027 | 0.0247 | 0.0067 | 0.0103 | −0.0333 | −0.0014 | 0.0124 | 0.0034 | 0.0052 | −0.0168 |

(*continued on next page*)

Table 3 (*continued*)

| | $n = 25$ | | | | | $n = 45$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | −0.0041 | 0.0378 | 0.0103 | 0.0158 | −0.0511 | −0.0021 | 0.0191 | 0.0052 | 0.0080 | −0.0257 |
| | 0.0133 | −0.1221 | −0.0333 | −0.0511 | 0.1648 | 0.0067 | −0.0615 | −0.0168 | −0.0257 | 0.0830 |
| $\sigma_\epsilon^2 = 1.0,\ \sigma_\phi^2 = 1.0,\ \sigma_\delta^2 = 1.0$ | | | | | | | | | | |
| $EB(b_K^{(2)})'$ | −0.0001 | 0.0007 | 0.0002 | 0.0003 | −0.0010 | 0.0006 | −0.0051 | −0.0014 | −0.0022 | 0.0069 |
| $EB(b_K^{(3)})'$ | 0 | −0.0001 | 0 | 0 | 0.0002 | 0.0006 | −0.0052 | −0.0014 | −0.0022 | 0.0070 |
| $EB(b_K^{(4)})'$ | −0.0008 | 0.0076 | 0.0021 | 0.0032 | −0.0103 | 0.0007 | −0.0062 | −0.0017 | −0.0026 | 0.0084 |
| | 0.0025 | −0.0230 | −0.0063 | −0.0096 | 0.0311 | 0.0014 | −0.0125 | −0.0034 | −0.0052 | 0.0168 |
| | −0.0230 | 0.2112 | 0.0576 | 0.0883 | −0.2852 | −0.0125 | 0.1142 | 0.0311 | 0.0478 | −0.1542 |
| $EMSEM(b_K^{(2)})$ | −0.0063 | 0.0576 | 0.0157 | 0.0241 | −0.0778 | −0.0034 | 0.0311 | 0.0085 | 0.0130 | −0.0420 |
| | −0.0096 | 0.0883 | 0.0241 | 0.0369 | −0.1193 | −0.0052 | 0.0478 | 0.0130 | 0.0200 | −0.0645 |
| | 0.0311 | −0.2852 | −0.0778 | −0.1193 | 0.3850 | 0.0168 | −0.1542 | −0.0420 | −0.0645 | 0.2081 |
| | 0.0028 | −0.0256 | −0.0070 | −0.0107 | 0.0345 | 0.0015 | −0.0137 | −0.0037 | −0.0057 | 0.0185 |
| | −0.0256 | 0.2343 | 0.0639 | 0.0980 | −0.3163 | −0.0137 | 0.1256 | 0.0343 | 0.0525 | −0.1696 |
| $EMSEM(b_K^{(3)})$ | −0.0070 | 0.0639 | 0.0174 | 0.0267 | −0.0863 | −0.0037 | 0.0343 | 0.0093 | 0.0143 | −0.0463 |
| | −0.0107 | 0.0980 | 0.0267 | 0.0410 | −0.1323 | −0.0057 | 0.0525 | 0.0143 | 0.0220 | −0.0709 |
| | 0.0345 | −0.3163 | −0.0863 | −0.1323 | 0.4270 | 0.0185 | −0.1696 | −0.0463 | −0.0709 | 0.2290 |
| | 0.0030 | −0.0272 | −0.0074 | −0.0114 | 0.0367 | 0.0015 | −0.0140 | −0.0038 | −0.0059 | 0.0190 |
| | −0.0272 | 0.2492 | 0.0680 | 0.1042 | −0.3364 | −0.0140 | 0.1287 | 0.0351 | 0.0538 | −0.1737 |
| $EMSEM(b_K^{(4)})$ | −0.0074 | 0.0680 | 0.0185 | 0.0284 | −0.0917 | −0.0038 | 0.0351 | 0.0096 | 0.0147 | −0.0474 |
| | −0.0114 | 0.1042 | 0.0284 | 0.0436 | −0.1407 | −0.0059 | 0.0538 | 0.0147 | 0.0225 | −0.0726 |
| | 0.0367 | −0.3364 | −0.0917 | −0.1407 | 0.4541 | 0.0190 | −0.1737 | −0.0474 | −0.0726 | 0.2345 |

**Table 4**

Empirical bias vectors and empirical mean squared error matrices of $b_\delta^{(2)}$, $b_\delta^{(3)}$ and $b_\delta^{(4)}$ when measurement errors and random error component follow normal distribution.

| | $n = 25$ | | | | | $n = 45$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma_\epsilon^2 = 0.4,\ \sigma_\phi^2 = 0.4,\ \sigma_\delta^2 = 0.4$ | | | | | | | | | | |
| $EB(b_\delta^{(2)})'$ | 0.0008 | −0.0071 | −0.0019 | −0.0030 | 0.0096 | −0.0001 | 0.0011 | 0.0003 | 0.0005 | −0.0015 |
| $EB(b_\delta^{(3)})'$ | 0.0006 | −0.0059 | −0.0016 | −0.0025 | 0.0079 | −0.0002 | 0.0019 | 0.0005 | 0.0008 | −0.0026 |
| $EB(b_\delta^{(4)})'$ | 0.0015 | −0.0134 | −0.0037 | −0.0056 | 0.0182 | 0.0005 | −0.0042 | −0.0011 | −0.0017 | 0.0056 |
| | 0.0012 | −0.0111 | −0.0030 | −0.0046 | 0.0150 | 0.0007 | −0.0060 | −0.0016 | −0.0025 | 0.0081 |
| | −0.0111 | 0.1019 | 0.0278 | 0.0426 | −0.1375 | −0.0060 | 0.0550 | 0.0150 | 0.0230 | −0.0742 |
| $EMSEM(b_\delta^{(2)})$ | −0.0030 | 0.0278 | 0.0076 | 0.0116 | −0.0375 | −0.0016 | 0.0150 | 0.0041 | 0.0063 | −0.0202 |
| | −0.0046 | 0.0426 | 0.0116 | 0.0178 | −0.0575 | −0.0025 | 0.0230 | 0.0063 | 0.0096 | −0.0310 |
| | 0.0150 | −0.1375 | −0.0375 | −0.0575 | 0.1857 | 0.0081 | −0.0742 | −0.0202 | −0.0310 | 0.1002 |
| | 0.0012 | −0.0109 | −0.0030 | −0.0046 | 0.0147 | 0.0006 | −0.0059 | −0.0016 | −0.0025 | 0.0080 |
| | −0.0109 | 0.1000 | 0.0273 | 0.0418 | −0.1349 | −0.0059 | 0.0544 | 0.0148 | 0.0227 | −0.0734 |
| $EMSEM(b_\delta^{(3)})$ | −0.0030 | 0.0273 | 0.0074 | 0.0114 | −0.0368 | −0.0016 | 0.0148 | 0.0040 | 0.0062 | −0.0200 |
| | −0.0046 | 0.0418 | 0.0114 | 0.0175 | −0.0564 | −0.0025 | 0.0227 | 0.0062 | 0.0095 | −0.0307 |
| | 0.0147 | −0.1349 | −0.0368 | −0.0564 | 0.1822 | 0.0080 | −0.0734 | −0.0200 | −0.0307 | 0.0991 |
| | 0.0016 | −0.0143 | −0.0039 | −0.0060 | 0.0193 | 0.0008 | −0.0071 | −0.0019 | −0.0030 | 0.0095 |
| | −0.0143 | 0.1308 | 0.0357 | 0.0547 | −0.1766 | −0.0071 | 0.0647 | 0.0176 | 0.0271 | −0.0874 |
| $EMSEM(b_\delta^{(4)})$ | −0.0039 | 0.0357 | 0.0097 | 0.0149 | −0.0482 | −0.0019 | 0.0176 | 0.0048 | 0.0074 | −0.0238 |
| | −0.0060 | 0.0547 | 0.0149 | 0.0229 | −0.0739 | −0.0030 | 0.0271 | 0.0074 | 0.0113 | −0.0365 |
| | 0.0193 | −0.1766 | −0.0482 | −0.0739 | 0.2384 | 0.0095 | −0.0874 | −0.0238 | −0.0365 | 0.1179 |
| $\sigma_\epsilon^2 = 1.0,\ \sigma_\phi^2 = 1.0,\ \sigma_\delta^2 = 0.4$ | | | | | | | | | | |
| $EB(b_\delta^{(2)})'$ | −0.0004 | 0.0041 | 0.0011 | 0.0017 | −0.0056 | 0.0002 | −0.0015 | −0.0004 | −0.0006 | 0.0020 |
| $EB(b_\delta^{(3)})'$ | −0.0005 | 0.0048 | 0.0013 | 0.0020 | −0.0065 | 0.0001 | −0.0011 | −0.0003 | −0.0004 | 0.0015 |
| $EB(b_\delta^{(4)})'$ | 0.0001 | −0.0014 | −0.0004 | −0.0006 | 0.0018 | 0.0005 | −0.0044 | −0.0012 | −0.0018 | 0.0059 |
| | 0.0009 | −0.0087 | −0.0024 | −0.0036 | 0.0117 | 0.0005 | −0.0046 | −0.0013 | −0.0019 | 0.0063 |
| | −0.0087 | 0.0794 | 0.0217 | 0.0332 | −0.1072 | −0.0046 | 0.0425 | 0.0116 | 0.0178 | −0.0574 |
| $EMSEM(b_\delta^{(2)})$ | −0.0024 | 0.0217 | 0.0059 | 0.0091 | −0.0292 | −0.0013 | 0.0116 | 0.0032 | 0.0049 | −0.0157 |
| | −0.0036 | 0.0332 | 0.0091 | 0.0139 | −0.0448 | −0.0019 | 0.0178 | 0.0049 | 0.0074 | −0.0240 |
| | 0.0117 | −0.1072 | −0.0292 | −0.0448 | 0.1447 | 0.0063 | −0.0574 | −0.0157 | −0.0240 | 0.0775 |

