

# On a Shrinkage Estimator of a Normal Common Mean Vector

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The problem of estimating the  $p \times 1$  mean vector  $\theta$  based on two independent normal vectors  $Y_1 \sim N_p(\theta, \sigma^2 I)$  and  $Y_2 \sim N_p(\theta, \xi \sigma^2 I)$  is considered. For  $p \geq 3$ , when  $\xi$  and  $\sigma^2$  are unknown, it was shown by George (1991, *Ann. Statist.*) that under certain conditions estimators of the form  $\delta_\eta = \eta Y_1 + (1 - \eta) Y_2$ , where  $\eta$  is a fixed number in  $(0, 1)$ , are uniformly dominated by a shrinkage estimator under the squared error loss. In this paper, George's result is improved by obtaining a simpler and better condition for the domination. © 1992 Academic Press, Inc.

## 1. INTRODUCTION

Let  $Y_1$  be an observation from a  $p$ -variate normal population with mean  $\theta$  and covariance matrix  $\sigma^2 I$ ,  $N_p(\theta, \sigma^2 I)$ . Let  $Y_2$  be an observation from  $N_p(\theta, \xi \sigma^2 I)$ . Assume that  $\sigma^2$ ,  $\xi$  are unknown positive scalars and  $Y_1$  and  $Y_2$  are independent. Let  $\Psi \equiv \{\psi: \theta \in R^p, \xi > 0, \sigma^2 > 0\}$  denote the parameter space. In this situation George [2] has considered the problem of estimating  $\theta$  based on  $(Y_1, Y_2)$  under the risk criterion of expected squared error loss,

$$R(\psi, \delta) = E_\psi \|\delta - \theta\|^2. \quad (1.1)$$

Under this criterion, it has been shown in George [2] that the estimator

$$\delta_\eta = \eta Y_1 + (1 - \eta) Y_2 \quad (1.2)$$

is dominated by the shrinkage estimator

$$\delta_\eta^c = \left(1 - c \frac{\|Y_1 - Y_2\|^2}{\|\delta_\eta\|^2}\right) \delta_\eta, \quad (1.3)$$

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where  $\eta$  is a fixed number in  $(0, 1)$ ,  $\kappa = \min(\eta^2, (1 - \eta)^2)$ ,  $0 < c < 2\kappa/13$  for  $p = 3$  and  $0 < c < 2(p - 2)\kappa/(p + 8)$  for  $p \geq 4$ . Based on strong simulation evidence, George [2] has conjectured that the upper bound for  $c$  can be increased to  $2\kappa/6$  for  $p = 3$  and to  $2(p - 2)\kappa/(p + 2)$  for  $p \geq 4$ .

In the following section we show that, for  $\delta_\eta^c$  to dominate  $\delta_\eta$ , the upper bound for  $c$  can in fact be increased to  $2(p - 2)\kappa/(p + 2)$  not only for  $p \geq 4$  but also for  $p = 3$ .

We note that although some expectations that are needed to prove Theorem 2.1 are evaluated in George [2], we derive them in Section 2 in a relatively easy manner using some standard methods. Also the present form of these expectations simplified the proof of Theorem 2.1 to some extent.

## 2. MAIN RESULTS

We now state the main results of this paper in the following theorem.

**THEOREM 2.1.** *Let  $\kappa \equiv \min(\eta^2, (1 - \eta)^2)$ . Then for  $p \geq 3$ ,  $\eta \in (0, 1)$  and  $0 < c < 2(p - 2)\kappa/(p + 2)$ , the estimator  $\delta_\eta^c$  in (1.3) uniformly dominates  $\delta_\eta$  under the risk criterion (1.1).*

We need the following lemmas to prove the theorem.

**LEMMA 2.1.** *For a Poisson random variable  $Z$  with mean  $\lambda$ , let  $\phi(q) \equiv E(q + 2Z)^{-1}$  for  $q \neq 0, -2, -4, \dots$ , then*

- (i)  $(q + 2\lambda)\phi(q) \geq 1$ ,
- (ii)  $2\lambda\phi(q + 2) = 1 - q\phi(q)$ , for  $q \neq 0, -2, -4, \dots$ ,
- (iii)  $2\lambda\phi(q + 2) \leq 1 - q\phi(q + 2)$ , for  $q \geq 0$ ,
- (iv)  $5\alpha[\phi(-1) - 3\phi(1)] + 3\alpha^2[\phi(1) - 2\phi(-1) + \phi(-3)] < 0$ , for  $0 \leq \alpha \leq 1$ .

*Proof.* (i) follows from Jensen's inequality. For proofs of (ii) and (iii), see George [2].

