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Parametric Families of Multivariate Distributions with Given Margins

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Parametric families of continuous bivariate distributions with given margins that include independence and perfect positive dependence are compared on the basis on some important properties. Since many such families exist, the comparisons are helpful for deciding on suitable models for multivariate data. The study of the properties has motivation from applications in extreme value inference. One property considered for bivariate families is whether they extend to multivariate families, and extensions are given when possible. Several new bivariate and multivariate families are included and some open research problems in the area of multivariate families are mentioned. © 1993 Academic Press, Inc.

1. INTRODUCTION

The purposes of this paper are to make contributions to the derivation of parametric families of continuous multivariate distributions with given univariate margins, to indicate how applications influence the type of properties desired, and to point out open problems in this area. There are (at least) two ways to look at parametric families of multivariate distributions. One approach is to take a characterizing property of a parametric family of univariate distributions and extend the property to the multivariate case to get a multivariate family with all univariate margins of the same class. Another approach is based on the well-known result of Sklar [31]. If $H(x_1, ..., x_p)$ is a continuous p-variate cumulative distribution function with univariate margins $F_j(x_j)$, $j = 1, ..., p$, then $C(u_1, ..., u_p) = H(F_1^{-1}(u_1), ..., F_p^{-1}(u_p))$ is a copula or a multivariate distribution with uniform (0, 1) margins and $C(G_1(y_1), ..., G_p(y_p))$ is a multivariate distribution with univariate margins G_j, $j = 1, ..., p$, where G_j are arbitrary

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continuous univariate distribution functions. This latter point of view is taken in this paper and the author's motivation for this approach is partly based on multivariate extremes (Joe [17, 19]).

An interpretation of Sklar's result is that, for continuous multivariate distributions, the multivariate structure is in the copula C and is "independent" of the univariate margins. Hence with the second approach, a parametric multivariate family corresponds to a parametric family of copulas and the parameters can often be interpreted as dependence parameters. In Section 2, families of one-parameter bivariate copulas that include independence, perfect positive dependence, and possibly perfect negative dependence are listed and compared on the basis on several properties. The properties include the potential extendability to a multivariate family with a wide range of dependence structures. The goal of getting general dependent structures is different from that of the recent book of Fang *et al.* [5] on symmetric multivariate distributions. In Section 3, families of two-parameter bivariate copulas are given; in one example, there is permutation asymmetry. In Section 4, a class of multivariate copulas with representations as mixtures is derived (compare Theorem 2.1 of Marshall and Olkin [22]). Several of the bivariate families extend to this class. However, the range of dependence structures for the class is not as extensive as that for the multivariate normal distribution. Section 5 consists of further discussion, applications, and some open research problems motivated in part by the applications. Note that as a result of the careful study in this paper, many relationships and similarities among the families will be seen.

2. One-Parameter Bivariate Families

In this section, we consider one-parameter bivariate families $H(x_1, x_2; \theta)$ with univariate margins $F_1(x_1)$ and $F_2(x_2)$ which include independence, $F_1(x_1)F_2(x_2)$, the Fréchet upper bound, $\min\{F_1(x_1), F_2(x_2)\}$, and possibly the Fréchet lower bound, $\max\{0, F_1(x_1)+F_2(x_2)-1\}$. Several new families have been discovered since the paper of Kimeldorf and Sampson [21] and a new family is included here. Because extendability to the multivariate situation is one of our considerations, we do not insist on the family including the Fréchet lower bound as in [21]. We consider only families that have (i) absolutely continuous copulas except for the bounds, and (ii) copulas with support on $[0, 1]^2$ for parameters corresponding to positive dependence. We do not consider families, such as that of Morgenstern [25], which are just perturbations of independence. Elliptically contoured families other than the normal are not included here because they do not include independence.

Because there are many families with the properties in the preceding paragraph, we compare the families through other properties which should help to decide on appropriate families as models for some given data. The families are differentiated as they do not all satisfy the same properties.

For the eight one-parameter families listed below via their copulas, the parametrization is such that the dependence increases as the parameter θ increases, according to the concept of dependence in Property 2. The bivariate normal copula is given first and then the others in chronological order according to their discovery. We let $C_-(u_1, u_2) = \max\{u_1 + u_2 - 1, 0\}$, $C_I(u_1, u_2) = u_1 u_2$, and $C_+(u_1, u_2) = \min\{u_1, u_2\}$ denote the copulas of the Fréchet lower bound, independence, and the Fréchet upper bound, respectively. There can be two associated family of copulas for each given family. If $C(u_1, u_2; \theta)$ is a family of copulas, then let $\bar{C}(u_1, u_2; \theta) = 1 - u_1 - u_2 + C(u_1, u_2; \theta)$ denote the bivariate survival function. Note that $C'(u_1, u_2; \theta) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2; \theta)$ is also a family of copulas with survival functions $\bar{C}'(u_1, u_2; \theta) = C(1 - u_1, 1 - u_2; \theta)$. This is different unless the family has symmetry about the medians; that is, if $(U_1, U_2) \sim C$, then $(1 - U_1, 1 - U_2) \sim C$. Another related family of copulas is obtained by considering uniform random variables U_1, U_2 such that $(1 - U_1, U_2)$ has the distribution $C(\cdot; \theta)$. This leads to the family $C''(u_1, u_2; \theta) = u_2 - C(1 - u_1, u_2; \theta)$; the family $u_1 - C(u_1, 1 - u_2; \theta)$ is not different for the families below because of permutation symmetry of u_1, u_2 in $C(u_1, u_2; \theta)$.

2.1. Let Φ be the standard normal distribution function and let Φ_Σ be the multivariate normal distribution function with mean vector zero and covariance matrix Σ. Then the bivariate normal copula is $C(u_1, u_2; \theta) = \Phi_{\Sigma(\theta)}(\Phi^{-1}(u_1), \Phi^{-1}(u_2))$, $\Sigma(\theta) = [\begin{smallmatrix} 1 & \theta \\ \theta & 1 \end{smallmatrix}]$, $-1 \leqslant \theta \leqslant 1$. The cases $-1, 0, 1$ correspond respectively to C_-, C_I, C_+.

2.2. The copula is $C(u_1, u_2; \theta) = \exp\{-[(-\log u_1)^\theta + (-\log u_2)^\theta]^{1/\theta}\}$, $\theta \geqslant 1$. The cases 1, ∞ correspond respectively to C_I, C_+. This family is due to Gumbel [11] in 1960.

2.3. The copula is $C(u_1, u_2; \theta) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}$, $\theta \geqslant 0$. The cases 0, ∞ correspond respectively to C_I, C_+. With appropriate margins, this family for $\theta = 1$ becomes the bivariate logistic distribution of Gumbel [12], the bivariate Pareto distribution of Mardia [23], and the bivariate Burr distribution of Takahasi [33]. The family appears in Clayton [1], Cook and Johnson [3], and Oakes [26]. Ruiz-Rivas [29] and Genest and Mackay [10] extend the family to $C(u_1, u_2; \theta) = [\max\{0, u_1^{-\theta} + u_2^{-\theta} - 1\}]^{-1/\theta}$, $-1 \leqslant \theta < 0$ to allow for negative dependence; the case $\theta = -1$ corresponds to C_-.

