

## Generalized Poisson Distributions as Limits of Sums for Arrays of Dependent Random Vectors

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Arrays of random vectors with values in  $R^d$ , stationary in rows, are investigated. By the assumptions related to the ones used in the extreme value theory a limit theorem for sums is proved. Necessary and sufficient conditions for the convergence in distribution of sums to some generalized Poisson distributions in  $m$ -dependent and  $\alpha$ -,  $\rho$ -,  $\phi$ -mixing cases are given. As a tool the point processes theory is used.

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### 1. INTRODUCTION

The present paper is devoted to the examination of the asymptotic behaviour of sums for arrays of random vectors with values in the finite dimensional real space  $R^d$  for some natural  $d$ . We assume that the arrays are strictly stationary in rows and infinitesimal, and the random vectors are dependent. Our interest is particularly devoted to the arrays  $\{X_{n,k}; k \in Z, n \in N\}$  (with  $Z$  denoting integers and  $N$  natural numbers) of random vectors, defined on a common probability space  $(\Omega, \mathcal{F}, P)$ , which are  $m$ -dependent,  $\alpha$ -,  $\rho$ -, or  $\phi$ -mixing in rows. However, these results are derived via some general theorems for dependent random vectors. As limits of sums the generalized Poisson distributions are obtained.

It may be instructive to demonstrate how such limits arise in the simplest case. An array  $\{X_{n,k}; k \in Z, n \in N\}$  is  $m$ -dependent in rows if for every  $n \in N$  the  $\sigma$ -algebras  $\sigma(\dots, X_{n-1}, X_n)$ ,  $\sigma(X_{n+k}, X_{n+k+1}, \dots)$  are independent for  $k > m$ .

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**THEOREM 1.1.** *Let  $\{X_{n,k}; k \in \mathbb{Z}, n \in \mathbb{N}\}$  be an array of Bernoulli random variables, strictly stationary and  $m$ -dependent in rows; and let  $\{k_n; n \in \mathbb{N}\}$  be a sequence of natural numbers such that  $k_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ .*

(A) *The following conditions are equivalent.*

(i) *The sequence  $\{S_n = \sum_{k=1}^{k_n} X_{n,k}; n \in \mathbb{N}\}$  is convergent in distribution.*

(ii) *The sequence  $\{k_n P(X_{n,0} = 1); n \in \mathbb{N}\}$  is bounded and for every  $s = 1, \dots, m+1$  there exists the limit*

$$\rho(s) = \lim_{n \rightarrow +\infty} k_n P\left(\sum_{j=-m}^0 X_{n,j} = s, X_{n,0} = 1, \sum_{j=1}^m X_{n,j} = 0\right) < +\infty.$$

(iii) *For every  $s = 1, \dots, m+1$  there exists*

$$\bar{\rho}(s) = \lim_{n \rightarrow +\infty} k_n \left[ P\left(\sum_{j=0}^m X_{n,j} = s\right) - P\left(\sum_{j=1}^m X_{n,j} = s\right) \right] < +\infty.$$

(iv) *For every  $s = 1, \dots, m+1$  there exists*

$$a(s) = \lim_{n \rightarrow +\infty} k_n P\left(X_{n,0} = 1, \sum_{j=1}^m X_{n,j} = s-1\right) < +\infty.$$

*If one of these conditions is satisfied, then  $\{S_n; n \in \mathbb{N}\}$  converges in distribution to a generalized Poisson distribution with the moment generating function*

$$M(t) = \exp \left[ \sum_{s=1}^{m+1} \rho(s)(e^{st} - 1) \right],$$

*where  $\rho(s) = \bar{\rho}(s) = a(s) - a(s+1)$ ,  $a(m+2) = 0$ .*

(B) *In particular,  $\{S_n; n \in \mathbb{N}\}$  is convergent to the Poisson distribution with the parameter  $\lambda > 0$  iff*

$$k_n P(X_{n,0} = 1) \rightarrow \lambda \quad \text{as } n \rightarrow +\infty$$

*and*

$$\lim_{n \rightarrow +\infty} k_n P(X_{n,0} = 1, X_{n,i} = 1) = 0 \quad \text{for every } i = 1, \dots, m.$$

The above theorem follows from Theorem 4.2 and Corollary 4.14. The equivalence of (A)(i)  $\Leftrightarrow$  (A)(ii) and (B) establish Theorems 1 and 2 of Hudson *et al.* [10] (see Remark 4.16). The property (iv) is much simpler than (ii).

Finally, property (iii) links this theorem with Theorem 5.3 [12], which we quote below, suitably transformed and in a very special case for the sake of brevity. In both of them the limit is determined by the distributional properties of the vectors  $(X_{n,0}, X_{n,1}, \dots, X_{n,m})$  only.

**THEOREM 1.2** [12, Theorem 5.3(i)]. *Let  $\{X_k; k \in \mathbb{Z}\}$  be a strictly stationary,  $m$ -dependent sequence of random variables. Let  $S_n = \sum_{k=1}^n X_k$ ,  $n \in \mathbb{N}$ . Assume that the joint distribution of the random vector  $(X_0, X_1, \dots, X_m)$  belongs to the domain of attraction of a nondegenerated  $(m+1)$ -dimensional strictly  $\alpha$ -stable distribution  $\mu$ ,  $0 < \alpha < 1$ , with the Lévy measure  $\nu$ .*

*Let  $\{a_n; n \in \mathbb{N}\}$  be a sequence of positive reals such that*

$$nP(a_n^{-1}(X_0, X_1, \dots, X_m) \in A) \rightarrow \nu(A)$$

*for every Borel subset  $A \subset \mathbb{R}^{m+1}$ ,  $0 \notin \bar{A}$ ,  $\nu(\partial A) = 0$ . (" $\bar{A}$ " denotes the closure of the set  $A$ .)*

*Then  $\{a_n^{-1}S_n; n \in \mathbb{N}\}$  converges in distribution to a strictly  $\alpha$ -stable distribution with the characteristic function*

$$\varphi(t) = \exp \left[ \int_{\mathbb{R}} (e^{itx} - 1) \nu_0(dx) \right],$$

*where*

$$\nu_0(dx) = (c_- 1(x < 0) + c_+ 1(x > 0)) \frac{dx}{|x|^{\alpha+1}},$$

*and the constants  $c_-$  and  $c_+$  are determined by the relations*

$$\begin{aligned} \frac{c_-}{\alpha} &= \lim_{n \rightarrow +\infty} n \left[ P \left( a_n^{-1} \sum_{k=0}^m X_k < -1 \right) - P \left( a_n^{-1} \sum_{k=1}^m X_k < -1 \right) \right] \\ \frac{c_+}{\alpha} &= \lim_{n \rightarrow +\infty} n \left[ P \left( a_n^{-1} \sum_{k=0}^m X_k > 1 \right) - P \left( a_n^{-1} \sum_{k=1}^m X_k > 1 \right) \right]. \end{aligned}$$

Theorem 1.2 has been obtained in [12] with the use of the point processes technique. The aim of the present paper is to develop and adapt this technique so that it would be possible to find a general theorem for arrays containing simultaneously both Theorems 1.1 and 1.2. This has been accomplished in Corollary 4.14. It turns out that it is possible to replace  $m$ -dependence by weaker assumptions such as  $\alpha$ -,  $\rho$ -, or  $\phi$ -mixing and still obtain limit theorems. As a corollary one can deduce the necessary and sufficient conditions for the weak invariance principle for sums in the case

of strictly stationary and  $\phi$ -mixing sequences  $\{X_k; k \in \mathbb{Z}\}$  with the marginal distribution  $\mathcal{L}(X_0)$  belonging to the domain of attraction of an  $\alpha$ -stable law,  $0 < \alpha < 2$  (Corollary 5.9). An especially simple case, in contrast to the unsolved Ibragimov–Iosifescu conjecture for  $\alpha = 2$  (e.g., [20]), is the situation  $0 < \alpha < 1$ , when under  $\phi$ -mixing with no restriction on the coefficients  $\phi(n)$

$$\sum_{k=1}^n X_k/a_n \xrightarrow{\mathcal{L}} v_\alpha$$

iff

$$\forall_{j \geq 1} \forall_{\varepsilon > 0} nP(|X_0| > a_n \varepsilon, |X_j| > a_n \varepsilon) \xrightarrow{n \rightarrow +\infty} 0,$$

where  $\{a_n\}$  is a respective norming sequence for the i.i.d. sequence  $\{\hat{X}_n\}$  such that  $\mathcal{L}(X_0) = \mathcal{L}(\hat{X}_0)$ .

The paper is organized as follows. In Section 2 we prove a general theorem about the convergence in distribution of certain point processes to a Poisson process. This convergence is the basis of Theorem 3.2 (Section 3) which gives sufficient conditions for convergence of sums to generalized Poisson distributions for arrays. Similar ideas can be found in [9, 21, and 14]. The dependence conditions demanded in Sections 2 and 3 are related to the theory of extremes [15, 16]. They are of two kinds: the first is weaker than  $\alpha$ -mixing (and hence also than  $m$ -dependence) and the second, giving some restrictions on the assemblies of great values, may be not fulfilled even for 1-dependent sequences. In Section 4, in the case of  $m$ -dependence, we show how to deal with the situation when the latter condition fails. Through a certain reduction of the problem amounting to consideration of more convenient arrays, we give in Theorem 4.2 sufficient conditions for the convergence in distribution of sums to a generalized Poisson distribution. It is also possible to give a characterization of all the limit points of centered sums under the boundedness of  $\{k_n P(\|X_{n,0}\| > \varepsilon)\}$  for every  $\varepsilon > 0$  (Theorem 4.11), and this allows us to obtain as a corollary Theorem 1.1. Finally, in Section 5 we relax the assumption of  $m$ -dependence to three kinds of strong mixing.

## 2. POISSON POINT PROCESSES AS LIMITS OF CERTAIN POINT PROCESSES GENERATED BY ARRAYS OF DEPENDENT RANDOM VECTORS

Let  $\{X_{n,k}; k \in \mathbb{Z}, n \in \mathbb{N}\}$  be an array of random vectors with values in  $\mathbb{R}^d$ ,  $d \in \mathbb{N}$ . Let  $\{k_n; n \in \mathbb{N}\}$  be a sequence of natural numbers,  $k_n \rightarrow +\infty$

as  $n \rightarrow +\infty$ . For such an array and every  $n \in N$  define the following point process

$$N_n(A) = \sum_{k=1}^{k_n} 1(X_{n,k} \in A), \quad (2.1)$$

where  $A \subset R^d \setminus \{0\} = E^d$  and  $1(U)$  is the indicator function of the set  $U$ .

The aim of this section is to prove a theorem similar to Lemma 3.3 in [12] about the convergence of  $\{N_n; n \in N\}$  to a Poisson process with the intensity measure  $\nu$ . The important difference between the theorem below and the lemma is that here we allow  $\nu$  to have atoms. The proof is different, since the Kallenberg criterion [13, Theorem 4.7], being the basic argument in [12, Lemma 3.3], works only for the atomless measures  $\nu$ . The price to pay for the change in the thesis of Theorem 2.1 is contained in the modification of the assumption B, which here is stronger than in Lemma 3.3. However, this condition is still weaker than  $\alpha$ -mixing. The second assumption concerning the dependence is a version of the Leadbetter condition  $D'$  used in the limit theory of extremes [15, 16]. The version we use here is more subtle than in [12] and this allows us to use Theorem 2.1 in the last chapter where mixing arrays are considered. Since in the sequel only the arrays stationary in rows will be considered, so Theorem 2.1 describes exactly that situation. It is possible to have a general version of this theorem (like [12, Lemma 3.3]).

In the following " $\|\cdot\|$ " denotes a norm in the Euclidean space whose dimension is obvious from the context.

For the definitions and any necessary information about point processes and their convergence in distribution we refer the reader to [13]. Here the general case of a Polish space is reduced to the space  $E^d = R^d \setminus \{0\}$ . The notion of a DC-semiring, appearing in the theorem, denotes a semiring  $\mathcal{J}$  contained in the family of relatively compact sets of  $E^d$  (i.e., separated from zero and bounded) with the property that for every  $\varepsilon > 0$  and every relatively compact set  $B$  there exists a finite set  $\{B_j; j \in J\} \subset \mathcal{J}$  such that every  $B_j$ ,  $j \in J$ , has a diameter smaller than  $\varepsilon$  (in any fixed metrization) and  $B \subset \bigcup_{j \in J} B_j$ . A good example of such a DC-semiring is the family of the sets  $B = \bigtimes_{i=1}^d ]a_i, b_i]$ , where  $a_i, b_i \in R \setminus \{0\}$ , and  $0 \notin \bar{B}$  (by " $\bar{B}$ " we denote the closure of the set  $B$ ). Of course there exist countable DC-semirings.

**THEOREM 2.1.** *Let  $\{X_{n,k}; k \in Z, n \in N\}$  be an array of random vectors with values in  $R^d$ ,  $d \in N$ , strictly stationary in rows, and let  $\{k_n; n \in N\}$  be a sequence of natural numbers such that  $k_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Assume that the following conditions are satisfied:*

(A) *For every  $\varepsilon > 0$ , the sequence*

$$\{k_n P(\|X_{n,0}\| > \varepsilon); n \in N\} \text{ is bounded.} \quad (2.2)$$

(B) For every measurable function  $f: R^d \rightarrow [0, 1]$  such that the support of  $1 - f$  is relatively compact in  $E^d$ ,

$$\sup_{1 \leq j < k \leq k_n} \left| E \prod_{1 \leq i \leq k} f(X_{n,i}) - E \prod_{1 \leq i \leq j} f(X_{n,i}) E \prod_{1 \leq i \leq k-j} f(X_{n,i}) \right| \rightarrow 0 \quad (2.3)$$

as  $n \rightarrow +\infty$ .

(D') For every  $\varepsilon > 0$ ,

$$\lim_{r \rightarrow +\infty} \limsup_{n \rightarrow +\infty} r \sum_{j=1}^{[k_n/r]-1} P \left( \{ \|X_{n,j}\| > \varepsilon \} \cap \bigcup_{i=j+1}^{[k_n/r]} \{ \|X_{n,i}\| > \varepsilon \} \right) = 0. \quad (2.4)$$

("[x]" denotes the greatest integer  $z$  such that  $z \leq x$ .)

Suppose that there exists a measure  $\nu$  on  $E^d$ , finite outside each neighbourhood of 0, and such that

$$k_n P(X_{n,0} \in A) \xrightarrow{n \rightarrow +\infty} \nu(A), \quad (2.5)$$

for every element  $A$  of a DC-semiring  $\mathcal{J}$  which consists of certain bounded in  $E^d$  and separated from 0 sets satisfying  $\nu(\partial A) = 0$ . Then

$$N_n \xrightarrow{\mathcal{J}} \Pi_\nu, \quad (2.6)$$

where  $N_n$  is defined by (2.1),  $\Pi_\nu$  is a Poisson process with the intensity measure  $\nu$ , and " $\xrightarrow{\mathcal{J}}$ " denotes the convergence in distribution of point processes.