(iv) As  $(2z - 1)^{-1} - 3(2z + 1)^{-1} \leq 0$  for  $z = 0, 1, 2, \dots$ , and  $\alpha \geq \alpha^2$ , we have

$$\begin{aligned} & 5\alpha[(2z - 1)^{-1} - 3(2z + 1)^{-1}] + 3\alpha^2[(2z + 1)^{-1} - 2(2z - 1)^{-1} + (2z - 3)^{-1}] \\ & \leq \alpha^2[5(4 - 4z)(2z - 3) + 24]/[(2z + 1)(2z - 1)(2z - 3)] \\ & = -4\alpha^2\gamma(z), \end{aligned}$$

where  $\gamma(z) = (10z^2 - 25z + 9)/[(2z - 1)(2z + 1)(2z - 3)]$ . Thus, to prove

(iv), it suffices to show that  $E(\gamma(Z)) > 0$ . Observing that  $\gamma(z) > 0$  for nonnegative integers  $z \neq 2$ , we prove that

$$E(\gamma(Z)) = \sum_{z \neq 2,3} \gamma(z) e^{-\lambda} \lambda^z / z! + [(8\lambda/7) - 1] e^{-\lambda} \lambda^2 / 30 > 0$$

for  $\lambda > 7/8$ . Similarly, adding the terms corresponding to  $z = 0$  and  $z = 2$ , we can show that  $E(\gamma(Z)) > 0$  for  $0 < \lambda \leq 7/8$ .

**LEMMA 2.2.** *Let  $U$  and  $V$  be  $p \times 1$  random vectors such that  $(U', V')' \sim N_{2p}((0', \mu')', \Sigma_\beta)$  with  $\Sigma_\beta = \begin{pmatrix} I & \beta I \\ \beta I & I \end{pmatrix}$  and  $-1 < \beta < 1$ . For a Poisson random variable  $Z$  with mean  $\lambda = \|\mu\|^2/2$ , as in Lemma 2.1, let  $\phi(q) \equiv E(q + 2Z)^{-1}$ . Then, for  $p \geq 3$ ,*

- (i)  $E(\|U\|^2 \|V\|^{-2}) = [\beta^2(p - 4 + 2\lambda) + p] \phi(p - 2) - \beta^2$
- (ii)  $E(\|U\|^4 \|V\|^{-2}) = [(p + 4) + \beta^2(p - 6 + 2\lambda)] E(\|U\|^2 \|V\|^{-2}) - [2(2\beta^4\lambda - \beta^2p + p) \phi(p - 2) + \beta^2(p - 2\beta^2)]$ .
- (iii)  $E(U' V \|V\|^{-2}) = \beta(p - 2) \phi(p - 2)$ .

*Proof.* All these expectations can be evaluated using the well-known identity that

$$E\|V\|^{-2} = E(p - 2 + 2Z)^{-1}. \quad (2.1)$$

Let  $f(U, V; \mu, \Sigma) = (2\pi)^{-p} \exp\{-1/2[U', (V - \mu)']' \Sigma^{-1}[U', (V - \mu)']'\}$ .

(i) First we note that for a real  $t$ ,

$$\begin{aligned} E(\|U\|^2 \|V\|^{-2}) &= -\frac{\partial}{\partial t} \left\{ (1 - \beta^2)^{-p/2} \int \|V\|^{-2} e^{-tU'U} f(U, V; \mu, \Sigma_\beta) d_U d_V \right\} \Big|_{t=0} \\ &= -\frac{\partial}{\partial t} \left\{ (1 - \beta^2)^{-p/2} \int \|V\|^{-2} f(U, V; \mu, \Sigma_{\beta,t}) d_U d_V \right\} \Big|_{t=0}, \end{aligned} \quad (2.2)$$

where

$$\Sigma_{\beta,t} = \begin{pmatrix} (1 + 2t)^{-1} I & \beta(1 + 2t)^{-1} I \\ \beta(1 + 2t)^{-1} I & (1 + 2t(1 - \beta^2))(1 + 2t)^{-1} I \end{pmatrix}.$$

Noting that  $|\Sigma_{\beta,t}| = (1 - \beta^2)^p (1 + 2t)^{-p}$ , it follows from (2.2) that

$$E(\|U\|^2 \|V\|^{-2}) = -\frac{\partial}{\partial t} \{ (1 + 2t)^{-p/2+1} (1 + 2t(1 - \beta^2))^{-1} E\|V_t\|^2 \} \Big|_{t=0}, \quad (2.3)$$

where  $V_t \sim N_p(\mu_t, I)$  and  $\mu_t = (1 + 2t)^{1/2} (1 + 2t(1 - \beta^2))^{-1/2} \mu$ . Thus, we obtain

$$\begin{aligned} E(\|U\|^2 \|V\|^{-2}) \\ = -\frac{\partial}{\partial t} \{ (1 + 2t)^{-p/2+1} (1 + 2t(1 - \beta^2))^{-1} E(p - 2 + 2Z_t)^{-1} \} \Big|_{t=0} \\ = (p - 2\beta^2 + 2\beta^2\lambda) E(p - 2 + 2Z)^{-1} - 2\beta^2 E(Z(p - 2 + 2Z)^{-1}), \end{aligned}$$

where  $Z_t$  in the first equality is a Poisson random variable with mean  $\lambda_t = \mu'_t \mu_t / 2$ . Now, using the fact that  $E(2Z(p - 2 + 2Z)^{-1}) = 1 - (p - 2)\phi(p - 2)$ , we prove (i).

