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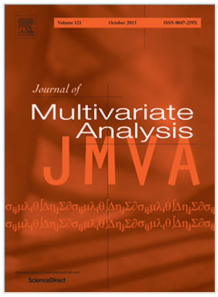

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Stochastic Orderings for Elliptical Random Vectors

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Abstract

The authors provide sufficient and/or necessary conditions for classifying multivariate elliptical random vectors according to the convex ordering and the increasing convex ordering. Their results generalize the corresponding ones for multivariate normal random vectors in the literature.

Keywords: Usual stochastic order, (Increasing) convex order, Multivariate normal distribution
2000 MSC: 60E15

1. Introduction

Stochastic orders offer more insights into the comparison of two random variables (vectors) than only through their means and variances, which may not exist. Stochastic orders have been applied successfully to queueing theory, reliability theory, economics, biomathematics, actuarial science, risk management and other related fields. This is documented, e.g., in the monographs of Denuit et al. [5], Müller and Stoyan [12], and Shaked and Shanthikumar [14].

It is natural to compare two normally distributed random variables (vectors) by some stochastic orders. However, necessary and sufficient conditions for stochastic ordering of multivariate normal random vectors could not be found until the work of Scarsini [13]. Müller [11] further discussed stochastic ordering characterizations of multivariate normal random vectors. Arlotto and Scarsini [1] unified and generalized several known results on comparisons of multivariate normal random vectors in the sense of different stochastic orders by introducing the so-called Hessian order. The formal definitions of the relevant stochastic orders are given in Section 2.

Theorem 1.1. *Let $\boldsymbol{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$ and $\boldsymbol{Y} \sim \mathcal{N}_n(\boldsymbol{\mu}', \Sigma')$ be two n-dimensional normally distributed random vectors. Then*

(1) $\boldsymbol{X} \leq_{\rm st} \boldsymbol{Y}$ *if and only if* $\boldsymbol{\mu} \leq \boldsymbol{\mu}'$ *and* $\Sigma = \Sigma'$ (Müller [11])*;*

(2) $\boldsymbol{X} \leq_{\rm cx} \boldsymbol{Y}$ *if and only if* $\boldsymbol{\mu} = \boldsymbol{\mu}'$ *and* $\Sigma' - \Sigma$ *is non-negative definite* (Scarsini [13]; Müller [11]).

Elliptical distributions are generalizations of the multivariate normal distribution and, therefore, share many of its tractable properties. This class of distributions, which was introduced by Kelker [9] and further discussed by Fang et al. [7], allows for the presence of heavy tails and asymptotic tail dependence. Ding and Zhang [6] extended Theorem 1.1 from multivariate normal distributions to Kotz-type distributions. The latter forms an important class of elliptically symmetric distributions. Besides, Landsman and Tsanakas [10] derived necessary and sufficient conditions for classifying bivariate elliptical distributions through the concordance ordering. In fact, the main results in Landsman and Tsanakas [10] are immediate consequences of the work of Block and Sampson [2].

It remains an interesting open problem whether necessary and sufficient conditions exist for stochastic ordering of multivariate elliptical distributions. Recently, Davidov and Peddada [4] obtained the following necessary and sufficient conditions for the usual stochastic ordering of multivariate elliptical random vectors.

Theorem 1.2. (Davidov and Peddada [4]) *Let* $\boldsymbol{X} \sim \mathcal{E}_n(\boldsymbol{\mu}, \Sigma, \phi)$ *and* $\boldsymbol{Y} \sim \mathcal{E}_n(\boldsymbol{\mu}', \Sigma', \phi)$ *be two* n*-dimensional elliptically distributed random vectors supported on* $\mathbb{R}^n$*. Then* $\boldsymbol{X} \leq_{\rm st} \boldsymbol{Y}$ *if and only if* $\boldsymbol{\mu} \leq \boldsymbol{\mu}'$ *and* $\Sigma = \Sigma'$*.*

However, few results can be found in the literature that characterize the convex ordering of multivariate elliptical distributions. The purpose of this paper is to obtain some sufficient and/or necessary conditions for convex ordering and increasing convex ordering of multivariate elliptical random vectors.

