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On the Hausdorff Dimension of the Set Generated by Exceptional Oscillations of a Two-Parameter Wiener Process

Zacharie Dindar

Université Paris VI, Paris, France

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S. Orey and S. J. Taylor (1974, *Proc. London Math. Soc.* **28**, 174–192) proved that for $0 \leqslant \lambda \leqslant 1$ the set $E(\lambda) = \{t \in [0, 1] : \limsup_{h \downarrow 0} (2h \log(1/h))^{-1/2} (W'(t+h) - W'(t)) \geqslant \lambda\}$ has Hausdorff dimension $\dim E(\lambda) = 1 - \lambda^2$ a.s. where $W'(t)$ is a standard Wiener process. A corresponding result is obtained when W' is replaced by a two-parameter Wiener process. © 2001 Academic Press

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1. INTRODUCTION AND STATEMENT OF MAIN RESULT

Let $\{W(s, t): 0 \leqslant s < \infty, 0 \leqslant t < \infty\}$ denote a standard two-parameter Wiener process, defined on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$.

In this paper, we investigate random fractals generated by large oscillations of W. For the statement of these results, the following notation is needed. Let $W(R)$ denote the white noise measure of W on the set $R = [s_1, s_2] \times [t_1, t_2] \subseteq \mathbb{R}^2_+$ defined by

$$W(R) = W(s_2, t_2) - W(s_1, t_2) - W(s_2, t_1) + W(s_1, t_1). \tag{1.1}$$

We refer to Csörgő and Révész (1981) and articles of Orey and Pruitt (1973), Lin (1985), Lu (1991, 1993), and Yeh (1963ab), among others, for properties and references concerning the two-parameter Wiener process.

Orey and Pruitt (1973) proved the following theorem

THEOREM A. *We have with probability* 1,

$$\lim_{h \to 0} \sup_{\substack{0 \leqslant s \leqslant 1-h, \\ 0 \leqslant t \leqslant 1-h}} \frac{|W([s, s+h] \times [t, t+h])|}{\sqrt{2h^2 \log(1/h^2)}} = 1. \tag{1.2}$$

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Theorem A is the analog of the following result proved by Lévy (1937, 1948). Consider a standard Wiener process $\{W'(t): t \geqslant 0\}$ on the probability space $(\Omega, \mathscr{A}, [\mathrm{p}])$. We have

THEOREM B. *With probability* 1,

$$\limsup_{h \downarrow 0} \frac{|W'(t+h) - W'(t)|}{\sqrt{2h \log(1/h)}} = 1. \tag{1.3}$$

For arbitrary $\lambda \in [0, 1]$, consider the set

$$B^{\pm}(\lambda) = \{t \in [0, 1) : \limsup_{h \downarrow 0} \pm (2h \log(1/h))^{-1/2} (W'(t+h) - W'(t)) \geqslant \lambda\}.$$

Orey and Taylor (1974) proved that $B^{\pm}(\lambda)$ is a.s. a random fractal. Their result, stated in Theorem C below, provides the Hausdorff dimension of the set $B^{\pm}(\lambda)$. Recall (see, e.g., Falconer (1985, 1990)) that the Hausdorff dimension dim A, of a subset A of $[0, 1]$, is defined by

$$\dim A = \inf \{c > 0 : s^c\text{-mes } A = 0\}, \tag{1.4}$$

where the s^c-measure of A (see, e.g., Falconer (1990)) is defined for each $c > 0$ by

$$s^c\text{-mes } A = \lim_{h \downarrow 0} \left(\left\{ \sum_i \|I_i\|^c : A \subseteq \bigcup_i I_i, \|I_i\| \leqslant h \right\} \right). \tag{1.5}$$

Here the I_i constitute an h-cover of A, that is a collection of sets with diameter not exceeding h, whose union includes A. We set $\|I_i\|$ for the diameter of I_i, in other words, the longest Euclidean distance between two elements of I_i.

THEOREM C. *With probability* 1,

$$\dim B^{\pm}(\lambda) = 1 - \lambda^2.$$

The aim of the paper is to prove that a related phenomenon occurs for the two-parameter Wiener process. Set for any $\alpha_0 \in [0, 1]$,

$$F(\alpha_0) = \left\{ 0 \leqslant s < 1, 0 \leqslant t < 1 : \limsup_{h \downarrow 0} \frac{|W(R)|}{\sqrt{2h^2 \log(1/h^2)}} \geqslant \alpha_0 \right\}, \tag{1.6}$$

where R denotes the set $[s, s+h] \times [t, t+h]$. We have

THEOREM 1.1. *For any* $\alpha_0 \in [0, 1]$, *with probability* 1,

$$\dim F(\alpha_0) = 2(1 - \alpha_0^2). \tag{1.7}$$

2. PROOF OF THEOREM 1.1

We postpone the proof of (1.7) until Sections 2.2 and 2.3. Below, we introduce some facts and notations suitable for our needs.

2.1. *Preliminary Facts and Notations*

Let X, a r.v. on $(\Omega, \mathcal{A}, \mathbb{P})$, follow a normal distribution $N(0, 1)$. We have

FACT 1. *For any* $x > 0$,

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x} - \frac{1}{x^3}\right)\exp(-x^2/2) \leqslant \mathbb{P}(X > x) \leqslant \frac{1}{x\sqrt{2\pi}}\exp(-x^2/2). \tag{2.1}$$

Proof. See, e.g., Csörgő and Révész (1981, p. 23) or Feller (1968, p. 175). ∎

Consider the set

$$T(h) = \{R = [s_1, s_2] \times [t_1, t_2] : 0 \leqslant s_1 < s_2 \leqslant 1, 0 \leqslant t_1 < t_2 \leqslant 1, \lambda(R) \leqslant h^2\},$$

where $\lambda(R) = (s_2 - s_1)(t_2 - t_1)$. Orey and Pruitt (1973) proved the following result.

FACT 2. *For any* $\varepsilon > 0$, *there exists a* $C = C(\varepsilon) < \infty$ *such that, for any* $v > 0$ *and* $0 < h < 1$,

$$\mathbb{P}\Big(\sup_{R \in T(h)} |W(R)| \geqslant vh\Big) \leqslant Ch^{-2}(\log h^{-2})^2 \exp(-v^2/(2+\varepsilon)). \tag{2.2}$$

Introduce the Chernoff function of a standard Poisson process (see, e.g., Shorack and Wellner (1986, pp. 432–433 and 856)) defined by

$$h(r) = \begin{cases} r\log r - r + 1 & \text{for} \quad r > 0, \\ 1 & \text{if} \quad r = 0, \\ \infty & \text{otherwise.} \end{cases}$$

FACT 3. *Let* S_N *follow a binomial distribution with parameters* N *and* p. *Then, for all* $r \in [1, \infty]$,

$$\mathbb{P}(S_n \geqslant Nrp) \leqslant \exp(-Nph(r)), \tag{2.3}$$

and for all $r \in [0, 1]$,

$$\mathbb{P}(S_n \leqslant Nrp) \leqslant \exp(-Nph(r)). \tag{2.4}$$

Proof. See, e.g., Deheuvels and Mason (1995, p. 373). ▮

For any measurable, relatively compact set $E \subseteq \mathbb{R}^2$, let $\mathrm{vol}(E)$ denote the Lebesgue measure of E. In what follows, a *square* A will denote any closed subset of the form $[s, s+h] \times [t, t+h] \subseteq \mathbb{R}^2$, with $s \in [0, 1]$, $t \in [0, 1]$, and $h \in (0, 1)$.