(ii) As in (i), we can write

$$\begin{aligned} E(\|U\|^4 \|V\|^{-2}) \\ = -\frac{\partial}{\partial t} \left\{ (1 - \beta^2)^{-p/2} \int \|U\|^2 \|V\|^{-2} f(U, V; \mu, \Sigma_{\beta, t}) d_U d_V \right\} \Big|_{t=0} \\ = -\frac{\partial}{\partial t} \{ (1 + 2t)^{-p/2} (1 + 2t(1 - \beta^2))^{-1} E(\|U_t\|^2 \|V_t\|^{-2}) \} \Big|_{t=0}, \end{aligned} \quad (2.4)$$

where  $(U'_t, V'_t)' \sim N_{2p}((0', \mu'_t)', \Sigma_{\beta, t})$ , and  $\Sigma_{\beta, t}$  equal to  $\Sigma_\beta$  with  $\beta$  replaced by  $\beta_t = \beta(1 + 2t(1 - \beta^2))^{-1/2}$ . Thus, using (i) in (2.4), taking derivative at  $t = 0$ , and after some simplification, we get (ii).

(iii)  $E(U' V \|V\|^{-2}) = EE((U' V \|V\|^{-2}) | V) = \beta E((V - \mu)' V \|V\|^{-2}) = \beta(1 - E(\mu' V \|V\|^{-2}))$  and  $E(\mu' V \|V\|^{-2}) = \mu' (\partial/\partial \mu) E \|V\|^{-2} + \|\mu\|^2 E \|V\|^{-2} = E(2Z(p - 2 + 2Z)^{-1})$ . Thus, we prove (iii).

We are now ready to prove Theorem 2.1.

*Proof of Theorem 2.1.* Using the Stein's [3] identity that  $E(X - \mu) h(X) = \sigma^2 E h'(X)$  for  $X \sim N(\mu, \sigma^2)$  and changing the variables, George [2] has expressed the risk difference as

$$R(\psi, \delta_\eta^c) - R(\psi, \delta_\eta) = c\sigma_\eta^2 (cA_1 - 2(p - 2)A_2 - 4\sigma_\eta^{-2}A_3), \quad (2.5)$$

where

$$A_1 = (1 + \xi)^2 \sigma^4 \sigma_\eta^{-4} E(\|U\|^4 \|V\|^{-2}),$$

$$A_2 = (1 + \xi) \sigma^2 \sigma_\eta^{-2} E(\|U\|^2 \|V\|^{-2}),$$

$$A_3 = (1 + \xi) \sigma^2 \beta E(U' V \|V\|^{-2}),$$

$$\begin{pmatrix} U \\ V \end{pmatrix} \sim N_{2p} \left( \begin{pmatrix} 0 \\ \mu \end{pmatrix}, \begin{pmatrix} I & \beta I \\ \beta I & I \end{pmatrix} \right),$$

$$\beta = \frac{(\eta - (1 - \eta)\xi)\sigma}{\sigma_\eta \sqrt{(1 + \xi)}}, \quad \mu = \theta/\sigma_\eta, \quad \sigma_\eta^2 = (\eta^2 + (1 - \eta)^2 \xi) \sigma^2.$$

In terms of these expectations, (2.5) may be expressed as

$$\begin{aligned} & R(\psi, \delta_\eta^c) - R(\psi, \delta_\eta) \\ &= c(1 + \xi) \sigma^2 [c(1 + \xi) \sigma^2 \sigma_\eta^{-2} E(\|U\|^4 \|V\|^{-2}) \\ &\quad - 2(p - 2) E(\|U\|^2 \|V\|^{-2}) - 4\beta E(U' V \|V\|^{-2})]. \end{aligned} \quad (2.6)$$