2.4. The copula $C(u_1, u_2; \theta)$ is defined through the equation $\theta = C(u_1, u_2; \theta)(1 - u_1 - u_2 + C(u_1, u_2; \theta))/(u_1 - C(u_1, u_2; \theta))(u_2 - C(u_1, u_2; \theta))$,

$\theta \geqslant 0$. The right-hand side of the equation is the odds ratio when $[0, 1]^2$ is split into 4 quadrants with centre (u_1, u_2). The case 0, 1, ∞ correspond respectively to C_-, C_I, C_+. The solution to the equation can be written as $C(u_1, u_2; \theta) = 0.5(\theta-1)^{-1}\{1+(\theta-1)(u_1+u_2)-[(1+(\theta-1)(u_1+u_2))^2+4\theta(1-\theta)u_1u_2]^{1/2}\}$, for $\theta \neq 1$. This family is due to Plackett [28] in 1965.

2.5. The copula is $C(u_1, u_2; \theta) = u_1u_2\exp\{[(-\log u_1)^{-\theta}+(-\log u_2)^{-\theta}]^{-1/\theta}\}$, $\theta \geqslant 0$. The cases 0, ∞ corresponds respectively to C_I, C_+. This family appears in Galambos [8] in 1975.

2.6. The copula is $C(u_1, u_2; \theta) = (-1/\theta)\log\{1-(1-e^{-\theta})^{-1}(1-e^{-\theta u_1})(1-e^{-\theta u_2})\}$, $-\infty \leqslant \theta \leqslant \infty$. The cases $-\infty$, 0, ∞ correspond respectively to C_-, C_I, C_+. This family is due to Frank [7] in 1979.

2.7. Let Φ be defined as in Family 2.1. The copula is $C(u_1, u_2; \theta) = (-\log u_2)\exp\{-\Phi(\theta^{-1}+0.5\theta\log[(-\log u_2)/(-\log u_1)])-(-\log u_1)\Phi(\theta^{-1}+0.5\theta\log[(-\log u_1)/(-\log u_2)])\}$, $\theta \geqslant 0$. The cases 0, ∞ correspond respectively to C_I, C_+. This family appears in Hüsler and Reiss [15] in 1989.

2.8. The copula is $C(u_1, u_2; \theta) = 1-[(1-u_1)^\theta+(1-u_2)^\theta-(1-u_1)^\theta(1-u_2)^\theta]^{1/\theta}$, $\theta \geqslant 1$. The cases 1, ∞ correspond respectively to C_I, C_+. This is a new family of copulas.

Next we list and discuss various properties that the families can possess.

Property 1: Permutation Symmetry. The family of copulas is permutation symmetric if $C(u, v; \theta) = C(v, u; \theta)$ for all $0 \leqslant u, v \leqslant 1$. This property is satisfied by all of the above families. We later also use the term "permutation symmetric" for multivariate copulas that are symmetric in all arguments.

Property 2: Ordered by Concordance. The family of copulas is increasing in the concordance ordering of Yanagimoto and Okamoto [37] and Tchen [35], if $C(u_1, u_2; \theta)$ is nondecreasing in θ for all u_1, u_2, or equivalently, $\bar{C}(u_1, u_2; \theta)$ is nondecreasing in θ for all u_1, u_2. It is known and/or easily checked that all of the above families are ordered by concordance.

For the associated families of copulas, the results are as follows. $C'(u_1, u_2; \theta)$ is nondecreasing in θ for all u_1, u_2 and $C''(u_1, u_2; \theta)$ is nonincreasing in θ for all u_1, u_2. If θ_0 is such that $C(\cdot; \theta_0)$ corresponds to C_I (respectively, C_+, C_-), then $C'(\cdot; \theta_0)$ corresponds to C_I (respectively, C_+, C_-) and $C''(\cdot; \theta_0)$ corresponds to C_I (respectively, C_-, C_+).

Property 3: Symmetry about Medians. The family of copulas has symmetry about medians if $C(u_1, u_2; \theta) = u_1+u_2-1+C(1-u_1, 1-u_2; \theta)$. This is satisfied for Families 2.1, 2.4, and 2.6.

Property 4: Mixtures of Powers. The subset of the family of copulas with positive dependence has a representation in terms of mixtures of powers of distributions if

$$C(u_1, u_2; \theta) = \int_0^\infty G_1^\alpha(u_1; \theta)\, G_2^\alpha(u_2; \theta)\, dM_\theta(\alpha), \tag{2.1}$$

where M_θ is a distribution function with $M_\theta(0) = 0$. Marshall and Olkin [22] show that this representation necessarily implies that $G_j(u_j; \theta) = \exp[-\phi^{-1}(u_j; \theta)]$, $j = 1, 2$, where $\phi(s; \theta) = \int_0 e^{-s\alpha}\, dM_\theta(\alpha)$ is the Laplace transformation (LT) of the mixture distribution M_θ.

Families 2.2, 2.3, 2.6 have this form with $\phi(s; \theta) = \exp(-s^{1/\theta})$ (corresponding to a positive stable distribution), $\phi(s; \theta) = (1+s)^{-1/\theta}$ (corresponding to a gamma distribution), and $\phi(s; \theta) = (-1/\theta) \log[1 - (1 - e^{-\theta}) e^{-s}]$ (corresponding to a logarithmic series distribution on positive integers with mass $(1 - e^{-\theta})^i/(i\theta)$ on the integer i), respectively. The parameter α is interpreted as a frailty parameter in Oakes [27]. Genest and Mackay [10], with necessary and sufficient conditions on ϕ, write (2.1) as

$$C(u_1, u_2) = \phi(\phi^{-1}(u_1) + \phi^{-1}(u_2)) \tag{2.2}$$

and call copulas of this form Archimedean copulas. They showed that the subset of Families 2.3 and 2.6 with negative dependence have the representation (2.2) even though ϕ is not a LT in this case. The three families with this mixture representation are quite different based on Property 6 below.

It will be shown in Section 3 that Family 2.8 is the limit of a two-parameter mixture family but is itself not a mixture family.

Property 5: Extreme Value Copula. From Pickand's result, which is in Chap. 5 of Galambos [9], a necessary and sufficient condition for $C(u_1, u_2)$ to be an extreme value copula is that $G(x_1, x_2) = C(e^{-x_1}, e^{-x_2})$ satisfy $\log G(tx_1, tx_2) = t \log G(x_1, x_2)$ for all $t > 0$, $x_1, x_2 \geq 0$. Also, extreme value copulas cannot have negative dependence. Families 2.2, 2.5, 2.7 are families of extreme value copulas. With each margin being a generalized extreme value distribution, these families are models for bivariate extremes. Family 2.7 results from an ingenious derivation involving a nonstandard limiting extreme of the bivariate normal distribution.

Property 6: Tail Dependence. A family of copulas has *upper tail dependence* if $\bar{C}(u, u; \theta)/(1-u)$ converges to a constant c in $(0, 1]$ as $u \to 1$. An interpretation of this condition is that if (U_1, U_2) have bivariate survival function $\bar{C}(u_1, u_2; \theta)$ then $\Pr(U_1 > u \mid U_2 > u)$ does not converge to zero as $u \to 1$. Equivalently, from Sibuya [30], the limiting (maxima) extreme value distribution of $C^n\ (F_1(a_{1n} + b_{1n}x_1),\ F_2(a_{2n} + b_{2n}x_2))$ does not consist of independent margins, where the sequences a_{1n}, b_{1n}, a_{2n}, b_{2n} and F_1, F_2 are such that marginal limiting extreme distributions $F_j^n(a_{jn}+b_{jn}x_j)$ exist for $j=1, 2$. Because the limiting extreme value copula does not depend on F_1, F_2 (see Galambos [9]), the limiting copula is often easily obtained from the limit of $C^n(1-n^{-1}e^{-x_1}, 1-n^{-1}e^{-x_2})$ followed by the transforms $-\log u_j = e^{-x_j}$, $j=1, 2$. Similarly, *lower tail dependence* holds if $C(u, u; \theta)/u$ converges to a constant c in $(0, 1]$ as $u \to 0$. We refer to c as a (bivariate) tail dependence parameter. A relation between the families $C(\cdot; \theta)$ and $C'(\cdot; \theta)$ is that if one has upper tail dependence, then the other has lower tail dependence.