Table 4 (*continued*)

| | $n = 25$ | | | | | $n = 45$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.0009 | −0.0085 | −0.0023 | −0.0036 | 0.0115 | 0.0005 | −0.0046 | −0.0013 | −0.0019 | 0.0062 |
| | −0.0085 | 0.0780 | 0.0213 | 0.0326 | −0.1052 | −0.0046 | 0.0422 | 0.0115 | 0.0177 | −0.0570 |
| EMSEM($b_\delta^{(3)}$) | −0.0023 | 0.0213 | 0.0058 | 0.0089 | −0.0287 | −0.0013 | 0.0115 | 0.0031 | 0.0048 | −0.0155 |
| | −0.0036 | 0.0326 | 0.0089 | 0.0136 | −0.0440 | −0.0019 | 0.0177 | 0.0048 | 0.0074 | −0.0238 |
| | 0.0115 | −0.1052 | −0.0287 | −0.0440 | 0.1421 | 0.0062 | −0.0570 | −0.0155 | −0.0238 | 0.0770 |
| | 0.0013 | −0.0117 | −0.0032 | −0.0049 | 0.0158 | 0.0006 | −0.0054 | −0.0015 | −0.0023 | 0.0073 |
| | −0.0117 | 0.1075 | 0.0293 | 0.0450 | −0.1451 | −0.0054 | 0.0496 | 0.0135 | 0.0207 | −0.0669 |
| EMSEM($b_\delta^{(4)}$) | −0.0032 | 0.0293 | 0.0080 | 0.0123 | −0.0396 | −0.0015 | 0.0135 | 0.0037 | 0.0057 | −0.0183 |
| | −0.0049 | 0.0450 | 0.0123 | 0.0188 | −0.0607 | −0.0023 | 0.0207 | 0.0057 | 0.0087 | −0.0280 |
| | 0.0158 | −0.1451 | −0.0396 | −0.0607 | 0.1959 | 0.0073 | −0.0669 | −0.0183 | −0.0280 | 0.0904 |
| $\sigma_\epsilon^2 = 1.0,\ \sigma_\phi^2 = 1.0,\ \sigma_\delta^2 = 1.0$ | | | | | | | | | | |
| EB($b_\delta^{(2)}$)′ | 0.0008 | −0.0074 | −0.0020 | −0.0031 | 0.0101 | 0.0005 | −0.0037 | −0.0010 | −0.0013 | 0.0041 |
| EB($b_\delta^{(3)}$)′ | −0.0009 | 0.0078 | 0.0021 | 0.0033 | −0.0105 | −0.0004 | 0.0038 | 0.0010 | 0.0015 | −0.0049 |
| EB($b_\delta^{(4)}$)′ | 0.0053 | −0.0489 | −0.0133 | −0.0205 | 0.0661 | 0.0018 | −0.0162 | −0.0044 | −0.0068 | 0.0219 |
| | 0.0069 | −0.0636 | −0.0173 | −0.0266 | 0.0858 | 0.0015 | −0.0142 | −0.0039 | −0.0059 | 0.0191 |
| | −0.0636 | 0.5826 | 0.1589 | 0.2436 | −0.7865 | −0.0142 | 0.1299 | 0.0354 | 0.0543 | −0.1754 |
| EMSEM($b_\delta^{(2)}$) | −0.0173 | 0.1589 | 0.0433 | 0.0664 | −0.2145 | −0.0039 | 0.0354 | 0.0097 | 0.0148 | −0.0478 |
| | −0.0266 | 0.2436 | 0.0664 | 0.1019 | −0.3289 | −0.0059 | 0.0543 | 0.0148 | 0.0227 | −0.0733 |
| | 0.0858 | −0.7865 | −0.2145 | −0.3289 | 1.0617 | 0.0191 | −0.1754 | −0.0478 | −0.0733 | 0.2367 |
| | 0.0031 | −0.0285 | −0.0078 | −0.0119 | 0.0385 | 0.0015 | −0.0134 | −0.0037 | −0.0056 | 0.0181 |
| | −0.0285 | 0.2611 | 0.0712 | 0.1092 | −0.3526 | −0.0134 | 0.1231 | 0.0336 | 0.0515 | −0.1662 |
| EMSEM($b_\delta^{(3)}$) | −0.0078 | 0.0712 | 0.0194 | 0.0298 | −0.0962 | −0.0037 | 0.0336 | 0.0092 | 0.0140 | −0.0453 |
| | −0.0119 | 0.1092 | 0.0298 | 0.0457 | −0.1474 | −0.0056 | 0.0515 | 0.0140 | 0.0215 | −0.0695 |
| | 0.0385 | −0.3526 | −0.0962 | −0.1474 | 0.4759 | 0.0181 | −0.1662 | −0.0453 | −0.0695 | 0.2243 |
| | 0.0537 | −0.4918 | −0.1341 | −0.2057 | 0.6640 | 0.0021 | −0.0189 | −0.0052 | −0.0079 | 0.0255 |
| | −0.4918 | 4.5084 | 1.2296 | 1.8853 | −6.0863 | −0.0189 | 0.1734 | 0.0473 | 0.0725 | −0.2341 |
| EMSEM($b_\delta^{(4)}$) | −0.1341 | 1.2296 | 0.3353 | 0.5142 | −1.6599 | −0.0052 | 0.0473 | 0.0129 | 0.0198 | −0.0638 |
| | −0.2057 | 1.8853 | 0.5142 | 0.7884 | −2.5452 | −0.0079 | 0.0725 | 0.0198 | 0.0303 | −0.0979 |
| | 0.6640 | −6.0863 | −1.6599 | −2.5452 | 8.2165 | 0.0255 | −0.2341 | −0.0638 | −0.0979 | 0.3160 |

**Table 5**

Empirical bias vectors and empirical mean squared error matrices of $b_\delta^{(2)}$, $b_\delta^{(3)}$ and $b_\delta^{(4)}$ when measurement errors and random error component follow $t$-distribution with 12 degrees of freedom.