As  $(1 + \xi) \sigma^2 \sigma_\eta^{-2} \leq \kappa^{-1}$ , for  $0 < c < 2(p - 2)\kappa/(p + 2)$ , it can be easily seen that  $R(\psi, \delta_\eta^c) - R(\psi, \delta_\eta) < 0$  if

$$\begin{aligned} g(\eta, \xi, \theta) &= E(\|U\|^4 \|V\|^{-2}) - (p + 2) E(\|U\|^2 \|V\|^{-2}) \\ &\quad - 2\beta(p - 2)^{-1} (p + 2) E(U' V \|V\|^{-2}) \leq 0 \end{aligned}$$

for all  $\eta$ ,  $\xi$ , and  $\theta$ . Using Lemma 2.2 and setting  $\alpha = \beta^2$ ,  $g(\eta, \xi, \theta)$  can be written as

$$\begin{aligned} g(\eta, \xi, \theta) &= [2 + \alpha(p - 6 + 2\lambda)] E\|U\|^2 \|V\|^{-2} - \alpha(p - 2\alpha) \\ &\quad - 2(2\alpha^2\lambda - \alpha p + p) \phi(p - 2) - 2\alpha(p + 2) \phi(p - 2). \end{aligned} \quad (2.7)$$

We now show that  $g(\eta, \xi, \theta) \leq 0$  for all  $\eta$ ,  $\xi$ , and  $\theta$ , separately for  $p \geq 4$  and  $p = 3$ .

Let  $p \geq 4$ . We first note that  $2 + \alpha(p - 6 + 2\lambda) \geq 0$  as  $0 \leq \alpha < 1$ . Using Lemma 2.1(iii) in Lemma 2.2(i), we obtain  $E\|U\|^2 \|V\|^{-2} \leq p\phi(p - 2)$ . Using this inequality in (2.7) and, after some minor simplification, we obtain

$$\begin{aligned} g(\eta, \xi, \theta) &\leq \alpha(p - 4 + 2\lambda) p\phi(p - 2) - 4\alpha^2\lambda\phi(p - 2) - \alpha p \\ &\quad + 2\alpha^2 - 2\alpha(p + 2) \phi(p - 2) \\ &\leq -4\alpha^2\lambda\phi(p - 2) + 2\alpha^2 - 2\alpha^2(p + 2) \phi(p - 2) \\ &= 2\alpha^2 - 8\alpha^2\phi(p - 2) - 2\alpha^2(p - 2 + 2\lambda) \phi(p - 2). \end{aligned} \quad (2.8)$$

The second inequality in the above expression follows from Lemma 2.1(iii)

and the relation  $\alpha \geq \alpha^2$ . Now the desired result follows from (2.8) and the fact that  $(p-2+2\lambda)\phi(p-2) \geq 1$  (Lemma 2.1(i)).

For  $p=3$ , using Lemma 2.1(ii) in Lemma 2.2(i), we obtain  $E\|U\|^2\|V\|^{-2} = \alpha\phi(-1) - \alpha\phi(1) + 3\phi(1)$ . Substituting this equality in (2.7) and, after some simplification, we can write

$$g(\eta, \xi, \theta) = 2\alpha\phi(-1) - 15\alpha\phi(1) + 3\alpha^2\phi(1) - 3\alpha^2\phi(-1) - 3\alpha \\ + 2\alpha^2 - 6\alpha^2\lambda\phi(1) + 6\alpha\lambda\phi(1) + 2\alpha^2\lambda\phi(-1). \quad (2.9)$$

Again from Lemma 2.1(ii), we obtain  $2\lambda\phi(1) = 1 + \phi(-1)$  and  $2\lambda\phi(-1) = 1 + 3\phi(-3)$ . Using these identities in (2.9), we can express

$$g(\eta, \xi, \theta) = 5\alpha[\phi(-1) - 3\phi(1)] + 3\alpha^2[\phi(1) - 2\phi(-1) + \phi(-3)].$$

Now, Lemma 2.1(iv) completes the proof.

*Remark 2.1.* Noting that  $\alpha$  becomes zero when  $\eta = \xi/(1+\xi)$ , it can be easily verified that  $R(\psi, \delta_\eta^c) - R(\psi, \delta_\eta) = 0$  when  $\xi = 1$  and  $c = 2(p-2)\kappa/(p+2)$ . This implies that, for  $\delta_\eta^c$  to dominate  $\delta_\eta$ , the least upper bound for  $c$  in  $\delta_\eta^c$  is  $2(p-2)\kappa/(p+2)$ .

*Remark 2.2.* For  $p \geq 3$ , it follows from (2.5) that an optimal choice for  $c$  in  $\delta_\eta^c$  is  $(p-2)\kappa/(p+2)$ .

*Remark 2.3.* The positive part version of  $\delta_\eta^c$ , that is,  $(1 - c\|Y_1 - Y_2\|^2\|\delta_\eta\|^{-2})^+ \delta_\eta$ , where  $a^+ = \max(0, a)$  improves  $\delta_\eta^c$  uniformly. The proof, noting the fact that  $g(Y_1, Y_2) = (1 - c\|Y_1 - Y_2\|^2\|\delta_\eta\|^{-2}) = g(FY_1, FY_2)$  for any orthogonal matrix  $F$ , is similar to the one given for one sample James-Stein estimation problem (for example, see Anderson [1, Lemma 3.5.2, p. 91]).

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