Here and elsewhere, $\# A$ will denote the cardinality of A. The following result will turn out to be an instrumental tool for our needs.

FACT 4. *Let* $\mathbb{K} \subseteq [0, 1] \times [0, 1]$ *be such that* $\mathbb{K} := \bigcap_{m=1}^{\infty} E_m$, *where* $E_1 \supseteq \cdots \supseteq E_m \supseteq \cdots$ *for* $m = 1, 2, ...,$ *and* $E_m = \bigcup_{\ell=1}^{M_m} I_{m,\ell}$ *with* $\{I_{m,\ell} : 1 \leqslant \ell \leqslant M_m\}$ *being, for each* $m \geqslant 1$, *a collection of disjoint closed squares of* $[0, 1] \times [0, 1]$ *such that* $\max_{1 \leqslant \ell \leqslant M_m} \|I_{m,\ell}\| \to 0$ *and* $M_m \to \infty$ *as* $m \to \infty$.

Assume that there exists three constants $\Delta > 0$, $d > 0$, *and* $c > 0$ *such that the following property holds.*

For every square $I \subseteq [0, 1] \times [0, 1]$ *with* $\|I\| \leqslant \Delta$, *there exists a constant* $m(I)$ *such that for all* $m \geqslant m(I)$,

$$M_m(I) := \#\{I_{m,\ell} \subseteq I : 1 \leqslant \ell \leqslant M_m\} \leqslant d \|I\|^c M_m. \tag{2.5}$$

Then we have, $\dim I\mathbb{K} \geqslant c$.

Proof. See, e.g., Deheuvels (1998). ▮

2.2. *Upper Bound*

In this Section, we prove that with probability 1,

$$\dim F(\alpha_0) \leqslant 2(1 - \alpha_0^2). \tag{2.6}$$

The proof of (2.6) is inspired by Orey and Taylor (1974). In short, the proof consists in showing that $F(\alpha_0)$ can be included in a set whose Hausdorff dimension is arbitrarily close to $2(1-\alpha_0^2)$. Consider two constants α_1 and α_2 with $0 < \alpha_1 < \alpha_2 < \alpha_0 < 1$ and define for any $k \geqslant 2$ the squares

$$I_{i,j;k} = [ih_k^2/\sqrt{2}, (i+1)h_k^2/\sqrt{2}] \times [jh_k^2/\sqrt{2}, (j+1)h_k^2/\sqrt{2}], \tag{2.7}$$

where i and j are two integers verifying $0 \leqslant i, j \leqslant \lfloor h_k^{-2}\sqrt{2} \rfloor$ with $\lfloor u \rfloor \leqslant u < \lfloor u \rfloor + 1$ denoting the integer part of $u \geqslant 0$. Let $m_k := \lfloor h_k^{-2}\sqrt{2} \rfloor$, with $h_k = (1+\gamma)^{-k}$ and $\gamma \in (0, 1)$, a constant whose value will be described later.

For any $i, j = 0, 1, ..., m_k$ and $k \geqslant 2$, consider the set

$$R_{i,j,k}(\alpha_1) = \begin{cases} I_{i,j,k} & \text{if} \quad 1_{i,j,k}(\alpha_1) = 1 \\ \varnothing & \text{if} \quad 1_{i,j,k}(\alpha_1) = 0, \end{cases}$$

where

$$1_{i,j,k}(\alpha_1) = 1_{\{\sup_{I \subseteq I_{i,j,k}} |W(I)| > \alpha_1 (2h_k^2 \log h_k^{-2})^{1/2}\}}. \tag{2.8}$$

Fix $(s, t) \in F(\alpha_0)$. It is easy to choose a sequence $\{u_n : n \geqslant 1\}$ decreasing to 0, such that for any n large enough,

$$|W([s, s+u_n] \times [t, t+u_n])| > \alpha_2 (2u_n^2 \log u_n^{-2})^{1/2}. \tag{2.9}$$

Moreover, for any n large enough, we can choose a $k \geqslant 2$ verifying $h_{k+2} < u_n \leqslant h_{k+1} < h_k$ and a square $I_{i,j,k}$ such that $[s, s+u_n] \times [t, t+u_n] \subseteq I_{i,j,k}$. Now, choose $\gamma \in (0, 1)$ small enough such that $0 < \gamma < \sqrt{\alpha_2/\alpha_1} - 1$. From (2.9), we obtain for any k large enough

$$\begin{aligned} |W([s, s+u_n] \times [t, t+u_n])| &> \alpha_2 (2u_n^2 \log u_n^{-2})^{1/2} \\ &> \alpha_1 (2h_k^2 \log h_k^{-2})^{1/2}. \end{aligned} \tag{2.10}$$

Therefore, (2.10) entails $(s, t) \in \bigcup_{k \geqslant 2} \bigcup_{i=1}^{m_k} \bigcup_{j=1}^{m_k} R_{i,j,k}(\alpha_1)$.

Next, using basic properties of the Hausdorff measure s^c-mes (see, e.g., Falconer (1990)), it is easy to see that

$$\begin{aligned} s^c\text{-mes}\,(F(\alpha_0)) &\leqslant s^c\text{-mes}\left(\bigcup_{k \geqslant 2} \bigcup_{i=1}^{m_k} \bigcup_{j=1}^{m_k} R_{i,j,k}(\alpha_1) \right) \\ &\leqslant \sum_{k \geqslant 2} \sum_{i=1}^{m_k} \sum_{j=1}^{m_k} h_k^{2c} 1_{i,j,k}(\alpha_1). \end{aligned} \tag{2.11}$$

Define $S_k = \sum_{i=1}^{m_k} \sum_{j=1}^{m_k} 1_{i,j,k}(\alpha_1)$. For any $k \geqslant 2$, observe that

$$\begin{aligned} \mathbb{E}(S_k) &= \sum_{i=1}^{m_k} \sum_{j=1}^{m_k} \mathbb{E}(1_{i,j,k}(\alpha_1)) \\ &= \sum_{i=1}^{m_k} \sum_{j=1}^{m_k} \mathbb{P}\left(\sup_{\substack{(s_1, t_1) \in I_{i,j,k}, \\ (s_2, t_2) \in I_{i,j,k}}} |W([s_1, s_2] \times [t_1, t_2])| > \alpha_1 (2h_k^2 \log h_k^{-2})^{1/2} \right). \end{aligned} \tag{2.12}$$

For any $\varepsilon > 0$, using Fact 2 and (2.2), we can choose a $C = C(\varepsilon)$ such that for any k sufficiently large

$$\mathbb{P}\Big(\sup_{R \in T(h_k^2/\sqrt{2})} |W(R)| \geqslant \alpha_1 (2h_k^2 \log h_k^{-2})^{1/2}\Big) \leqslant 2\, C h_k^{-4} \times (\log 2h_k^{-4})^2 \exp(-2\alpha_3^2 h_k^{-2} \log h_k^{-2}), \tag{2.13}$$

where we set $\alpha_3^2 := 2\alpha_1^2/(2+\varepsilon)$. Hence, (2.12) and (2.13) jointly imply

$$\begin{aligned}\mathbb{E}(S_k) &\leqslant 2C\, m_k^2 (\log 2h_k^{-4})^2\, h_k^{4\alpha_3^2} \\ &\leqslant 4C (\log 2h_k^{-4})^2\, h_k^{4\alpha_3^2 - 4}.\end{aligned} \tag{2.14}$$

The Markov inequality, combined with (2.14), and the definition of h_k jointly entail

$$\begin{aligned}\mathbb{P}(S_k \geqslant h_k^{-4+4\alpha_3^2-\varepsilon}) &\leqslant h_k^{4-4\alpha_3^2+\varepsilon}\, \mathbb{E}(S_k) \leqslant 4C (\log 2h_k^{-4})^2\, h_k^{\varepsilon} \\ &= \mathrm{O}(k(1+\gamma)^{-k\varepsilon}).\end{aligned} \tag{2.15}$$

Assertion (2.15) and the Borel–Cantelli lemma imply that almost surely, for all k sufficiently large, $S_k \leqslant h_k^{-4+4\alpha_3^2-\varepsilon}$.