The rest of this paper is organized as follows. In Section 2, we recall some important concerned concepts, including stochastic orders and elliptical distributions. Sections 3 and 4 present convex orderings of univariate and multivariate elliptical distributions, respectively.

2. Preliminaries

For ease of reference, in this section we recall the definitions of some stochastic orders and elliptical random vectors. Throughout the paper, the terms "increasing" and "decreasing" are used to mean "non-decreasing" and "non-increasing", respectively. All integrals and expectations are implicitly assumed to exist whenever they are written.

2.1. Stochastic orders

In this study, we will employ the following stochastic orders. Standard references for stochastic orders include Denuit et al. [5], Müller and Stoyan [12], and Shaked and Shanthikumar [14].

Definition 2.1. *For two random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ on $\mathbb{R}^n$, we say that $\boldsymbol{X}$ is smaller than $\boldsymbol{Y}$*

(1) *in the (multivariate) usual stochastic order, denoted by $\boldsymbol{X} \leq_{\rm st} \boldsymbol{Y}$, if $\mathrm{E}\{\phi(\boldsymbol{X})\} \leq \mathrm{E}\{\phi(\boldsymbol{Y})\}$ for all increasing functions ϕ;*

(2) *in the (multivariate) convex order, denoted by $\boldsymbol{X} \leq_{\rm cx} \boldsymbol{Y}$, if $\mathrm{E}\{\phi(\boldsymbol{X})\} \leq \mathrm{E}\{\phi(\boldsymbol{Y})\}$ for all convex functions ϕ;*

(3) *in the (multivariate) increasing convex order, denoted by $\boldsymbol{X} \leq_{\rm icx} \boldsymbol{Y}$, if $\mathrm{E}\{\phi(\boldsymbol{X})\} \leq \mathrm{E}\{\phi(\boldsymbol{Y})\}$ for all increasing convex functions ϕ;*

(4) *in the (multivariate) linear convex order, denoted by $\boldsymbol{X} \leq_{\rm lcx} \boldsymbol{Y}$, if $\boldsymbol{a}^\top \boldsymbol{X} \leq_{\rm cx} a^\top \boldsymbol{Y}$ for all $\boldsymbol{a} \in \mathbb{R}^n$;*

(5) *in the (multivariate) increasing linear convex order, denoted by $\boldsymbol{X} \leq_{\rm ilcx} \boldsymbol{Y}$, if $\boldsymbol{a}^\top \boldsymbol{X} \leq_{\rm icx} a^\top \boldsymbol{Y}$ for all $\boldsymbol{a} \in \mathbb{R}^n$.*

The following implications are well known (see, e.g., Scarsini [13]):

$$\begin{array}{ccccc} & & \boldsymbol{X} \leq_{\rm cx} \boldsymbol{Y} & \Longrightarrow & \boldsymbol{X} \leq_{\rm lcx} \boldsymbol{Y} \\ & & \Downarrow & & \Updownarrow \\ \boldsymbol{X} \leq_{\rm st} \boldsymbol{Y} & \Longrightarrow & \boldsymbol{X} \leq_{\rm icx} \boldsymbol{Y} & \not\Leftarrow\!\Rightarrow & \boldsymbol{X} \leq_{\rm ilcx} \boldsymbol{Y}. \end{array}$$

Moreover, if $\boldsymbol{X} \leq_{\rm icx} \boldsymbol{Y}$ and $\mathrm{E}(\boldsymbol{X}) = \mathrm{E}(\boldsymbol{Y})$, then $\boldsymbol{X} \leq_{\rm cx} \boldsymbol{Y}$.

2.2. Elliptical distributions

Elliptical distributions, introduced by Kelker [9] and further discussed by Fang et al. [7], constitute generalizations of the multivariate normal family. We briefly recall below the basic definition of an elliptical distribution.

Definition 2.2. *Let $\boldsymbol{X} = (X_1, \ldots, X_n)^\top$ be an n-dimensional random vector. We say that $\boldsymbol{X}$ has a multivariate elliptical distribution, denoted by $\boldsymbol{X} \sim \mathcal{E}_n(\boldsymbol{\mu}, \Sigma, \phi)$, if its characteristic function can be expressed as*

$$\mathrm{E}\left(e^{i\boldsymbol{t}^\top \boldsymbol{X}}\right) = \exp(i\boldsymbol{t}^\top \boldsymbol{\mu})\phi\left(\frac{1}{2}\,\boldsymbol{t}^\top \Sigma \boldsymbol{t}\right),$$

where ϕ is an n-dimensional characteristic function.