*Proof.* According to one of the equivalent definitions of the convergence in distribution for point processes [13, Theorem 4.2(iii)] it is sufficient to show that

$$(N_n(B_1), \dots, N_n(B_k)) \xrightarrow{\mathcal{J}} (\Pi_\nu(B_1), \dots, \Pi_\nu(B_k)) \quad (2.7)$$

for every  $k \in \mathbb{N}$  and every collection  $B_1, \dots, B_k$  of pairwise disjoint elements of  $\mathcal{J}$ . We will show that

$$E \exp \left( - \sum_{m=1}^k \lambda_m N_n(B_m) \right) \rightarrow \prod_{m=1}^k \exp(\nu(B_m)(e^{-\lambda_m} - 1)) \quad (2.8)$$

for every  $\lambda_1, \dots, \lambda_k \geq 0$ . The following lemma will be useful in the sequel.

LEMMA 2.2. Let  $a_1, \dots, a_k \in R$ .

(i) There holds the equality

$$\prod_{j=1}^k (1 - a_j) = 1 - \sum_{j=1}^k a_j + \sum_{j=1}^{k-1} a_j \left( 1 - \prod_{i=j+1}^k (1 - a_i) \right). \quad (2.9)$$

(ii) If  $0 \leq a_j \leq 1$  for  $j = 1, \dots, k$ , then

$$0 \leq 1 - \prod_{j=1}^k (1 - a_j) \leq \sum_{j=1}^k a_j \quad (2.10)$$

and

$$0 \leq \prod_{j=1}^k (1 - a_j) - 1 + \sum_{j=1}^k a_j \leq \frac{1}{2} \sum_{\substack{i \neq j \\ 1 \leq i, j \leq k}} a_i a_j. \quad (2.11)$$

*Proof.* (i) is proved by induction, and (ii) follows from (i). ■

We have for  $\lambda_1, \dots, \lambda_k \geq 0$ ,

$$E \exp \left( - \sum_{m=1}^k \lambda_m N_n(B_m) \right) = E \prod_{j=1}^{k_n} g(X_{n,j}), \quad (2.12)$$

where

$$g : R^d \rightarrow [0, 1],$$

$$g(x) = \prod_{m=1}^k [1 + (e^{-\lambda_m} - 1) 1(x \in B_m)]. \quad (2.13)$$

The function  $g$  belongs to the class of functions described in (2.3). Thus, by the stationarity, (A) and (B), one can show that for every  $q \in N$

$$\left| E \left( \prod_{1 \leq j \leq k_n} g(X_{n,j}) \right) - \left( E \prod_{1 \leq j \leq [k_n/q]} g(X_{n,j}) \right)^q \right| \xrightarrow{n \rightarrow +\infty} 0. \quad (2.14)$$

Let  $\lambda, \varepsilon > 0$  be such that  $B_1, \dots, B_m \subset \{\varepsilon < \|x\| \leq \lambda\}$  and  $v(\|x\| = \varepsilon) = v(\|x\| = \lambda) = 0$ .

By (2.2) for every  $q \in N$

$$qP(\|X_{n,0}\| > \varepsilon) \xrightarrow{n \rightarrow +\infty} 0. \quad (2.15)$$

Let  $\{M_r; r \in N\}$  be a sequence of natural numbers such that  $M_{r-1} < M_r$  and for  $n \geq M_r$ ,

$$\left| E \prod_{1 \leq j \leq k_n} g(X_{n,j}) - \left( E \prod_{1 \leq j \leq [k_n/r]} g(X_{n,j}) \right)^r \right| < 1/r,$$

$$rP(\|X_{n,0}\| > \varepsilon) < 1/r.$$

Define  $r_n := r$  for  $M_r \leq n < M_{r+1}$ . We have  $r_n \rightarrow +\infty$ ,

$$\left| E \prod_{1 \leq j \leq k_n} g(X_{n,j}) - \left( E \prod_{1 \leq j \leq [k_n/r_n]} g(X_{n,j}) \right)^{r_n} \right| \rightarrow 0 \quad (2.16)$$

and

$$r_n P(\|X_{n,0}\| > \varepsilon) \xrightarrow{n \rightarrow +\infty} 0. \quad (2.17)$$

Also,

$$[k_n/r_n] P(\varepsilon < \|X_{n,0}\| \leq \lambda) \xrightarrow{n \rightarrow +\infty} 0, \quad (2.18)$$

since  $\{k_n P(\varepsilon < \|X_{n,0}\| \leq \lambda); n \in N\}$  is bounded due to (2.5). Now, using the inequality

$$|\exp(-x) - 1 + x| \leq \frac{1}{2}x^2 \quad \text{for } x \geq 0,$$

we obtain

$$\left| \left( E \prod_{1 \leq j \leq [k_n/r_n]} g(X_{n,j}) \right)^{r_n} - \exp \left[ r_n \left( E \prod_{1 \leq j \leq [k_n/r_n]} g(X_{n,j}) - 1 \right) \right] \right|$$

$$\leq \frac{1}{2} r_n \left( 1 - E \prod_{1 \leq j \leq [k_n/r_n]} g(X_{n,j}) \right)^2. \quad (2.19)$$

But from Lemma 2.2(ii),

$$0 \leq 1 - E \prod_{1 \leq j \leq [k_n/r_n]} g(X_{n,j})$$

$$= E \left[ 1 - \prod_{\substack{1 \leq j \leq [k_n/r_n] \\ 1 \leq m \leq k}} (1 + (e^{-\lambda_m} - 1) 1(X_{n,j} \in B_m)) \right]$$

$$\leq E \left[ \sum_{\substack{1 \leq j \leq [k_n/r_n] \\ 1 \leq m \leq k}} (1 - e^{-\lambda_m}) 1(X_{n,j} \in B_m) \right]$$

$$\leq k [k_n/r_n] P(\varepsilon < \|X_{n,0}\| \leq \lambda).$$



Coming back to (2.19) we can see that the whole expression tends to 0 when  $n \rightarrow +\infty$  because

$$\frac{1}{2}r_n k^2 [k_n/r_n]^2 P^2(\varepsilon < \|X_{n,0}\| \leq \lambda) \rightarrow 0.$$

Next we shall prove

$$\left| \exp \left[ r_n \left( E \prod_{1 \leq j \leq [k_n/r_n]} g(X_{n,j}) - 1 \right) \right] - \exp \left[ r_n \left( \sum_{\substack{1 \leq j \leq [k_n/r_n] \\ 1 \leq m \leq k}} (e^{-\lambda_m} - 1) P(X_{n,j} \in B_m) \right) \right] \right| \rightarrow 0. \quad (2.20)$$

Let us first consider the difference

$$\begin{aligned} & r_n \left( E \prod_{1 \leq j \leq [k_n/r_n]} g(X_{n,j}) - 1 \right) - r_n \sum_{\substack{1 \leq j \leq [k_n/r_n] \\ 1 \leq m \leq k}} (e^{-\lambda_m} - 1) P(X_{n,j} \in B_m) \\ &= r_n E \left( -1 + \sum_{\substack{1 \leq j \leq [k_n/r_n] \\ 1 \leq m \leq k}} (1 - e^{-\lambda_m}) P(X_{n,j} \in B_m) \right. \\ &\quad \left. + \prod_{\substack{1 \leq j \leq [k_n/r_n] \\ 1 \leq m \leq k}} [1 + (e^{-\lambda_m} - 1) 1(X_{n,j} \in B_m)] \right) = r_n E T^{(n)}. \end{aligned}$$

We have

$$T^{(n)} = -1 + \sum_{(j,m)} \alpha(j, m) + \prod_{(j,m)} (1 - \alpha(j, m)), \quad (2.21)$$

where

$$\alpha(j, m) = (1 - e^{-\lambda_m}) 1(X_{n,j} \in B_m)$$

and  $(j, m)$  belong to the matrix

$$M = \begin{pmatrix} (1, 1) & (1, 2) & \cdots & (1, k) \\ (2, 1) & (2, 2) & \cdots & (2, k) \\ \vdots & \vdots & \vdots & \vdots \\ ([k_n/r_n], 1) & ([k_n/r_n], 2) & \cdots & ([k_n/r_n], k) \end{pmatrix}.$$

Let us introduce the linear order among the elements of  $M$  such that

$$\begin{aligned} (j, m) &< (j, m+1) && \text{for } 1 \leq m < k, 1 \leq j \leq [k_n/r_n], \\ (j, k) &< (j+1, 1) && \text{for } 1 \leq j < [k_n/r_n]. \end{aligned}$$

Due to Lemma 2.2(i) and (2.21),

$$\begin{aligned}
 0 \leq T^{(n)} &= \sum_{\substack{(j,m) \in M \\ (j,m) \neq ([k_n/r_n], k)}} \alpha(j, m) \left[ 1 - \prod_{(j,m) < (i,p) \in M} (1 - \alpha(i, p)) \right] \\
 &= \sum_{\substack{(j,m) \in M \\ 1 \leq j < [k_n/r_n]}} \alpha(j, m) \left[ 1 - \prod_{(j,m) < (i,p) \in M} (1 - \alpha(i, p)) \right] \\
 &\quad + \sum_{m=1}^{k-1} \alpha([k_n/r_n], m) \left[ 1 - \prod_{p=m+1}^k (1 - \alpha([k_n/r_n], p)) \right] \\
 &= T^{(n)}(1) + T^{(n)}(2).
 \end{aligned} \tag{2.22}$$

Since

$$\alpha(j, m) = (1 - e^{-\lambda_m}) 1(X_{n,j} \in B_m) \leq 1(X_{n,j} \in B_m)$$

and for every subset  $G$  of the elements of the matrix  $M$

$$\begin{aligned}
 &1 - \prod_{(i,p) \in G} (1 - \alpha(i, p)) \\
 &= 1 - \prod_{(i,p) \in G} [1 + (e^{-\lambda_p} - 1) 1(X_{n,i} \in B_p)] \\
 &\leq 1 - \prod_{(i,p) \in G} 1(X_{n,i} \in B_p^c) = 1 \left( \bigcup_{(i,p) \in G} \{X_{n,i} \in B_p\} \right),
 \end{aligned}$$

so due to the disjointness of  $B_m$ ,  $m = 1, \dots, k$ , and the inclusion  $\bigcup_{m=1}^k B_m \subset \{\varepsilon < \|x\| \leq \lambda\}$  we have

$$\begin{aligned}
 0 \leq T^{(n)}(2) &\leq \sum_{m=1}^{k-1} 1(X_{n,[k_n/r_n]} \in B_m) \\
 &\quad \times 1 \left( \bigcup_{p=m+1}^k \{X_{n,[k_n/r_n]} \in B_p\} \right) = 0
 \end{aligned} \tag{2.23}$$

and

$$\begin{aligned}
 0 \leq T^{(n)}(1) &\leq \sum_{1 \leq j < [k_n/r_n]} \sum_{m=1}^k 1(X_{n,j} \in B_m) \\
 &\quad \times 1 \left( \bigcup_{(j,m) < (i,p) \in M} \{X_{n,i} \in B_p\} \right)
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{1 \leq j < [k_n/r_n]} \sum_{m=1}^k 1(X_{n,j} \in B_m) \\
&\quad \times 1 \left( \bigcup_{p=m+1}^k \{X_{n,j} \in B_p\} \cup \bigcup_{i=j+1}^{[k_n/r_n]} \bigcup_{p=1}^k \{X_{n,i} \in B_p\} \right) \\
&= \sum_{1 \leq j < [k_n/r_n]} \sum_{m=1}^k 1(X_{n,j} \in B_m) 1 \left( \bigcup_{i=j+1}^{[k_n/r_n]} \{X_{n,i} \in \bigcup_{p=1}^k B_p\} \right) \\
&\leq \sum_{1 \leq j < [k_n/r_n]} 1 \left( \{\|X_{n,j}\| > \varepsilon\} \cap \bigcup_{i=j+1}^{[k_n/r_n]} \{\|X_{n,i}\| > \varepsilon\} \right). \quad (2.24)
\end{aligned}$$

Equations (2.22), (2.23), and (2.24) give

$$\begin{aligned}
0 &\leq r_n E T^{(n)} = r_n E(T^{(n)}(1) + T^{(n)}(2)) = r_n E T^{(n)}(1) \\
&\leq r_n \sum_{1 \leq j < [k_n/r_n]} P \left( \{\|X_{n,j}\| > \varepsilon\} \cap \bigcup_{i=j+1}^{[k_n/r_n]} \{\|X_{n,i}\| > \varepsilon\} \right). \quad (2.25)
\end{aligned}$$

The right-hand side of (2.25) tends to 0 as  $n \rightarrow +\infty$  because  $r_n \rightarrow +\infty$  and (2.4) holds. This together with (2.5) gives (2.20). Finally, (2.12), (2.16), (2.19), and (2.20) imply (2.8). ■

If we assume the infinitesimality of  $\{X_{n,k}\}$  then the assumption (2.3) describing certain asymptotic independence in rows is weaker than  $\alpha$ -mixing by the Hoeffding inequality (see [6]).

*Remark 2.3.* In the case when  $\nu$  is atomless the proof of Lemma 3.3 in [12] can be modified so that the weaker versions of (2.2) and (2.3) (conditions  $A_0$  and  $B_0$  in [12]) and the more subtle assumption (2.4) are sufficient for the convergence of  $N_n$  to a Poisson process with the intensity measure  $\nu$ .

*Remark 2.4.* It is possible to obtain also non-stationary versions of Theorem 2.1 and the modification described in Remark 2.3.

### 3. GENERALIZED POISSON LAWS AS LIMITS OF SUMS OF DEPENDENT RANDOM VECTORS

In the present chapter we formulate a theorem which is similar to Theorem 3.1 of [12] in the same way as Theorem 2.1 and Lemma 3.3 of [12] are. Again, we consider an array which is strictly stationary in rows, although it is also possible to have a more general non-stationary version. The new thing is that now  $\nu$  may have atoms.

We adopt the following convention which can be found in the book of Araujo and Giné [1].

For  $0 < \tau < +\infty$ , the distribution  $\mu$  on  $R^d$ , given by the characteristic function

$$\hat{\mu}(y) = \exp \left[ \int (e^{i(y,x)} - 1 - i(y,x) 1(\|x\| \leq \tau)) v(dx) \right],$$

where  $v$  is a Lévy measure on  $R^d$ ,  $v(\|x\| = \tau) = 0$ , will be denoted by  $c_\tau\text{-Pois}(v)$ . Also,

$$c_0\text{-Pois}(v) := (c_\tau\text{-Pois}(v)) * \delta_{b_1(\tau)},$$

where  $b_1(\tau) = \int x 1(\|x\| \leq \tau) v(dx)$  if  $\int \|x\| 1(\|x\| \leq \tau) v(dx) < +\infty$ , and

$$c_\infty\text{-Pois}(v) := (c_\tau\text{-Pois}(v)) * \delta_{b_2(\tau)},$$

where  $b_2(\tau) = -\int x 1(\|x\| > \tau) v(dx)$  if  $\int \|x\| 1(\|x\| > \tau) v(dx) < +\infty$ . (“ $\delta_b$ ” denotes a probability measure concentrated in the point  $b$ , “ $*$ ”-convolution.)

The following result belongs to the group of non-normal limit theorems which assume some mixing conditions and also some limitation concerning the frequency of big values ([2, 7, 8, 12, 14, 22]).