Thus, (2.11) and (2.16) imply, almost surely, s^c-mes $(F(\alpha_0)) < \infty$ under the condition $2c - 4 + 4\alpha_3^2 - \varepsilon > 0$. Therefore, $\dim F(\alpha_0) \leqslant 2 - 2\alpha_3^2 + \varepsilon/2$.

We conclude by choosing α_1 arbitrary close to α_0 and ε arbitrary small to obtain (2.6).

2.3. *Lower Bound*

The aim of this section is to prove that, with probability 1, we have

$$\dim F(\alpha_0) \geqslant 2(1-\alpha_0^2). \tag{2.16}$$

2.3.1. *First step.* Set $\alpha_0 < \alpha < 1$. Let $\{h_k : k \geqslant 1\}$ denote a sequence of constants such that conditions (H1) and (H2) below are verified.

(H1) $0 < h_k < 1$, $h_k \to 0$ and $kh_k \to \infty$.

(H2) For any $0 < c < 1$ and any k large enough, $h_k^{-2} \exp(-h_k^{-c}/2) < 2^{-k}$.

For any $k \geqslant 1$, let $m_k := \lfloor h_k^{-1} \rfloor - 1$. For any $k \geqslant 1$ and $i = 0, ..., m_k$, we set $t_k(i) = ih_k$. Define the set

$$\begin{aligned}W_k = \{&(t_k(i), t_k(j)),\ i, j = 0, ..., m_k : \\ &|W([t_k(i), t_k(i+1)] \times [t_k(j), t_k(j+1)])| > \alpha(2h_k^2 \log h_k^{-2})^{1/2}\}.\end{aligned} \tag{2.17}$$

For any square $E \subseteq [0, 1] \times [0, 1]$, set

$$m_k(E) := \#\{(i, j) : i = 0, 1, ..., m_k,\ j = 0, 1, ..., m_k,\ [t_k(i), t_k(i+1)] \times [t_k(j), t_k(j+1)] \subseteq E\}, \tag{2.18}$$

and

$$N_k(E) := \#\{(i, j): i = 0, 1, ..., m_k,\ j = 0, 1, ..., m_k,\ [t_k(i), t_k(i+1)] \times [t_k(j), t_k(j+1)] \subseteq E,\ |W([t_k(i), t_k(i+1)] \times [t_k(j), t_k(j+1)])| > \alpha(2h_k^2 \log h_k^{-2})^{1/2}\}. \tag{2.19}$$

For any $i = 0, 1, ..., m_k$ and $j = 0, 1, ..., m_k$, we define

$$X_{i,j} = 1_{\{|W([t_k(i), t_k(i+1)] \times [t_k(j), t_k(j+1)])| > \alpha(2h_k^2 \log h_k^{-2})^{1/2}\}}. \tag{2.20}$$

Note for further use that for $i = 0, 1, ..., m_k$ and $j = 0, 1, ..., m_k$, the random variables $X_{i,j}$ are independent, identically distributed, and follow a Bernoulli law with parameter

$$p_k(\alpha) := \mathbb{P}(|W([0, h_k] \times [0, h_k])| > \alpha(2h_k^2 \log h_k^{-2})^{1/2}). \tag{2.21}$$

LEMMA 2.1. *$\forall \varepsilon > 0$, there exists with probability 1, a $k_0(\varepsilon)$ such that $\forall k \geqslant k_0(\varepsilon)$, we have*

$$h_k^{2\alpha^2 + \varepsilon} \leqslant p_k(\alpha). \tag{2.22}$$

Proof. It is easy to see that

$$\mathbb{P}(|W([0, h_k] \times [0, h_k])| > \alpha(2h_k^2 \log h_k^{-2})^{1/2}) = 2\mathbb{P}(N(0, 1) > \alpha(2 \log h_k^{-2})^{1/2}).$$

By combining Fact 1 and the last equality, we show that for any $\varepsilon > 0$, there exists with probability 1 a $k_0(\varepsilon) < \infty$ such that, for all $k \geqslant k_0(\varepsilon)$, (2.22) is verified. ∎

Consider a disjoint union of squares $E \subseteq [0, 1] \times [0, 1]$ with Lebesgue measure $\mathrm{vol}(E)$ greater than $S \geqslant 3h_k^2$. The following result is verified.

LEMMA 2.2. *For all $k \geqslant 1$,*

$$\frac{\mathrm{vol}(E)}{3h_k^2} \leqslant \left(1 - 2\frac{h_k^2}{S}\right)\frac{\mathrm{vol}(E)}{h_k^2} \leqslant m_k(E) \leqslant \frac{\mathrm{vol}(E)}{h_k^2}. \tag{2.23}$$

Proof. Observe that $m_k(.)$ is an additive set function. Consequently, it suffices to prove (2.23) when E is replaced by a square $I \subseteq [0,1] \times [0,1]$ with Lebesgue measure greater than $S \geqslant 3h_k^2$. The rest of the proof is trivial and therefore omitted. ▌

LEMMA 2.3. *$\forall \delta > 0$, $\forall \sigma \in (0,1)$, there exists a.s. a $k_1(\delta, \sigma) < \infty$ such that $\forall k \geqslant k_1(\delta, \sigma)$*

$$|N_k(E) - m_k(E)\, p_k(\alpha)| \leqslant \sigma m_k(E)\, p_k(\alpha), \qquad \forall E \subseteq [0,1] \times [0,1], \tag{2.24}$$

where E is a disjoint union of squares with Lebesgue measure greater than $h_k^{2-2\alpha^2-2\delta}$.

Proof. Set $\delta > 0$ and $\sigma \in (0,1)$. It is easy to see that for all $k \geqslant 1$, $N_k(.)$ and $m_k(.)$ are additive set functions. Therefore, it suffices to prove (2.24) when $E = I$ is a square with Lebesgue measure $\mathrm{vol}(I) \geqslant h_k^{2-2\alpha^2-2\delta}$.

Fix δ' and σ' such that $0 < \delta' < \delta$ and $0 < \sigma' < \sigma$. In a first step, we prove the existence of disjoint squares J_ℓ, $\ell \in F$ for which $\bigcup_{\ell \in O \subset F} J_\ell \subseteq I \subseteq \bigcup_{\ell \in F} J_\ell$ and F is a finite subset of IN. If it is assumed that (2.24) holds with the formal replacement of E by J_ℓ, Lemma 2.3 will be proved when $E = I$. In a second step, we show that (2.24) is verified with the formal replacement of E by J_ℓ, $\ell \in F$.