| | $n = 25$ | | | | | $n = 45$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma_\epsilon^2 = 0.4,\ \sigma_\phi^2 = 0.4,\ \sigma_\delta^2 = 0.4$ | | | | | | | | | | |
| EB($b_\delta^{(2)}$)′ | 0.0008 | −0.0078 | −0.0021 | −0.0033 | 0.0105 | −0.0008 | 0.0078 | 0.0021 | 0.0033 | −0.0105 |
| EB($b_\delta^{(3)}$)′ | 0.0007 | −0.0066 | −0.0018 | −0.0028 | 0.0089 | −0.0010 | 0.0087 | 0.0024 | 0.0037 | −0.0118 |
| EB($b_\delta^{(4)}$)′ | 0.0017 | −0.0155 | −0.0042 | −0.0065 | 0.0209 | −0.0002 | 0.0019 | 0.0005 | 0.0008 | −0.0026 |
| | 0.0013 | −0.0115 | −0.0031 | −0.0048 | 0.0156 | 0.0006 | −0.0058 | −0.0016 | −0.0024 | 0.0078 |
| | −0.0115 | 0.1057 | 0.0288 | 0.0442 | −0.1427 | −0.0058 | 0.0528 | 0.0144 | 0.0221 | −0.0713 |
| EMSEM($b_\delta^{(2)}$) | −0.0031 | 0.0288 | 0.0079 | 0.0121 | −0.0389 | −0.0016 | 0.0144 | 0.0039 | 0.0060 | −0.0195 |
| | −0.0048 | 0.0442 | 0.0121 | 0.0185 | −0.0597 | −0.0024 | 0.0221 | 0.0060 | 0.0092 | −0.0298 |
| | 0.0156 | −0.1427 | −0.0389 | −0.0597 | 0.1927 | 0.0078 | −0.0713 | −0.0195 | −0.0298 | 0.0963 |
| | 0.0012 | −0.0113 | −0.0031 | −0.0047 | 0.0153 | 0.0006 | −0.0057 | −0.0016 | −0.0024 | 0.0077 |
| | −0.0113 | 0.1036 | 0.0282 | 0.0433 | −0.1398 | −0.0057 | 0.0523 | 0.0143 | 0.0219 | −0.0706 |
| EMSEM($b_\delta^{(3)}$) | −0.0031 | 0.0282 | 0.0077 | 0.0118 | −0.0381 | −0.0016 | 0.0143 | 0.0039 | 0.0060 | −0.0192 |
| | −0.0047 | 0.0433 | 0.0118 | 0.0181 | −0.0585 | −0.0024 | 0.0219 | 0.0060 | 0.0091 | −0.0295 |
| | 0.0153 | −0.1398 | −0.0381 | −0.0585 | 0.1888 | 0.0077 | −0.0706 | −0.0192 | −0.0295 | 0.0953 |
| | 0.0016 | −0.0150 | −0.0041 | −0.0063 | 0.0203 | 0.0008 | −0.0069 | −0.0019 | −0.0029 | 0.0093 |
| | −0.0150 | 0.1377 | 0.0375 | 0.0576 | −0.1858 | −0.0069 | 0.0634 | 0.0173 | 0.0265 | −0.0856 |
| EMSEM($b_\delta^{(4)}$) | −0.0041 | 0.0375 | 0.0102 | 0.0157 | −0.0507 | −0.0019 | 0.0173 | 0.0047 | 0.0072 | −0.0234 |
| | −0.0063 | 0.0576 | 0.0157 | 0.0241 | −0.0777 | −0.0029 | 0.0265 | 0.0072 | 0.0111 | −0.0358 |
| | 0.0203 | −0.1858 | −0.0507 | −0.0777 | 0.2509 | 0.0093 | −0.0856 | −0.0234 | −0.0358 | 0.1156 |
| $\sigma_\epsilon^2 = 1.0,\ \sigma_\phi^2 = 1.0,\ \sigma_\delta^2 = 0.4$ | | | | | | | | | | |
| EB($b_\delta^{(2)}$)′ | 0.0012 | −0.0076 | −0.0029 | −0.0023 | 0.0093 | −0.0009 | 0.0068 | 0.0019 | 0.0027 | −0.0093 |

(*continued on next page*)

Table 5 (*continued*)

| | $n=25$ | | | | | $n=45$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $EB(b_\delta^{(3)})'$ | 0.0013 | −0.0054 | −0.0019 | −0.0033 | 0.0075 | −0.0015 | 0.0082 | 0.0032 | 0.0027 | −0.0099 |
| $EB(b_\delta^{(4)})'$ | 0.0015 | −0.0149 | −0.0037 | −0.0054 | 0.0201 | −0.0006 | 0.0021 | 0.0009 | 0.0004 | −0.0017 |
| $EMSEM(b_\delta^{(2)})$ | 0.0008 | −0.0106 | −0.0025 | −0.0062 | 0.0155 | 0.0005 | −0.0079 | 0.0025 | −0.0018 | 0.0053 |
| | −0.0106 | 0.1041 | 0.0277 | 0.0468 | −0.1426 | −0.0079 | 0.0510 | 0.0180 | 0.0226 | −0.0735 |
| | −0.0025 | 0.0277 | 0.0072 | 0.0138 | −0.0388 | 0.0025 | 0.0180 | 0.0035 | 0.0049 | −0.0152 |
| | −0.0062 | 0.0468 | 0.0138 | 0.0144 | −0.0599 | −0.0018 | 0.0226 | 0.0049 | 0.0090 | −0.0291 |
| | 0.0155 | −0.1426 | −0.0388 | −0.0599 | 0.1927 | 0.0053 | −0.0735 | −0.0152 | −0.0291 | 0.0937 |
| $EMSEM(b_\delta^{(3)})$ | 0.0003 | −0.0117 | −0.0025 | −0.0042 | 0.0144 | 0.0005 | −0.0053 | −0.0011 | −0.0024 | 0.0072 |
| | −0.0117 | 0.1034 | 0.0285 | 0.0436 | −0.1403 | −0.0053 | 0.0501 | 0.0116 | 0.0218 | −0.0679 |
| | −0.0025 | 0.0285 | 0.0072 | 0.0114 | −0.0374 | −0.0011 | 0.0116 | 0.0006 | 0.0058 | −0.0159 |
| | −0.0042 | 0.0436 | 0.0114 | 0.0178 | −0.0579 | −0.0024 | 0.0218 | 0.0058 | 0.0091 | −0.0293 |
| | 0.0144 | −0.1403 | −0.0374 | −0.0579 | 0.1878 | 0.0072 | −0.0679 | −0.0159 | −0.0293 | 0.0920 |
| $EMSEM(b_\delta^{(4)})$ | 0.0008 | −0.0199 | −0.0012 | −0.0071 | 0.0216 | 0.0008 | −0.0097 | 0.0010 | −0.0024 | 0.0082 |
| | −0.0199 | 0.1277 | 0.0434 | 0.0560 | −0.1832 | −0.0097 | 0.0603 | 0.0205 | 0.0271 | −0.0868 |
| | −0.0012 | 0.0434 | 0.0067 | 0.0166 | −0.0523 | 0.0010 | 0.0205 | 0.0015 | 0.0066 | −0.0221 |
| | −0.0071 | 0.0560 | 0.0166 | 0.0238 | −0.0773 | −0.0024 | 0.0271 | 0.0066 | 0.0110 | −0.0356 |
| | 0.0216 | −0.1832 | −0.0523 | −0.0773 | 0.2502 | 0.0082 | −0.0868 | −0.0221 | −0.0356 | 0.1151 |
| $\sigma_\epsilon^2=1.0,\ \sigma_\phi^2=1.0,\ \sigma_\delta^2=1.0$ | | | | | | | | | | |
| $EB(b_\delta^{(2)})'$ | 0.0003 | −0.0027 | −0.0007 | −0.0011 | 0.0036 | 0.0005 | −0.0046 | −0.0013 | −0.0019 | 0.0062 |
| $EB(b_\delta^{(3)})'$ | 0.0006 | −0.0054 | −0.0015 | −0.0023 | 0.0073 | 0.0001 | −0.0006 | −0.0002 | −0.0002 | 0.0008 |
| $EB(b_\delta^{(4)})'$ | −0.0007 | 0.0068 | 0.0019 | 0.0028 | −0.0092 | 0.0017 | −0.0154 | −0.0042 | −0.0065 | 0.0209 |
| $EMSEM(b_\delta^{(2)})$ | 0.0056 | −0.0511 | −0.0139 | −0.0214 | 0.0690 | 0.0017 | −0.0158 | −0.0043 | −0.0066 | 0.0214 |
| | −0.0511 | 0.4683 | 0.1277 | 0.1958 | −0.6322 | −0.0158 | 0.1452 | 0.0396 | 0.0607 | −0.1960 |
| | −0.0139 | 0.1277 | 0.0348 | 0.0534 | −0.1724 | −0.0043 | 0.0396 | 0.0108 | 0.0166 | −0.0535 |
| | −0.0214 | 0.1958 | 0.0534 | 0.0819 | −0.2644 | −0.0066 | 0.0607 | 0.0166 | 0.0254 | −0.0820 |
| | 0.0690 | −0.6322 | −0.1724 | −0.2644 | 0.8534 | 0.0214 | −0.1960 | −0.0535 | −0.0820 | 0.2647 |
| $EMSEM(b_\delta^{(3)})$ | 0.0031 | −0.0287 | −0.0078 | −0.0120 | 0.0388 | 0.0016 | −0.0148 | −0.0040 | −0.0062 | 0.0200 |
| | −0.0287 | 0.2633 | 0.0718 | 0.1101 | −0.3554 | −0.0148 | 0.1355 | 0.0370 | 0.0567 | −0.1830 |
| | −0.0078 | 0.0718 | 0.0196 | 0.0300 | −0.0969 | −0.0040 | 0.0370 | 0.0101 | 0.0155 | −0.0499 |
| | −0.0120 | 0.1101 | 0.0300 | 0.0460 | −0.1486 | −0.0062 | 0.0567 | 0.0155 | 0.0237 | −0.0765 |
| | 0.0388 | −0.3554 | −0.0969 | −0.1486 | 0.4798 | 0.0200 | −0.1830 | −0.0499 | −0.0765 | 0.2470 |
| $EMSEM(b_\delta^{(4)})$ | 0.0269 | −0.2465 | −0.0672 | −0.1031 | 0.3328 | 0.0024 | −0.0217 | −0.0059 | −0.0091 | 0.0293 |
| | −0.2465 | 2.2600 | 0.6164 | 0.9451 | −3.0510 | −0.0217 | 0.1986 | 0.0542 | 0.0831 | −0.2681 |
| | −0.0672 | 0.6164 | 0.1681 | 0.2578 | −0.8321 | −0.0059 | 0.0542 | 0.0148 | 0.0227 | −0.0731 |
| | −0.1031 | 0.9451 | 0.2578 | 0.3952 | −1.2759 | −0.0091 | 0.0831 | 0.0227 | 0.0347 | −0.1121 |
| | 0.3328 | −3.0510 | −0.8321 | −1.2759 | 4.1189 | 0.0293 | −0.2681 | −0.0731 | −0.1121 | 0.3620 |