For example, extreme value copulas have upper tail dependence; for Families 2.2, 2.5, 2.7, the limits of $\bar{C}(u, u; \theta)/(1-u)$ are respectively $2-2^{1/\theta}$, $2^{-1/\theta}$ and $2-2\Phi(\theta^{-1})$. Family 2.8 also has upper tail dependence with $2-2^{1/\theta}$ as the limit and Family 2.2 is the extreme value limit of Family 2.8. In addition, Family 2.3 has lower tail dependence; the limit of $C(u, u; \theta)/u$ is $2^{-1/\theta}$. Family 2.5 can be derived as an extreme value limit from Family 2.3 in its alternate form, $C'(u_1, u_2; \theta) = u_1 + u_2 - 1 + ((1-u_1)^{-\theta} + (1-u_2)^{-\theta} - 1)^{-1/\theta}$. The details are not given here, but it can be shown that no other upper or lower tail dependence exists in Families 2.1 to 2.8.

Within mixture families that have the form of (2.2), lower tail dependence essentially implies Family 2.3 and upper tail dependence essentially implies Family 2.2. For lower tail dependence, (2.2) implies $\phi(2\phi^{-1}(u))/u \sim c$ as $u \to 0$ $(0 < c < 1)$ or $\phi^{-1}(cu) \sim 2\phi^{-1}(u)$ or $\phi^{-1}(u) \sim u^{-\beta}$, $u \to 0$, where $\beta > 0$. Hence, $\phi(s) \sim s^{-1/\beta}$ as $s \to \infty$ or $\phi(s) = (1+s)^{-1/\beta}$. For upper tail dependence, (2.2) implies $1-2u+\phi(2\phi^{-1}(u)) \sim c(1-u)$ as $u \to 1$ or $g(v) = \phi^{-1}(1-v) \sim 0.5\phi^{-1}(1-(2-c)\,v) = 0.5g((2-c)\,v)$ as $v = 1-u \to 0$. Hence $\phi^{-1}(1-v) \sim v^{\beta}$, where $\beta > 1$, or $\phi(s) \sim 1 - s^{1/\beta} \sim \exp[-s^{1/\beta}]$ as $s \to 0$.

Property 7: Extendability to Multivariate Family. A multivariate extension of a given bivariate family would be such that each marginal distribution or order 2 or more has the same form and each bivariate margin is within the given bivariate family. Of special interest is whether the extension allows copulas that are not permutation symmetric and whether the extension allows negative dependence.

Family 2.1, of course, extends to the multivariate normal family with a parameter for each bivariate margin. Hüsler and Reiss [15] also extend their approach to obtain a multivariate distribution; since the distribution is from a limit of multivariate normal distribution, there is a (bivariate) parameter for each bivariate margin. The distribution involves a bit of notation to write so it will not be repeated here.

No extension of Family 2.4 to 3 dimensions or higher is known. Families 2.2, 2.3, and 2.6 extend to permutation symmetric copulas using

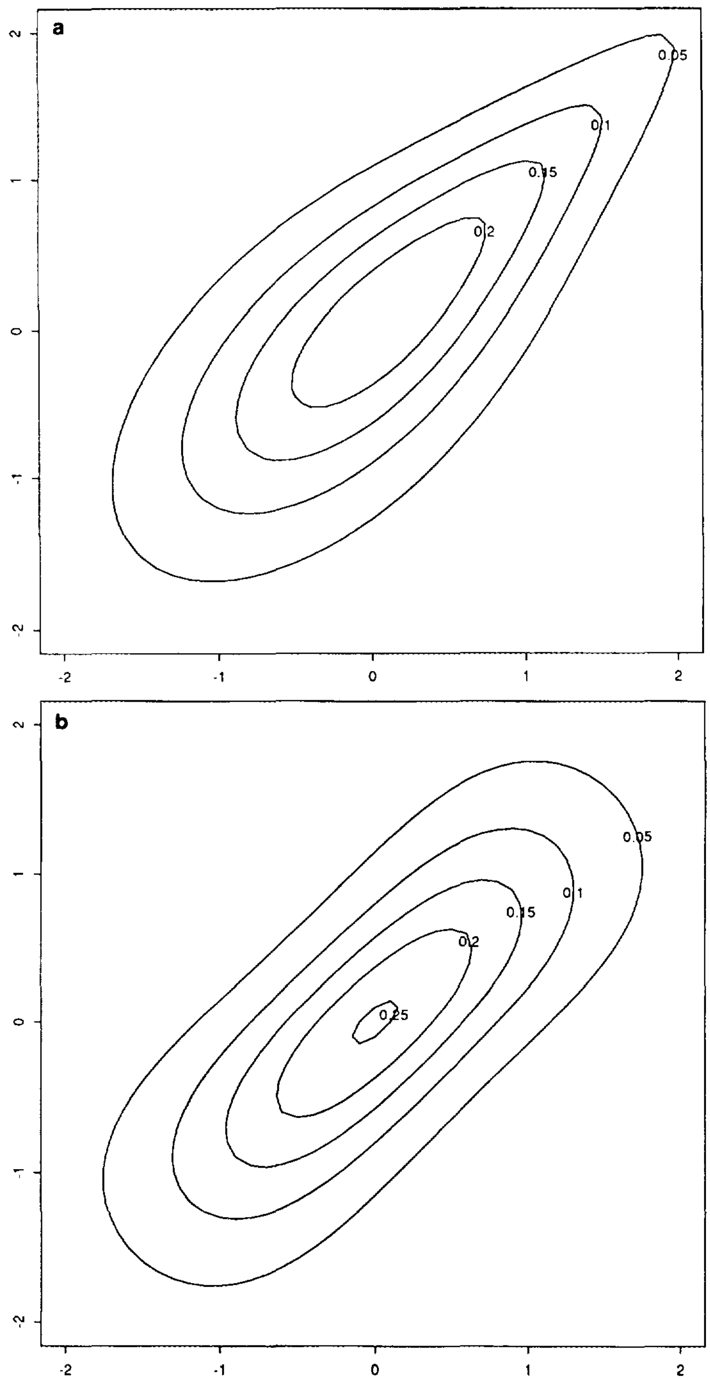

FIG. 1. (a) Contours of density for Family 2.2 with normal margins, $\theta = 2$; (b) contours of density for Family 2.6 with normal margins, $\theta = 5.7$.

the mixture representation. The result is $C(u_1, ..., u_p) = \phi(\sum_{j=1}^{p} \phi^{-1}(u_j))$. In Section 3, a more general extension that allows copulas without permutation symmetry is given; the number of parameters is $p-1$, so that there is not a distinct parameter for each bivariate margin. By taking a limiting extreme of the extension of Family 2.3, Family 2.5 also extends to a multivariate family with $p-1$ parameters. Family 2.8 extends in a similar way because it is a limit of mixtures.