Let J denote a square of the form $[ih_k, (i + l(k))\, h_k] \times [jh_k, (j + l(k))\, h_k]$, defined for any $k \geqslant 1$ with $0 \leqslant i \leqslant 2m_k$, $0 \leqslant j \leqslant 2m_k$, and $l(k) := \lfloor h_k^{-\alpha^2-\delta'} \rfloor$. Observe that

$$\mathrm{vol}(J) = (l(k)\, h_k)^2 \in [h_k^{-2\alpha^2-2\delta'+2} + h_k^2 - 2h_k^{-\alpha^2-\delta'+2},\ h_k^{-2\alpha^2-2\delta'+2}). \tag{2.25}$$

Assume that the statement (2.24) is true with $E = J$, $\sigma = \sigma'$, and $\delta = \delta'$. Let $K(k) := \lfloor h_k^{2\delta'-2\delta} \rfloor$. By (H1), we can choose $k(\delta, \sigma) < \infty$ such that, for all $k \geqslant k(\delta, \sigma)$, we have

$$h_k^{2-2\alpha^2-2\delta'} < h_k^{2-2\alpha^2-2\delta}, \tag{2.26}$$

$$9 \leqslant K(k), \tag{2.27}$$

$$3h_k^2 < h_k^{2-2\alpha^2-2\delta}, \tag{2.28}$$

$$(1+\sigma') \left\{ \frac{(1 + 4/(\lfloor \sqrt{K(k)} \rfloor - 2))^2}{1 - 2h_k^{\alpha^2+2\delta}} \right\} \leqslant 1 + \sigma, \tag{2.29}$$

$$1 - \sigma \leqslant (1-\sigma') \left\{ \frac{1 - 2h_k^{2\alpha^2+2\delta'}}{1 + 4/(\lfloor \sqrt{K(k)} \rfloor - 2))^2} \right\}. \tag{2.30}$$

For any $k \geqslant 1$, let $\mathscr{K} := K(I,k) = \lfloor \mathrm{vol}(I)/h_k^{2-2\alpha^2-2\delta'} \rfloor$. Observe that $\mathscr{K} \geqslant K(k)$. Next, set $J(i,j) := [ih_k,(i+l(k))\,h_k] \times [jh_k,(j+l(k))\,h_k]$, with $0 \leqslant i \leqslant 2m_k$ and $0 \leqslant j \leqslant 2m_k$. Let $L := (\lfloor \sqrt{\mathscr{K}} \rfloor - 2)^2$ and $\mathrm{M} := (\lfloor \sqrt{\mathscr{K}} \rfloor + 2)^2$.

From (2.27), for any $k \geqslant k(\delta,\sigma)$, we can choose L disjoint squares $J_1, ..., J_L$ of the form $J(i,j)$ and M disjoint squares $J'_1, ..., J'_M$ of the form $J(i,j)$ such that

$$\bigcup_{\ell=1}^{L} J_\ell \subseteq I \subseteq \bigcup_{m=1}^{M} J'_m.$$

By (2.25), it is easy to see that $\mathrm{vol}(J_1) \sim h_k^{-2\alpha^2-2\delta'+2}$. Moreover, we have the inequalities $L\,\mathrm{vol}(J_1) \leqslant \mathrm{vol}(I) \leqslant M\,\mathrm{vol}(J_1)$. From (2.26) and (2.28), the use of Lemmas 2.2 and 2.3 applied to $E = J'_n$ entail with probability 1, for all k sufficiently large,

$$\begin{aligned} N_k(I) &\leqslant \sum_{m=1}^{M} N_k(J'_m) \leqslant (1+\sigma') \left\{ \sum_{m=1}^{M} m_k(J'_m) \right\} p_k(\alpha) \\ &\leqslant M(1+\sigma')\,\mathrm{vol}(J_1)\,h_k^{-2} p_k(\alpha) \\ &\leqslant \frac{M}{L}(1+\sigma')\,\mathrm{vol}(I)\,h_k^{-2} p_k(\alpha) \\ &\leqslant (1+\sigma') \frac{(1+4/(\lfloor \sqrt{\mathscr{K}} \rfloor - 2))^2}{1-2\,h_k^{2\alpha^2+2\delta}} m_k(I)\,p_k(\alpha) \\ &\leqslant (1+\sigma)\,m_k(I)\,p_k(\alpha), \end{aligned} \tag{2.31}$$

where (2.29) entails the last statement.

Likewise, a similar argument shows with probability 1, for all k large enough,

$$\begin{aligned} N_k(I) &\geqslant N_k\left(\bigcup_{\ell=1}^{L} J_\ell \right) \\ &\geqslant L(1-\sigma')(1-2h_k^2 h_k^{-2+2\alpha^2+2\delta'})\,\mathrm{vol}(J_1)\,h_k^{-2} p_k(\alpha) \\ &\geqslant (1-\sigma') \frac{1-2h_k^{2\alpha^2+2\delta'}}{(1+4/(\lfloor \sqrt{\mathscr{K}} \rfloor - 2))^2} m_k(I)\,p_k(\alpha) \\ &\geqslant (1-\sigma)\,m_k(I)\,p_k(\alpha), \end{aligned} \tag{2.32}$$

where (2.30) entails the last statement. Thus, (2.31) and (2.32) jointly imply (2.24) with $E = I$ and $\mathrm{vol}(I) \geqslant h_k^{2-2\alpha^2-2\delta}$.

Now let us show that (2.24) is verified when E is replaced by a square J of the form $[ih_k, (i+l(k))\,h_k] \times [jh_k, (j+l(k))\,h_k]$ with $0 \leqslant i \leqslant 2m_k$ and

$0 \leqslant j \leqslant 2m_k$. We note that the total number of such squares is less than $4h_k^{-2}$. For any $k \geqslant 1$, let $Q_k = IP(N_k(J) > r_{k,1})$ where $r_{k,1} = (1+\sigma')\, m_k(J)\, p_k(\alpha)$. In view of Fact 3 and (2.3), for all k sufficiently large, we have

$$Q_k \leqslant \exp(-m_k(J)\, p_k(\alpha)\, h(1+\sigma')).$$

Moreover, for all k large enough, we have $m_k(J) \geqslant \frac{1}{2} h_k^{-2\alpha^2 - 2\delta'}$. Thus, Lemma 2.1 implies for k sufficiently large

$$h_k^{-2} Q_k \leqslant 4h_k^{-2} \exp(-\tfrac{1}{2} h_k^{-\delta'} h(1+\sigma')).$$

We use (H2) to obtain $\sum_{k=1}^{\infty} h_k^{-2} Q_k < \infty$. Therefore, the Borel–Cantelli lemma implies with probability 1 that, for all k sufficiently large, we have

$$N_k(J) \leqslant (1+\sigma')\, m_k(J)\, p_k(\alpha). \tag{2.33}$$

Define likewise $R_k = \mathbb{P}(N_k(J) \leqslant r_{k,2})$ where $r_{k,2} = (1-\sigma')\, m_k(J)\, p_k(\alpha)$. By a similar argument as above, we show that with probability 1, for all k sufficiently large,

$$N_k(J) \geqslant (1-\sigma')\, m_k(J)\, p_k(\alpha). \tag{2.34}$$

Finally (2.24) is proved for any square of the form $[ih_k, (i+l(k))\, h_k] \times [jh_k, (j+l(k))\, h_k]$ with $0 \leqslant i \leqslant 2m_k$, $0 \leqslant j \leqslant 2m_k$. This concludes the proof of (2.24) when $E = I$ and hence when E is a disjoint union of squares with Lebesgue measure greater than $h_k^{2-2\alpha^2-2\delta}$. ∎

Let $\eta \in (0, (1/3)\min(2\alpha^2, 2-2\alpha^2))$. To prove (2.16), we show the existence of a sequence of sets $\{E_m : m \geqslant 1\}$ verifying the assumptions of Fact 4 and such that $\mathbb{K} = \bigcap_{m=1}^{\infty} E_m \subseteq F(\alpha)$. The following arguments are devoted to the construction of these sets. Let $\{\sigma_m : m \geqslant 1\}$ and $\{\delta_m : m \geqslant 0\}$ denote two sequences of positive constants such that

(1i) $0 < \sigma_m < 1/2$, for any $m \geqslant 1$,

(1ii) $\prod_{m=1}^{\infty} (1+\sigma_m)^2/(1-\sigma_m)^2 \leqslant 2$,

(1iii) $\delta_0 = 0$, $\delta_m \geqslant 0$, for any $m \geqslant 1$,

(1iv) $\sum_{m=1}^{\infty} \delta_m \leqslant \eta/8 < (1/24)\min(2\alpha^2, 2-2\alpha^2)$.