**Table 6**

Empirical bias vectors and empirical mean squared error matrices of $b_\delta^{(2)}$, $b_\delta^{(3)}$ and $b_\delta^{(4)}$ when measurement errors and random error component follow gamma distribution.

| | $n=25$ | | | | | $n=45$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma_\epsilon^2=0.4,\ \sigma_\phi^2=0.4,\ \sigma_\delta^2=0.4$ | | | | | | | | | | |
| $EB(b_\delta^{(2)})'$ | 0.0010 | −0.0088 | −0.0024 | −0.0037 | 0.0118 | −0.0007 | 0.0068 | 0.0018 | 0.0028 | −0.0091 |
| $EB(b_\delta^{(3)})'$ | 0.0008 | −0.0077 | −0.0021 | −0.0032 | 0.0104 | −0.0006 | 0.0070 | 0.0020 | 0.0029 | −0.0099 |
| $EB(b_\delta^{(4)})'$ | 0.0016 | −0.0147 | −0.0040 | −0.0062 | 0.0199 | −0.0009 | 0.0045 | 0.0011 | 0.0022 | −0.0057 |
| $EMSEM(b_\delta^{(2)})$ | 0.0012 | −0.0112 | −0.0031 | −0.0047 | 0.0151 | 0.0006 | −0.0059 | −0.0016 | −0.0025 | 0.0079 |
| | −0.0112 | 0.1028 | 0.0280 | 0.0430 | −0.1388 | −0.0059 | 0.0539 | 0.0147 | 0.0225 | −0.0727 |
| | −0.0031 | 0.0280 | 0.0076 | 0.0117 | −0.0379 | −0.0016 | 0.0147 | 0.0040 | 0.0061 | −0.0198 |
| | −0.0047 | 0.0430 | 0.0117 | 0.0180 | −0.0580 | −0.0025 | 0.0225 | 0.0061 | 0.0094 | −0.0304 |
| | 0.0151 | −0.1388 | −0.0379 | −0.0580 | 0.1874 | 0.0079 | −0.0727 | −0.0198 | −0.0304 | 0.0982 |
| $EMSEM(b_\delta^{(3)})$ | 0.0012 | −0.0111 | −0.0030 | −0.0046 | 0.0149 | 0.0006 | −0.0058 | −0.0016 | −0.0024 | 0.0079 |
| | −0.0111 | 0.1013 | 0.0276 | 0.0424 | −0.1368 | −0.0058 | 0.0534 | 0.0146 | 0.0223 | −0.0721 |
| | −0.0030 | 0.0276 | 0.0075 | 0.0116 | −0.0373 | −0.0016 | 0.0146 | 0.0040 | 0.0061 | −0.0197 |
| | −0.0046 | 0.0424 | 0.0116 | 0.0177 | −0.0572 | −0.0024 | 0.0223 | 0.0061 | 0.0093 | −0.0301 |
| | 0.0149 | −0.1368 | −0.0373 | −0.0572 | 0.1847 | 0.0079 | −0.0721 | −0.0197 | −0.0301 | 0.0973 |

Table 6 (*continued*)