**THEOREM 3.1.** *Let us suppose that all the assumptions of Theorem 2.1 are satisfied and let*

$$S_n = \sum_{k=1}^{k_n} X_{n,k},$$

$$S_n ]\delta, \eta] = \sum_{k=1}^{k_n} X_{n,k} 1(\delta < \|X_{n,k}\| \leq \eta).$$

Assume that

$$\lim_{\lambda \rightarrow +\infty} \limsup_{n \rightarrow +\infty} k_n P(\|X_{n,0}\| \geq \lambda) = 0. \quad (3.1)$$

(i) If

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} k_n E(\|X_{n,0}\| 1(\|X_{n,0}\| \leq \delta)) = 0, \quad (3.2)$$

then

$$\int \|x\| 1(\|x\| \leq 1) v(dx) < +\infty \quad \text{and} \quad S_n \xrightarrow{\mathcal{L}} c_0\text{-Pois}(v).$$

(ii) If  $\nu$  is a Lévy measure (i.e.,  $\int \min(1, \|x\|^2) \nu(dx) < +\infty$ ) and for every  $\varepsilon > 0$ ,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} P(\|S_n\|0, \delta] - ES_n\|0, \delta\| > \varepsilon) = 0, \quad (3.3)$$

then for every  $0 < \tau < +\infty$  such that  $\nu(\|x\| = \tau) = 0$ ,

$$S_n - ES_n\|0, \tau\| \xrightarrow{\mathcal{D}} c_\tau\text{-Pois}(\nu).$$

(iii) If  $\nu$  is a Lévy measure, (3.3) holds and additionally

$$\lim_{\lambda \rightarrow +\infty} \limsup_{n \rightarrow +\infty} k_n E(\|X_{n,0}\| \mathbf{1}(\|X_{n,0}\| > \lambda)) = 0, \quad (3.4)$$

then

$$S_n - ES_n \xrightarrow{\mathcal{D}} c_\infty\text{-Pois}(\nu).$$

*Proof.* The proof is exactly the same as the proof of Theorem 3.1 of [12]. The only difference is that we use Theorem 2.1 instead of Lemma 3.3 of [12]. ■

*Remark 3.2.* The modification of Lemma 3.3 of [12], described in Remark 2.3, leads also to the version of Theorem 3.1 of [12] with the weaker local dependence condition  $D'_0$ . In particular, as in [12] corollaries concerning  $\alpha$ -stable limits can be obtained.

#### 4. SUMS FOR STATIONARY AND $m$ -DEPENDENT ARRAYS

Since  $m$ -dependence is much stronger than the condition (B), one can trivially obtain versions of Theorems 2.1 and 3.1 in such a case. However, it is easy to find examples of  $m$ -dependent arrays for which (D') is not fulfilled. In this chapter we deal with such a situation. The main idea of Theorem 5.3 of [12] and its proof remains valid, but now it is described in the context of general arrays (not only generated by stationary sequences) and thus limits become generalized Poisson distributions. Despite the similarity with the proof of Theorem 5.3 of [12] the main steps are repeated to make the proofs in Chapter 5 clearer.

*Remark 4.1.* Let  $\nu$  be a Lévy measure in  $E^{d(m+1)} = R^{d(m+1)} \setminus \{0\}$ ,  $d, m \in \mathbb{N}$ . There exists a time-homogeneous process with independent increments  $\mathbf{Y} = \{Y(s); s \in [0, 1]\}$  such that  $Y(0) = 0$  and  $\mathcal{L}(Y(1)) =$



(i) If

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} k_n E[\|X_{n,0}\|^2 1(\|X_{n,0}\| \leq \delta)] = 0, \quad (4.4)$$

then  $\nu_0$  is a Lévy measure and for  $0 < \tau < +\infty$  such that  $\nu_0(\|x\| = \tau) = 0$ ,

$$\begin{aligned} S_n - ES_n ]0, \tau] &= \sum_{k=1}^{k_n} (X_{n,k} - EX_{n,k} 1(\|X_{n,k}\| \leq \tau)) \\ &\xrightarrow{\mathcal{D}} c_\tau\text{-Pois}(\nu_0) * \delta_{b(\tau)}, \end{aligned} \quad (4.5)$$

where

$$\begin{aligned} b(\tau) &= \int \left[ \sum_{i=0}^m x_i 1\left(\left\|\sum_{i=0}^m x_i\right\| \leq \tau\right) \right. \\ &\quad \left. - \sum_{i=1}^m x_i 1\left(\left\|\sum_{i=1}^m x_i\right\| \leq \tau\right) - x_0 1(\|x_0\| \leq \tau) \right] \nu(dx). \end{aligned} \quad (4.6)$$

(i)' If (4.4) is satisfied and instead of  $ES_n ]0, \tau]$  we take the centering sequence

$$\begin{aligned} b_n(\tau) &= k_n E \left[ \left( \sum_{i=0}^m X_{n,i} \right) 1\left(\left\|\sum_{i=0}^m X_{n,i}\right\| \leq \tau\right) \right. \\ &\quad \left. - \left( \sum_{i=1}^m X_{n,i} \right) 1\left(\left\|\sum_{i=1}^m X_{n,i}\right\| \leq \tau\right) \right], \end{aligned} \quad (4.7)$$

then  $\nu$  is a Lévy measure and

$$S_n - b_n(\tau) \xrightarrow{\mathcal{D}} c_\tau\text{-Pois}(\nu_0) \quad (4.8)$$

for  $0 < \tau < +\infty$  such that  $\nu_0(\|x\| = \tau) = 0$ .

(ii) If

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} k_n E[\|X_{n,0}\| 1(\|X_{n,0}\| \leq \delta)] = 0, \quad (4.9)$$

then  $\int \|x\| 1(\|x\| \leq 1) \nu_0(dx) < +\infty$  and

$$S_n \xrightarrow{\mathcal{D}} c_0\text{-Pois}(\nu_0). \quad (4.10)$$

(iii) If (4.4) holds and

$$\lim_{\lambda \rightarrow +\infty} \limsup_{n \rightarrow +\infty} k_n E[\|X_{n,0}\| 1(\|X_{n,0}\| \geq \lambda)] = 0, \quad (4.11)$$

then  $\int \|x\| 1(\|x\| > 1) v_0(dx) < +\infty$  and

$$S_n - ES_n \xrightarrow{\mathcal{L}} c_\infty\text{-Pois}(v_0). \quad (4.12)$$

*Proof.* Let us note that due to (4.4) also

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} k_n E[\|(X_{n,0}, \dots, X_{n,m})\|^2 1(\|(X_{n,0}, \dots, X_{n,m})\| \leq \delta)] = 0.$$

Thus, using (4.3) we obtain

$$\int \|x\|^2 1(\|x\| \leq 1) v(dx) < +\infty,$$

which, together with the boundedness of  $v$  outside neighbourhoods of 0, means that  $v$  is a Lévy measure.

According to Remark 4.1  $v_{\sum_{k \in K} X_k}$  is also a Lévy measure for every  $K \subseteq \{0, 1, \dots, m\}$ .

Let  $\eta > 0$  be such that  $v_{x_0}(\|x\| = \eta) = 0$ . Define

$$X_{n,k}^* = X_{n,k} 1(B_{n,k}),$$

$$B_{n,k} = \{\|X_{n,k}\| > \eta\} \cap \left[ \bigcup_{i=k}^{k+m} \left( \{\|X_{n,i}\| > \eta\} \cap \bigcap_{j=1}^m \{\|X_{n,i+j}\| \leq \eta\} \right) \right]$$

$$S_n^* = \sum_{k=1}^{k_n} X_{n,k}^*$$

$$S_n ]\eta, +\infty[ = \sum_{k=1}^{k_n} X_{n,k} 1(\|X_{n,k}\| > \eta).$$

LEMMA 4.3.

$$P(S_n^* \neq S_n ]\eta, +\infty[) \leq k_n m P^2(\|X_{n,0}\| > \eta) \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Define

$$X_{n,k}^{**} = \sum_{j=0}^m X_{n,k-j} 1(\|X_{n,k-j}\| > \eta) \\ \times 1(\|X_{n,k}\| > \eta) \times 1(\|X_{n,k+i}\| \leq \eta, i = 1, \dots, m)$$

$$S_n^{**} = \sum_{k=1}^{k_n} X_{n,k}^{**}.$$

LEMMA 4.4.  $P(S_n^{**} \neq S_n^*) \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$



LEMMA 4.5. The array  $\{X_{n,k}^{**}; k \in \mathbb{Z}, n \in \mathbb{N}\}$  and the sequence  $\{k_n; n \in \mathbb{N}\}$  satisfy the assumptions of Theorem 3.2(i), namely:

- (i)  $\{X_{n,k}^{**}\}$  is stationary and  $(3m+1)$ -dependent in rows.
- (ii)  $\{X_{n,k}^{**}\}$  is infinitesimal (i.e., (2.2) holds).
- (iii) For every  $\varepsilon > 0$

$$\lim_{r \rightarrow +\infty} \limsup_{n \rightarrow +\infty} k_n \sum_{i=2}^{[k_n/r]} P(\|X_{n,i}^{**}\| > \varepsilon, \|X_{n,i}^{**}\| > \varepsilon) = 0.$$

(iv)

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} k_n E[\|X_{n,0}^{**}\| 1(\|X_{n,0}^{**}\| \leq \delta)] = 0.$$

(v) For every  $A \subset E^d$ , such that  $0 \notin \bar{A}$ ,  $v_{\sum_{k \in K} x_k}(\partial A) = 0$  if only  $K \subset \{0, \dots, m\}$ , there holds the convergence

$$\begin{aligned} k_n P(X_{n,0}^{**} \in A) &\rightarrow v_\eta(A) \\ &= \sum_{J \subset \{1, \dots, m\}} \sum_{\emptyset \neq K \subset J \cup \{0\}}^{|J|+1-|K|} (-1)^{|J|+1-|K|} \\ &\quad \times v \left\{ (x_0, \dots, x_m); \sum_{k \in K} x_k \in A, \|x_j\| > \eta, j \in J \cup \{0\} \right\}. \end{aligned}$$

COROLLARY 4.6.  $\int \|x\| 1(\|x\| \leq 1) v_\eta(dx) < +\infty$  and

$$S_n ]\eta, +\infty[ \xrightarrow[\mathcal{L}]{} c_0\text{-Pois}(v_\eta).$$

*Proof.* The corollary is proved by Lemmas 4.3, 4.4, 4.5, and Theorem 3.1(i). ■

LEMMA 4.7. If  $A \subset E^d$  is a Borel set such that  $0 \notin \bar{A}$ ,  $v_{\sum_{k \in K} x_k}(\partial A) = 0$  for every  $K \subset \{0, 1, \dots, m\}$ , then

$$v_\eta(A) \rightarrow v_0(A) = v_{x_0 + \dots + x_m}(A) - v_{x_1 + \dots + x_m}(A) \quad \text{as } \eta \rightarrow 0.$$

LEMMA 4.8.  $v_0$  is a Lévy measure and for every  $0 < \tau < +\infty$  such that  $v_0(\|x\| = \tau) = 0$ ,

$$c_\tau\text{-Pois}(v_\eta) \xrightarrow[w]{} c_\tau\text{-Pois}(v_0) \quad \text{as } \eta \rightarrow 0.$$

Moreover, if

$$\int \|x\| \mathbf{1}(\|x\| \leq 1) \nu(dx) < +\infty,$$

then

$$\int \|x\| \mathbf{1}(\|x\| \leq 1) \nu_0(dx) < +\infty, \quad (4.13)$$

and thus

$$c_0\text{-Pois}(\nu_\eta) \xrightarrow{w} c_0\text{-Pois}(\nu_0), \quad \text{as } \eta \rightarrow 0; \quad (4.14)$$

if

$$\int \|x\| \mathbf{1}(\|x\| > 1) \nu(dx) < +\infty,$$

then

$$\int \|x\| \mathbf{1}(\|x\| > 1) \nu_\eta(dx) < +\infty, \quad (4.15)$$

$$\int \|x\| \mathbf{1}(\|x\| > 1) \nu_0(dx) < +\infty, \quad (4.16)$$

and thus

$$c_x\text{-Pois}(\nu_\eta) \rightarrow c_x\text{-Pois}(\nu_0), \quad \text{as } \eta \xrightarrow{w} 0. \quad (4.17)$$

*Proof.* In the proof we use Remark 4.1 and the fact that  $\int \|x\|^2 \mathbf{1}(\|x\| \leq \delta) \nu_\eta(dx)$  ( $\int \|x\| \mathbf{1}(\|x\| \leq \delta) \nu_\eta(dx)$  or  $\int \|x\| \mathbf{1}(\|x\| > \lambda) \nu_\eta(dx)$ , respectively) is dominated by a finite number of integrals of the analogical form with respect to the measures  $\nu_{\sum_{k \in K} \lambda_k x_k}$ , where  $K \subset \{0, \dots, m\}$ . Hence

$$\lim_{\delta \rightarrow 0} \limsup_{\eta \rightarrow 0} \int \|x\|^2 \mathbf{1}(\|x\| \leq \delta) \nu_\eta(dx) = 0$$

$$\left( \lim_{\delta \rightarrow 0} \limsup_{\eta \rightarrow 0} \int \|x\| \mathbf{1}(\|x\| \leq \delta) \nu_\eta(dx) = 0 \right.$$

or

$$\left. \lim_{\lambda \rightarrow \infty} \limsup_{\eta \rightarrow 0} \int \|x\| \mathbf{1}(\|x\| > \lambda) \nu_\eta(dx) = 0, \text{ respectively} \right).$$

Thus  $\int \|x\|^2 \mathbf{1}(\|x\| \leq 1) \nu_0(dx) < +\infty$  ((4.13) or (4.15), (4.16), respectively.)

LEMMA 4.9. *If  $\{Z_n = (Z_n^1, \dots, Z_n^d); n \in N\}$  is a sequence of random vectors,  $m$ -dependent, square integrable, and such that  $\mathcal{L}(Z_n) = \mathcal{L}(Z_1)$ ,  $n \in N$ , then there exists a constant  $K$  for which the inequality*

$$E \|S_n - ES_n\|^2 \leq K k_n E \|Z_1\|^2$$

holds, where  $S = \sum_{k=1}^{k_n} Z_k$ .

*Proof of Theorem 4.2.* (i) Let  $0 < \eta < \tau$ ,  $\nu_0(\|x\| = \eta) = \nu_0(\|x\| = \tau) = 0$ .

$$S_n - ES_n \rfloor \eta, \tau \rfloor = (S_n \rfloor 0, \eta \rfloor - ES_n \rfloor 0, \eta \rfloor) + S_n \rfloor \eta, +\infty \rfloor + ES_n \rfloor \eta, \tau \rfloor.$$

We have

1. By Corollary 4.6,  $S_n \rfloor \eta, +\infty \rfloor \xrightarrow{\mathcal{D}} c_0\text{-Pois}(\nu_\eta)$ .
2. Due to (4.3) for every  $\delta > 0$ ,

$$k_n P(X_{n,0} \in (\cdot)) \big|_{\{\|x\| > \delta\}} \xrightarrow{w} \nu_{x_0}(\cdot) \big|_{\{\|x\| > \delta\}}, \quad n \rightarrow +\infty.$$

Hence

$$ES_n \rfloor \eta, \tau \rfloor \rightarrow \int x \mathbf{1}(\eta < \|x\| \leq \tau) \nu_{x_0}(dx) \stackrel{\text{df}}{=} b_\eta(\tau).$$

Let

$$h_\eta(\tau) = \int x \mathbf{1}(\|x\| \leq \tau) \nu_\eta(dx) - b_\eta(\tau).$$

So, if  $n \rightarrow +\infty$ , then

$$\begin{aligned} & S_n \rfloor \eta, +\infty \rfloor - ES_n \rfloor \eta, \tau \rfloor \\ & \xrightarrow{\mathcal{D}} c_0\text{-Pois}(\nu_\eta) * \delta_{-b_\eta(\tau)} = c_\tau\text{-Pois}(\nu_\eta) * \delta_{h_\eta(\tau)}. \end{aligned} \quad (4.18)$$

As in the proof of Theorem 5.3(iii) of [12], there exists the finite limit

$$\begin{aligned} b(\tau) = \lim_{\eta \rightarrow 0} h_\eta(\tau) &= \int [(x_0 + \dots + x_m) \mathbf{1}(\|x_0 + \dots + x_m\| \leq \tau) \\ & \quad - (x_1 + \dots + x_m) \mathbf{1}(\|x_1 + \dots + x_m\| \leq \tau) \\ & \quad - x_0 \mathbf{1}(\|x_0\| \leq \tau)] d\nu(x_0, \dots, x_m). \end{aligned} \quad (4.19)$$

3. Lemma 4.9 and (4.4) imply that for every  $\varepsilon > 0$ ,

$$\lim_{\eta \rightarrow 0} \limsup_{n \rightarrow +\infty} P(\|S_n\|0, \eta] - ES_n\|0, \eta] \| > \varepsilon) = 0, \quad (4.20)$$

Theorem 4.2 of [3], Lemma 4.8, (4.18), (4.19), and (4.20) finish the proof of (4.5).