Moreover, we choose two decreasing sequences of constants $\{\varepsilon_m : m \geqslant 1\}$ and $\{\theta_m : m \geqslant 1\}$ such that

(2i) $\varepsilon_1 < 1$, $\varepsilon_m \downarrow 0$, $\varepsilon_m < \delta_m$,

(2ii) $\theta_m < (1/16)\, \varepsilon_m^2$,

(2iii) $(1 + 4\theta_{m-1}^2 + 4\theta_{m-1})^{1-\alpha^2} < (1+\sigma_m)(1+(1/2)\,\sigma_m)$,

(2iv) $(1 - 4\theta_{m-1}^2 - 4\theta_{m-1})^{\alpha^2 - 1} < (1+\sigma_m)/(1+(1/2)\,\sigma_m)$.

Set $\Delta_m = \sum_{k=1}^{m} \delta_k$. Observe that, for any $m \geqslant 1$, $\Delta_m < (1/24)\min(2\alpha^2, 2-2\alpha^2)$ and $0 < \delta_m < \Delta_m < 1$. For any $m \geqslant 1$, by Lemma 2.1 we can choose $k_0(\varepsilon_m)$ such that (2.22) holds for $k \geqslant k_0(\varepsilon_m)$. Next, using Lemma 2.3 we can choose $k_1(\delta_m, \sigma_m) \geqslant k_0(\varepsilon_m)$ such that (2.24) holds for any $k \geqslant k_1(\delta_m, \sigma_m)$.

The sets E_m are constructed by an induction argument. Given E_{m-1} and $\{M_{m-1}, k_{m-1}, E^*_{m-1}\}$, we construct E_m and $\{M_m, k_m, E^*_m\}$. The constants M_{m-1}, k_{m-1}, and the set E^*_{m-1} will be defined below.

Let

$$k_0 = 1, \qquad L_0 = 1, \qquad M_0 = 1, \tag{2.35}$$

for $m \geqslant 1$,

$$L_m = \theta_m h_{k_m} \qquad \text{and} \qquad L^*_{m-1} = L_{m-1} - L_m. \tag{2.36}$$

Choose a positive integer k_m such that the following assumptions hold.

(3i) $k_m > \max\{k_{m-1}, k_1(\delta_m, \sigma_m)\}$,

(3ii) $3L_m < 3h_{k_m} \leqslant h_{k_m}^{1-\alpha^2} \leqslant h_{k_m}^{1-\alpha^2-\delta_m} \leqslant L^*_{m-1} \leqslant L_{m-1}$,

(3iii) $L^2_{m-1}/(L^*_{m-1})^2 \leqslant 1 + (1/2)\,\sigma_m$,

(3iv) $1 - \sigma_m \leqslant 1 - 2h^2_{k_m}/(L^*_{m-1})^2 < 1$,

(3v) $\sqrt{3}\, h^{\delta_m}_{k_m}/h^{\alpha^2+\delta_{m-1}/2}_{k_{m-1}} \leqslant \theta_m$, for $m \geqslant 2$,

(3vi) $3h^{\delta_m}_{k_m} \leqslant \theta_m$, for $m \geqslant 1$,

(3vii) $6h^{\delta_m - \varepsilon_m}_{k_m} \leqslant \theta^2_m\, M_{m-1}(L^*_{m-1})^2$, for $m \geqslant 1$.

Using (H1), it is easy to choose k_m so large that the statements (3i–3vii) hold. Now, for any $m \geqslant 1$, given k_m and $\{M_{m-1}, k_{m-1}, E^*_{m-1}, E_{m-1}\}$, let us define the sets E_m, E^*_m, and the positive integer M_m. We set

$$E_m = \bigcup_{\substack{1 \leqslant i \leqslant m_{k_m},\ 1 \leqslant j \leqslant m_{k_m}: \\ (tk_m(i),\, t_{k_m}(j)) \in W_{k_m} \cap E^*_{m-1}}} [t_{k_m}(i),\, t_{k_m}(i) + L_m] \times [t_{k_m}(j),\, t_{k_m}(j) + L_m] \tag{2.37}$$

with $E_0 = [0, 1] \times [0, 1]$.

$$E^*_m = \bigcup_{\substack{1 \leqslant i \leqslant m_{k_m},\ 1 \leqslant j \leqslant m_{k_m}: \\ (tk_m(i),\, t_{k_m}(j)) \in W_{k_m} \cap E^*_{m-1}}} [t_{k_m}(i),\, t_{k_m}(i) + L^*_m] \times [t_{k_m}(j),\, t_{k_m}(j) + L^*_m] \tag{2.38}$$

with $E^*_0 = [0, L^*_0] \times [0, L^*_0]$.

Making use of (2.19) and (2.38), we set $M_0 = 1$ and for any $m \geqslant 1$,

$$M_m := N_{k_m}(E^*_{m-1})$$
$$= \#\{(i, j)\colon i, j = 1, ..., m_{k_m}, (t_{k_m}(i), t_{k_m}(j)) \in W_{k_m} \cap E^*_{m-1}\}. \tag{2.39}$$

By (2.37), (2.38), and (2.39), we observe that for all $m \geqslant 0$, the sets E_m (resp. E^*_m) are union of M_m squares. Hence, for $1 \leqslant \ell \leqslant M_m$, we can write

$$E_m = \bigcup_{\ell=1}^{M_m} I_{m,\ell} \qquad \left(\text{resp. } E^*_m = \bigcup_{\ell=1}^{M_m} I^*_{m,\ell}\right). \tag{2.40}$$

Next we use (3ii) combined with (2.36) to observe that the squares $I_{m,\ell}$ (resp. $I^*_{m,\ell}$) are disjoint. Finally, we use (3ii) and $3L_m < L^*_{m-1} \leqslant L_{m-1}$ to check that, for any $m \geqslant 1$,

$$E_m \subseteq E_{m-1} \qquad \text{and} \qquad E^*_m \subseteq E^*_{m-1}. \tag{2.41}$$

In order to prove that the induction process used to construct the sets E_m carries over from stage $m-1$ to stage m, it is important to verify that $M_m \geqslant 1$ for all $m \geqslant 1$. This is shown with the help of Lemma 2.4 below.