Property 8: Contours. A way to compare families is through contours of the density. Since the elliptic contours of the bivariate normal density are familiar, we make comparisons of contours of densities when the marginal distributions F_1, F_2 are standard normal, that is, for the densities $c(\Phi(x_1), \Phi(x_2); \theta)\, \Phi'(x_1)\, \Phi'(x_2)$, where Φ' is the standard normal density and $c(u_1, u_2; \theta) = [\partial^2/\partial u_1\, \partial u_2]\, C(u_1, u_2; \theta)$. To compare different families, we used θ's such that Kendall's tau, $4 \int C(u_1, u_2)\, dC(u_1, u_2) - 1$, is fixed. Figure 1 shows the contours of the (normal) density when Kendall's tau is 0.5 for Families 2.2 and 2.6. Note that the contours for Family 2.2 are more pointed in the upper quadrant. All families with upper tail dependence have densities with contours like those for Family 2.2. Family 2.3, which has lower tail dependence, has contours with the pointed ends in the opposite direction. The contours for the density of Family 2.6 are a bit more rounded compared with ellipses; the contours for the density of Family 2.4 are closer to ellipses.

3. Two-Parameter Bivariate Families

In this section, we consider two-parameter bivariate families with given margins that include some of the one-parameter families of Section 2 as special cases. With one-parameter families, the parameter should be interpretable as a dependence parameter and this is the case in Section 2. With two-parameter families, the second parameter could be an additional dependence parameter or a parameter representing permutation asymmetry. Examples are given for both cases.

3.1. *A New Family with Two Dependence Parameters*

We present a two-parameter family $C(\cdot; \theta, \delta)$ that is increasing in concordance as θ or δ increases and that includes Families 2.6 and 2.8 on boundaries of the parameter space. The family is

$$C(u_1, u_2; \theta, \delta) = \delta^{-1}[1 - \{1 - [1 - (1-\delta)^\theta]^{-1}\, [1 - (1 - \delta u_1)^\theta] \times [1 - (1 - \delta u_2)^\theta]\}^{1/\theta}], \qquad 0 \leqslant \delta \leqslant 1,\ \theta \geqslant 1. \tag{3.1}$$

The limit as $\delta \to 0$ or $\theta \to 1$ corresponds to C_1; Family 2.8 is obtained when $\delta = 1$, and Family 2.6 is obtained as $\theta \to \infty$ with $\gamma = 1-(1-\delta)^\theta$ held constant (or with $\delta = 1-(1-\gamma)^{1/\theta}$). The family was derived as a power mixture family with the LT $\phi(s) = \delta^{-1}[1-\{1-[1-(1-\delta)^\theta]\,e^{-s}\}^{1/\theta}]$. This is the LT of a discrete power series distribution and it appears in a different parametrization in Family 4.3 in Section 4. The proof of the concordance ordering is nontrivial and is outlined in the appendix. Considering other properties in Section 2, (3.1) is permutation symmetric and the family does not have tail dependence except when $\delta = 1$. This family extends to the multivariate case for each fixed δ (Family 4.4).

3.2. *Another Family with Two Dependence Parameters*

Another two-parameter family of LT's is $\phi(s) = \exp\{-(\alpha+s)^{1/\theta} + \alpha^{1/\theta}\}$, $\alpha \geqslant 0$, $\theta \geqslant 1$. The resulting family of copulas is

$$C(u_1, u_2; \theta, \alpha) = \exp\{-[(\alpha^{1/\theta} - \log u_1)^\theta + (\alpha^{1/\theta} - \log u_2)^\theta - \alpha]^{1/\theta} + \alpha^{1/\theta}\}. \tag{3.2}$$

It is not difficult to check that (3.2) is increasing in concordance as either θ increases or as α decreases. C_1 is obtained as $\alpha \to \infty$ or for $\theta = 1$, and C_+ is obtained as $\theta \to \infty$. Family 2.2 is a subfamily when $\alpha = 0$. The upper tail dependence parameter is $2 - 2^{1/\theta}$, independently of α. Therefore for fixed θ, as α increases, measures of concordance, such as Kendall's tau, decrease but the tail dependence remains the same.

3.3. *Other Mixture Families*

The approach in Section 4.3 of Marshall and Olkin [22] leading to something like their Eq. (4.7) is a general way of adding a second dependence parameter to one-parameter power mixture families. The mixture form of the families, generalizing (2.1), is

$$C(u_1, u_2; \theta, \delta) = \int_0^\infty K(G_1^\alpha(u_1; \theta), G_2^\alpha(u_2; \theta); \delta)\, dM_\theta(\alpha), \tag{3.3}$$

where $K(\cdot\,; \delta)$ is a family of copulas that is increasing in concordance as δ increases. Whether (3.3) is increasing in θ would have to be checked on an individual basis.

3.4. *Families with Bivariate Permutation Asymmetry*

In the context of bivariate extremes, Smith [32] and Joe *et al.* [20] have used an extreme value copula that generalizes Family 2.2 and that does not have permutation symmetry except when the two parameters are equal. We derive a similar extreme value copula that generalizes Family 2.5. These families do not have closed forms and are more simply given in a form with exponential survival functions for the univariate margins.

The two-parameter family that generalizes Family 2.2 is

$$H(x_1, x_2; \alpha, \beta) = \exp\left\{-\int_0^1 \max[x_1 g_1(z), x_2 g_2(z)]\, dz\right\}, \tag{3.4}$$

with $g_1(z) = (1-\alpha) z^{-\alpha}$, $g_2(z) = (1-\beta)(1-z)^{-\beta}$, where $0 < \alpha, \beta < 1$. From the representation of deHaan [13] for multivariate extreme value distributions, the general conditions on g_1, g_2 are that they are probability density functions on $[0, 1]$. Hence the range of α, β can actually be extended to $\alpha < 0$, $\beta < 0$. The family with $\alpha_0 = -\alpha > 0$ and $\beta_0 = -\beta > 0$ is a two-parameter extension of Family 2.5. Families 2.2 and 2.5 result when $\alpha = \beta$. To study properties of (3.4), alternative forms are given next. For positive parameters, expansion of the integral in (3.4) leads to

$$H(x_1, x_2; \alpha, \beta) = \exp\{-(x_1 + x_2)\, A(x_2/(x_1 + x_2); \alpha, \beta)\},$$
$$0 < \alpha < 1, 0 < \beta < 1, \tag{3.5}$$

where $A(w; \alpha, \beta) = (1-w)\, u^{1-\alpha} + w(1-u)^{1-\beta}$, $0 \leqslant w \leqslant 1$, and $u = u(w; \alpha, \beta)$ is the root of the equation $(1-\alpha)(1-w)(1-u)^{\beta} - (1-\beta)\, wu^{\alpha} = 0$. For negative parameters in (3.4), one obtains

$$H(x_1, x_2; \alpha_0, \beta_0) = \exp\{-(x_1 + x_2)\, A(x_2/(x_1 + x_2); \alpha_0, \beta_0)\},$$
$$\alpha_0 > 0, \beta_0 > 0, \tag{3.6}$$

where $A(w; \alpha_0, \beta_0) = 1 - w(1-u)^{1+\beta_0} - (1-w)\, u^{1+\alpha_0}$, $0 \leqslant w \leqslant 1$, and $u = u(w; \alpha_0, \beta_0)$ is the root of the equation $(1+\alpha_0)(1-w)\, u^{\alpha_0} - (1+\beta_0)\, w(1-u)^{\beta_0} = 0$.