Besides multivariate normal distributions, obtained by choosing $\phi(t) = e^{-t}$, Laplace distributions, t-Student distributions, Cauchy distributions, logistic distributions and symmetric stable distributions are examples of elliptical distributions.

One useful characterization of the elliptical distribution is as follows. Let $\boldsymbol{X} \sim \mathcal{E}_n(\boldsymbol{\mu}, \Sigma, \phi)$, and let A be an $n \times n$ matrix such that $AA^\top = \Sigma$. Then $\boldsymbol{X}$ has the following stochastic representation

$$\boldsymbol{X} \stackrel{\mathrm{d}}{=} \boldsymbol{\mu} + RA\boldsymbol{U}, \tag{2.1}$$

where $\stackrel{\mathrm{d}}{=}$ means equality in distribution, $\boldsymbol{U}$ is uniformly distributed on the unit hypersphere $\mathcal{S}^{n-1} = \left\{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{u}^\top \boldsymbol{u} = 1\right\}$, and R is a non-negative random variable, independent of $\boldsymbol{U}$. The distribution function F_0 of R is related to ϕ by the following relation: for any $r \in \mathbb{R}_+$,

$$F_0(r) = \int_{B(r)} \mathrm{d}G(\boldsymbol{y}), \tag{2.2}$$

where $B(r) = \{\boldsymbol{y} \in \mathbb{R}^n : \boldsymbol{y}^\top \boldsymbol{y} \le r\}$, and G is the distribution function of a random vector whose characteristic function is given by $g(\boldsymbol{t}) = \phi(\boldsymbol{t}^\top \boldsymbol{t})$ for $\boldsymbol{t} \in \mathbb{R}^n$ (see the proof of Theorem 2.2 in [7]).

For any $\boldsymbol{X} \sim \mathcal{E}_n(\boldsymbol{\mu}, \Sigma, \phi)$, it is easy to see from (2.1) that $\mathrm{E}(\boldsymbol{X}) = \boldsymbol{\mu}$ if and only if $\mathrm{E}(R) < \infty$. When $\Sigma \neq 0$, all components of $\boldsymbol{X}$ have finite second moment if and only if $\mathrm{E}(R^2) < \infty$. From Theorem 4 in Cambanis et al. [3], it follows that the covariance matrix of $\boldsymbol{X}$ exists if and only if the right-hand derivative of $\phi(u)$ at $u = 0$, denoted $\phi'_+(0)$, exists and is finite, in which case $\mathrm{cov}(\boldsymbol{X}) = -2\phi'_+(0)\boldsymbol{\Sigma}$. The characteristic generator ϕ can be chosen such that $-2\phi'_+(0) = 1$, so that $\mathrm{cov}(\boldsymbol{X}) = \Sigma$. The random vector $\boldsymbol{X}$ does not, in general, possess a density but if it does, it is given, for all $\boldsymbol{x} \in \mathbb{R}^n$, by

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{c_n}{\sqrt{|\Sigma|}}\, g_n\left\{(\boldsymbol{x} - \boldsymbol{\mu})^\top \Sigma^{-1} (\boldsymbol{x} - \boldsymbol{\mu})\right\},$$

where the non-negative function g_n is called the density generator and c_n is a normalizing constant. For simplicity, we denote $\boldsymbol{X} \sim \mathcal{E}_n(\boldsymbol{\mu}, \Sigma, g_n)$. We refer to Fang et al. [7] for more details about elliptical distributions.

Throughout this paper, all elliptical distributions are assumed to be non-degenerate, i.e., $\Pr(R > 0) > 0$, where R is given in (2.1). Also, assume that $\mathrm{E}(R^2) < \infty$.

3. Convex ordering for univariate elliptical distributions

In this section, we discuss the characterization of the convex ordering for univariate elliptical distributions with the same generator.