| | $n = 25$ | | | | | $n = 45$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.0015 | −0.0142 | −0.0039 | −0.0059 | 0.0191 | 0.0008 | −0.0070 | −0.0019 | −0.0029 | 0.0094 |
| | −0.0142 | 0.1299 | 0.0354 | 0.0543 | −0.1754 | −0.0070 | 0.0638 | 0.0174 | 0.0267 | −0.0861 |
| EMSEM($b_\delta^{(4)}$) | −0.0039 | 0.0354 | 0.0097 | 0.0148 | −0.0478 | −0.0019 | 0.0174 | 0.0047 | 0.0073 | −0.0235 |
| | −0.0059 | 0.0543 | 0.0148 | 0.0227 | −0.0733 | −0.0029 | 0.0267 | 0.0073 | 0.0112 | −0.0360 |
| | 0.0191 | −0.1754 | −0.0478 | −0.0733 | 0.2367 | 0.0094 | −0.0861 | −0.0235 | −0.0360 | 0.1162 |
| $\sigma_\epsilon^2 = 1.0,\ \sigma_\phi^2 = 1.0,\ \sigma_\delta^2 = 0.4$ | | | | | | | | | | |
| EB($b_\delta^{(2)}$)′ | −0.0003 | 0.0025 | 0.0007 | 0.0010 | −0.0033 | −0.0003 | 0.0024 | 0.0007 | 0.0010 | −0.0033 |
| EB($b_\delta^{(3)}$)′ | −0.0004 | 0.0032 | 0.0009 | 0.0014 | −0.0044 | −0.0003 | 0.0032 | 0.0009 | 0.0013 | −0.0043 |
| EB($b_\delta^{(4)}$)′ | 0.0004 | −0.0035 | −0.0010 | −0.0015 | 0.0047 | 0.0004 | −0.0034 | −0.0009 | −0.0014 | 0.0046 |
| | 0.0010 | −0.0090 | −0.0025 | −0.0038 | 0.0122 | 0.0005 | −0.0045 | −0.0012 | −0.0019 | 0.0060 |
| | −0.0090 | 0.0825 | 0.0225 | 0.0345 | −0.1114 | −0.0045 | 0.0410 | 0.0112 | 0.0171 | −0.0553 |
| EMSEM($b_\delta^{(2)}$) | −0.0025 | 0.0225 | 0.0061 | 0.0094 | −0.0304 | −0.0012 | 0.0112 | 0.0030 | 0.0047 | −0.0151 |
| | −0.0038 | 0.0345 | 0.0094 | 0.0144 | −0.0466 | −0.0019 | 0.0171 | 0.0047 | 0.0072 | −0.0231 |
| | 0.0122 | −0.1114 | −0.0304 | −0.0466 | 0.1504 | 0.0060 | −0.0553 | −0.0151 | −0.0231 | 0.0746 |
| | 0.0010 | −0.0089 | −0.0024 | −0.0037 | 0.0120 | 0.0005 | −0.0044 | −0.0012 | −0.0019 | 0.0060 |
| | −0.0089 | 0.0814 | 0.0222 | 0.0341 | −0.1099 | −0.0044 | 0.0407 | 0.0111 | 0.0170 | −0.0549 |
| EMSEM($b_\delta^{(3)}$) | −0.0024 | 0.0222 | 0.0061 | 0.0093 | −0.0300 | −0.0012 | 0.0111 | 0.0030 | 0.0046 | −0.0150 |
| | −0.0037 | 0.0341 | 0.0093 | 0.0142 | −0.0460 | −0.0019 | 0.0170 | 0.0046 | 0.0071 | −0.0230 |
| | 0.0120 | −0.1099 | −0.0300 | −0.0460 | 0.1484 | 0.0060 | −0.0549 | −0.0150 | −0.0230 | 0.0741 |
| | 0.0013 | −0.0116 | −0.0032 | −0.0049 | 0.0157 | 0.0006 | −0.0053 | −0.0014 | −0.0022 | 0.0071 |
| | −0.0116 | 0.1065 | 0.0290 | 0.0445 | −0.1437 | −0.0053 | 0.0484 | 0.0132 | 0.0202 | −0.0653 |
| EMSEM($b_\delta^{(4)}$) | −0.0032 | 0.0290 | 0.0079 | 0.0121 | −0.0392 | −0.0014 | 0.0132 | 0.0036 | 0.0055 | −0.0178 |
| | −0.0049 | 0.0445 | 0.0121 | 0.0186 | −0.0601 | −0.0022 | 0.0202 | 0.0055 | 0.0085 | −0.0273 |
| | 0.0157 | −0.1437 | −0.0392 | −0.0601 | 0.1941 | 0.0071 | −0.0653 | −0.0178 | −0.0273 | 0.0882 |
| $\sigma_\epsilon^2 = 1.0,\ \sigma_\phi^2 = 1.0,\ \sigma_\delta^2 = 1.0$ | | | | | | | | | | |
| EB($b_\delta^{(2)}$)′ | 0.0009 | −0.0077 | −0.0021 | −0.0032 | 0.0103 | 0.0008 | −0.0067 | −0.0018 | −0.0028 | 0.0090 |
| EB($b_\delta^{(3)}$)′ | −0.0009 | 0.0081 | 0.0022 | 0.0034 | −0.0110 | 0.0004 | −0.0036 | −0.0010 | −0.0015 | 0.0048 |
| EB($b_\delta^{(4)}$)′ | 0.0053 | −0.0482 | −0.0132 | −0.0202 | 0.0651 | 0.0021 | −0.0192 | −0.0052 | −0.0080 | 0.0259 |
| | 0.0061 | −0.0559 | −0.0152 | −0.0234 | 0.0755 | 0.0016 | −0.0149 | −0.0041 | −0.0062 | 0.0201 |
| | −0.0559 | 0.5125 | 0.1398 | 0.2143 | −0.6919 | −0.0149 | 0.1367 | 0.0373 | 0.0572 | −0.1845 |
| EMSEM($b_\delta^{(2)}$) | −0.0152 | 0.1398 | 0.0381 | 0.0585 | −0.1887 | −0.0041 | 0.0373 | 0.0102 | 0.0156 | −0.0503 |
| | −0.0234 | 0.2143 | 0.0585 | 0.0896 | −0.2893 | −0.0062 | 0.0572 | 0.0156 | 0.0239 | −0.0772 |
| | 0.0755 | −0.6919 | −0.1887 | −0.2893 | 0.9341 | 0.0201 | −0.1845 | −0.0503 | −0.0772 | 0.2491 |
| | 0.0010 | −0.0089 | −0.0024 | −0.0037 | 0.0120 | 0.0016 | −0.0143 | −0.0039 | −0.0060 | 0.0193 |
| | −0.0089 | 0.0814 | 0.0222 | 0.0341 | −0.1099 | −0.0143 | 0.1308 | 0.0357 | 0.0547 | −0.1766 |
| EMSEM($b_\delta^{(3)}$) | −0.0024 | 0.0222 | 0.0061 | 0.0093 | −0.0300 | −0.0039 | 0.0357 | 0.0097 | 0.0149 | −0.0482 |
| | −0.0037 | 0.0341 | 0.0093 | 0.0142 | −0.0460 | −0.0060 | 0.0547 | 0.0149 | 0.0229 | −0.0739 |
| | 0.0120 | −0.1099 | −0.0300 | −0.0460 | 0.1484 | 0.0193 | −0.1766 | −0.0482 | −0.0739 | 0.2385 |
| | 0.0330 | −0.3026 | −0.0825 | −0.1265 | 0.4085 | 0.0021 | −0.0196 | −0.0053 | −0.0082 | 0.0264 |
| | −0.3026 | 2.7737 | 0.7565 | 1.1599 | −3.7445 | −0.0196 | 0.1794 | 0.0489 | 0.0750 | −0.2422 |
| EMSEM($b_\delta^{(4)}$) | −0.0825 | 0.7565 | 0.2063 | 0.3163 | −1.0212 | −0.0053 | 0.0489 | 0.0133 | 0.0205 | −0.0661 |
| | −0.1265 | 1.1599 | 0.3163 | 0.4851 | −1.5659 | −0.0082 | 0.0750 | 0.0205 | 0.0314 | −0.1013 |
| | 0.4085 | −3.7445 | −1.0212 | −1.5659 | 5.0551 | 0.0264 | −0.2422 | −0.0661 | −0.1013 | 0.3270 |

variabilities of $b_K^{(l)}$ and $b_\delta^{(l)}$ $(l = 2, 3, 4)$ increase significantly, and vice-versa. When $\sigma_\phi^2$ increases, keeping the other two variances fixed, then the variabilities $b_K^{(l)}$ and $b_\delta^{(l)}$ $(l = 2, 3, 4)$ decrease. However this decrement is very small. The effect of reliability matrix $K_X$ is also observed on the variability of $b_K^{(l)}$ $(l = 2, 3, 4)$, and it is seen that the variability decreases when the reliability ratio increases.

The pattern of effect of variances $\sigma_\epsilon^2$, $\sigma_\phi^2$ and $\sigma_\delta^2$ on the variability of $b_K^{(l)}$ $(l = 2, 3, 4)$ is the same, but different for $b_\delta^{(l)}$ $(l = 2, 3, 4)$ under the normal, $t$ and gamma distributed measurement errors. The variabilities $b_\delta^{(l)}$ $(l = 2, 3, 4)$ increase with $\sigma_\delta^2$, and vice-versa under normal distribution. When $\sigma_\delta^2$ increases, then the variabilities of all the estimators under $t$ distributed measurement errors increase, whereas it decreases in case of gamma distributed measurement errors. On the other hand, when $\sigma_\phi^2$ increases, then the variabilities of all the estimators decrease under the $t$ as well as gamma distributed measurement errors.

Further, we compare the corresponding values in the EMSEMs with the same parameters under different distributions of measurement errors. The difference in such values may be contributed as due to the departure from normality, in particular

due to the effect of skewness and kurtosis of the distribution of measurement errors. No significant difference among the variabilities of $b_K^{(l)}$ and $b_\delta^{(l)}$ $(l = 2, 3, 4)$ is observed when $\sigma_\delta^2$, $\sigma_\phi^2$ and $\sigma_\epsilon^2$ are small. When $\sigma_\delta^2$ increases, the variabilities of $b_K^{(l)}$ $(l = 2, 3, 4)$ under $t$ distribution are higher than under normality. However this difference is very small. Such a difference becomes higher when the degree of freedom of $t$ distribution are decreased, which in turn increases the coefficient of kurtosis of $t$ distribution. We have obtained results for $t$ distribution with 12 and 8 degree of freedom. On the other hand, the variability is smaller under gamma distribution than under $t$ and normal distributions. This decrement becomes very high when the shape parameter of gamma distribution is decreased from 5 to 2. Note that the skewness of gamma distribution increases when shape parameter is decreased. Also, $\sigma_\phi^2$ does not significantly affect the variability of $b_K^{(l)}$ $(l = 2, 3, 4)$ under different distributions of measurement errors. These changes are significant when sample size is small. This clearly shows the effect of the coefficients of skewness and kurtosis on the variability of these estimators. Such an effect is also explained by the asymptotic theory.