We mention some properties next, some of which are easier to determine numerically. The distributions in (3.5) are increasing in concordance as α or β decreases; the Fréchet upper bound obtains in the limit as $\alpha = \beta \to 0$ and independence obtains as $\alpha = \beta \to 1$. The distributions in (3.6) are also increasing in concordance as α_0 or β_0 decreases; the Fréchet upper bound obtains in the limit as $\alpha_0 = \beta_0 \to 0$ and independence obtains as $\alpha_0 = \beta_0 \to \infty$. Independence obtains more generally as one of α, β (or α_0, β_0) is fixed and the other approaches 1 (∞). A different limit occurs as one of the parameters approaches 0. For examples, as $\beta \to 0$,

$$A(w; \alpha, \beta) = \begin{cases} 1 - w, & \text{if } 0 \leqslant w \leqslant (1-\alpha)/(2-\alpha), \\ w + \alpha[w/(1-\alpha)]^{1-1/\alpha}\, (1-w)^{1/\alpha}, & \text{if } (1-\alpha)/(2-\alpha) < w \leqslant 1, \end{cases}$$

and as $\beta_0 \to 0$,

$$A(w; \alpha_0, \beta_0) = \begin{cases} 1 - w + \alpha_0 [w/(1+\alpha_0)]^{1+1/\alpha_0} (1-w)^{-1/\alpha_0}, \\ \qquad \text{if} \quad 0 \leqslant w < (1+\alpha_0)/(2+\alpha_0), \\ w, \qquad \text{if} \quad (1+\alpha_0)/(2+\alpha_0) \leqslant w \leqslant 1. \end{cases}$$

The following parametrization is useful to interpret the parameters. From theory for min-stable bivariate exponential distributions, in (3.5) and (3.6), A is convex and bounded above by $A \equiv 1$ and below by $A(w) = \max\{w, 1-w\}$ (corresponding to the Fréchet upper bound). For both families, let $\gamma = w_0 - 0.5$ and $\theta = 2(1 - A(w_0))$, where w_0 is the point at which A is minimized. θ takes value in $[0, 1]$ and is a dependence parameter since through implicit derivatives it can be shown that (3.5) and (3.6) are increasing in concordance as θ increases with γ fixed. The parameter γ takes values in a subinterval of $(\theta - 1, 1 - \theta)/2$ with θ fixed and can be interpreted as a permutation asymmetry parameter since it can be shown numerically that $\mathrm{E}[X_2/(X_1 + X_2)]$, with (X_1, X_2) having distribution (3.5) or (3.6), increases as γ increases with θ fixed. (Note that $\mathrm{E}[X_2/(X_1 + X_2)] = \int_0^1 \Pr(X_2/(X_1 + X_2) > s)\, ds = \int_0^1 \int_0^\infty \Pr((1-s)\, X_2 > sy \mid X_1 = y)\, e^{-y}\, dy\, ds = 0.5 - \int_0^1 s(1-s)[A'(s)/A(s)]\, ds$.) The inverse transforms from θ, γ to α, β and α_0, β_0 are the following. For given $w_0 = \gamma + 0.5$, let u_0 be the root of $(1 - w_0)(1 - u) \log(1 - u) - w_0 u \log u = 0$. Then for a feasible (θ, γ), $1 - \alpha = \log(1 - \theta/2)/\log u_0$ and $1 - \beta = \log(1 - \theta/2)/\log(1 - u_0)$ for (3.5), and $1 + \alpha_0 = \log(\theta/2)/\log u_0$ and $1 + \beta_0 = \log(\theta/2)/\log(1 - u_0)$ for (3.6). For fixed θ, as γ increases over its possible range, u_0 increases, α decreases and β increases for (3.5), and α_0 increases and β_0 decreases for (3.6).

The multivariate extension of (3.4), (3.5), or (3.6) has a parameter for each dimension, which makes the parameters hard to interpret for dependence.

Several questions arise from the study of the families given by (3.5) and (3.6). First, there is no general known way of adding a permutation asymmetry parameter to a bivariate family. Second, the measure of asymmetry used, $\mathrm{E}[X_2/(X_1 + X_2)]$, was chosen mainly for convenience because of the form of the distributions. There is the question of how to choose a measure of asymmetry and whether there are inequalities like those resulting from the concordance ordering for dependence. In the multivariate case, there are even more problems with permutation asymmetry, since there is even the problem of constructing families with a wide range of dependence structure (see the next section).

4. Multivariate Mixture Families

A nontrivial generalization of Property 4 of Section 2 is given in this section. We give details for three dimensions and state the generalizations to four dimensions from which the p-variate result will be apparent. The notation to state everything in p dimensions would be too messy. A condition and a theorem needed for the generalization are given first.

Condition A. Let τ be a monotone increasing function on $[0, \infty)$ that satisfies $\tau(0)=0$ and $\lim_{s\to\infty} \tau(s)=\infty$. The function τ is such that $e^{-\alpha\tau}$ is a LT for all $\alpha>0$.

From Feller [6], condition A is satisfied if $\chi^\alpha = e^{-\alpha\tau}$ is completely monotone for all $\alpha>0$; that is, $(-1)^k (d^k\chi^\alpha(s)/ds^k) \geqslant 0$ for $k=1, 2, \ldots$. A condition that may be easier to verify is given in the following theorem.

Theorem B. *χ^α is completely monotone for all $\alpha>0$ if and only if $\log \chi = -\tau$ is completely monotone.*

Proof. First suppose that $-\tau$ is completely monotone. Then it is easily checked that $d\chi^\alpha(s)/ds \leqslant 0$ and that, by induction, the derivative of each summand of $(-1)^k (d^k\chi^\alpha(s)/ds^k)$ is opposite in sign. Hence sufficiency has been proved.

Next we show necessity. Let $\sigma=-\tau$. $(\chi^\alpha)' = \alpha\sigma'\chi^\alpha$ and $(\chi^\alpha)^{(2)} = \alpha\sigma''\chi^\alpha + \alpha^2(\sigma')^2\chi^\alpha$ which has the form $\alpha\sigma^{(k)}\chi^\alpha + \alpha^2\chi^\alpha \sum_l C_{kl}(s)\,\alpha^{m_{kl}}$ for $k=2$ with m_{kl} being nonnegative integers for all l. Suppose the kth derivative of χ^α has this form; then the $(k+1)$st derivative of χ^α is $\alpha\sigma^{(k+1)}\chi^\alpha + \alpha^2\sigma^{(k)}\sigma'\chi^\alpha + \alpha^3\sigma'\chi^\alpha \sum_l C_{kl}(s)\,\alpha^{m_{kl}} + \alpha^2\chi^\alpha \sum_l C'_{kl}(s)\,\alpha^{m_{kl}}$ which has the form $\alpha\sigma^{(k+1)}\chi^\alpha + \alpha^2\chi^\alpha \sum_l C_{k+1,l}(s)\,\alpha^{m_{k+1,l}}$, where $m_{k+1,l}\geqslant 0$ for all l. Therefore $\lim_{\alpha\to 0}(\chi^\alpha)^{(k)}/\alpha = \sigma^{(k)}$ for $k=1, 2, \ldots$. Hence the complete monotonicity of $\sigma=-\tau$ is necessary for χ^α to be completely monotone for all α near 0. ∎

The generalization of representation (2.2) to three dimensions is

$$C(u_1, u_2, u_3) = \psi(\psi^{-1}\circ\phi[\phi^{-1}(u_1)+\phi^{-1}(u_2)] + \psi^{-1}(u_3)), \tag{4.1}$$

where ψ, ϕ are Laplace transforms (LT) and $\tau=\psi^{-1}\circ\phi$ satisfies Condition A. The mixture representation for (4.1) that generalizes (2.1) is

$$C(u_1, u_2, u_3) = \int_0^\infty \int_0^\infty G_1^\beta(u_1)\, G_2^\beta(u_2)\, dM_2(\beta; \alpha)\, G_3^\alpha(u_3)\, dM_1(\alpha), \tag{4.2}$$

where the relationships with (4.1) and the definitions of G_1, G_2, G_3, M_1, M_2 will be seen from the derivation given below. Since $\phi^{-1}(1) = \psi^{-1}(1)=0$, note that the $(1, 2)$ bivariate margin has form (2.2) with LT ϕ and the $(1, 3)$ and $(2, 3)$ have form (2.2) with LT ψ.