LEMMA 2.4. *We have* $M_0(L^*_0)^2 \geqslant 4/9$. *Moreover, if* $M_{m-1} \geqslant 1$ *for some* $m \geqslant 1$, *then*

$$M_m(L^*_m)^2 \geqslant h_{k_m}^{2\alpha^2 + \delta_m}. \tag{2.42}$$

Proof. The first inequality is obvious from (2.35), (2.36), and (3ii). Now, suppose that $\{M_k, j_k, E_k, E^*_k\}$ are defined for any $k \in \{0, ..., m-1\}$ and suppose that $M_{m-1} \geqslant 1$.

First, apply Lemma 2.3 with $\varepsilon = \varepsilon_m$, $\delta = \delta_m$ (given (1iii)), $\sigma = \sigma_m$, $E = E^*_{m-1}$ (by (3ii)), and use the fact that $M_{m-1}(L^*_{m-1})^2 \geqslant (L^*_{m-1})^2 \geqslant h_{k_m}^{2-2\alpha^2-2\delta_m}$ to obtain by (2.39),

$$1 - \sigma_m < \frac{M_m}{m_{k_m}(E^*_{m-1})\, p_{k_m}(\alpha)}. \tag{2.43}$$

We can apply Lemma 2.2 with $k = k_m$ and $S = (L^*_{m-1})^2$ to obtain

$$\left(1 - 2\frac{h^2_{k_m}}{(L^*_{m-1})^2}\right)\frac{\mathrm{vol}(E^*_{m-1})}{h^2_{k_m}} \leqslant m_{k_m}(E^*_{m-1}) \leqslant \frac{\mathrm{vol}(E^*_{m-1})}{h^2_{k_m}}. \tag{2.44}$$

Assertion (3iv) implies

$$(1 - \sigma_m)\, h_{k_m}^{-2} \leqslant \frac{m_{k_m}(E^*_{m-1})}{M_{m-1}(L^*_{m-1})^2} \leqslant h_{k_m}^{-2}. \tag{2.45}$$

The use of (3i) and Lemma 2.1 with $k = k_m$ and $\varepsilon = \varepsilon_m$ jointly imply

$$h_{k_m}^{2\alpha^2+\varepsilon_m} \leqslant p_{k_m}(\alpha). \tag{2.46}$$

Making use of (2.43) and (2.45), we obtain

$$(1-\sigma_m)^2\, h_{k_m}^{-2+2\alpha^2+\varepsilon_m} \leqslant \frac{M_m}{M_{m-1}(L_{m-1}^*)^2}. \tag{2.47}$$

By (1i), note that for any $m \geqslant 1$, we have the inequalities $1/2 \leqslant 1-\sigma_{m+1}$ and $1+\sigma_{m+1} \leqslant 3/2$. Therefore, we get

$$\frac{1}{6} \leqslant (1-\sigma_m)^2 \frac{1}{1+\sigma_{m+1}}. \tag{2.48}$$

Making use of (2.47), (2.48), (3ii), and (3iii), we obtain

$$\begin{aligned}
&\frac{1}{6}\,\theta_m^2\, M_{m-1}(L_{m-1}^*)^2\, h_{k_m}^{2\alpha^2+\varepsilon_m} \\
&\quad\leqslant \frac{(1-\sigma_m)^2}{1+\sigma_{m+1}}\, M_{m-1}(L_{m-1}^*)^2\, h_{k_m}^{-2+2\alpha^2+\varepsilon_m} h_{k_m}^2\, \theta_m^2 \\
&\quad\leqslant \frac{1}{1+\sigma_{m+1}}\, M_m h_{k_m}^2\, \theta_m^2 \\
&\quad\leqslant M_m (L_m^*)^2.
\end{aligned} \tag{2.49}$$

This last statement combined with (3vii) entails (2.42) and completes the proof of Lemma 2.4. ▮

With the help of Lemma 2.4, we now prove that $M_m \geqslant 1$, for any $m \geqslant 1$.

Given (1iv), it is easy to see that $2\alpha^2 - 2 + \delta_m < -(23/24)(2-2\alpha^2)$. Besides, (2ii) implies $\theta_m < (1/16)\,\varepsilon_m^2 < 1/16$. Therefore, $L_m^* \leqslant L_m < h_{k_m}/16$. We next use (2.42) to obtain $M_m \geqslant 256 h_{k_m}^{2\alpha^2+\delta_m-2} \geqslant 256 h_{k_m}^{-(23/24)(2-2\alpha^2)}$. This last inequality, combined with (H1) and $h_{k_m} < 1$ jointly imply $M_m \geqslant 256 \geqslant 1$. Next, recall that $M_0 = 1$ to obtain by induction for all $m \geqslant 0$, $M_m \geqslant 1$.

This last property establishes the existence of $\{M_m, k_m, E_m^*, E_m\}$ for any $m \geqslant 0$.

2.3.2. *Second step.* The aim of this section is to prove the existence of constants c, d, and Δ such that the assertion (2.5) of Fact 4 is verified for any square $I \subseteq [0,1] \times [0,1]$ with $\|I\| \leqslant \Delta$.

Recalling (2.5), (2.17), (2.19), and (2.37), it is easy to see that, for any square $I \subseteq [0,1] \times [0,1]$,

$$M_m(I) \leqslant N_{k_m}(I). \tag{2.50}$$

The proof of (2.5) is achieved by considering several cases, depending on the nature of the square $I \subseteq [0, 1] \times [0, 1]$. These cases are

Case 1. For some $m \geqslant 1$, $I \subseteq I_{m-1, \ell_0}$, for some $\ell_0 \in \{1, ..., M_{m-1}\}$.

Case 2. For some $m \geqslant 1$, $I \cap I_{m-1, \ell_0} \neq \varnothing$, for some $\ell_0 \in \{1, ..., M_{m-1}\}$.

Case 3. For some $m \geqslant 1$, $I \cap I_{m-1, \ell} = \varnothing$, $\forall \ell \in \{1, ..., M_{m-1}\}$.

a. Case 1. For $m \geqslant 1$, assume that $I \subseteq I_{m-1, \ell_0}$ for some $\ell_0 \in \{1, ..., M_{m-1}\}$. Three possibilities have to be considered.

Case 1a. We suppose that $\text{vol}(I) \geqslant h_{k_m}^{2-2\alpha^2-2\delta_m}$ (This is possible by (3ii)). We may apply Lemma 2.3 with $\delta = \delta_m$, $\sigma = \sigma_m$, $E = I$, $k = k_m$, combined with (2.23) and (2.50) to obtain

$$M_m(I) \leqslant (1 + \sigma_m)\, h_{k_m}^{-2}\, \text{vol}(I)\, p_{k_m}(\alpha). \tag{2.51}$$

This last inequality, combined with (2.46) and (2.47) imply for any $m \geqslant 1$

$$\begin{aligned} \frac{M_m(I)}{M_m} &= \frac{M_{m-1}(L_{m-1}^*)^2}{M_m}\, M_m(I)\, \frac{1}{M_{m-1}(L_{m-1}^*)^2} \\ &\leqslant \frac{1 + \sigma_m}{(1 - \sigma_m)^2}\, \text{vol}(I)\, \frac{1}{M_{m-1}(L_{m-1}^*)^2}. \end{aligned} \tag{2.52}$$

Lemma 2.4 and (2.52) jointly imply for any $m \geqslant 2$

$$\frac{M_m(I)}{M_m} \leqslant \frac{1 + \sigma_m}{(1 - \sigma_m)^2}\, \text{vol}(I)\, h_{k_{m-1}}^{-2\alpha^2-\delta_{m-1}}. \tag{2.53}$$