The difference in the variability of $b_K^{(l)}$ $(l = 2, 3, 4)$ under different distributions is not very significant. While this is not so, the case with the estimators $b_\delta^{(l)}$ $(l = 2, 3, 4)$. When the EMSEM's of the estimators $b_\delta^{(l)}$ $(l = 2, 3, 4)$ under normal distribution case are compared under the same sample size and same variance with the corresponding EMSEMs in the gamma distribution, we find that the variability of $b_\delta^{(4)}$ decreases significantly when the sample size is 25. This change is not significant when the sample size is large. The changes in the variabilities of $b_\delta^{(2)}$ and $b_\delta^{(3)}$ are not significant. The difference between the variabilities of $b_\delta^{(2)}$ and $b_\delta^{(4)}$ in case of the $t$ distribution, is higher than under the gamma distribution of measurement errors. The difference in the variability of $b_\delta^{(3)}$ under $t$ and gamma distributed measurement errors is not significant. In large samples, the variability of $b_\delta^{(2)}$, $b_\delta^{(3)}$ and $b_\delta^{(4)}$ are almost the same under the $t$ and gamma distributed measurement errors. Thus, the difference in the EMSEM of different estimators under the $t$ and gamma distributed measurement errors with their corresponding values under normally distributed measurement errors, is significant. The magnitude of such difference essentially depends on the direction and magnitude of the departure from symmetry and peakedness of the distribution of measurement errors. Such changes are also explained by the asymptotic theory. The effect of departure from normality is seen to be more prominent in small samples than in large samples.

## 6. Conclusions

We have proposed three methodologies to utilize the prior information in constructing the consistent estimators that also satisfy the linear restrictions in a measurement error model. Utilizing the additional knowledge of the reliability matrix of the explanatory variables and covariance matrix of the measurement errors in explanatory variables, we have derived three different estimators in each case. Without assuming any distributional form of measurement errors and random error component, we have derived the asymptotic distribution of the estimators under a multivariate ultrastructural model. The dominance conditions of the estimators under each specification of additional knowledge, and between the two types of information, are derived under the criterion of Löwner ordering based on the asymptotic covariance matrices. Such dominance conditions can be checked by finding the eigenvalues of a certain matrix. The findings from a Monte-Carlo simulation experiment shed some light on the finite sample properties of the estimators. Some ideas about the role and effect of individual measurement error variance and dominance of estimators is reported, which is based on simulated results. The effect of non-normality is more prevalent in small samples than in large samples and when the coefficients of skewness and kurtosis are high.

We have used a very general framework for incorporating the non-stochastic linear restrictions in the measurement error model. Such framework of linear restrictions is also used in developing the tests of hypothesis in a general framework, developing pre-test estimators under the theory of preliminary test estimation, studying the structural change in the parameters etc. So our methodology can also be used in such a setting. The set-up of measurement error model considered in this paper is a particular case of structural equation model (SEM). So the approaches presented in this paper can also be extended to SEM. These are some areas for the further research in this direction.

## Appendix

The following lemma will be useful in deriving orders of various expressions in the proofs of the Theorems.

**Lemma 3.** *Let $C = (c_{ij})$ be a $m \times m$ matrix and let $\|C\|_1 = \max_{1\le i\le m} \sum_{j=1}^m |c_{ij}|$ and $\|C\|_2 = \max_{1\le j\le m} \sum_{i=1}^m |c_{ij}|$ be the maximum column sum and maximum row sum matrix norms, respectively. If $\|C\|_1 < 1$ and/or $\|C\|_2 < 1$, then $(I_m - C)$ is invertible and*

$$(I_m - C)^{-1} = \sum_{i=0}^{\infty} C^i$$

*where $C^0 = I_m$.*

**Proof.** see e.g., [37]. □

Let us define,

$$H := \frac{1}{\sqrt{n}}S - \sqrt{n}\Sigma_X, \tag{A.1}$$

$$h := \frac{1}{\sqrt{n}}X'(\epsilon - \Delta\beta) + \sqrt{n}\sigma_\delta^2\beta. \tag{A.2}$$

Various expectations used in the results are presented in the following lemma.

**Lemma 4.** *For a non stochastic $p \times p$ matrix $C$ and a non stochastic $1 \times p$ vector $d$, using the distributional properties of $\epsilon$, $\Delta$ and $\Phi$, we have*

$$\begin{aligned}
E(HCH) &= (\sigma_\delta^2 + \sigma_\phi^2)\left\{\Sigma_X\left(C' + (\operatorname{tr} C)I_p\right) + C'S_{\mu\mu} + (\operatorname{tr}\frac{1}{n}MCM')I_p\right\} \\
&\quad + (\gamma_{1\delta}\sigma_\delta^3 + \gamma_{1\phi}\sigma_\phi^3)\left\{f(s_\mu e_p', C) + \left(f(s_\mu e_p', C)\right)' + 2f(I_p, e_p s_\mu' C)\right\} + (\gamma_{2\delta}\sigma_\delta^4 + \gamma_{2\phi}\sigma_\phi^4)f(I_p, C), \\
E(hdH) &= -\sigma_\delta^2\Big[\Sigma_X\left\{d'\beta' + (\operatorname{tr}\beta d)I_p\right\} + \gamma_{1\delta}\sigma_\delta\left\{f(s_\mu e_p', d'\beta') + \left(f(s_\mu e_p', d'\beta')\right)' + f(I_p, e_p s_\mu' d'\beta')\right\} \\
&\quad + \gamma_{2\delta}\sigma_\delta^2 f(I_p, d'\beta')\Big], \\
E(hh') &= \sigma_\epsilon^2\Sigma_X + \sigma_\delta^2(\operatorname{tr}\beta\beta')\Sigma_X + \sigma_\delta^4\beta\beta' + \gamma_{1\delta}\sigma_\delta^3\left\{f(s_\mu e_p', \beta\beta') + \left(f(s_\mu e_p', \beta\beta')\right)'\right\} + \gamma_{2\delta}\sigma_\delta^4 f(I_p, \beta\beta'),
\end{aligned}$$

*where $f(Z_1, Z_2) = Z_1 * (Z_2 * I_p)\ \ \forall\, Z_1, Z_2 \in \mathbb{R}^{p\times p}$ as in (4.1).*

**Proof.** Proof follows by using the distributional properties of $\epsilon$, $\Delta$, $\Phi$ and following [38]. □

With the help of Lemma 1, it can be shown that $h = O_P(1)$ and $H = O_P(1)$. The estimation errors of the estimators $b_K^{(1)}$, $b_K^{(2)}$, $b_K^{(3)}$ and $b_K^{(4)}$ are expanded using the large sample asymptotic approximation theory to find the asymptotic distributions.

From (3.5),

$$b_K^{(1)} = \Sigma_T^{-1}\Sigma_X\left(\beta + S^{-1}X'(\epsilon - \Delta\beta)\right).$$

Let $\bar{K}_X := I_p - K_X$. Using Lemma 3, we can write

$$\sqrt{n}(b_K^{(1)} - \beta) = \Sigma_T^{-1}\{h + H\bar{K}_X\beta\} + O_P(n^{-\frac{1}{2}}). \tag{A.3}$$

The estimation error of $b_K^{(2)}$

$$b_K^{(2)} - \beta = (b_K^{(1)} - \beta) + S^{-1}R'(RS^{-1}R')^{-1}(r - Rb_K^{(1)}). \tag{A.4}$$

Since $\Sigma_X^{-1} = O(1)$ and plim $(n^{-\frac{1}{2}}\Sigma_X^{-1}H) = 0$, using Lemma 3, we have

$$S^{-1} = \frac{1}{n}(I_p - n^{-\frac{1}{2}}\Sigma_X^{-1}H)\Sigma_X^{-1} + O_P(n^{-2}), \tag{A.5}$$

for sufficiently large $n$.

Let $R_X = R\Sigma_X^{-1}R'$ so that $R_X^{-1} = O(1)$. Moreover, since plim$(n^{-\frac{1}{2}}R_X^{-1}R\Sigma_X^{-1}H\Sigma_X^{-1}R') = 0$, from (A.5), we have

$$\frac{1}{n}(RS^{-1}R')^{-1} = \left(I_p + n^{-\frac{1}{2}}R_X^{-1}R\Sigma_X^{-1}H\Sigma_X^{-1}R'\right)R_X^{-1} + O_P(n^{-1}). \tag{A.6}$$

Also,

$$\left(r - Rb_K^{(1)}\right) = -n^{-\frac{1}{2}}R\Sigma_T^{-1}\{h + H\bar{K}_X\beta\} + n^{-1}R\Sigma_T^{-1}H\Sigma_X^{-1}\{h + H\bar{K}_X\beta\} + O_P(n^{-\frac{3}{2}}). \tag{A.7}$$

Using (A.3)–(A.7), we have

$$\sqrt{n}(b_K^{(2)} - \beta) = (I_p - Q\Sigma_X)\Sigma_T^{-1}\{h + H\bar{K}_X\beta\} + O_P(n^{-\frac{1}{2}}),$$

where $Q = \Sigma_X^{-1}R'(R\Sigma_X^{-1}R')^{-1}R\Sigma_X^{-1}$. Clearly $Q\Sigma_X = \bar{K}_X R'\{R\bar{K}_X R'\}^{-1}R$, we get

$$\sqrt{n}(b_K^{(2)} - \beta) = A_{(2,X)}\Sigma_T^{-1}\{h + H\bar{K}_X\beta\} + O_P(n^{-\frac{1}{2}}), \tag{A.8}$$

where $A_{(2,X)} = [I_p - \bar{K}_X R'\{R\bar{K}_X R'\}^{-1}R]$.