Before the derivation, we state the two generalizations or nestings of LT's for four dimensions. In higher dimensions, there are many possible nestings. At each level of nesting, Condition A must be satisfied in order for the result to be a multivariate distribution. Let ϕ_1, ϕ_2, ϕ_3 be LT's. The first LT representation is

$$C(u_1, u_2, u_3, u_4) = \phi_3[\phi_3^{-1} \circ \phi_2(\phi_2^{-1} \circ \phi_1[\phi_1^{-1}(u_1) + \phi_1^{-1}(u_2)] + \phi_2^{-1}(u_3)) + \phi_3^{-1}(u_4)], \tag{4.3}$$

where $\phi_3^{-1} \circ \phi_2$ and $\phi_2^{-1} \circ \phi_1$ satisfy Condition A. A second distinct LT representation is

$$C(u_1, u_2, u_3, u_4) = \phi_3(\phi_3^{-1} \circ \phi_1[\phi_1^{-1}(u_1) + \phi_1^{-1}(u_2)] + \phi_3^{-1} \circ \phi_2[\phi_2^{-1}(u_3) + \phi_2^{-1}(u_4)]), \tag{4.4}$$

where $\phi_3^{-1} \circ \phi_1$ and $\phi_3^{-1} \circ \phi_2$ satisfy Condition A. Note that all trivariate margins of (4.3) and (4.4) have form (4.1) and all bivariate margins of (4.3) and (4.4) have form (2.2). For (4.3), the (i, j) has LT ϕ_{j-1} for $i < j$. For (4.4), the (1, 2) margin has LT ϕ_1, the (3, 4) margin has LT ϕ_2, and the remaining four bivariate margins have LT ϕ_3. Clearly the idea of (4.3) and (4.4) generalizes to higher dimensions.

The mixture representations for (4.3) and (4.4) are respectively

$$\int_0^\infty \int_0^\infty \int_0^\infty G_1^\gamma G_2^\gamma \, dM_3(\gamma; \beta) \, G_3^\beta \, dM_2(\beta; \alpha) \, G_4^\alpha \, dM_1(\alpha)$$

and

$$\int_0^\infty \int_0^\infty G_1^\beta G_2^\beta \, dM_2(\beta; \alpha) \int_0^\infty G_3^\gamma G_4^\gamma \, dM_3(\gamma; \alpha) \, dM_1(\alpha).$$

Derivation for (4.1) *and* (4.2). Equation (4.1) has the formal representation

$$\int_0^\infty G_{12}^\alpha(u_1, u_2) \, G_3^\alpha(u_3) \, dM_1(\alpha),$$

where M_1 is the distribution corresponding to ψ, $G_3(u_3) = \exp[-\psi^{-1}(u_3)]$, $G_{12}(u_1, u_2) = \exp\{-\tau[\phi^{-1}(u_1) + \phi^{-1}(u_2)]\}$, and $\tau = \psi^{-1} \circ \phi$. Note that in the bivariate case, the power of a distribution function need not be a distribution function (a necessary and sufficient condition for all powers to be distributions is that the distribution is max-infinitely divisible), so that it must be proved that G_{12}^α is a distribution function. We show this by giving $F_\alpha = G_{12}^\alpha$ the mixture representation in (4.2).

The univariate margins of F_α are $F_{j\alpha}(u_j) = \exp\{-\alpha\psi^{-1}(u_j)\}$, $j = 1, 2$. Hence $u_j = \psi(-\alpha^{-1} \log F_{j\alpha})$, $j = 1, 2$, $F_\alpha = \exp\{-\alpha\tau[\tau^{-1}(-\alpha^{-1} \log F_{1\alpha}) + \tau^{-1}(-\alpha^{-1} \log F_{2\alpha})]\}$, and

$$\tau^{-1}(-\alpha^{-1} \log F_\alpha) = \tau^{-1}(-\alpha^{-1} \log F_{1\alpha}) + \tau^{-1}(-\alpha^{-1} \log F_{2\alpha}).$$

Let χ_α be defined by $\chi_\alpha^{-1}(z) = \tau^{-1}(-\alpha^{-1} \log z)$ so that

$$F_\alpha = \chi_\alpha[\chi_\alpha^{-1}(F_{1\alpha}) + \chi_\alpha^{-1}(F_{2\alpha})]. \tag{4.5}$$

Therefore $\chi_\alpha = e^{-\alpha\tau} = \chi^\alpha$, where $\chi = \chi_1$. From (2.1) and (2.2), (4.5) is a distribution function for all $\alpha > 0$ if χ_α is a LT for all $\alpha > 0$ or if Condition A holds.

Assuming Condition A holds, representation (4.2) holds with $M_2(\cdot\,; \alpha)$ being the distribution with LT χ^α and, for all $\alpha > 0$ and $j = 1, 2$, $G_j = \exp[-\chi_\alpha^{-1}(F_{j\alpha})] = \exp[-\tau^{-1}(-\alpha^{-1} \log F_{j\alpha})] = \exp[-\tau^{-1} \circ \psi^{-1}] = \exp[-\phi^{-1}]$.

Some examples of parametric families of the form (4.1) are listed next; these generalize Families 2.2, 2.3, 2.6, and 3.1.

4.1. *A Generalization of Family* 2.2

Let $\phi(s; \theta) = \exp\{-s^{1/\theta}\}$, $\theta \geqslant 1$. For $\theta_1 \leqslant \theta_2$, by Theorem B, Condition A is satisfied for $\tau(s) = \phi^{-1}(\phi(s; \theta_2); \theta_1) = s^\rho$, where $\rho = \theta_1/\theta_2$. Equation (4.1) becomes

$$C(u_1, u_2, u_3; \theta_1, \theta_2) = \exp\{-([(-\log u_1)^{\theta_2} + (-\log u_2)^{\theta_2}]^{\theta_1/\theta_2} + (-\log u_3)^{\theta_1})^{1/\theta_1}\}. \tag{4.6}$$

The LT $\chi^\alpha(s)$ is $\exp\{-\alpha s^\rho\}$ corresponding to a scaled positive stable random variable. The generalizations to higher dimensions are such that the parameter which is further nested is always larger. Family (4.6) has been used recently as a model for multivariate extremes (Tawn [34], Coles and Tawn [2], Joe [19]). The mixture represention via positive stable random variables is given in Hougaard [14]. The family also appears in McFadden [24].