Recall that $\text{vol}(I) \leqslant \text{vol}(I_{m-1, \ell_0}) \leqslant h_{k_{m-1}}^2$ and use (2.53) to obtain for any $m \geqslant 2$,

$$\begin{aligned} \frac{M_m(I)}{M_m} &\leqslant \frac{1 + \sigma_m}{(1 - \sigma_m)^2}\, (\text{vol}(I))^{1-\alpha^2-(\delta_{m-1})/2} \\ &\leqslant \frac{1 + \sigma_m}{(1 - \sigma_m)^2}\, \|I\|^{2-2\alpha^2-\delta_{m-1}}. \end{aligned} \tag{2.54}$$

When $m = 1$ observe that $M_0(L_0^*)^2 \geqslant 4/9$ to obtain by (2.52)

$$\frac{M_1(I)}{M_1} \leqslant \frac{9}{4}\, \frac{1 + \sigma_1}{(1 - \sigma_1)^2}\, \text{vol}(I) \tag{2.55}$$

$$\leqslant 3\, \frac{1 + \sigma_1}{(1 - \sigma_1)^2}\, \|I\|^{2(1-\alpha^2)}. \tag{2.56}$$

Case 1b. Consider the possibility where $(\theta_m h_{k_m})^2/3 \leqslant \mathrm{vol}(I) \leqslant h_{k_m}^{2-2\alpha^2-2\delta_m}$. In this case, it is always possible to choose a square I' such that $I \subseteq I' \subseteq [0,1]\times[0,1]$ with $\mathrm{vol}(I') = h_{k_m}^{2-2\alpha^2-2\delta_m}$. It is easy to see that $h_{k_m} \leqslant \theta_m^{-1}(3\,\mathrm{vol}(I))^{1/2}$ and therefore,

$$h_{k_m}^{2-2\alpha^2-4\delta_m} \leqslant 3\theta_m^{-2}(\mathrm{vol}(I))^{1-\alpha^2-2\delta_m}. \tag{2.57}$$

For any $m \geqslant 2$, apply (2.53) with the formal replacement of I by I'. This, combined with (2.57) and (3v), implies

$$\begin{aligned}
\frac{M_m(I)}{M_m} &\leqslant \frac{M_m(I')}{M_m} \leqslant \frac{1+\sigma_m}{(1-\sigma_m)^2}\,\mathrm{vol}(I')\,h_{k_{m-1}}^{-2\alpha^2-\delta_{m-1}} \\
&\leqslant \frac{1+\sigma_m}{(1-\sigma_m)^2}\,h_{k_m}^{2-2\alpha^2-4\delta_m}\,\frac{h_{k_m}^{2\delta_m}}{h_{k_{m-1}}^{2\alpha^2+\delta_{m-1}}} \\
&\leqslant \frac{1+\sigma_m}{(1-\sigma_m)^2}\,(\mathrm{vol}(I))^{1-\alpha^2-2\delta_m} \\
&\leqslant \frac{1+\sigma_m}{(1-\sigma_m)^2}\,\|I\|^{2-2\alpha^2-4\delta_m}.
\end{aligned} \tag{2.58}$$

For $m=1$, apply (2.55) with the formal replacement of I by I' and use (2.57) with (3vi) to obtain

$$\begin{aligned}
\frac{M_1(I)}{M_1} &\leqslant \frac{9}{4}\frac{1+\sigma_1}{(1-\sigma_1)^2}\,\mathrm{vol}(I') \\
&\leqslant \frac{9}{4}\frac{1+\sigma_1}{(1-\sigma_1)^2}\,3\theta_1^{-2}(\mathrm{vol}(I))^{1-\alpha^2-2\delta_1}h_{k_1}^{2\delta_1} \\
&\leqslant \frac{1+\sigma_1}{(1-\sigma_1)^2}\,\|I\|^{2-2\alpha^2-4\delta_1}.
\end{aligned} \tag{2.59}$$

Case 1c. Finally, suppose that $\mathrm{vol}(I) < (\theta_m h_{k_m})^2/3$. Recall that for any $\ell = 1, \ldots, M_m$, we have $\mathrm{vol}(I_{m,\ell}) = (\theta_m h_{k_m})^2$. Therefore, for any $m \geqslant 1$,

$$\frac{M_m(I)}{M_m} = 0. \tag{2.60}$$

In view of (2.54), (2.56), (2.58), (2.59), and (2.60), we conclude that for any $I \subseteq [0,1]\times[0,1]$ verifying $I \subseteq I_{m-1,\ell_0}$ for some $\ell_0 \in \{1, \ldots, M_{m-1}\}$, we have the inequalities

$$\frac{M_1(I)}{M_1} \leqslant \frac{1+\sigma_1}{(1-\sigma_1)^2} \|I\|^{2-2\alpha^2-4\delta_1}, \tag{2.61}$$

$$\frac{M_m(I)}{M_m} \leqslant \frac{1+\sigma_m}{(1-\sigma_m)^2} \|I\|^{2-2\alpha^2-4\delta_m} \qquad \text{for any} \quad m \geqslant 2. \tag{2.62}$$

b. Case 2. We now turn to the case where I is not necessarily a subset of I_{m-1,ℓ_0}.

Case 2a. Assume that $\ell_0 \in \{1, ..., M_{m-1}\}$ is the unique integer verifying $I \cap I_{m-1,\ell_0} \neq \varnothing$. Let $I' := I \cap I_{m-1,\ell_0}$. It is easy to see that $M_m(I) = M_m(I')$. Moreover, I' verifies the assumptions of Case 1. Therefore, we use (2.62) with $I = I'$ to obtain for any $m \geqslant 2$,

$$\frac{M_m(I)}{M_m} \leqslant \frac{1+\sigma_m}{(1-\sigma_m)^2} \|I'\|^{2-2\alpha^2-4\delta_m} \leqslant \frac{1+\sigma_m}{(1-\sigma_m)^2} \|I\|^{2-2\alpha^2-4\delta_m}. \tag{2.63}$$

Case 2b. Suppose that I verifies $I \cap I_{m-1,\ell} \neq \varnothing$ for some $\ell \in \{1, ..., M_{m-1}\}$.

For any $m \geqslant 1$, let $H(m)$ denote the statement that, for any $J \subseteq [0,1] \times [0,1]$,

$$\frac{M_m(J)}{M_m} \leqslant \prod_{i=1}^{m} \left(\frac{1+\sigma_i}{1-\sigma_i}\right)^2 \|J\|^{2-2\alpha^2-4\Delta_m}, \qquad \text{where } \Delta_m := \sum_{k=1}^{m} \delta_k. \tag{2.64}$$

Remark 2.1. For $m = 1$ and any square $J \subseteq [0,1] \times [0,1]$, assertions (2.40) and (2.61) jointly imply (2.64). For any $m \geqslant 1$, with J verifying assumptions of Case 1, (2.61) and (2.62) entail (2.64). Moreover, (2.64) is true for any $m \geqslant 1$ with J verifying assumptions of Case 2a.

Here and elsewhere, let $\mathrm{Fr}(A)$ and $\mathring{A}$ denote respectively the frontier and the open set of $A \subseteq IR^k$, $k \geqslant 1$.