(i)' It is enough to note that

$$k_n E \left[ \sum_{i=0}^m X_{n,i} 1 \left( \left\| \sum_{i=0}^m X_{n,i} \right\| \leq \tau \right) - \sum_{i=1}^m X_{n,i} 1 \left( \left\| \sum_{i=1}^m X_{n,i} \right\| \leq \tau \right) \right] \rightarrow b(\tau)$$

when  $n \rightarrow +\infty$  (compare with Corollary 5.4 of [12]).

(ii) Assume (4.9). Then

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} k_n [ \| (X_{n,0}, \dots, X_{n,m}) \| 1 ( \| (X_{n,0}, \dots, X_{n,m}) \| \leq \delta ) ] = 0,$$

and hence  $\int \|x\| 1(\|x\| \leq 1) \nu(dx) < +\infty$ . Lemma 4.8 implies

$$c_0\text{-Pois}(v_\eta) \xrightarrow{w} c_0\text{-Pois}(v_0) \quad \text{as } \eta \rightarrow 0.$$

Since  $S_n = S_n\|0, \eta] + S_n\|\eta, +\infty[$  and, due to (4.9),

$$\lim_{\eta \rightarrow 0} \limsup_{n \rightarrow +\infty} P(\|S_n\|0, \eta] \| > \varepsilon) = 0,$$

(4.10) holds according to Theorem 4.2 of [3].

(iii) Let (4.4) and (4.11) be satisfied. Then

$$\int \|x\| 1(\|x\| > 1) \nu_{x_0}(dx) < +\infty$$

and

$$\lim_{\lambda \rightarrow +\infty} \limsup_{n \rightarrow +\infty} k_n E [ \| (X_{n,0}, \dots, X_{n,m}) \| 1 ( \| (X_{n,0}, \dots, X_{n,m}) \| > \lambda ) ] = 0.$$

Hence also

$$\int \|x\| 1(\|x\| > 1) \nu(dx) < +\infty.$$

By Lemma 4.8,

$$\int \|x\| 1(\|x\| > 1) v_\eta(dx) < +\infty, \quad \int \|x\| 1(\|x\| > 1) v_0(dx) < +\infty$$

and

$$c_\infty\text{-Pois}(v_\eta) \xrightarrow{w} c_\infty\text{-Pois}(v_0). \quad (4.21)$$

On the other hand, from (4.3) and (4.11),

$$ES_n] \eta, +\infty[ \rightarrow \int x 1(\|x\| > \eta) v_{x_0}(dx).$$

As in the proof of Theorem 5.3(ii) of [12],

$$\int x v_\eta(dx) = \int x 1(\|x\| \geq \eta) v_{x_0}(dx).$$

We have

$$\begin{aligned} S_n] \eta, +\infty[ - ES_n] \eta, +\infty[ \\ \xrightarrow{\mathcal{L}} c_0\text{-Pois}(v_\eta) * \delta_{-\int x 1(\|x\| > \eta) v_{x_0}(dx)} \\ = c_\infty\text{-Pois}(v_\eta) * \delta_{[\int x v_\eta(dx) - \int x 1(\|x\| > \eta) v_{x_0}(dx)]} = c_\infty\text{-Pois}(v_\eta). \end{aligned}$$

This fact, the convergence (4.21), and Lemma 4.9 and (4.4), both implying (4.20), finish the proof of (4.12) by Theorem 4.2 of [3]. ■

With the array  $\{X_{n,k}\}$  from Theorem 4.2 one can associate the array  $\{\hat{Y}_{n,k}; k \in \mathbb{Z}, n \in \mathbb{N}\}$  of random vectors with values in  $\mathbb{R}^{d(m+1)}$ , independent and identically distributed in rows, with  $\mathcal{L}(\hat{Y}_{n,0}) = \mathcal{L}(X_{n,0}, \dots, X_{n,m})$ . The following corollary shows that the key to Theorem 4.2 is the comparison of the behaviour of the sums  $\sum_{k=1}^{k_n} X_{n,k}$  and  $\sum_{k=1}^{k_n} \hat{Y}_{n,k}$ .

**COROLLARY 4.10.** *Let  $\{X_{n,k}\}$  and  $\{k_n\}$  satisfy (4.1) and (4.2) and  $\{\hat{Y}_{n,k}; k \in \mathbb{Z}, n \in \mathbb{N}\}$  be an array whose properties have been described above. Let us suppose that there exist a Lévy measure  $v$  on  $\mathbb{R}^{d(m+1)}$  and a sequence  $\{z_n; n \in \mathbb{N}\}$  of vectors from  $\mathbb{R}^{d(m+1)}$  such that*

$$\sum_{k=1}^{k_n} \hat{Y}_{n,k} - z_n \xrightarrow{\mathcal{L}} c_v\text{-Pois}(v) * \delta_z. \quad (4.22)$$

for some  $0 < \vartheta < +\infty$  and  $z \in R^{d(m+1)}$ . Then (4.5) with (4.6) holds and, for  $\{h_n(\tau); n \in N\}$  defined in (4.7), the convergence (4.8) takes place.

*Proof.* We need the modification in Lemma 4.9:

$$E \|S_n - ES_n\|^2 \leq K k_n E \|Z_1 - EZ_1\|^2.$$

Instead of (4.4) we have as a necessary condition of (4.22)

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} k_n E \|(X_{n,0} 1(\|X_{n,0}\| \leq \delta) - EX_{n,0} 1(\|X_{n,0}\| \leq \delta))\|^2 = 0.$$

Of course, (4.3) holds and  $\nu$  is a Lévy measure. ■

Theorem 4.2 permits us to characterize all the limit points of centered sums if for every  $\varepsilon > 0$  the set  $\{k_n P(\|X_{n,0}\| > \varepsilon); n \in N\}$  is bounded.

**THEOREM 4.11.** *Let  $\{X_{n,k}; k \in Z, n \in N\}$  and  $\{k_n; n \in N\}$  be as in (4.1), (4.2), and for every  $\varepsilon > 0$*

$$\sup_{n \in N} k_n P(\|X_{n,0}\| > \varepsilon) < M_\varepsilon \quad \text{and} \quad 0 < M_\varepsilon < +\infty. \quad (4.23)$$

*Assume that one of the following assumptions is fulfilled:*

- (i) (4.4) is true,
- (ii) (4.9) is true,
- (iii) (4.4) and (4.11) are true.

*Then all the limit points of the sequence  $\{S_n - h_n; n \in N\}$ , where*

$$h_n = \begin{cases} h_n(\tau), & \text{as defined in (4.7), in the case (i)} \\ 0 & \text{in the case (ii)} \\ k_n EX_{n,0} & \text{in the case (iii)} \end{cases} \quad (4.24)$$

*are convolutions of  $\delta_a$  for some  $a \in R^d$  and a generalized Poisson distribution  $c_\beta$ -Pois( $\rho$ ), where  $0 \leq \beta \leq +\infty$  and  $\rho$  is a Lévy measure on  $R^d$ . For each such limit point there exists a sequence  $\varphi: N \rightarrow N$ ,  $\varphi(n) \rightarrow +\infty$  as  $n \rightarrow +\infty$ , and a measure  $\nu^\varphi$  on  $E^{d(m+1)}$  such that for every  $\delta > 0$ ,*

$$k_{\varphi(n)} P((X_{\varphi(n),0}, \dots, X_{\varphi(n),m}) \in (\cdot))|_{\{\|\cdot\| > \delta\}} \xrightarrow{w} \nu^\varphi(\cdot)|_{\{\|\cdot\| > \delta\}} \quad (4.25)$$

and then

$$\begin{aligned} \rho(\cdot) &= (w\text{-}\lim_{\eta \rightarrow 0})(w\text{-}\lim_{n \rightarrow +\infty}) k_{\varphi(n)} \\ &\times P\left(\sum_{j=-m}^0 X_{\varphi(n),j} 1(\|X_{\varphi(n),j}\| > \eta) \in (\cdot), \right. \\ &\quad \left. \|X_{\varphi(n),0}\| > \eta, \|X_{\varphi(n),j}\| < \eta, j=1, \dots, m\right), \end{aligned} \quad (4.26)$$

$$\begin{aligned} \rho(\cdot) &= w\text{-}\lim_{\eta \rightarrow 0} \nu^\varphi \left\{ \left( \sum_{j=-m}^0 x_j 1(\|x_j\| > \eta) \right) \in (\cdot), \right. \\ &\quad \left. \|x_0\| > \eta, \|x_j\| \leq \eta, j=1, \dots, m \right\}, \end{aligned} \quad (4.27)$$

$$\rho(\cdot) = \nu_{x_0 + \dots + x_m}^\varphi(\cdot) - \nu_{x_1 + \dots + x_m}^\varphi(\cdot) \quad (4.28)$$

("w-lim" denotes the weak limit of measures considered outside the neighbourhoods of 0).

*Proof.* Note that (4.23) guarantees infinitesimality. Suppose that  $\psi: N \rightarrow N$ ,  $\psi(n) \rightarrow +\infty$ , and  $(S_{\psi(n)} - h_{\psi(n)}) \rightarrow_{\mathcal{D}} \mu$  as  $n \rightarrow +\infty$ . One can choose a further subsequence  $\chi \circ \psi: N \rightarrow N$ ,  $\chi \circ \psi(n) \rightarrow +\infty$  such that for every  $\delta > 0$ ,

$$k_{\chi \circ \psi(n)} P((X_{\chi \circ \psi(n),0}, \dots, X_{\chi \circ \psi(n),m}) \in (\cdot))|_{\{\|x\| > \delta\}} \xrightarrow{w} \nu^{\chi \circ \psi}(\cdot)|_{\{\|x\| > \delta\}}$$

and  $\nu^{\chi \circ \psi}$  is finite outside every neighbourhood of 0. Due to Theorem 4.2 in every case (i), (ii), or (iii), the sequence  $\{S_{\chi \circ \psi(n)} - h_{\chi \circ \psi(n)}; n \in N\}$  tends to a convolution of  $\delta_a$  and a generalized Poisson distribution  $c_\beta\text{-Pois}(\rho)$  for some  $a \in R^d$  and  $0 \leq \beta \leq +\infty$ ; moreover, (4.26), (4.27), and (4.28) hold with the sequence  $\varphi$  replaced by  $\chi \circ \psi$ . This limit coincides with  $\mu$ . The equivalence between (4.27) and (4.28) is not trivial. This fact is hidden in Lemma 4.7 and is implied by  $m$ -dependence, stationarity, and Lemma 6.5 in [12]. ■

**EXAMPLE 4.12.** Let  $\mathcal{F} = \{f: R^{m+1} \rightarrow R^{\geq 0}; f \text{ measurable, continuous at } 0 \text{ with } f(0)=0, \text{ and such that } \exists_{a_0>0} \exists_{b_0>0} (|x_0| > a_0 \text{ or } \dots \text{ or } |x_m| > a_0) \Rightarrow (f(x_0, \dots, x_m) > b_0)\}$ . Suppose that  $\{X_{n,k}; k \in \mathbb{Z}, n \in N\}$  is an array of random variables whose rows form i.i.d. sequences and such that for a certain sequence  $\{k_n; n \in N\}$  of natural numbers,  $k_n \rightarrow +\infty$

$$\forall_{\varepsilon>0} k_n P(X_{n,0} \in (\cdot))|_{\{|x| > \varepsilon\}} \xrightarrow{w} \nu(\cdot)|_{\{|x| > \varepsilon\}},$$

where  $\nu$  is a measure in  $R \setminus \{0\}$ , finite outside each neighbourhood of 0 and  $\nu(|x| > a_0) > 0$ . For some  $f \in \mathcal{F}$  define  $Y_{n,k} = f(X_{n,k}, \dots, X_{n,k+m})$ . The new array is strictly stationary and  $m$ -dependent in rows. The laws of the sums  $\sum_{k=1}^{k_n} Y_{n,k} - h_n$  ( $h_n$  is a special centering sequence) have a nonzero limit point which is a convolution of a generalized Poisson distribution and a measure concentrated in a certain point. To show this we use Theorem 4.11, noting that the inequalities below guarantee that the limit distribution is not trivial.

For  $c_0, c_1, \dots, c_m \geq 0$  with  $c_i > 0$  for some  $i$ , there exists  $\delta_{c_i} > 0$ :

$$\begin{aligned} & \limsup_{n \rightarrow \infty} k_n P(Y_{n,0} > c_0, Y_{n,1} > c_1, \dots, Y_{n,m} > c_m) \\ & \leq \limsup_{n \rightarrow \infty} k_n P(Y_{n,i} > c_i) \leq \limsup_{n \rightarrow \infty} k_n(m+1) P(|X_{n,0}| > \delta_{c_i}) \\ & \leq (m+1) \nu(|X_{n,0}| \geq \delta_{c_i}) < +\infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \liminf_{n \rightarrow \infty} k_n P(Y_{n,0} > b_0, Y_{n,1} > b_0, \dots, Y_{n,m} > b_0) \\ & \geq \liminf_{n \rightarrow \infty} k_n P\left(\bigcup_{i=0}^m \{|X_{n,i}| > a_0\} \cap \bigcup_{i=1}^{m+1} \{|X_{n,i}| > a_0\} \cap \dots \cap \bigcup_{i=m}^{2m} \{|X_{n,i}| > a_0\}\right) \\ & \geq \liminf_{n \rightarrow \infty} k_n P(|X_{n,m}| > a_0) \geq \nu(|x| \geq a_0) > 0. \end{aligned}$$

**EXAMPLE 4.13.** Let  $\{X_{n,k}; k \in \mathbb{Z}, n \in \mathbb{N}\}$  be an array of nonnegative random variables, independent and identically distributed in rows. Let  $\{k_n; n \in \mathbb{N}\}$  be a sequence of natural numbers and  $k_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ .

Suppose that

$$k_n P(X_{n,0} > a) \rightarrow \nu(]a, +\infty[) < +\infty$$

for all  $a$  belonging to a dense subset  $D$  of  $R^+$ . Define

$$Y_{n,k} = X_{n,k} \vee X_{n,k+1} \vee \dots \vee X_{n,k+m}.$$

The array  $\{Y_{n,k}; k \in \mathbb{Z}, n \in \mathbb{N}\}$  is stationary and  $m$ -dependent in rows.