Now, consider the estimator $b_K^{(3)}$,

$$b_K^{(3)} - \beta = A_{K_X}^{-1} b_R - \beta, \tag{A.9}$$

where

$$A_{K_X} = \left[K_X + \bar{K}_X R' \left(R\bar{K}_X R'\right)^{-1} R\bar{K}_X\right]$$

and

$$b_R = \beta + [I_p - S^{-1}R'R_S^{-1}R](nS^{-1})\left(\frac{1}{\sqrt{n}}h - \sigma_\delta^2\beta\right). \tag{A.10}$$

Since $\bar{K}_X = \sigma_\delta^2\Sigma_X^{-1} \Longrightarrow A_{K_X} = I_p - \sigma_\delta^2(\Sigma_X^{-1} - Q)$ and using (A.5) and (A.6) in (A.10), we get

$$b_R = A_{K_X}\beta + \frac{1}{\sqrt{n}}(\Sigma_X^{-1} - Q)\{h + H(I_p - A_{K_X})\beta\} + O_P(n^{-1}). \tag{A.11}$$

Thus,

$$\begin{aligned}\sqrt{n}(b_K^{(3)} - \beta) &= A_{K_X}^{-1}(\Sigma_X^{-1} - Q)\{h + H(I_p - A_{K_X})\beta\} + O_P(n^{-\frac{1}{2}}) \\ &= A_{K_X}^{-1}A_{(2,X)}\Sigma_X^{-1}\{h + H(I_p - A_{K_X})\beta\} + O_P(n^{-\frac{1}{2}}).\end{aligned} \tag{A.12}$$

Next, the estimation error of $b_K^{(4)}$

$$\begin{aligned}b_K^{(4)} - \beta &= (b_K^{(1)} - \beta) + R'(RR')^{-1}(r - Rb_K^{(1)}) \\ &= \left[I_p - R'(RR')^{-1}R\right](b_K^{(1)} - \beta).\end{aligned} \tag{A.13}$$

From (A.3),

$$\sqrt{n}(b_K^{(4)} - \beta) = \left[I_p - R'(RR')^{-1}R\right]\Sigma_T^{-1}\{h + H\bar{K}_X\beta\} + O_P(n^{-\frac{1}{2}}). \tag{A.14}$$

**Proof of Theorem 1.** Let $m_i$, $\phi_i$ and $\delta_i$ be the $i$th rows of the matrices $M$, $\Phi$ and $\Delta$ respectively; $i = 1, 2, \ldots, n$. We can write,

$$\begin{aligned}H &= \frac{1}{\sqrt{n}}S - \sqrt{n}\Sigma_X \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}[m_i'(\phi_i + \delta_i) + (\phi_i' + \delta_i')m_i + (\delta_i'\phi_i + \phi_i'\delta_i) + (\delta_i'\delta_i - \sigma_\delta^2 I_p) + (\phi_i'\phi_i - \sigma_\phi^2 I_p)], \\ H\bar{K}_X\beta &= \frac{1}{\sqrt{n}}\left(I_p \otimes (\beta'\bar{K}_X),\quad I_p \otimes (\beta'\bar{K}_X),\quad I_p \otimes (\beta'\bar{K}_X),\quad I_p \otimes (\beta'\bar{K}_X),\quad I_p \otimes (\beta'\bar{K}_X)\right) \\ &\quad \times \sum_{i=1}^{n}\left(\text{vec}\left(m_i'(\phi_i + \delta_i)\right), \text{vec}\left((\phi_i' + \delta_i')m_i\right), \text{vec}(\delta_i'\phi_i + \phi_i'\delta_i), \text{vec}(\delta_i'\delta_i - \sigma_\delta^2 I_p), \text{vec}(\phi_i'\phi_i - \sigma_\phi^2 I_p)\right)' \\ &= \sum_{i=1}^{n} C_{1in}w_{1i},\end{aligned} \tag{A.15}$$

where

$$\begin{aligned}C_{1in} &= \frac{1}{\sqrt{n}}\left(I_p \otimes (\beta'\bar{K}_X),\quad I_p \otimes (\beta'\bar{K}_X),\quad I_p \otimes (\beta'\bar{K}_X),\quad I_p \otimes (\beta'\bar{K}_X),\quad I_p \otimes (\beta'\bar{K}_X)\right) \\ &\quad \times \left((I_p \otimes m_i'),\quad (I_{p^2} \otimes m_i),\quad I_{p^2},\quad I_{p^2},\quad I_{p^2}\right)\end{aligned}$$

and $w_{1i} = \left((\phi_i' + \delta_i')\ \text{vec}\left(I_p \otimes (\phi_i' + \delta_i')\right)\ \text{vec}(\delta_i'\phi_i + \phi_i'\delta_i)\ \text{vec}(\delta_i'\delta_i - \sigma_\delta^2 I_p)\ \text{vec}(\phi_i'\phi_i - \sigma_\phi^2 I_p)\right)'$.

Now consider

$$\begin{aligned}h &= \frac{1}{\sqrt{n}}X'\epsilon - \frac{1}{\sqrt{n}}(X'\Delta - n\sigma_\delta^2 I_p)\beta \\ &= \left(I_p, -(I_p \otimes \beta')\right)\sum_{i=1}^{n}\frac{1}{\sqrt{n}}\left[(I_{p+1} \otimes m_i')\begin{pmatrix}\epsilon_i \\ \delta_i'\end{pmatrix} + \begin{pmatrix}\phi_i'\epsilon_i \\ \text{vec}(\phi_i'\delta_i)\end{pmatrix} + \begin{pmatrix}\delta_i'\epsilon_i \\ \text{vec}(\delta_i'\delta_i - \sigma_\delta^2 I_p)\end{pmatrix}\right] \\ &= \sum_{i=1}^{n} C_{2in}\, w_{2i},\end{aligned} \tag{A.16}$$

where,

$$C_{in} = \frac{1}{\sqrt{n}}\left(I_p\,, \quad -(I_p \otimes \beta')\right)\left((I_{p+1} \otimes m_i') \quad I_{p^2+p} \quad I_{p^2+p}\right)$$

are $p \times (2p^2 + 3p + 1)$ non-stochastic matrices and

$$w_{2i} = \left(\epsilon_i, \delta_i', \phi_i'\epsilon_i, \mathrm{vec}(\phi_i'\delta_i), \delta_i\epsilon_i, \mathrm{vec}(\delta_i'\delta_i - \sigma_\delta^2 I_p)\right)$$

are $(2p^2 + 3p + 1) \times 1$ independent and identically distributed random vectors, $i = 1, 2, \ldots, n$.