Theorem 3.1. *Let $X \sim \mathcal{E}_1(\mu, \sigma^2, \phi)$ and $Y \sim \mathcal{E}_1(\mu', (\sigma')^2, \phi)$. Then $X \le_{\mathrm{cx}} Y$ if and only if $\mu = \mu'$ and $\sigma \le \sigma'$.*

Proof. Given that X and Y have the same characteristic generator ϕ, it follows from (2.1) and (2.2) that they have the following stochastic representations:

$$X \stackrel{\mathrm{d}}{=} \mu + \sigma RU, \qquad Y \stackrel{\mathrm{d}}{=} \mu' + \sigma' RU, \tag{3.1}$$

where R is a non-negative random variable, independent of U with $\Pr(U = \pm 1) = 1/2$. Denote by F_0 the distribution function of R. Then, for all convex functions $g : \mathbb{R} \to \mathbb{R}$, we have

$$\begin{aligned}
\mathrm{E}\{g(Y)\} - \mathrm{E}\{g(X)\} &= \mathrm{E}\left\{g\left(\mu' + \sigma' RU\right)\right\} - \mathrm{E}\{g(\mu + \sigma RU)\} \\
&= \int_0^\infty \left[\mathrm{E}\left\{g\left(\mu' + \sigma' rU\right)\right\} - \mathrm{E}\{g(\mu + \sigma rU)\}\right] \mathrm{d}F_0(r) \\
&= \int_0^\infty \frac{1}{2}\left\{g(\mu' + \sigma' r) - g(\mu + \sigma r) + g(\mu' - \sigma' r) - g(\mu - \sigma r)\right\} \mathrm{d}F_0(r).
\end{aligned}$$

If $\mu = \mu'$ and $\sigma \le \sigma'$, then $X \le_{\mathrm{cx}} Y$ holds due to the convexity of g. On the other hand, if $X \le_{\mathrm{cx}} Y$, then $\mu = \mathrm{E}(X) = \mathrm{E}(Y) = \mu'$ and

$$\sigma^2 = \frac{\mathrm{var}(X)}{\mathrm{var}(RU)} \le \frac{\mathrm{var}(Y)}{\mathrm{var}(RU)} = \left(\sigma'\right)^2.$$

Therefore, the desired result follows. ∎

From the representation (3.1), it is known that a univariate elliptical random variable is distributed symmetrically about its mean. Davidov and Peddada [4] proved that, under the assumption of Theorem 3.1, $X \le_{\mathrm{st}} Y$ if and only if $\mu \le \mu'$ and $\sigma = \sigma'$. They also gave an example to show that, for the usual stochastic ordering, this result may not be true when X and Y have finite supports. However, the next example shows that Theorem 3.1 may hold when X and Y have finite supports.

Example 3.2. *If $R \sim \mathcal{U}(0, 1)$ in (3.1), then $X \sim \mathcal{U}(\mu - \sigma, \mu + \sigma)$ and $Y \sim \mathcal{U}(\mu' - \sigma', \mu' + \sigma')$. It is easy to see that $X \le_{\mathrm{cx}} Y$ if and only if $\mu = \mu'$ and $\sigma \le \sigma'$.*

Theorem 3.3. *Let $X \sim \mathcal{E}_1(\mu, \sigma^2, \phi)$ and $Y \sim \mathcal{E}_1(\mu', (\sigma')^2, \phi)$. Then $X \le_{\mathrm{icx}} Y$ if and only if $\mu \le \mu'$ and $\sigma \le \sigma'$.*

Proof. First, X and Y have the stochastic representation (3.1) with $\Pr(R > 0) > 0$. If $X \le_{\mathrm{icx}} Y$, then $\mu = \mathrm{E}(X) \le \mathrm{E}(Y) = \mu'$ and, for all t,

$$\mathrm{E}(X - t)_+ \le \mathrm{E}(Y - t)_+. \tag{3.2}$$

Note that

$$\mathrm{E}(X - t)_+ = \int_t^\infty \Pr\left(RU > \frac{x - \mu}{\sigma}\right) \mathrm{d}x \ge \int_t^\infty \Pr\left(RU > \frac{x - \mu'}{\sigma}\right) \mathrm{d}x,$$

$$\mathrm{E}(Y-t)_+ \;=\; \int_t^\infty \Pr\left(RU > \frac{x-\mu'}{\sigma'}\right) \mathrm{d}x.$$

If $\sigma > \sigma'$, there exists $x_0 > \mu'$ such that

$$\Pr\left(RU > \frac{x_0-\mu'}{\sigma}\right) > \Pr\left(RU > \frac{x_0-\mu'}{\sigma'}\right),$$

which implies $\mathrm{E}(X-\mu')_+ > \mathrm{E}(Y-\mu')_+$, violating (3.2). Therefore, $\sigma \le \sigma'$.