PROPOSITION. For  $a_0, \dots, a_m > 0$ ,

$$k_n P(Y_{n,0} > a_0, Y_{n,1} > a_1, \dots, Y_{n,m} > a_m) \\ \xrightarrow{n \rightarrow +\infty} v(\lceil a_0 \vee a_1 \vee \dots \vee a_m, +\infty \rceil).$$

*Proof.*

$$P(Y_{n,0} > a_0, Y_{n,1} > a_1, \dots, Y_{n,m} > a_m) \\ = \sum_{k=1}^{m+1} (-1)^{k-1} \sum_{0 \leq i_1 < i_2 < \dots < i_k \leq m} (1 - P(Y_{n,i_1} \leq a_{i_1}, \dots, Y_{n,i_k} \leq a_{i_k})). \quad (*)$$

Note that

$$P(Y_{n,i_1} \leq a_{i_1}, \dots, Y_{n,i_k} \leq a_{i_k}) \\ = F_n(a_{i_1})^{i_2-i_1} \cdot F_n(a_{i_1} \wedge a_{i_2})^{i_3-i_2} \cdot \dots \cdot F_n(a_{i_1} \wedge \dots \wedge a_{i_k})^{i_1+m-i_k+1} \\ \times F_n(a_{i_2} \wedge \dots \wedge a_{i_k})^{i_2-i_1} \cdot \dots \cdot F_n(a_{i_k})^{i_k-i_{k-1}-1} = \prod_{j=1}^s F_n(b_j),$$

where  $F_n(x) = P(X_{n,0} \leq x)$ . Due to Lemma 2.2,

$$1 - \prod_{j=1}^s F_n(b_j) = \sum_{j=1}^s (1 - F_n(b_j)) - \sum_{j=1}^{s-1} (1 - F_n(b_j)) \left(1 - \prod_{i=j+1}^s F_n(b_i)\right).$$

Hence

$$\lim_{n \rightarrow +\infty} k_n \left(1 - \prod_{j=1}^s F_n(b_j)\right) = \sum_{j=1}^s v(\lceil b_j, +\infty \rceil).$$

Now we count the coefficient which appears by

$$\lim_{n \rightarrow +\infty} k_n (1 - F_n(a_{i_1} \wedge \dots \wedge a_{i_k})) = v(\lceil a_{i_1} \wedge \dots \wedge a_{i_k}, +\infty \rceil)$$

in (\*) after multiplying both sides by  $k_n$  and tending with  $n$  to  $+\infty$ :

$$(-1)^{k-1} (m - i_k + 1) + \sum_{s=1}^{m-i_k} \sum_{i_k < j_1 < \dots < j_s \leq m} (-1)^{k+s} (j_1 - i_k) \\ = (-1)^{k-1} (m - i_k + 1) + (-1)^k \sum_{s=1}^{m-i_k} (-1)^s \left[ 1 \cdot \binom{m-i_k-1}{s-1} \right. \\ \left. + 2 \cdot \binom{m-i_k-2}{s-1} + \dots + (m-s+1-i_k) \binom{s-1}{s-1} \right] \\ = (-1)^{k-1} (m - i_k + 1) + (-1)^k (m - i_k) = (-1)^{k-1}.$$

If we denote  $v(\cdot)a, +\infty[\cdot) = G(a)$ , then we have

$$\begin{aligned} & \lim_{n \rightarrow +\infty} k_n P(Y_{n,0} > a_0, Y_{n,1} > a_1, \dots, Y_{n,m} > a_m) \\ &= \sum_{i=0}^m G(a_i) - \sum_{0 \leq i < j \leq m} G(a_i \wedge a_j) \\ & \quad + \dots + (-1)^{k-1} \sum_{0 \leq i_1 < \dots < i_k \leq m} G(a_{i_1} \wedge a_{i_2} \wedge \dots \wedge a_{i_k}) \\ & \quad + \dots + (-1)^{m-1} G(a_0 \wedge a_1 \wedge \dots \wedge a_m) \\ &= G(a_0 \vee a_1 \vee \dots \vee a_m). \quad \blacksquare \end{aligned}$$

According to Theorem 4.2(ii) and (iii), if

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} k_n E[X_{n,0} 1(X_{n,0} \leq \delta)] = 0,$$

then

$$\sum_{i=1}^{k_n} Y_{n,i} \xrightarrow{\mathcal{L}} c_0\text{-Pois}(\mu_0);$$

if

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} k_n E[X_{n,0}^2 1(X_{n,0} \leq \delta)] = 0$$

and

$$\lim_{\lambda \rightarrow +\infty} \limsup_{n \rightarrow +\infty} k_n E[X_{n,0} 1(X_{n,0} > \lambda)] = 0,$$

then

$$\sum_{i=1}^{k_n} (Y_{n,i} - EY_{n,i}) \xrightarrow{\mathcal{L}} c_{\infty}\text{-Pois}(\mu_0),$$

where  $\mu_0 = \mu_{x_0} + \dots + \mu_{x_m} - \mu_{x_1} - \dots - \mu_{x_m}$  and  $\mu$  is defined by the formulae

$$\begin{aligned} & \mu(\cdot)a_0, +\infty[\cdot \times \cdot]a_1, +\infty[\cdot \times \dots \times \cdot]a_m, +\infty[\cdot) \\ &= v(\cdot)a_0 \vee a_1 \vee \dots \vee a_m, +\infty[\cdot). \quad \blacksquare \end{aligned}$$

We complete this section with two corollaries, giving the necessary and sufficient conditions for the convergence in law of centered sums under the assumption (4.23). They lead to the explanation of the equivalence of the four conditions in Theorem 1.1. The specification of three different cases

in Corollary 4.14 is perhaps not elegant, but as shown for the theory of  $\alpha$ -stable limit theorems, the situations when  $0 < \alpha < 1$ ,  $\alpha = 1$ , and  $1 < \alpha < 2$  seem to differ much and require different ways of centering.

**COROLLARY 4.14.** *Let  $\{X_{n,k}; k \in \mathbb{Z}, n \in \mathbb{N}\}$  and  $\{k_n; n \in \mathbb{N}\}$  be as in (4.1), (4.2), and for every  $\varepsilon > 0$  let (4.23) be satisfied.*

(i) *If (4.4) is true, then the following conditions are equivalent:*

(i<sub>1</sub>) *The sequence  $\{S_n - b_n(\tau); n \in \mathbb{N}\}$ , where  $b_n(\tau)$  is defined in (4.7), is convergent in distribution for some  $\tau > 0$ .*

(i<sub>2</sub>) *There exists a measure  $\rho$  on  $E^d$ , finite outside every neighbourhood of 0, satisfying the following condition:*

$$\text{if } \varphi : \mathbb{N} \rightarrow \mathbb{N}, \quad \varphi(n) \rightarrow +\infty \text{ as } n \rightarrow +\infty,$$

*and there exists a measure  $\nu^\varphi$  on  $E^{d(m+1)}$ , finite outside every neighbourhood of 0, such that (4.25) is fulfilled, then  $\rho$  is defined by the equality (4.27).*

(i<sub>3</sub>) *There exists a measure  $\rho$  on  $E^d$ , finite outside every neighbourhood of 0, satisfying the following condition:*

$$\text{if } \varphi : \mathbb{N} \rightarrow \mathbb{N}, \quad \varphi(n) \rightarrow +\infty \text{ as } n \rightarrow +\infty,$$

*and there exists a measure  $\nu^\varphi$  on  $E^{d(m+1)}$ , finite outside every neighbourhood of 0, such that (4.25) is fulfilled, then  $\rho$  is defined by the equality (4.28).*

*If one of these conditions is satisfied, then the measures in (i<sub>2</sub>) and (i<sub>3</sub>) coincide,  $\rho$  is a Lévy measure, and for  $0 < \tau < +\infty$  such that  $\rho(\|x\| = \tau) = 0$ ,*

$$S_n - b_n(\tau) \xrightarrow{\mathcal{L}} c_\tau\text{-Pois}(\rho).$$

(ii) *If (4.9) is true, then the following conditions are equivalent:*

(ii<sub>1</sub>) *The sequence  $\{S_n; n \in \mathbb{N}\}$  is convergent in distribution.*

(i<sub>2</sub>) *holds.*

(i<sub>3</sub>) *holds.*

*If one of these conditions is satisfied, then the measures in (i<sub>2</sub>) and (i<sub>3</sub>) coincide,  $\rho$  is a Lévy measure,*

$$\int \|x\| \mathbf{1}(\|x\| \leq 1) \rho(dx) < +\infty,$$

*and*

$$S_n \xrightarrow{\mathcal{L}} c_0\text{-Pois}(\rho).$$

(iii) If (4.4) and (4.11) are true then the following conditions are equivalent:

- (iii<sub>1</sub>) The sequence  $\{S_n - ES_n; n \in N\}$  is convergent in distribution.
- (i<sub>2</sub>) holds.
- (i<sub>3</sub>) holds.

If one of these conditions is satisfied, then the measures in (i<sub>2</sub>) and (i<sub>3</sub>) coincide,  $\rho$  is a Lévy measure,

$$\int \|x\| 1(\|x\| > 1) \rho(dx) < +\infty,$$

and

$$S_n - ES_n \xrightarrow{\mathcal{D}} c_{+\infty}\text{-Pois}(\rho).$$

In (i<sub>2</sub>) and (i<sub>3</sub>) such  $\varphi$  and  $v^\varphi$  always exist.

*Proof.* The equivalence of (i<sub>2</sub>) and (i<sub>3</sub>) in all the cases is true according to the remark at the end of the proof of Theorem 4.11. Theorem 4.11 implies the implications (i<sub>2</sub>)  $\Rightarrow$  (i<sub>1</sub>), (i<sub>2</sub>)  $\Rightarrow$  (ii<sub>1</sub>), (i<sub>2</sub>)  $\Rightarrow$  (iii<sub>1</sub>). Finally, if centered sums converge, then all the accumulation points are equal and (i<sub>2</sub>) is true in all three cases. ■

**COROLLARY 4.15.** Let  $\{X_{n,k}; k \in Z, n \in N\}$  and  $\{k_n; n \in N\}$  be as in (4.1) and (4.2). Suppose that  $X_{n,0}$  takes values in the finite set  $A = \{a_0 = 0, a_1, \dots, a_p\} \subset R^d$ ,  $p \in N$ . Assume that

$$\{k_n P(X_{n,0} \neq 0); n \in N\} \quad \text{is a bounded set.} \quad (4.29)$$

The following conditions are equivalent:

- (i)  $S_n \xrightarrow{\mathcal{D}} c_0\text{-Pois}(\rho)$ .
- (ii) There exist the limits

$$\bar{\rho}(s) = \lim_{n \rightarrow +\infty} k_n P\left(\sum_{j=-m}^0 X_{n,j} = s, X_{n,0} \neq 0, X_{n,1} = 0, \dots, X_{n,m} = 0\right) \quad (4.30)$$

for every  $s \in V = \{\sum_{k=0}^m a_{i_k}; a_{i_k} \in \{a_0, \dots, a_p\}, \sum_{k=0}^m a_{i_k} \neq 0\}$ .

- (iii) There exist the limits

$$\bar{\rho}(s) = \lim_{n \rightarrow +\infty} k_n \left[ P\left(\sum_{i=0}^m X_{n,i} = s\right) - P\left(\sum_{i=1}^m X_{n,i} = s\right) \right] \quad (4.31)$$

for every  $s \in V$  defined in (ii).

If one of these conditions is true, then for every  $s \in V$ ,

$$\rho(s) = \bar{\rho}(s) = \tilde{\rho}(s) \quad (4.32)$$

and

$$S_n \xrightarrow{\mathcal{Q}} c_0\text{-Pois}(\rho).$$

*Proof.* The condition (4.29) implies infinitesimality of the array. We can use Corollary 4.14(ii), since if  $\varepsilon = \min\{\|a_i\|; i = 1, \dots, p\}$ , then  $\varepsilon > 0$  and

$$k_n E[\|X_{n,0}\| 1(\|X_{n,0}\| < \varepsilon)] = 0,$$

and hence (4.9) holds. Moreover, if  $\varphi: N \rightarrow N$ ,  $\varphi(n) \rightarrow +\infty$  as  $n \rightarrow +\infty$ , and

$$k_{\varphi(n)} P((X_{\varphi(n),0}, \dots, X_{\varphi(n),m}) = (a_{i_0}, \dots, a_{i_m})) \rightarrow v^\varphi(a_{i_0}, \dots, a_{i_m}) < +\infty$$

for every  $(a_{i_0}, \dots, a_{i_m}) \in A^{\{0, \dots, m\}} \setminus \{(0, \dots, 0)\}$ , then  $v_\eta^\varphi = v_0^\varphi$  if only  $0 < \eta \leq \varepsilon$  (compare the proof of Theorem 4.2). Hence in this case

$$v_0^\varphi(s) = \lim_{n \rightarrow +\infty} k_{\varphi(n)} \times P\left(\sum_{j=-m}^0 X_{\varphi(n),j} = s, X_{\varphi(n),0} \neq 0, X_{\varphi(n),1} = 0, \dots, X_{\varphi(n),m} = 0\right)$$

and also

$$v_{x_0 + \dots + x_m}^\varphi(s) = \lim_{n \rightarrow +\infty} k_{\varphi(n)} P\left(\sum_{j=0}^m X_{n,j} = s\right)$$

$$v_{x_1 + \dots + x_m}^\varphi(s) = \lim_{n \rightarrow +\infty} k_{\varphi(n)} P\left(\sum_{j=1}^m X_{n,j} = s\right)$$

for  $s \in V$ .

We use Corollary 4.14(ii) and Theorem 4.11. The existence of limits in (ii), (iii) can be deduced by the subsequence argument. ■

Corollary 4.15 obtains a particularly simple form of Theorem 1.1, when we consider an array of Bernoulli random variables.