Thus from (A.15) and (A.16),

$$h + H\bar{K}_X\beta = \sum_{i=1}^{n}\left(C_{2in} \quad C_{1in}\right)\begin{pmatrix} w_{2i} \\ w_{1i}\end{pmatrix}. \tag{A.17}$$

Therefore by central limit theorem, $h + H\bar{K}_X\beta$ has a limiting normal distribution with mean vector $E(h + H\bar{K}_X\beta) = 0$ and covariance matrix

$$\begin{aligned}\Omega_K &:= \lim_{n\to\infty} E\left[(h + H\bar{K}_X\beta)(h + H\bar{K}_X\beta)'\right] \\ &= (\sigma_\delta^2 + \sigma_\phi^2)[\sigma_\delta^2\{(\beta\beta' - K\beta\beta'K) + (\mathrm{tr}\,\bar{K}\beta\beta'K)I_p\} + (\mathrm{tr}\,\bar{K}\beta\beta'\bar{K})\Sigma] \\ &\quad + \{\sigma_\epsilon^2 + \sigma_\delta^2(\mathrm{tr}(2K - I_p)\beta\beta')\}\Sigma - \sigma_\delta^4\beta\beta' + N(K),\end{aligned}$$

where

$$\begin{aligned}N(K) &= \gamma_{1\delta}\sigma_\delta^3\left[f\left(\boldsymbol{\sigma}_\mu e_p',\ K\beta\beta'K\right) + \left\{f\left(\boldsymbol{\sigma}_\mu e_p',\ K\beta\beta'K\right)\right\}' - 2f\left(I_p,\ e_p\boldsymbol{\sigma}_\mu'\bar{K}\beta\beta'K\right)\right] \\ &\quad + \gamma_{1\phi}\sigma_\phi^3\left[f\left(\boldsymbol{\sigma}_\mu e_p',\ \bar{K}\beta\beta'\bar{K}\right) + \left\{f\left(\boldsymbol{\sigma}_\mu e_p',\ \bar{K}\beta\beta'\bar{K}\right)\right\}' + 2f\left(I_p,\ e_p\boldsymbol{\sigma}_\mu'\bar{K}\beta\beta'\bar{K}\right)\right] \\ &\quad + \gamma_{2\delta}\sigma_\delta^4 f\left(I_p,\ K\beta\beta'K\right) + \gamma_{2\phi}\sigma_\phi^4 f\left(I_p,\ \bar{K}\beta\beta'\bar{K}\right),\end{aligned} \tag{A.18}$$

and $\bar{K} = I_p - K$, $K = \lim_{n\to\infty} K_X$.

Now from (A.3), it is clear that the asymptotic distribution of $\sqrt{n}(b_K^{(1)} - \beta)$ is the same as the asymptotic distribution of $\Sigma_T^{-1}(h + \sigma_\delta^2 H\Sigma_X^{-1}\beta)$. Since $\lim_{n\to\infty}\Sigma_T \lim_{n\to\infty}\Sigma_X K_X = \Sigma K$, the asymptotic distribution of $\sqrt{n}(b_K^{(1)} - \beta)$ is normal with mean vector 0 and covariance matrix given by

$$\Omega_K^{(1)} = (\Sigma K)^{-1}\Omega_K(\Sigma K)^{-1}. \tag{A.19}$$

Asymptotic normality of $\sqrt{n}(b_K^{(2)} - \beta)$, $\sqrt{n}(b_K^{(3)} - \beta)$ and $\sqrt{n}(b_K^{(4)} - \beta)$ can be shown in a similar fashion using (A.8), (A.12) and (A.14).

The proof of the results in Theorem 2 can be followed from [14,35]. □

## References


[1] T. Wansbeek, E. Meijer, Measurement Error and Latent Variables in Econometrics, North Holland, 2000.
[2] C.L. Cheng, J.W. Van Ness, Statistical Regression with Measurement Errors, Arnold Publishers, 1999.
[3] W.A. Fuller, Measurement Error Models, John Wiley, 1987.
[4] H. Toutenburg, Prior Information in Linear Models, John Wiley, 1982.
[5] C.R. Rao, H. Toutenburg, Shalabh, C. Heumann, Linear Models and Generalizations: Least Squares and Alternatives, 3rd ed., Springer, New York, 2008.
[6] Bukhard Schaffrin, A note on constrained total least-squares estimation, Linear Algebra and its Applications 417 (2006) 245–258.
[7] G.R. Dolby, The ultrastructural relation: A synthesis of the functional and structural relations, Biometrika 63 (1976) 39–50.
[8] A.K. Srivastava, Shalabh, Consistent estimation for the non-normal ultrastructural model, Statistics & Probability Letters 34 (1997) 67–73.
[9] A.K. Srivastava, Shalabh, Improved estimation of slope parameter in a linear ultrastructural model when measurement errors are not necessarily normal, Journal of Econometrics 78 (1997) 153–157.
[10] H. Schneeweiss, Consistent estimation of a regression with errors in the variables, Metrika 23 (1976) 101–115.
[11] L.J. Gleser, The Importance of assessing measurement reliability in multivariate regression, Journal of American Statistical Association 87 (419) (1992) 696–707.
[12] L.J. Gleser, Estimators of slopes in linear errors-in-variables regression models when the predictors have known reliability matrix, Statistics & Probability Letters 17 (1993) 113–121.
[13] Shalabh, Improved estimation in measurement error models through Stein-rule procedure, Journal of Multivariate Analysis 67 (1998) 35–48; Journal of Multivariate Analysis 74 (2000) 162 (corrigendum).
[14] Shalabh, Consistent estimation of coefficients in measurement error models under non-normality, Journal of Multivariate Analysis 86 (2) (2003) 227–241.
[15] Shalabh, L.J. Gleser, O. Rosen, On the usefulness of the knowledge of error variances in the consistent estimation of an unreplicated ultrastructural model, Journal of Statistical Computation & Simulation 74 (6) (2004) 391–417.
[16] S.Y. Lee, Constrained estimation in covariance structure analysis, Biometrika 66 (1979) 539–545.
[17] S.Y. Lee, P.M. Bentler, Some asymptotic properties of constrained generalized least squares estimation in covariance structure models, South African Statistical Journal 14 (1980) 121–136.
[18] C.W. Caroll, The created response surface technique for optimizing nonlinear, restrained systems, Operations Research 9 (1961) 169–185.

[19] A.V. Fiacco, G.P. McCormick, Nonlinear Programming: Sequential Unconstrained Minimization Techniques, John Wiley, 1968.
[20] S.Y. Lee, W.Y. Poon, Further developments on constrained estimation in analysis of covariance structures, Statistician 34 (1985) 305–316.
[21] S.Y. Lee, On testing functional constraints in structural equation models, Biometrika 72 (1985) 125–131.
[22] Vincent J. Geraci, Identification of simultaneous equation models with measurement error, Journal of Econometrics 4 (3) (1976) 263–283.
[23] Vincent J. Geraci, Estimation of simultaneous equation models with measurement error, Econometrica 45 (5) (1977) 1243–1255.
[24] Cheng Hsiao, Identification and estimation of simultaneous equation models with measurement error, International Economic Review 18 (1) (1976) 252. International Economic Review 17 (2) (1972) 319–339 (Corrigendum).
[25] Vincent J. Geraci, Errors in variables and the individual structural equation, International Economic Review 24 (1) (1983) 217–236.
[26] T.W. Anderson, Multivariate linear relations, in: Proceedings of Second Tampere Conference in Statistics, Tarmo Pukkila, Simo Putanen (Eds.), Department of Mathematical Sciences, University of Tampere, Tampere, Finland, 1–4 June 1987, pp. 9–36.
[27] T.W. Anderson, Linear latent variable models and covariance structures, Journal of Econometrics 41 (1989) 91–119.
[28] A.S. Goldberger, Maximum likelihood estimation of regressions containing unobservable independent variables, International Economic Review 13 (1972) 1–15.
[29] A.S. Goldberger, Structural equation methods in the social sciences, Econometrics 40 (1972) 979–1001.
[30] A.S. Goldberger, Unobservable variables in econometrics, in: P. Zarembka (Ed.), Frontiers of Econometrics, Academic Press, New York, 1974, pp. 193–213.
[31] L.J. Gleser, A note on G.R. Dolby's ultrastructural model, Biometrika 72 (1984) 117–124.
[32] H. Schneeweiss, Note on a linear model with errors in the variables and with trend, Statistical Papers 32 (1991) 261–264.
[33] G. Dunn, Design and Analysis of Reliability Studies, Oxford University Press, New York, 1989.
[34] de D.N.M. Gruijter, van der L.J.T. Kamp, Statistical Test Theory for the Behavioral Sciences, Chapman & Hall/ CRC, 2008.
[35] Shalabh, G. Garg, N. Misra, Restricted regression estimation in measurement error models, Computational Statistics & Data Analysis 52 (2007) 1149–1166.
[36] C.L. Cheng, A. Kukush, Non-existence of the first moment of the adjusted least squares estimator in multivariate errors-in-variables model, Metrika 64 (1) (2006) 41–46.
[37] C.R. Rao, M.B. Rao, Matrix Algebra and its Applications to Statistics and Econometrics, World Scientific, 1998.
[38] V.K. Srivastava, R. Tiwari, Evaluation of expectation of product of stochastic matrices, Scandinavian Journal of Statistics 3 (1976) 135–138.