On the other hand, suppose that $Z \sim \mathcal{E}_1(\mu', \sigma^2, \phi)$. If $\mu \le \mu'$ and $\sigma \le \sigma'$, then

$$X \le_{\mathrm{st}} Z \le_{\mathrm{cx}} Y,$$

implying $X \le_{\mathrm{icx}} Y$. This completes the proof of the theorem. ∎

4. Convex ordering for multivariate elliptical distributions

In this section, we mainly discuss characterizations of the convex ordering between two n-dimensional elliptically distributed random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, where $\boldsymbol{X} \sim \mathcal{E}_n(\boldsymbol{\mu}, \Sigma, \phi)$ and $\boldsymbol{Y} \sim \mathcal{E}_n(\boldsymbol{\mu}', \Sigma', \phi)$.

From (2.1) and (2.2), $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the following stochastic representations:

$$\boldsymbol{X} \overset{\mathrm{d}}{=} \boldsymbol{\mu} + RA\boldsymbol{U}, \qquad \boldsymbol{Y} \overset{\mathrm{d}}{=} \boldsymbol{\mu}' + RA'\boldsymbol{U}. \tag{4.1}$$

Here $\Sigma = AA^\top$, $\Sigma' = A'(A')^\top$, R is a nonnegative random variable, independent of $\boldsymbol{U}$, and $\boldsymbol{U}$ is uniformly distributed on the unit hypersphere $\mathcal{S}^{n-1} \equiv \left\{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{u}^\top \boldsymbol{u} = 1\right\}$. Given that $\mathrm{E}(R^2) < \infty$, $\phi'_+(0) < 0$ exists and thus

$$\mathrm{cov}(\boldsymbol{Y}) - \mathrm{cov}(\boldsymbol{X}) = -2\phi'_+(0)\,(\Sigma' - \Sigma).$$

Then $\boldsymbol{\Sigma}' - \boldsymbol{\Sigma}$ is positively semi-definite if and only if $\mathrm{cov}(\boldsymbol{Y}) - \mathrm{cov}(\boldsymbol{X})$ is positively semi-definite.

The main result of this section is the following theorem, which generalizes Theorem 3.3 from the univariate to the multivariate case.

Theorem 4.1. *Let $\boldsymbol{X} \sim \mathcal{E}_n(\boldsymbol{\mu}, \Sigma, \phi)$ and $\boldsymbol{Y} \sim \mathcal{E}_n(\boldsymbol{\mu}', \Sigma', \phi)$. Then the following statements are equivalent:*

(1) $\boldsymbol{\mu} = \boldsymbol{\mu}'$ *and* $\Sigma' - \Sigma$ *is positively semi-definite;*

(2) $\boldsymbol{X} \le_{\mathrm{lcx}} \boldsymbol{Y}$*;*

(3) $\boldsymbol{X} \le_{\mathrm{cx}} \boldsymbol{Y}$.

To prove Theorem 4.1, we need the following two lemmas.

Lemma 4.2. *Let* $\boldsymbol{U} = (U_1, \ldots, U_n)^\top$ *be a random vector uniformly distributed on the unit hypersphere* $\mathcal{S}^{n-1} \equiv \{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{u}^\top \boldsymbol{u} = 1\}$. *Then, for any fixed* $\boldsymbol{r} = (r_1, \ldots, r_n) \in [0,1]^n$,

$$(r_1 U_1, \ldots, r_n U_n)^\top \leq_{\mathrm{cx}} \boldsymbol{U}.$$

Proof. It suffices to prove that, for all $r \in [0,1]$,

$$(rU_1, U_2, \ldots, U_n)^\top \leq_{\mathrm{cx}} \boldsymbol{U} \tag{4.2}$$

and for any random vector $\boldsymbol{U}$ with the property

$$\boldsymbol{U} \stackrel{\mathrm{d}}{=} (-U_1, U_2, \ldots, U_n)^\top. \tag{4.3}$$