*Proof of Theorem 1.1.* Each of the conditions (i)–(iv) implies the boundedness of the set  $\{k_n P(X_{n,0} = 1); n \in N\}$  and hence infinitesimality. In the case (i) compare Lemma 2 of [10]; (ii) is trivial; in (iv),

$$\lim_{n \rightarrow +\infty} k_n P(X_{n,0} = 1) = a(1) + a(2) + \dots + a(m+1) < +\infty.$$

If (iii) is satisfied, then since

$$\begin{aligned} P\left(\sum_{j=0}^m X_{n,j} = s\right) - P\left(\sum_{j=1}^m X_{n,j} = s\right) \\ = P\left(X_{n,0} = 1, \sum_{j=1}^m X_{n,j} = s-1\right) - P\left(X_{n,0} = 1, \sum_{j=1}^m X_{n,j} = s\right) \end{aligned}$$

for  $s = 1, \dots, m$  and

$$\begin{aligned} P\left(\sum_{j=0}^m X_{n,j} = m+1\right) - P\left(\sum_{j=1}^m X_{n,j} = m+1\right) \\ = P(X_{n,j} = 1, j = 0, \dots, m), \end{aligned}$$

hence (iv) also is fulfilled for every  $s = 1, \dots, m+1$ . We have

$$\begin{aligned} a(m+1) &= \bar{\rho}(m+1) \\ a(s) &= \bar{\rho}(s) + \bar{\rho}(s+1) + \dots + \bar{\rho}(m+1), \quad s = 1, \dots, m. \end{aligned}$$

The above remark shows also that (iii)  $\Rightarrow$  (iv). The implication (iv)  $\Rightarrow$  (iii) is implied by the equations

$$\begin{aligned} \bar{\rho}(m+1) &= a(m+1) \\ \bar{\rho}(s) &= a(s) - a(s+1), \quad s = 1, \dots, m. \end{aligned}$$

The equivalence of (i), (ii), and (iii) is just the consequence of Corollary 4.15. Moreover, let us note that

$$\begin{aligned} c_0\text{-Pois}(\rho) &= c_0\text{-Pois}\left(\sum_{s=1}^{m+1} \rho(s) \delta_s\right) \\ &= c_0\text{-Pois}(\rho(1) \delta_1) * \dots * c_0\text{-Pois}(\rho(m+1) \delta_{m+1}). \end{aligned}$$

Thus  $S_n \xrightarrow{\mathcal{L}} c_0\text{-Pois}(\lambda \delta_1)$  iff  $\rho(1) = \lambda$  and  $\rho(j) = 0$  for  $j = 2, \dots, m+1$ . This means that

$$\begin{aligned} \rho(1) = a(1) &= \lim_{n \rightarrow +\infty} k_n P\left(X_{n,0} = 1, \sum_{j=1}^m X_{n,j} = 0\right) \\ &= \lim_{n \rightarrow +\infty} k_n P(X_{n,0} = 1) \end{aligned}$$

and hence

$$k_n P(X_{n,0} = 1, X_{n,j} = 1) \rightarrow 0 \quad \text{for } j = 1, \dots, m.$$

Finally, these last convergences together with the condition

$$\lim_{n \rightarrow +\infty} k_n P(X_{n,0} = 1) = \lambda$$

imply

$$\rho(j) = 0 \quad \text{for } j = 2, \dots, m+1. \quad \blacksquare$$

*Remark 4.16.* Theorem 1.1,  $A(i) \Leftrightarrow A(ii)$ , differs apparently from Theorem 2 in [10], but it is not important whether we consider finite or infinite sequences in rows; moreover, in the proof of Theorem 4.2 we can instead of  $X_{n,k}^{**}$  consider

$$\begin{aligned} X_{n,k}^{***} &= \sum_{j=1}^m X_{n,k+j} 1(\|X_{n,k+j}\| > \eta) 1(\|X_{n,k}\| > \eta) \\ &\quad \times 1(\|X_{n,k-j}\| \leq \eta, j = 1, \dots, m) \end{aligned}$$

and repeat all the proofs. Hence Theorem 1.1 covers exactly both theorems in [10].  $\blacksquare$

One remark should still be made. Theorem 4.11 is similar in some sense to a special case of Theorem 4.1 by Chen [7], if one considers in particular arrays strictly stationary and  $m$ -dependent in rows. Chen gives necessary and sufficient conditions for the convergence of sums to infinitely divisible distributions possessing the Kolmogorov representation and assumes the existence of finite variance for  $X_{n,k}$ ; here we consider the analogical problem but with general Poisson laws as limits. However, in both cases one can obtain as a corollary Theorem 1.1,  $A(i) \Leftrightarrow A(ii)$ , and B.

## 5. SUMS FOR STRICTLY STATIONARY AND STRONGLY MIXING ARRAYS

Now we consider an array  $\{X_{n,k}; k \in \mathbb{Z}, n \in \mathbb{N}\}$  of random vectors with values in  $R^d$ , which is strictly stationary and  $\alpha$ -,  $\rho$ -, or  $\phi$ -mixing in rows.  $\{X_{n,k}\}$  is

$\alpha$ -mixing if  $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \alpha_n(k) = 0$ , where

$$\alpha_n(k) = \sup \left\{ |P(A \cap B) - P(A)P(B)|; \begin{array}{l} A \in \sigma(\dots, X_{n,-1}, X_{n,0}), \\ B \in \sigma(X_{n,k}, X_{n,k+1}, \dots) \end{array} \right\},$$

$\rho$ -mixing if  $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \rho_n(k) = 0$ , where

$$\rho_n(k) = \sup \left\{ \frac{|E(f \cdot g) - EfEg|}{(Ef^2)^{1/2}(Eg^2)^{1/2}}; 0 < Ef^2, Eg^2 < \infty, \right. \\ \left. \begin{array}{l} f: \Omega \rightarrow R \text{ measurable with respect to } \sigma(\dots, X_{n-1}, X_{n,0}) \\ g: \Omega \rightarrow R \text{ measurable with respect to } \sigma(X_{n,k}, X_{n,k+1}, \dots) \end{array} \right\};$$

$\phi$ -mixing if  $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \phi_n(k) = 0$ , where

$$\phi_n(k) = \sup \{ |P(B/A) - P(B)|; \\ A \in \sigma(\dots, X_{n-1}, X_{n,0}), B \in \sigma(X_{n,k}, X_{n,k+1}, \dots) \}.$$

When  $n$  is fixed the sequences  $\alpha_n(\cdot)$ ,  $\rho_n(\cdot)$ ,  $\phi_n(\cdot)$  are non-negative and nonincreasing.  $\phi$ -mixing implies  $\rho$ -mixing, and  $\rho$ -mixing is stronger than  $\alpha$ -mixing. Moreover, if we assume the infinitesimality of the array, then the condition (2.3) is weaker than  $\alpha$ -mixing. Let us note that if  $\{X_{n,k}\}$  is  $\alpha$ -mixing then for every sequence  $\{m_n\}$  of natural numbers such that  $\lim_{n \rightarrow \infty} m_n = +\infty$ ,  $\limsup_{n \rightarrow \infty} \alpha_n(m_n) = 0$ .

In the following we consider

- an array  $\{X_{n,k}; k \in \mathbb{Z}, n \in \mathbb{N}\}$  of random vectors with values in  $R^d$  strictly stationary in rows

(5.1)

- the sequence  $\{k_n; n \in \mathbb{N}\}$  of natural numbers such that  $\lim_{n \rightarrow \infty} k_n = +\infty$ .

(5.2)

We will use the following conditions

$$\forall \varepsilon > 0 \quad \exists M_\varepsilon < +\infty \quad \forall n \in \mathbb{N} \quad k_n P(\|X_{n,0}\| > \varepsilon) < M_\varepsilon \quad (5.3)$$

$$\exists m \in \mathbb{N} \cup \{0\} \quad \forall j > m \quad k_n P(\|X_{n,0}\| > \varepsilon, \|X_{n,j}\| > \varepsilon) \xrightarrow{n \rightarrow +\infty} 0. \quad (5.4)$$

We use the following convention: if  $\mu$  is a Lévy measure on  $E^{d(m+1)}$ ,  $m \in \mathbb{N} \cup \{0\}$ , then

$$\mu_0 = \begin{cases} \mu_{x_0 + \dots + x_m} & \text{if } m \geq 1 \\ \mu & \text{if } m = 0. \end{cases}$$



THEOREM 5.1. Let  $\{X_{n,k}\}$  and  $\{k_n\}$  be as in (5.1) and (5.2). Assume that (5.4) holds for certain  $m \in N \cup \{0\}$  and there exists a measure  $\nu$  on  $E^{d(m+1)}$ , finite outside every neighbourhood of 0, such that for every  $\delta > 0$ ,

$$k_n P((X_{n,0}, \dots, X_{n,m}) \in (\cdot))|_{\{\|x\| > \delta\}} \xrightarrow{w} \nu(\cdot)|_{\{\|x\| > \delta\}}, \quad n \rightarrow +\infty. \quad (5.5)$$

(I) Let  $\{X_{n,k}\}$  be  $\phi$ -mixing.

(i) If (4.4) holds and  $\limsup_{n \rightarrow \infty} \sum_{k=1}^{\infty} \rho_n(2^k) < +\infty$ , then  $\nu$  and  $\nu_0$  are Lévy measures and

$$S_n - b_n(\tau) \xrightarrow{\mathcal{L}} c_\tau\text{-Pois}(\nu_0)$$

for  $0 < \tau < +\infty$  such that  $\nu_0(\|x\| = \tau) = 0$ , where  $b_n(\tau)$  are defined by (4.7).

(ii) If (4.9) holds then  $\nu$  and  $\nu_0$  are Lévy measures,  $\int \|x\| 1(\|x\| \leq 1) \nu_0(dx) < +\infty$  and

$$S_n \xrightarrow{\mathcal{L}} c_0\text{-Pois}(\nu_0).$$

(iii) If (4.4), (4.11) hold and  $\limsup_{n \rightarrow \infty} \sum_{k=1}^{\infty} \rho_n(2^k) < +\infty$ , then  $\nu$  and  $\nu_0$  are Lévy measures  $\int \|x\| 1(\|x\| > 1) \nu_0(dx) < +\infty$  and

$$S_n - ES_n \xrightarrow{\mathcal{L}} c_\infty\text{-Pois}(\nu_0).$$

(II) Let  $\{X_{n,k}\}$  be  $\rho$ -mixing and there exists a sequence  $\{m_n; n \in N\}$  of natural numbers such that  $m_n \rightarrow \infty$ ,  $m_n = o(k_n)$ ,  $k_n^{1/2} \rho_n(m_n) \rightarrow 0$ , and for every  $\varepsilon > 0$

$$\sum_{j=m+1}^{m_n} k_n P(\|X_{n,0}\| > \varepsilon, \|X_{n,j}\| > \varepsilon) \xrightarrow{n \rightarrow +\infty} 0. \quad (5.6)$$

Then the thesis of (I) is true.

(III) Let  $\{X_{n,k}\}$  be  $\alpha$ -mixing and there exists a sequence  $\{m_n; n \in N\}$  of natural numbers such that  $m_n \rightarrow \infty$ ,  $m_n = o(k_n)$ ,  $k_n \alpha_n(m_n) \rightarrow 0$ , and for every  $\varepsilon > 0$ ,

$$\sum_{j=m+1}^{m_n} k_n P(\|X_{n,0}\| > \varepsilon, \|X_{n,j}\| > \varepsilon) \xrightarrow{n \rightarrow +\infty} 0. \quad (5.7)$$

If (4.9) holds then  $\nu$  and  $\nu_0$  are Lévy measures,  $\int \|x\| 1(\|x\| \leq 1) \nu_0(dx) < +\infty$ , and

$$S_n \xrightarrow{\mathcal{L}} c_0\text{-Pois}(\nu_0).$$

*Proof.* The proof is a modification of the proof of Theorem 4.2. We give the sketch of it and the estimations in all three cases simultaneously.

Let  $\eta > 0$  and  $\{m_n; n \in N\}$  be a sequence of natural numbers such that

$$\lim_{n \rightarrow \infty} m_n = +\infty, \quad m_n = o(k_n),$$

$$k_n \sum_{j=m+1}^{m_n} P(\|X_{n,0}\| > \eta, \|X_{n,j}\| > \eta) \xrightarrow{n \rightarrow +\infty} 0$$

(in the case of  $\rho$ - and  $\alpha$ -mixing  $(m_n)$  is given in the assumptions). Define

$$X_{n,k}^* = X_{n,k} 1(B_{n,k})$$

$$B_{n,k} = \{\|X_{n,k}\| > \eta\} \cap \left[ \bigcup_{i=k}^{k+m_n} \left( \{\|X_{n,i}\| > \eta\} \cap \bigcap_{j=1}^{m_n} \{\|X_{n,i+j}\| \leq \eta\} \right) \right]$$

$$S_n^* = \sum_{k=1}^{k_n} X_{n,k}^*$$

$$S_n ] \eta, +\infty [ = \sum_{k=1}^{k_n} X_{n,k} 1(\|X_{n,k}\| > \eta).$$

LEMMA 5.2.

$$P(S_n^* \neq S_n ] \eta, +\infty [)$$

$$\begin{aligned} &\leq k_n P\left(\{\|X_{n,0}\| > \eta\} \cap \bigcup_{j=1}^{m_n} \{\|X_{n,m_n+j}\| > \eta\}\right) \\ &\leq \begin{cases} k_n P(\|X_{n,0}\| > \eta) \phi_n(m_n) + k_n m_n P^2(\|X_{n,0}\| > \eta) \\ k_n (m_n)^{1/2} P(\|X_{n,0}\| > \eta) \rho_n(m_n) + k_n m_n P^2(\|X_{n,0}\| > \eta). \\ k_n \alpha_n(m_n) + k_n m_n P^2(\|X_{n,0}\| > \eta) \end{cases} \end{aligned}$$

We define

$$\begin{aligned} X_{n,k}^{**} &= \sum_{j=0}^{m_n} X_{n,k-j} 1(\|X_{n,k-j}\| > \eta) \\ &\quad \times 1(\|X_{n,k}\| > \eta) \times 1(\|X_{n,k+i}\| \leq \eta, i=1, \dots, m_n) \\ S_n^{**} &= \sum_{k=1}^{k_n} X_{n,k}^{**}. \end{aligned}$$

LEMMA 5.3.

$$P(S_n^{**} \neq S_n^*) \leq 2m_n P(\|X_{n,0}\| > \eta) \xrightarrow{n \rightarrow +\infty} 0.$$

LEMMA 5.4. *The array  $\{X_{n,k}^{**}; k \in \mathbb{Z}, n \in \mathbb{N}\}$  and the sequence  $\{k_n; n \in \mathbb{N}\}$  satisfy the assumptions of Theorem 2.1(i), namely*

(i)  $\{X_{n,k}^{**}\}$  is stationary and in all the three cases (2.3), (2.2), and (2.4) are fulfilled.

(ii)

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow +\infty} k_n E(\|X_{n,0}^{**}\| \mathbf{1}(\|X_{n,0}^{**}\| \leq \delta)) = 0.$$

(iii) *For every  $A \subset E^d$ , such that  $0 \notin \bar{A}$ ,  $v_{\sum_{k \in K} x_k}(\partial A) = 0$ , and every  $K \subset \{0, \dots, m\}$ , there holds the convergence*

$$\begin{aligned} k_n P(X_{n,0}^{**} \in A) &\xrightarrow{n \rightarrow +\infty} v_\eta(A) \\ &= \sum_{J \subset \{1, \dots, m\}} \sum_{\emptyset \neq K \subset J \cup \{0\}} (-1)^{|J|+1-|K|} \\ &\quad \times v\left(\left\{(x_0, \dots, x_m); \sum_{k \in K} x_k \in A, \|x_j\| > \eta, j \in J \cup \{0\}\right\}\right). \end{aligned}$$

*Proof.* Stationarity and (2.2) are obvious. There holds

$$\sigma(X_{n,k}^{**}) \subset \sigma(X_{n,k-m_n}, \dots, X_{n,k}, \dots, X_{n,k+m_n}).$$

If we define the mixing coefficients  $\phi_n^{**}$ ,  $\rho_n^{**}$ ,  $\alpha_n^{**}$  for  $\{X_{n,k}^{**}\}$  analogically as at the beginning of Section 5, then

$$\begin{aligned} \phi_n^{**}(s+2m_n) &\leq \phi_n(s), \\ \rho_n^{**}(s+2m_n) &\leq \rho_n(s), \\ \alpha_n^{**}(s+2m_n) &\leq \alpha_n(s), \end{aligned}$$

hence in each case (2.3) holds.