4.2. *A Generalization of Family* 2.3

Let $\phi(s; \theta) = (1 + s)^{1/\theta}$, $\theta \geqslant 0$. For $\theta_1 \leqslant \theta_2$, Condition A is satisfied for $\tau(s) = \phi^{-1}(\phi(s; \theta_2); \theta_1) = (1 + s)^\rho - 1$, where $\rho = \theta_1/\theta_2$. Equation (4.1) becomes

$$C(u_1, u_2, u_3; \theta_1, \theta_2) = [(u_1^{-\theta_2} + u_2^{-\theta_2} - 1)^{\theta_1/\theta_2} + u_3^{-\theta_1} - 1]^{-1/\theta_1}. \tag{4.7}$$

The LT $\chi^\alpha(s)$ is $\exp\{-\alpha[(1+s)^\rho - 1]\}$ which also corresponds to a positive stable random variable. As in Family 4.1, the generalizations to higher dimensions are such that the parameter which is further nested is always larger. The multivariate extreme value limit from using (4.7) as a copula for survival functions leads to a generalization of Family 2.5 (Joe [19]). This multivariate extreme value family has also been useful for multivariate extreme value data.

4.3. *A Generalization of Family* 2.6

Family 2.6 fits within the same form as Families 2.2 and 2.3, and it turns out that it generalizes in the same way to the multivariate case, although the proof that Condition A holds is more difficult.

Let $\phi(s;\theta) = -\theta^{-1}\log(1-(1-e^{-\theta})e^{-s})$, $\theta \geqslant 0$. For $\theta_1 < \theta_2$, we show that Condition A holds by showing that $\chi(s) = e^{-\tau(s)}$ is a LT of an infinitely divisible discrete distribution, where $\tau(s) = \phi^{-1}(\phi(s;\theta_2);\theta_1) = -\log\{(1-[1-ce^{-s}]^\rho)/(1-e^{-\theta_1})\}$, $\rho = \theta_1/\theta_2$, and $c = 1-e^{-\theta_2}$. Replacing e^{-s} by z, $\chi(-\log z) = (1-[1-cz]^\rho)/(1-e^{-\theta_1})$ is the probability generating function of a power series discrete distribution on 1, 2, ... The probability mass at the positive integer i is $p_i = \rho c/(1-e^{-\theta_1})$ for $i = 1$ and $\rho[\prod_{j=1}^{i-1}(j-\rho)]\,c^i/[i!(1-e^{-\theta_1})]$ for $i > 1$. Now, $g(z) = \chi(-\log z)/z$ is a probability generating function of a discrete random variable with mass p_{i+1} at i, $i = 0, 1, 2, \ldots$. By the sufficient condition of Warde and Katti [26], since $p_{i+1}/p_i = (i-\rho)\,c/(i+1)$ is increasing in i, g corresponds to an infinitely divisible distribution and, for $\alpha > 0$, $g^\alpha(z)$ is the probability generating function of a discrete random variable on the integers 0, 1, 2, ... Therefore $\chi^\alpha(-\log z) = z^\alpha g^\alpha(z)$ is the probability generating function of a discrete random variable on $\alpha, \alpha+1, \alpha+2, \ldots$. Hence Condition A is satisfied.

The family (4.1) becomes

$$C(u_1,u_2,u_3;\theta_1,\theta_2) = -\theta_1^{-1}\log\{1-c_1^{-1}(1-[1-c_2^{-1}(1-e^{-\theta_2 u_1}) \times (1-e^{-\theta_2 u_2})]^{\theta_1/\theta_2})(1-e^{-\theta_1 u_3})\}, \tag{4.8}$$

where $c_1 = 1-e^{-\theta_1}$ and $c_2 = 1-e^{-\theta_2}$.

4.4. *A Generalization of Family* 3.1

With δ fixed in (0, 1), Family 3.1 generalizes in the same way to the multivariate case. Let $\phi(s;\theta) = \delta^{-1}[1-(1-c(\theta)\,e^{-s})^{1/\theta}]$, $\theta \geqslant 1$, where $c(\theta) = 1-(1-\delta)^\theta$. For $\theta_1 < \theta_2$, $\chi(s) = \exp\{-\phi^{-1}(\phi(s;\theta_2);\theta_1)\} = [c(\theta_1)]^{-1}\,\{1-[1-c(\theta_2)\,e^{-s}]^\rho\}$, where $\rho = \theta_1/\theta_2$. This is the LT of an infinitely divisible discrete distribution that appears in the Family 4.3 and hence Condition A is satisfied. Equation (4.1) becomes

$$\begin{aligned} C(u_1, u_2, u_3; \theta_1, \theta_2) \\ &= \delta^{-1}(1 - [1 - \{1 - [1 - A(u_1, \theta_2)\, A(u_2, \theta_2)/c(\theta_2)]^{\theta_1/\theta_2}\} \\ &\quad \times A(u_3, \theta_1)/c(\theta_1)]^{1/\theta_1}), \end{aligned} \tag{4.9}$$

where $A(u, \theta) = 1 - (1 - \delta u)^{\theta}$. The limit as $\delta \to 1$ results in the family

$$C(u_1, u_2, u_3; \theta_1, \theta_2) = 1 - \{[v_1^{\theta_2}(1 - v_2^{\theta_2}) + v_2^{\theta_2}]^{\theta_1/\theta_2}(1 - v_3^{\theta_1}) + v_3^{\theta_1}\}^{1/\theta_1}, \tag{4.10}$$

where $v_j = 1 - u_j$, $j = 1, 2, 3$. The copula resulting from the extreme value limit of (4.10) is (4.6).

In each of the families given in this section and their multivariate extensions, the parameters can be interpreted as dependence parameters since the bivariate margins are increasing in concordance as the parameters increase. Joe [18] defines a multivariate concordance ordering generalizing that of Yanagimoto and Okamoto [37], with two multivariate distributions being ordered if the cumulative distribution functions and the survival functions are both ordered. However, the inequalities and monotonicities involved in the multivariate concordance ordering are difficult to check analytically, because for a family given via cumulative distributions functions, the survival functions are sums of marginal cumulative distributions with alternating signs.

5. Applications, Discussion, and Open Problems

It should be apparent from the presentation here that it is difficult to come up with a multivariate family that covers a wide range of dependence structure. This is partly due to the possibly unsolvable problem of constructing a multivariate family with *given compatible bivariate margins*. The successful application of the multivariate normal copula or distribution is really because of its flexibility in dependence structures rather than for physical reasons or as an approximation from the Central Limit Theorem. It is well known that the multivariate normal distribution is maximum entropy (maximizing $-\int f \log f$ over multivariate densities f) subject to constraints of given (compatible) means, variances, and covariances. Perhaps less well known is the fact that the multivariate normal distribution is also the maximum entropy distribution given its set of bivariate margins.

The use of multivariate extreme value distributions is based on an asymptotic approximation, where it is assumed that extremes are componentwise maxima (or minima) of a "large" number of observations.

This is a natural application of a multivariate non normal distribution. The Hüsler–Reiss family, which is constructed from the multivariate normal distribution, has almost as much flexibility in dependence as the multivariate normal distribution. There is a simpler form for the family than that given in Hüsler and Reiss [15] to make numerical implementation of the likelihood easier, but dimensions greater than 5 would be too time-consuming because of computations involving the integrals of multivariate normal distribution functions. Joe [17] has studied families of multivariate extreme value copulas (with univariate exponential margins) that have a parameter for each marginal distribution of order 2 or more and that have the Marshall–Olkin multivariate exponential distributions at the boundary. However, these models have not been successfully applied to multivariate extremes (Joe [19]), and the reason seems to be that the parameters represent partly dependence and partly permutation asymmetry and they change for marginal distributions even though the copulas for the margins are within the same multivariate family. Hence it seems important that the parameters of a multivariate distribution have simple interpretations, which is a reason for the emphasis on properties in Section 2.