To see it, define $\theta = (1+r)/2$. Note that, for any $r \in [0,1]$,

$$(rU_1, U_2, \ldots, U_n)^\top = \theta \boldsymbol{U} + (1-\theta)(-U_1, U_2, \ldots, U_n)^\top.$$

Then, for any convex function $\phi : \mathbb{R}^n \to \mathbb{R}$, we have

$$\mathrm{E}\{\phi(rU_1, U_2, \ldots, U_n)\} \leq \theta \mathrm{E}\{\phi(\boldsymbol{U})\} + (1-\theta)\mathrm{E}\{\phi(-U_1, U_2, \ldots, U_n)\} = \mathrm{E}\{\phi(\boldsymbol{U})\},$$

where the equality follows from (4.3). This implies (4.2). This completes the proof. ∎

It should be pointed out that Lemma 4.2 holds for all random vectors $\boldsymbol{U}$ which are such that, for all $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)^\top \in \{-1, 1\}^n$,

$$\boldsymbol{U} \stackrel{\mathrm{d}}{=} (\delta_1 U_1, \ldots, \delta_n U_n)^\top.$$

Recall that an $n \times n$ matrix C is said to be a contraction if $\sigma_1(C) \leq 1$, where $\sigma_1(C)$ is the largest singular value of C.

Lemma 4.3. (Horn and Johnson [8], Theorem 7.7.3) *Let A and B be two non-negative definite $n \times n$ matrices. Then $A^2 - B^2$ is non-negative definite if and only if there exists a contraction matrix C such that $B = AC$.*

Proof of Theorem 4.1. Since "(3) ⇒ (2)" is obvious, we only need to prove "(2) ⇒ (1)" and "(1) ⇒ (3)".

For "(2) ⇒ (1)", $\boldsymbol{X} \leq_{\mathrm{lcx}} \boldsymbol{Y}$ means that $\boldsymbol{a}^\top \boldsymbol{X} \leq_{\mathrm{cx}} \boldsymbol{a}^\top \boldsymbol{Y}$ for all $\boldsymbol{a} \in \mathbb{R}^n$ and, hence, $\boldsymbol{a}^\top \boldsymbol{\mu} = \boldsymbol{a}^\top \boldsymbol{\mu}'$ and

$$\boldsymbol{a}^\top \mathrm{cov}(\boldsymbol{X}) \boldsymbol{a} = \mathrm{var}(\boldsymbol{a}^\top \boldsymbol{X}) \leq \mathrm{var}(\boldsymbol{a}^\top \boldsymbol{Y}) = \boldsymbol{a}^\top \mathrm{cov}(\boldsymbol{Y}) \boldsymbol{a}.$$

Thus, $\boldsymbol{\mu} = \boldsymbol{\mu}'$ and $\mathrm{cov}(\boldsymbol{Y}) - \mathrm{cov}(\boldsymbol{X})$ is non-negative definite. That is, statement (1) holds.

For "(1) ⇒ (3)", without loss of generality assume that $\boldsymbol{\mu} = \boldsymbol{\mu}' = \mathbf{0}$. First note that $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the stochastic representation (4.1) with $A = \Sigma^{1/2}$ and $A' = (\Sigma')^{1/2}$. By Lemma 4.3, there exists a contraction matrix C such that $A = A'C$. By the singular value decomposition, there exist two orthogonal $n \times n$ matrices S_1 and S_2 such that

$$C = S_1 \Delta S_2,$$

where $\Delta = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $0 \le \sigma_n \le \cdots \le \sigma_1 \le 1$. For any convex function $h : \mathbb{R}^n \to \mathbb{R}$, define, for all $\boldsymbol{x} \in \mathbb{R}^n$,

$$g(\boldsymbol{x}) = h(A' S_1 \boldsymbol{x}).$$

Then g is convex. Given that $\boldsymbol{U}$ is uniformly distributed on $\mathcal{S}^{n-1}$, it follows that

$$\boldsymbol{U} \stackrel{\mathrm{d}}{=} S_2 \boldsymbol{U} \stackrel{\mathrm{d}}{=} S_1^\top \boldsymbol{U}.$$