To obtain (2.4) let us note

$$\begin{aligned} r \sum_{i=1}^{\lfloor k_n/r \rfloor + r - 1} P\left(\left\{\|X_{n,i}^{**}\| > \varepsilon\right\} \cap \bigcup_{j=i+1}^{\lfloor k_n/r \rfloor + r} \left\{\|X_{n,j}^{**}\| > \varepsilon\right\}\right) \\ \leq r \sum_{i=1}^{\lfloor k_n/r \rfloor + r - m_n - 1} P\left(\left\{\|X_{n,i}^{**}\| > \varepsilon\right\} \cap \bigcup_{j=i+m_n+1}^{\lfloor k_n/r \rfloor + r} \left\{\|X_{n,j}^{**}\| > \varepsilon\right\}\right) \end{aligned}$$

$$\begin{aligned}
&\leq r \sum_{i=1}^{[k_n/r] + r - m_n - 1} P \left( \{ \|X_{n,i}\| > \eta \} \cap \bigcup_{j=i+m_n+1}^{[k_n/r] + r} \{ \|X_{n,j}\| > \eta \} \right) \\
&\leq \begin{cases} 2k_n P(\|X_{n,0}\| > \eta) \phi_n(m_n) + 2k_n [k_n/r] P^2(\|X_{n,0}\| > \eta) \\ 2k_n P(\|X_{n,0}\| > \eta) [k_n/r]^{1/2} \rho_n(m_n) + 2k_n [k_n/r] P^2(\|X_{n,0}\| > \eta) \\ 2k_n \alpha_n(m_n) + 2k_n [k_n/r] P^2(\|X_{n,0}\| > \eta). \end{cases}
\end{aligned}$$

To show (ii) we use the inequality

$$k_n E[\|X_{n,0}^{**}\| 1(\|X_{n,0}^{**}\| \leq \delta)] \leq \delta k_n P(\|X_{n,0}\| > \eta).$$

To check (iii) let us introduce the following notation:

$$\begin{aligned}
X_{n,k}^{** (s)} &= \sum_{j=0}^s X_{n,k-j} 1(\|X_{n,k-j}\| > \eta) \\
&\quad \times 1(\|X_{n,k}\| > \eta) \times 1(\|X_{n,k+i}\| > \eta, i = 1, \dots, s).
\end{aligned}$$

Let  $A \subset E^d$  be such that  $v_{\sum_{k \in K} X_k}(\partial A) = 0$  for every  $K \subset \{0, \dots, m\}$ . As in Chapter 4, one can show that

$$k_n P(X_{n,k}^{** (m)} \in A) \xrightarrow{n \rightarrow +\infty} v_\eta(A)$$

using the assumption (5.4) instead of  $m$ -dependence (see Lemma 4.5). All the properties of  $v_\eta$  from the proof of Theorem 4.2 are true. We show that

$$k_n P(X_{n,k}^{** (m_n)} \in A) \xrightarrow{n \rightarrow +\infty} v_\eta(A).$$

We have

$$\begin{aligned}
&|P(X_{n,0}^{** (s+1)} \in A) - P(X_{n,0}^{** (s)} \in A)| \\
&= \left| P \left( \{ \|X_{n,0}\| > \eta \} \cap \bigcap_{i=1}^{s+1} \{ \|X_{n,i}\| \leq \eta \} \cap \left\{ \sum_{j=0}^{s+1} X_{n,-j} 1(\|X_{n,-j}\| > \eta) \in A \right\} \right) \right. \\
&\quad \left. - P \left( \{ \|X_{n,0}\| > \eta \} \cap \bigcap_{i=1}^s \{ \|X_{n,i}\| \leq \eta \} \cap \left\{ \sum_{j=0}^s X_{n,-j} 1(\|X_{n,-j}\| > \eta) \in A \right\} \right) \right|
\end{aligned}$$

$$\begin{aligned}
&= \left| P \left( \{ \|X_{n,0}\| > \eta \} \cap \bigcap_{i=1}^{s+1} \{ \|X_{n,i}\| \leq \eta \} \right. \right. \\
&\quad \left. \cap \left\{ \sum_{j=0}^{s+1} X_{n,-j} 1(\|X_{n,-j}\| > \eta) \in A \right\} \cap \{ \|X_{n,-s-1}\| > \eta \} \right) \\
&\quad + P \left( \{ \|X_{n,0}\| > \eta \} \cap \bigcap_{i=1}^{s+1} \{ \|X_{n,i}\| \leq \eta \} \right. \\
&\quad \left. \cap \left\{ \sum_{j=0}^s X_{n,-j} 1(\|X_{n,-j}\| > \eta) \in A \right\} \cap \{ \|X_{n,-s-1}\| \leq \eta \} \right) \\
&\quad - P \left( \{ \|X_{n,0}\| > \eta \} \cap \bigcap_{i=1}^s \{ \|X_{n,i}\| \leq \eta \} \right. \\
&\quad \left. \cap \left\{ \sum_{j=0}^s X_{n,-j} 1(\|X_{n,-j}\| > \eta) \in A \right\} \cap \{ \|X_{n,s+1}\| > \eta \} \right) \\
&\quad - P \left( \{ \|X_{n,0}\| > \eta \} \cap \bigcap_{i=1}^s \{ \|X_{n,i}\| \leq \eta \} \right. \\
&\quad \left. \cap \left\{ \sum_{j=0}^s X_{n,-j} 1(\|X_{n,-j}\| > \eta) \in A \right\} \cap \{ \|X_{n,s+1}\| \leq \eta \} \right) \Big| \\
&= |\beta_n(1) + \beta_n(2) - \beta_n(3) - \beta_n(4)| \\
&\leq |\beta_n(2) - \beta_n(4)| + \beta_n(1) + \beta_n(3).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\beta_n(1) &\leq P(\|X_{n,0}\| > \eta, \|X_{n,-s-1}\| > \eta) \\
\beta_n(3) &\leq P(\|X_{n,0}\| > \eta, \|X_{n,s+1}\| > \eta) \\
0 &\leq \beta_n(4) - \beta_n(2) \leq P(\|X_{n,0}\| > \eta, \|X_{n,-s-1}\| > \eta).
\end{aligned}$$

Thus

$$\begin{aligned}
&|k_n [P(X_{n,k}^{** (m_n)} \in A) - P(X_{n,k}^{** (m)} \in A)]| \\
&\leq \sum_{j=m+1}^{m_n} k_n |P(X_{n,k}^{** (j)} \in A) - P(X_{n,k}^{** (j-1)} \in A)| \\
&\leq 3 \sum_{j=m+1}^{m_n} k_n P(\|X_{n,0}\| > \eta, \|X_{n,j}\| > \eta)
\end{aligned}$$

and the last term tends to 0 due to the properties of  $(m_n)$ . ■

COROLLARY 5.5.  $\int \|x\| 1(\|x\| \leq 1) v_\eta(dx) < +\infty$  and

$$S_n ]\eta, +\infty[ \xrightarrow{\mathcal{L}} c_0\text{-Pois}(v_\eta).$$

*Proof.* We use Theorem 2.1. To obtain the full proof of Theorem 5.1 we still need Lemma 4.7 and Lemma 4.8 without any changes and instead of Lemma 4.9 we need the finite-dimensional version of Lemma 3.4 of [17].

$$E \|S_n ]0, \delta] - ES_n ]0, \delta]\|^2 \leq Kk_n E(\|X_{n,0}\|^2 1(\|X_{n,0}\| \leq \delta)).$$

The other arguments are exactly the same as in the proof of Theorem 4.2. ■

It is possible to deduce the analogs of Theorem 4.11 and Corollary 4.14.

COROLLARY 5.6. Let  $\{X_{n,k}; k \in \mathbb{Z}, n \in \mathbb{N}\}$  and  $\{k_n; n \in \mathbb{N}\}$  satisfy (5.1) and (5.2). Suppose that (5.3) and (5.4) hold for some  $m \in \mathbb{N} \cup \{0\}$ .

(I) Assume that one of the following conditions holds:

- (1)  $\{X_{n,k}\}$  is  $\phi$ -mixing.
- (2)  $\{X_{n,k}\}$  is  $\rho$ -mixing and (5.6) is fulfilled.

Then

(i) If  $\limsup_{n \rightarrow \infty} \sum_{k=1}^{\infty} \rho_n(2^k) < +\infty$  and (4.4) holds then (i<sub>1</sub>), (i<sub>2</sub>), (i<sub>3</sub>) of Corollary 4.14 are equivalent.

(ii) If (4.9) holds, then (ii<sub>1</sub>), (ii<sub>2</sub>), (ii<sub>3</sub>) of Corollary 4.14 are equivalent.

(iii) If  $\limsup_{n \rightarrow \infty} \sum_{k=1}^{\infty} \rho_n(2^k) < +\infty$  and (4.4), (4.11) hold then (iii<sub>1</sub>), (iii<sub>2</sub>), (iii<sub>3</sub>) of Corollary 4.14 are equivalent.

(II) Assume that  $\{X_{n,k}\}$  is  $\alpha$ -mixing and (5.7) holds. If (4.9) is true, then (ii<sub>1</sub>), (ii<sub>2</sub>), (ii<sub>3</sub>) of Corollary 4.14 are equivalent.

At the end of this chapter let us have a look at the case of arrays generated by a strictly stationary sequence.

COROLLARY 5.7. Let  $\{X_k; k \in \mathbb{Z}\}$  be a strictly stationary sequence such that  $\mathcal{L}(X_0)$  belongs to the domain of attraction of a  $d$ -dimensional  $\alpha$ -stable law  $c_\tau\text{-Pois}(\mu_\alpha)$ ,  $0 < \alpha < 2$ . Let  $\{a_n; n \in \mathbb{N}\}$  be a norming sequence for the sequence of  $\{\tilde{X}_k; k \in \mathbb{Z}\}$  of i.i.d. random vectors with  $\mathcal{L}(\tilde{X}_0) = \mathcal{L}(X_0)$  such that the sums  $\sum_{k=1}^n \tilde{X}_k/a_n$  suitably centered tend in distribution to  $c_\tau\text{-Pois}(\mu_\alpha)$ .

Assume that

$$\forall_{\varepsilon > 0} \forall_{j \geq 1} k_n P(\|X_0\| > a_n \varepsilon, \|X_j\| > a_n \varepsilon) \xrightarrow{n \rightarrow +\infty} 0. \quad (5.8)$$

(i) Let  $0 < \alpha < 1$ . If one of the conditions

- (1)  $\{X_k; k \in \mathbb{Z}\}$  is  $\phi$ -mixing,
- (2)  $\{X_n; k \in \mathbb{Z}\}$  is  $\rho$ -mixing and (5.6) holds for  $m=0$ , or
- (3)  $\{X_n; k \in \mathbb{Z}\}$  is  $\alpha$ -mixing and (5.7) holds for  $m=0$

is satisfied then

$$\sum_{k=1}^n X_k/a_n \xrightarrow{\mathcal{D}} c_0\text{-Pois}(\mu_\alpha). \quad (5.9)$$

(ii) Let  $1 \leq \alpha < 2$ . If one of the conditions

- (1)  $\{X_k; k \in \mathbb{Z}\}$  is  $\phi$ -mixing and  $\sum_{k=1}^{\infty} \rho(2^k) < +\infty$ , or
- (2)  $\{X_k; k \in \mathbb{Z}\}$  is  $\rho$ -mixing,  $\sum_{k=1}^{\infty} \rho(2^k) < +\infty$ , and (5.6) holds for  $m=0$ ,

is satisfied, then

$$\sum_{k=1}^n X_k/a_n - b_n(\tau) \xrightarrow{\mathcal{D}} c_\tau\text{-Pois}(\mu_\alpha) \quad (5.10)$$

where  $b_n(\tau)$  is given by (4.7).

If  $1 < \alpha < 2$ , then by the same assumptions (ii)

$$\sum_{k=1}^n (X_k - EX_k)/a_n \xrightarrow{\mathcal{D}} c_\infty\text{-Pois}(\mu_\alpha). \quad (5.11)$$

**Remark 5.8.** A comparison of Corollary 5.7 with Theorems 4.1 and 4.2 of [12] shows that the reduction described in Chapters 4 and 5 allows to relax the condition (D') at the cost of the asymptotic independence condition (B). An especially simple form obtains the theorem in the case of  $\phi$ -mixing, which seems to be interesting in the context of the question which has remained unsolved for many years, that of whether a stationary  $\phi$ -mixing sequence  $\{X_n\}$  with  $EX_0=0$  and  $\text{Var } X_0=1$  satisfies a central limit theorem (e.g. [18–20]).

The marginal distribution must belong to the domain of attraction of an  $\alpha$ -stable distribution,  $0 < \alpha \leq 2$ . Four natural cases can be enumerated:  $0 < \alpha < 1$ ,  $\alpha = 1$ ,  $1 < \alpha < 2$ , and  $\alpha = 2$ . For  $\alpha = 2$  the central limit question has many partial answers [5, 19] which require e.g. some more assumptions on the moments or the rate of convergence of  $\phi(n)$ . The cases

$0 < \alpha < 2$  were considered in [8, 11, 12, 14, 23]. As Corollary 5.7 shows, the simplest situation is when  $0 < \alpha < 1$  and the sums need not be centered—then the rate of  $\phi$ -mixing can be undetermined and the very weak condition (D') in the form (5.8) is sufficient. In the case  $1 < \alpha < 2$  the sums may be centered by their respective expectations, but our method delivers a limit theorem under (5.8) and special rate of convergence ( $\sum_{n=1}^{\infty} \rho(2^n) < +\infty$ ). The most awkward case is  $\alpha = 1$  since the proper centering is more complicated, while the weak convergence of laws requires the same assumptions as for  $1 < \alpha < 2$ .

A composition of our Corollary 5.7 and Theorem 3.2 of [23] leads to the following functional limit theorem.

**COROLLARY 5.9.** *Let  $\{X_k; k \in \mathbb{Z}\}$  be a strictly stationary sequence of random vectors with the marginal distribution belonging to the domain of attraction of an  $\alpha$ -stable law  $c_{\gamma(\alpha)}\text{-Pois}(\mu_\alpha)$ . Let  $W_\alpha$  denote a homogeneous process with independent increments on  $D[0, 1]$  generated by  $c_{\gamma(\alpha)}\text{-Pois}(\mu_\alpha)$ . We consider a sequence of positive reals  $\{a_n; n \in \mathbb{N}\}$  and a sequence  $\{b_n(\alpha); n \in \mathbb{N}\}$  of vectors in  $\mathbb{R}^d$  such that*

$$\sum_{k=1}^n (\hat{X}_k - b_n(\alpha))/a_n \xrightarrow{\mathcal{L}} c_{\gamma(\alpha)}\text{-Pois}(\mu_\alpha), \quad (5.12)$$

where  $\{\hat{X}_k; k \in \mathbb{Z}\}$  is an i.i.d. sequence of random vectors with  $\mathcal{L}(\hat{X}_0) = \mathcal{L}(X_0)$ .

Define

$$S_n(t) = \sum_{k=1}^{[nt]} (X_k - b_n(\alpha))/a_n.$$

If one of the conditions

- (i)  $0 < \alpha < 1$ ,  $\{X_n; n \in \mathbb{Z}\}$  is  $\phi$ -mixing, or
- (ii)  $1 \leq \alpha < 2$ ,  $\{X_n; n \in \mathbb{Z}\}$  is  $\phi$ -mixing, and additionally

$$\sum_{k=1}^{\infty} \rho(2^k) < +\infty,$$

is satisfied, then there holds the equivalence

$$S_n(\cdot) \xrightarrow{\mathcal{L}} W_\alpha$$

iff

$$\bigvee_{j \in \mathbb{N}} \bigvee_{\varepsilon > 0} nP(\|X_0\| > a_n \varepsilon, \|X_j\| > a_n \varepsilon) \rightarrow 0. \quad (5.13)$$



*Proof.* It is well known that the sequence  $\{a_n\}$  is  $(1/\alpha)$ -regularly varying (in the sense that the function  $f_{\{a_n\}}(x) = a_{[x]}$  is an  $(1/\alpha)$ -regularly varying function). Due to [4, Theorem 1.53, p. 23] there exists a sequence  $\{\tilde{a}_n; n \in N\}$  of positive reals which is nondecreasing and  $\lim_{n \rightarrow \infty} a_n/\tilde{a}_n = 1$ .