For the families given in this paper that have closed forms for the cumulative distribution functions, we remark here that likelihood inference is straightforward given computer resources. The density function for a copula can be obtained with a symbolic manipulation software and the log likelihood can then be maximized with a quasi-Newton routine. For the multivariate families in Section 4, the parameters retain their interpretations for margins, so that the estimates based on separate bivariate likelihoods can be used as initial guesses for the multivariate likelihood (see Joe [19] for further details). Since likelihood inference can be straightforward, the further research problems are to construct multivariate families that have desirable properties.

As a further application to demonstrate the importance of properties, note that a Markov chain stationary time series model (of order 1) can be derived from each bivariate family. Let F be the univariate marginal distribution of the time series $X_1, X_2, \ldots$ and assume that F is absolutely continuous. Let $C(\cdot\,;\theta)$ be a family of bivariate copulas and let $H(x, y; \theta) = C(F(x), F(y); \theta)$. The density is $h(x, y; \theta) = c(F(x), F(y); \theta)$ $f(x)\,f(y)$, where c is the density corresponding to C and $f = F'$. A Markov transition function can be defined as $p(x_t \mid x_{t-1}; \theta) = h(x_{t-1}, x_t; \theta)/f(x_{t-1})$ for $t > 1$. (Note that this is a simple construction of a stationary times series that can have an arbitrary univariate marginal. This family of times series generalizes the normal AR(1) time series since the latter obtains when F is normal and the copula is bivariate normal.) Some environmental time series exhibit the behaviour of clustering of consecutive observations above high thresholds. This behaviour is expected for a Markov chain stationary time series where the family of copulas has upper tail dependence (see Joe [16]). Hence to model a time series for extreme value inference, the copulas with strong tail dependence as well as the extreme value copulas are useful.

Possibly a Markov chain time series of order 1 is enough for some extreme value inferences. However, if a Markov chain time series of order $p-1$, $p>2$, based on a p-dimensional copula is desired, then the families of copulas in Section 4 probably do not provide adequate dependence structure. Certainly a p-dimensional copula with permutation symmetry is not a reasonable model. So for both multivariate extremes and Markov chain stationary time series, the first problem given in the next paragraph is important.

In conclusion, some difficult open problems are:

1. Are there families of absolutely continuous multivariate copulas that have (i) simple forms, (ii) all bivariate margins in the same family, (iii) a wider range of dependence structures than those given in Section 4, and (iv) bivariate tail dependence?

2. Is there a simple family of bivariate copulas with both upper and lower tail dependence?

3. Are there general approaches other than the mixture or Laplace transform approach? Note that for the Laplace transform approach, there does not seem to be a way to tell from a family of LT's whether the resulting family of copulas will interpolate between independence and the Fréchet upper bound.

For Problem 1, the family of multivariate distributions in Cuadras and Augé [4] satisfies all the conditions except absolute continuity. The limiting multivariate extreme value distribution from this family is not interesting. Using an approach different from that in this paper, the author has obtained families that satisfy all but one of the conditions in Problem 1; the multivariate families have only $p-1$ of $p(p-1)/2$ bivariate margins in the same bivariate family.

As several new families of multivariate distributions with given margins were discovered for this paper, there is no doubt that more will be found in the future.

APPENDIX: Outline of Proof That (3.1) is Increasing in θ and δ

We first fix δ and show that (3.1) is increasing in θ, and then fix θ and show that (3.1) is increasing in δ.

With δ fixed, let $A_\theta(u)=1-(1-\delta u)^\theta$ so that (3.1) becomes

$$C(u_1, u_2; \theta, \delta)=\delta^{-1}[1-\{1-[A_\theta(1)]^{-1} A_\theta(u_1) A_\theta(u_2)\}^{1/\theta}].$$

Note that $\partial A_\theta(u)/\partial\theta = -(1-\delta u)^\theta \log(1-\delta u)$. Let $v_j = (1-\delta u_j)^\theta$, $\bar{v}_j = 1 - v_j$, $j = 1, 2$, and let $c = 1-(1-\delta)^\theta$. Then $\delta[\partial C/\partial\theta] = \theta^{-2}(1-\bar{v}_1\bar{v}_2/c)^{1/\theta-1}$ $g(v_1, v_2)$, where $g(v_1, v_2) = (1-\bar{v}_1\bar{v}_2/c)\log(1-\bar{v}_1\bar{v}_2/c) - c^{-1}(v_1\bar{v}_2\log v_1 + \bar{v}_1 v_2 \log v_2) + c^{-2}\bar{v}_1\bar{v}_2(1-c)\log(1-c)$, $1-c \leqslant v_1$, $v_2 \leqslant 1$. At the boundaries with $v_1 = 1-c$ and $v_1 = 1$, $g(1-c, v_2) = g(1, v_2) = 0$, and $g(\cdot, v_2)$ is concave since $\partial^2 g/\partial v_1^2 \leqslant 0$. Hence $g \geqslant 0$ and $\partial C/\partial\theta \geqslant 0$.

With $\theta > 1$ fixed, an indirect method is needed to show that (3.1) is increasing in δ. Let $0 < \delta_1 < \delta_2 < 1$ and define inverse LT's $\phi_j^{-1}(z) = -\log\{[1-(1-\delta_j z)^\theta]/[1-(1-\delta_j)^\theta]\}$, $j = 1, 2$. By the lemma in the Appendix of Genest and Mackay [10], it suffices to show that $\alpha_1(z)/\alpha_2(z)$ is increasing in z, where $\alpha_j(z) = -\theta^{-1}\, d\phi_j^{-1}(z)/dz = \delta_j(1-\delta_j z)^{\theta-1}/[1-(1-\delta_j z)^\theta]$. The monotonicity of α_1/α_2 is satisfied if $S(w_1)/S(w_2)$ is increasing in z, where $w_j = 1-\delta_j z$ and $S(w) = w^{\theta-1}/(1-w^\theta)$. The derivative of $S(w_1)/S(w_2)$ is nonnegative if

$$S'(w_2)\, w_2'/S(w_2) \leqslant S'(w_1)\, w_1'/S(w_1), \tag{A1}$$

where $w_j' = w_j'(z) = -\delta_j$ and $S'(w) = (1-w^\theta)^{-2} w^{\theta-2}(\theta - 1 + w^\theta)$. The inequality (A1) holds if $T(\delta) = \delta[\theta-1+(1-\delta z)^\theta](1-\delta z)^{-1}[1-(1-\delta z)^\theta]^{-1}$ is increasing in $\delta \in (0, 1)$. Straightforward calculations yield that $T'(\delta)$ is equal to sign to $h(\theta, w) = \theta - 1 + (2-\theta-\theta^2)\, w^\theta + \theta^2 w^{\theta+1} - w^{2\theta}$, where $w = 1-\delta z$. Based on analyzing h at the boundaries $\theta = 1$, $\theta \to \infty$, $w = 0$ and $w = 1$ and numerically computing h otherwise, it has been shown that $h \geqslant 0$.

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