Then

$$\begin{aligned}
\mathrm{E}\{h(\boldsymbol{X})\} &= \mathrm{E}\{g(R\Delta S_2 \boldsymbol{U})\} = \mathrm{E}\{g(R\Delta \boldsymbol{U})\} \\
&= \mathrm{E}\left[\mathrm{E}\{g(R\Delta \boldsymbol{U}) | R\}\right] \\
&\le \mathrm{E}\left[\mathrm{E}\{g(R \boldsymbol{U}) | R\}\right] \qquad \text{(by Lemma 4.2)} \\
&= \mathrm{E}\left\{g(R S_1^\top \boldsymbol{U})\right\} = \mathrm{E}\{h(\boldsymbol{Y})\}.
\end{aligned}$$

That is, $\boldsymbol{X} \le_{\mathrm{cx}} \boldsymbol{Y}$ and statement (3) holds. This completes the proof of the theorem. ∎

Since $\boldsymbol{X} \le_{\mathrm{cx}} \boldsymbol{Y}$ is equivalent to $\boldsymbol{X} \le_{\mathrm{icx}} \boldsymbol{Y}$ and $\mathrm{E}(\boldsymbol{X}) = \mathrm{E}(\boldsymbol{Y})$ (Shaked and Shanthikumar [14]), we have the following corollaries of Theorem 4.1.

Corollary 4.4. *Let $\boldsymbol{X} \sim \mathcal{E}_n(\boldsymbol{\mu}, \Sigma, \phi)$ and $\boldsymbol{Y} \sim \mathcal{E}_n(\boldsymbol{\mu}, \Sigma', \phi)$. Then*

$$\boldsymbol{X} \le_{\mathrm{cx}} \boldsymbol{Y} \quad \Leftrightarrow \quad \boldsymbol{X} \le_{\mathrm{lcx}} \boldsymbol{Y} \quad \Leftrightarrow \quad \boldsymbol{X} \le_{\mathrm{ilcx}} \boldsymbol{Y} \quad \Leftrightarrow \quad \boldsymbol{X} \le_{\mathrm{icx}} \boldsymbol{Y}.$$

Corollary 4.5. *Let $\boldsymbol{X} \sim \mathcal{E}_n(\boldsymbol{\mu}, \Sigma, \phi)$ and $\boldsymbol{Y} \sim \mathcal{E}_n(\boldsymbol{\mu}', \Sigma', \phi)$. Then*

$$\boldsymbol{X} \le_{\mathrm{ilcx}} \boldsymbol{Y} \quad \Rightarrow \quad \boldsymbol{X} \le_{\mathrm{icx}} \boldsymbol{Y}.$$

For general random vectors, there is no relationship between the orders $\le_{\mathrm{icx}}$ and $\le_{\mathrm{ilcx}}$. Example 3.5.4 in Müller and Stoyan [12] shows that $\boldsymbol{X} \le_{\mathrm{ilcx}} \boldsymbol{Y} \not\Rightarrow \boldsymbol{X} \le_{\mathrm{icx}} \boldsymbol{Y}$. It is shown in Example 4.7 below that the order $\le_{\mathrm{icx}}$ does not imply the order $\le_{\mathrm{ilcx}}$ even for elliptical distributions. Before we state Example 4.7, we first give a sufficient condition for the increasing convex order.

Theorem 4.6. *Let $\boldsymbol{X} \sim \mathcal{E}_n(\boldsymbol{\mu}, \Sigma, \phi)$ and $\boldsymbol{Y} \sim \mathcal{E}_n(\boldsymbol{\mu}', \Sigma', \phi)$.*

(1) *If* $\boldsymbol{\mu} \le \boldsymbol{\mu}'$ *and* $\Sigma' - \Sigma$ *is positively semi-definite, then* $\boldsymbol{X} \le_{\rm icx} \boldsymbol{Y}$.