In the following we can assume that  $\{a_n\}$  is nondecreasing.

LEMMA 5.10. *Let  $X_0$  belong to the domain of attraction of an  $\alpha$ -stable distribution  $c_{\gamma(x)}\text{-Pois}(\mu_\alpha)$  and the sequences  $\{a_n; n \in N\}$  and  $\{b_n(\alpha); n \in N\}$  be as in Corollary 5.9;  $\{a_n\}$  is nondecreasing. Then*

- (a) (i)  $\lim_{n \rightarrow \infty} a_n = +\infty$   
 (ii) for every natural sequence  $\{r_n; n \in N\}$ ,  $r_n = o(n)$

$$\lim_{n \rightarrow \infty} \frac{a_{r_n}}{a_n} = 0, \quad (5.14)$$

- (b) there hold the following convergences:

- (i) if  $0 < \alpha < 1$  then there exists  $a \in R^d$  such that

$$\frac{nb_n(\alpha)}{a_n} \xrightarrow{n \rightarrow +\infty} a \quad (5.15)$$

- (ii) if  $1 < \alpha < 2$  then there exists  $a \in R^d$  such that

$$\frac{n(EX_0 - b_n(\alpha))}{a_n} \xrightarrow{n \rightarrow +\infty} a \quad (5.16)$$

- (iii) if  $\alpha = 1$  then there exists  $a \in R^d$  such that

$$\frac{n(EX_0 1(\|X_0\| \leq a_n) - b_n(\alpha))}{a_n} \xrightarrow{n \rightarrow +\infty} a \quad (5.17)$$

and for every natural sequence  $\{r_n; n \in N\}$ ,  $r_n = o(n)$

$$r_n \cdot E(X_0 1(a_{r_n} \leq \|X_0\| \leq a_n))/a_n \xrightarrow{n \rightarrow +\infty} 0. \quad (5.18)$$

*Proof of Lemma 5.10.* Since

$$a_n = n^{1/\alpha} L(n), \quad (5.19)$$

where  $L(n)$  is slowly varying, so by the Karamata representation

$$a_n = n^{1/\alpha} \exp \left[ \eta(n) + \int_{\beta}^n \frac{\varepsilon(t)}{t} dt \right], \quad (5.20)$$

where  $\eta(n)$  is a sequence convergent to some  $\eta \in R^+$ ,  $\beta > 0$ , and  $\varepsilon(\cdot)$  is a continuous function on  $[\beta, +\infty[$  such that  $\lim_{t \rightarrow \infty} \varepsilon(t) = 0$ .

The part (a) of Lemma 5.10 follows easily from (5.20).

To prove (b) let us consider a sequence  $\{\hat{X}_k; k \in Z\}$  of i.i.d. random vectors with  $\mathcal{L}(\hat{X}_0) = \mathcal{L}(X_0)$ .

We have

$$\sum_{k=1}^n (\hat{X}_k - \hat{b}_n(\alpha))/a_n \rightarrow c_{\tau(\alpha)}\text{-Pois}(\mu_\alpha), \quad (5.21)$$

where

$$\hat{b}_n(\alpha) = \begin{cases} 0 & 0 < \alpha < 1 \\ EX_0 1(\|X_0\| \leq a_n) & \alpha = 1 \\ EX_0 & 1 < \alpha < 2 \end{cases}$$

and

$$\tau(\alpha) = \begin{cases} 0 & 0 < \alpha < 1 \\ 1 & \alpha = 1 \\ +\infty & 1 < \alpha < 2. \end{cases}$$

Equations (5.12) and (5.21) give (5.15)–(5.17).

It remains to be shown that (5.18) holds. Let  $\alpha = 1$  and  $F$  be the distribution function of  $\|X_0\|$ . Let us note that

$$\begin{aligned} & \left| \frac{r_n}{a_n} E[\|X_0\| 1(a_{r_n} \leq \|X_0\| \leq a_n)] - \frac{r_n}{a_n} \int_{a_{r_n}}^{a_n} (1 - F(x)) dx \right| \\ & \leq \frac{r_n}{a_n} a_{r_n} P(a_{r_n} \leq \|X_0\| \leq a_n) + \frac{r_n}{a_n} (a_n - a_{r_n}) P(\|X_0\| > a_n) \\ & \leq \frac{a_{r_n}}{a_n} r_n P(\|X_0\| > a_{r_n}) + \frac{r_n}{n} n P(\|X_0\| > a_n), \end{aligned}$$

where  $r_n = o(n)$ ,  $\lim(a_{r_n}/a_n) = 0$  by (a)(ii), and  $\{nP(\|X_0\| > a_n)\}$  is bounded due to (5.12).

Thus (5.18) will be proved if we show that

$$\frac{r_n}{a_n} \int_{a_{r_n}}^{a_n} (1 - F(x)) dx \xrightarrow{n \rightarrow \infty} 0.$$

We have

$$\begin{aligned} \frac{r_n}{a_n} \int_{a_{r_n}}^{a_n} (1 - F(x)) dx &\leq \frac{r_n}{a_n} \sum_{k=r_n}^{n-1} (1 - F(a_k))(a_{k+1} - a_k) \\ &= \frac{r_n}{a_n} \sum_{k=r_n}^{n-1} [k(1 - F(a_k))](a_{k+1} - a_k) \frac{1}{k}. \end{aligned}$$

Since  $\{k(1 - F(a_k)); k \in N\}$  is a bounded sequence as  $\lim_{n \rightarrow \infty} nP(\|X_0\| > a_n) = \mu_x(\|x\| > 1) < +\infty$ , so due to (5.19) it is sufficient to show that

$$\lim_{n \rightarrow \infty} \frac{r_n}{nL(n)} \sum_{k=r_n}^{n-1} \left[ L(k+1) - L(k) + \frac{L(k+1)}{k} \right] = 0. \quad (5.22)$$

We have

$$\begin{aligned} \frac{r_n}{nL(n)} \sum_{k=r_n}^{n-1} \left[ L(k+1) - L(k) + \frac{L(k+1)}{k} \right] \\ = \frac{r_n}{n} - \frac{a_{r_n}}{a_n} + \frac{r_n}{n} \sum_{k=r_n}^{n-1} \frac{L(k+1)}{L(n)} \cdot \frac{1}{k}. \end{aligned} \quad (5.23)$$

Denote  $c_n = \exp \eta(n)$  and take  $t_0$  such that for  $t > t_0$   $|\varepsilon(t)| < \frac{1}{2}$ . For  $n$  satisfying  $r_n \geq t_0$  we have by (5.20)

$$\frac{r_n}{n} \sum_{k=r_n}^{n-1} \frac{L(k+1)}{L(n)} \cdot \frac{1}{k} = \frac{r_n}{n} \sum_{k=r_n}^{n-1} \frac{c_{k+1}}{c_n} \cdot \exp \left[ - \int_{(k+1)}^n \frac{\varepsilon(t)}{t} dt \right] \frac{1}{k}.$$

Since

$$\sup_{n,k} \frac{c_{k+1}}{c_n} \leq M_1,$$

for some  $M_1 > 0$ , thus for  $n$  large enough

$$\begin{aligned} \frac{r_n}{n} \sum_{k=r_n}^{n-1} \frac{L(k+1)}{L(n)} \cdot \frac{1}{k} &\leq M_1 \frac{r_n}{n} \sum_{k=r_n}^{n-1} \frac{1}{k} \exp \left[ \frac{1}{2} \int_{(k+1)}^n \frac{dt}{t} \right] \\ &\leq M_1 \frac{r_n}{n^{1/2}} \sum_{k=r_n}^{n-1} \frac{1}{k^{3/2}} \leq 4M_1 \left( \frac{r_n}{n} \right)^{1/2}. \end{aligned}$$

The last term tends to 0 as  $n \rightarrow +\infty$ , so due to (5.23) and (5.14) the property (5.22) is true. ■

*Proof of the Corollary 5.9.* Let  $X_{n,k} = (X_k - b_n(\alpha))/a_n$ . By Theorem 3.2 of [23], in all three cases  $S_n(\cdot) \xrightarrow{\mathcal{D}} W_\alpha$  iff

- (1)  $S_n(1) \xrightarrow{\mathcal{D}} c_{\gamma(\alpha)}\text{-Pois } \mu(\alpha)$
- (2) for every natural sequence  $r_n = o(n)$ ,

$$\sum_{k=1}^{r_n} X_{n,k} \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty$$

$$(3) \quad \forall_{\varepsilon > 0} \quad \sup_n P(\|X_{n,0}\| > \varepsilon) < +\infty$$

$$(4) \quad \forall_{j \geq 1} \forall_{\varepsilon > 0} \quad nP(\|X_{n,0}\| > \varepsilon, \|X_{n,j}\| > \varepsilon) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

By Lemma 5.10 we have  $b_n(\alpha)/a_n \rightarrow 0$  in all the three cases (for  $\alpha = 1$  take  $r_n = 1$  in (5.18) and (5.17)), so in the properties (3) and (4) the inequalities  $\|X_{n,k}\| > \varepsilon$  can be changed into  $\|X_0\| > a_n \varepsilon$ .

Due to Corollary 5.7 we can omit the conditions (1) and (3) (which are the consequences of the assumptions of Corollary 5.9 and the property (4)). It remains to be shown that (2) can also be omitted. Let  $r_n = o(n)$ . Note that

for  $0 < \alpha < 1$

$$\sum_{j=1}^{r_n} X_{n,j} = \left[ \frac{1}{a_{r_n}} \sum_{j=1}^{r_n} X_j \right] \frac{a_{r_n}}{a_n} + \frac{r_n}{n} \cdot \frac{nb_n(\alpha)}{a_n}, \quad (5.24)$$

for  $1 < \alpha < 2$

$$\sum_{j=1}^{r_n} X_{n,j} = \left[ \frac{1}{a_{r_n}} \sum_{j=1}^{r_n} (X_j - EX_0) \right] \frac{a_{r_n}}{a_n} + \frac{r_n}{n} \cdot \frac{n(EX_0 - b_n(\alpha))}{a_n}, \quad (5.25)$$

and for  $\alpha = 1$

$$\begin{aligned} \sum_{j=1}^{r_n} X_{n,j} &= \left[ \frac{1}{a_{r_n}} \sum_{j=1}^{r_n} (X_j - EX_0 1(\|X_0\| \leq a_{r_n})) \right] \frac{a_{r_n}}{a_n} \\ &\quad + \frac{r_n E[X_0(1(\|X_0\| \leq a_{r_n}) - 1(\|X_0\| \leq a_n))]}{a_n} \\ &\quad + \frac{r_n}{n} \cdot \frac{n[EX_0 1(\|X_0\| \leq a_n) - b_n(\alpha)]}{a_n}. \end{aligned} \quad (5.26)$$

The sequences of random vectors in (5.24)–(5.26) are convergent to  $c_{\tau(\alpha)}$ -Pois  $\mu(\alpha)$  if  $r_n \rightarrow +\infty$  (due to Corollary 5.7) and are bounded if  $r_n$  is bounded.

This and (5.14)–(5.19) give the convergence of the expressions (5.24)–(5.26) to 0. ■

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#### REFERENCES

1. ARAUJO, A., AND GINÉ, E. (1980). *The Central Limit Theorem for Real and Banach Valued Random Variables*. Wiley, New York.
2. BERMAN, S. M. (1980). A compound Poisson limit for stationary sums and sojourns of Gaussian processes. *Ann. Probab.* **8** 511–538.
3. BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.
4. BINGHAM, N. H., GOLDIE, C. M., AND TEUGELS, J. L. (1987). *Regular Variation*. Cambridge Univ. Press, Cambridge.
5. BRADLEY, R. C. (1986). Basic properties of strong mixing conditions. In *Dependence in Probability and Statistics (Oberwolfach, 1985)*, Progress in Probability and Statistics, Vol. 11 (E. Eberlein and M. S. Taqqu, Eds.), pp. 165–192. Birkhäuser, Boston.
6. BRADLEY, R. C., AND PELIGRAD, M. (1986). In variance principles under two-part mixing assumption. *Stochastic Processes Appl.* **22** 271–289.
7. CHEN, L. H. Y. (1978). Two central limit problems for dependent random variables. *Z. Wahrsch. Verw. Gebiete* **43** 223–243.
8. DAVIS, R. A. (1983). Stable limits for partial sums of dependent random variables. *Ann. Probab.* **11** 262–269.
9. DURRETT, R., AND RESNICK, S. I. (1978). Functional limit theorems for dependent variables. *Ann. Probab.* **6** 829–846.
10. HUDSON, W. N., TUCKER, H. G., AND VEEH, J. A. (1989). Limit distributions of sums of  $m$ -dependent Bernoulli random variables, *Probab. Theory Related Fields* **82** 9–17.
11. JAKUBOWSKI, A. (1991). *Asymptotic Independent Representations for Sums and Order Statistics of Stationary Sequences*, Uniwersytet Mikołaja Kopernika, Toruń.
12. JAKUBOWSKI, A., AND KOBUS, M. (1989).  $\alpha$ -stable limit theorems for sums of dependent random vectors. *J. Multivariate Anal.* **29** 219–251.
13. KALLENBERG, O. (1975). *Random Measures*. Akademie-Verlag, Berlin.
14. KOBUS, M. (1990). *Zastosowanie procesów punktowych w twierdzeniach granicznych dla zależnych wektorów losowych*, Ph.D. thesis, Toruń.
15. LEADBETTER, M. R. (1974). On extreme values in stationary sequences, *Z. Wahrsch. Verw. Gebiete* **28** 289–303.
16. LEADBETTER, M. R., LINDGREN, G., AND ROOTZEN, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer, Berlin.
17. PELIGRAD, M. (1982). Invariance principles for mixing sequences of random variables. *Ann. Probab.* **10** 968–981.
18. PELIGRAD, M. (1985). An invariance principle for  $\phi$ -mixing sequences. *Ann. Probab.* **13** 1304–1313.

19. PELIGRAD, M. (1986). Recent advances in the central limit theorem and its weak invariance principle for mixing sequences of random variables (a survey). In *Dependence in Probability and Statistics (Oberwolfach, 1985)*, Progress in Probability and Statistics, Vol. 11 (E. Eberlein and M. S. Taquq, Eds.), pp. 225–268. Birkhäuser, Boston.
20. PELIGRAD, M. (1990). On Ibragimov-Iosifescu conjecture for  $\phi$ -mixing sequences. *Stochastic Processes Appl.* **35** 293–308.
21. RESNICK, S. I. (1986). Point processes, regular variation and weak convergence, *Adv. Appl. Probab.* **18** 66–138.
22. SAMUR, J. D. (1984). Convergence of sums of mixing triangular arrays of random vectors with stationary laws. *Ann. Probab.* **12** 390–426.
23. SAMUR, J. D. (1987). On the invariance principle for stationary  $\phi$ -mixing triangular arrays with infinitely divisible limits. *Probab. Theory Related Fields* **75** 245–259.