LEMMA 2.5. *Let $m \geqslant 2$. Suppose that $H(m-1)$ holds and assume that $J \subseteq [0,1] \times [0,1]$ is a square such that*

$$\mathrm{Fr}(J) \cap \bigcup_{\ell=1}^{M_{m-1}} \mathring{I}_{m-1,\ell} = \varnothing. \tag{2.65}$$

Then, we have

$$\frac{M_m(J)}{M_m} \leqslant \frac{1+(1/2)\,\sigma_m}{1+\sigma_m} \prod_{i=1}^{m} \left(\frac{1+\sigma_i}{1-\sigma_i}\right)^2 \|J\|^{2-2\alpha^2-4\Delta_{m-1}}. \tag{2.66}$$

Proof. By (2.65), it is easy to see that

$$M_m(J) = \sum_{\substack{1 \leqslant \ell \leqslant M_{m-1}: \\ I_{m-1,\ell} \subseteq J}} M_m(I_{m-1,\ell}). \tag{2.67}$$

This last equality, combined with (2.52), (3iii), and $H(m-1)$, entails

$$\begin{aligned}
\frac{M_m(J)}{M_m} &= \sum_{\substack{1 \leqslant l \leqslant M_{m-1}: \\ I_{m-1,l} \subseteq J}} \frac{M_m(I_{m-1,l})}{M_m} \leqslant M_{m-1}(J) \max_{\substack{1 \leqslant l \leqslant M_{m-1}: \\ I_{m-1,l} \subseteq J}} \frac{M_m(I_{m-1,l})}{M_m} \\
&\leqslant \max_{\substack{1 \leqslant l \leqslant M_{m-1}: \\ I_{m-1,l} \subseteq J}} \mathrm{vol}(I_{m-1,l}) \times M_{m-1}(J) \frac{1+\sigma_m}{(1-\sigma_m)^2} \frac{1}{M_{m-1}(L^*_{m-1})^2} \\
&\leqslant \frac{1+(1/2)\,\sigma_m}{1+\sigma_m} \prod_{i=1}^{m} \left(\frac{1+\sigma_i}{1-\sigma_i}\right)^2 \|J\|^{2-2\alpha^2-4\Delta_{m-1}}.
\end{aligned} \tag{2.68}$$

This achieves the proof of Lemma 2.5. ∎

Let I' denote the smallest square which contains the sets $I_{m-1,\ell}$ verifying $1 \leqslant \ell \leqslant M_{m-1}$ with $I_{m-1,\ell} \cap I \neq \varnothing$. Let $I'' = I \cap I'$. It is easy to see that

$$M_m(I) = M_m(I'') \leqslant M_m(I'). \tag{2.69}$$

In the following, we use an induction argument to prove statement (2.66) for any $m \geqslant 1$ and I verifying assumptions of Case 2b.

Using Remark 2.1, we already know that $H(1)$ is true. Now assume that $H(m-1)$ is verified for any $m \geqslant 2$. It is easy to see that

$$\mathrm{Fr}(I') \cap \bigcup_{\ell=1}^{M_{m-1}} \mathring{I}_{m-1,\ell} = \varnothing.$$

Therefore, Lemma 2.5 entails

$$\begin{aligned}
\frac{M_m(I')}{M_m} &\leqslant \prod_{i=1}^{m} \left(\frac{1+\sigma_i}{1-\sigma_i}\right)^2 2(\mathrm{vol}(I))^{1-\alpha^2-2\Delta_m} \\
&\quad \times \frac{1+(1/2)\,\sigma_m}{1+\sigma_m} \left(\frac{\mathrm{vol}(I')}{\mathrm{vol}(I)}\right)^{1-\alpha^2-2\Delta_m}.
\end{aligned} \tag{2.70}$$

It is easy to see that

$$h_{k_{m-1}}^2 < \mathrm{vol}(I') \text{ and} \tag{2.71}$$

$$\mathrm{vol}(I') \leqslant \mathrm{vol}(I) + 4L_{m-1}^2 + 4L_{m-1}(\mathrm{vol}(I))^{1/2}. \tag{2.72}$$

Next, assume $\mathrm{vol}(I) \geqslant h_{k_{m-1}}^2$ and see that (2.36) and (2.72) jointly entail

$$\frac{\mathrm{vol}(I')}{\mathrm{vol}(I)} \leqslant 1 + 4\theta_{m-1}^2 + 4\theta_m.$$

This last inequality combined with (2.69), (2.70), and (2iii) jointly imply

$$\frac{M_m(I)}{M_m} \leqslant \frac{M_m(I')}{M_m} \leqslant \prod_{i=1}^{m} \left(\frac{1+\sigma_i}{1-\sigma_i} \right)^2 \|I\|^{2-2\alpha^2-4\Delta_m}. \tag{2.73}$$

On the other hand, if $\mathrm{vol}(I) < h_{k_{m-1}}^2$, assumptions (2.71) and (2.72) entail

$$\begin{aligned} \mathrm{vol}(I) &\geqslant h_{k_{m-1}}^2 - 4\theta_{m-1}^2 h_{k_{m-1}}^2 - 4\theta_{m-1} h_{k_{m-1}}^2 \\ &= h_{k_{m-1}}^2 (1 - 4\theta_{m-1}^2 - 4\theta_{m-1}). \end{aligned}$$

Therefore, (2.72) and $\mathrm{vol}(I) > L_{m-1}^2$ imply

$$\left(\frac{\mathrm{vol}(I')}{\mathrm{vol}(I)} \right)^{1-\alpha^2-2\Delta_m} \leqslant \left(\frac{1}{1 - 4\theta_{m-1}^2 - 4\theta_{m-1}} \right)^{1-\alpha^2}. \tag{2.74}$$

This, combined with (2.69) and (2iv), jointly implies

$$\begin{aligned} \frac{M_m(I)}{M_m} &\leqslant \prod_{i=1}^{m} \left(\frac{1+\sigma_i}{1-\sigma_i} \right)^2 2(\mathrm{vol}(I))^{1-\alpha^2-2\Delta_m} \\ &= \prod_{i=1}^{m} \left(\frac{1+\sigma_i}{1-\sigma_i} \right)^2 \|I\|^{2-2\alpha^2-4\Delta_m}. \end{aligned} \tag{2.75}$$

Hence, (2.66) is verified for I verifying assumptions of Case 2b.

c. Case 3. We now consider the case where $I \cap I_{m-1,\ell} = \varnothing$ for any $\ell \in \{1, ..., M_{m-1}\}$. Observe that

$$\frac{M_m(I)}{M_m} = 0. \tag{2.76}$$

This last statement implies (2.66) for $J = I$ and any $m \geqslant 1$.

Assertions (2.73), (2.75) and (2.76) and Remark 2.1 entail that $H(m)$ is true for any $m \geqslant 1$. By (1ii), we conclude that, for any square $I \subseteq [0, 1] \times [0, 1]$ and $m \geqslant 1$, we have

$$M_m(I) \leqslant 2 \|I\|^{2(1-\alpha^2)-4\Delta_m} M_m \leqslant 2 \|I\|^{2(1-\alpha^2)-\eta} M_m. \tag{2.77}$$

Making use of Fact 4 with IK previously described and $\Delta = 1$, $d = 2$, $c = 2(1-\alpha^2)-\eta$, we get $\dim_H \mathbb{K} \geqslant 2(1-\alpha^2)-\eta$. This, combined with $\mathbb{K} = \bigcap_{m=1}^{\infty} E_m \subseteq F(\alpha) \subseteq F(\alpha_0)$, entails (2.16).

Finally the use of (2.6) and (2.16) conclude the proof of Theorem 1.1.

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