(2) *If* $\boldsymbol{X} \le_{\rm icx} \boldsymbol{Y}$, *then* $\boldsymbol{\mu} \le \boldsymbol{\mu}'$ *and* $\Sigma' - \Sigma$ *is copositive, i.e.,* $\boldsymbol{a}^\top(\Sigma' - \Sigma)\boldsymbol{a} \ge 0$ *for all* $\boldsymbol{a} \ge \boldsymbol{0}$.

Proof. (1) Let $\boldsymbol{Z} \sim \mathcal{E}_n(\boldsymbol{\mu}', \Sigma, \phi)$. Then $\boldsymbol{X} \le_{\rm st} \boldsymbol{Z}$. By Theorem 4.1, we have $\boldsymbol{Z} \le_{\rm cx} \boldsymbol{Y}$. Due to Theorem 7.A.3 in Shaked and Shanthikumar [14], we have $\boldsymbol{X} \le_{\rm icx} \boldsymbol{Y}$.

(2) On one hand, it is easy to see that $\boldsymbol{\mu} \le \boldsymbol{\mu}'$ given that the functions $g_i(\boldsymbol{x}) = x_i$ are increasing and convex. On the other hand, $g_{\boldsymbol{a}}(\boldsymbol{x}) = g(\boldsymbol{a}^\top \boldsymbol{x})$ is increasing convex for any increasing convex function g and $\boldsymbol{a} \ge \boldsymbol{0}$. Since $\boldsymbol{X} \le_{\rm icx} \boldsymbol{Y}$ implies $\boldsymbol{a}^\top \boldsymbol{X} \le_{\rm icx} \boldsymbol{a}^\top \boldsymbol{Y}$, it follows that

$$\mathrm{var}(\boldsymbol{a}^\top \boldsymbol{X}) = \boldsymbol{a}^\top \mathrm{var}(\boldsymbol{X})\boldsymbol{a} \le \boldsymbol{a}^\top \mathrm{var}(\boldsymbol{Y})\boldsymbol{a} = \mathrm{var}(\boldsymbol{a}^\top \boldsymbol{Y}),$$

implying $\boldsymbol{a}^\top(\Sigma' - \Sigma)\boldsymbol{a} \ge 0$. ∎

It is known from Arlotto and Scarsini [1] that, in Theorem 4.6, the conditions that $\boldsymbol{\mu} \le \boldsymbol{\mu}'$ and $\Sigma' - \Sigma$ is copositive characterize an order called the completely positive order between two multivariate normal distributions. It is still unknown whether such a characterization holds for multivariate elliptical distributions.

Example 4.7. (The order $\le_{\rm icx}$ does not imply the order $\le_{\rm ilcx}$) *Let* R *in* (2.1) *have a uniform distribution on the interval* $[0, 1]$, *i.e.,* $\mathcal{U}(0, 1)$, *and let*

$$\begin{aligned} \boldsymbol{X} &= \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{E}_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \phi \right), \\ \boldsymbol{Y} &= \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim \mathcal{E}_2\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \phi \right). \end{aligned}$$

Then, by Theorem 4.6, $\boldsymbol{X} \le_{\rm icx} \boldsymbol{Y}$. *Due to* (2.1), *we have*

$$\boldsymbol{X} \stackrel{\rm d}{=} \begin{pmatrix} R\cos\Theta \\ R\sin\Theta \end{pmatrix}, \quad \boldsymbol{Y} \stackrel{\rm d}{=} \begin{pmatrix} 2R\cos\Theta + 3 \\ 2R\sin\Theta + 3 \end{pmatrix}$$

with $R \sim \mathcal{U}(0, 1)$ *and* $\Theta \sim \mathcal{U}(0, 2\pi)$. *Set* $\boldsymbol{a}^\top = (-1, 0)$ *and* $g(x) = \max(x, 0)$. *Then*

$$\mathrm{E}\left\{g(\boldsymbol{a}^\top \boldsymbol{X})\right\} = \mathrm{E}\{g(-R\cos\Theta)\} > 0 = \mathrm{E}\{g(-2R\cos\Theta - 3)\} = \mathrm{E}\left\{g(\boldsymbol{a}^\top \boldsymbol{Y})\right\},$$

which means $\boldsymbol{X} \not\le_{\rm ilcx} \boldsymbol{Y}$.

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