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Journal of Multivariate Analysis 89 (2004) 135–147

Journal of
Multivariate
Analysis

<http://www.elsevier.com/locate/jmva>

Partial autocorrelation functions of the fractional ARIMA processes with negative degree of differencing

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Received 6 December 2000

Abstract

Let $\{X_n : n \in \mathbf{Z}\}$ be a fractional ARIMA(p, d, q) process with partial autocorrelation function $\alpha(\cdot)$. In this paper, we prove that if $d \in (-1/2, 0)$ then $|\alpha(n)| \sim |d|/n$ as $n \rightarrow \infty$. This extends the previous result for the case $0 < d < 1/2$.

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AMS 2000 subject classifications: primary 62M10; 60G10

Keywords: Partial autocorrelation function; Fractional ARIMA process; Stationary process; Long memory; Prediction error

1. Introduction

The fractional ARIMA or autoregressive integrated moving-average processes were introduced independently by Granger and Joyeux [4] and Hosking [6], and have been used as a useful parametric family of long-memory stationary processes. In [8], the first author has proved an asymptotic formula for the partial autocorrelation functions of fractional ARIMA processes with positive degree of differencing. Our purpose in this article is to extend this result to those with *negative* degree of differencing.

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We recall the definition of a fractional ARIMA process. Let $\{X_n : n \in \mathbf{Z}\}$ be a real, zero-mean, weakly stationary process, defined on a probability space (Ω, \mathcal{F}, P) , which we shall simply call a *stationary process*. We write $\gamma(\cdot)$ for the autocovariance function of $\{X_n\}$:

$$\gamma(n) := E[X_n X_0] \quad (n \in \mathbf{Z}).$$

If there exists an even, nonnegative, and integrable function $\Delta(\cdot)$ on $(-\pi, \pi)$ such that

$$\gamma(n) = \int_{-\pi}^{\pi} e^{in\lambda} \Delta(\lambda) d\lambda \quad (n \in \mathbf{Z}),$$

then $\Delta(\cdot)$ is called the *spectral density* of $\{X_n\}$. For $d \in (-1/2, 1/2)$ and $p, q \in \mathbf{N} \cup \{0\}$, $\{X_n\}$ is said to be a fractional ARIMA(p, d, q) process if it has a spectral density $\Delta(\cdot)$ of the form

$$\Delta(\lambda) = \frac{1}{2\pi} \frac{|\Theta(e^{i\lambda})|^2}{|\Phi(e^{i\lambda})|^2} |1 - e^{i\lambda}|^{-2d} \quad (-\pi < \lambda < \pi), \quad (1.1)$$

where $\Phi(z)$ and $\Theta(z)$ are polynomials with real coefficients of degrees p, q , respectively, satisfying the following condition:

$\Phi(z)$ and $\Theta(z)$ have no common zeros, and $\Phi(z) \neq 0$

$$\Theta(z) \neq 0 \text{ for all } z \text{ in the closed unit disk } \{z \in \mathbf{C} : |z| \leq 1\}. \quad (A1)$$

We also assume, without loss of generality, that

$$\Theta(0)/\Phi(0) > 0. \quad (A2)$$

Note that (A1) and (A2) imply $\Theta(1)/\Phi(1) > 0$.

The fractional ARIMA process $\{X_n\}$ satisfies a difference equation of the form

$$\Phi(B)\nabla^d X_n = \Theta(B)Z_n \quad (n \in \mathbf{Z}), \quad (1.2)$$

where B is the backward shift operator, i.e., $BX_m = X_{m-1}$, ∇ is the differencing operator defined by $\nabla := 1 - B$, and $\{Z_n\}$ is a zero-mean process such that $E[Z_n Z_m] = \delta_{nm}$. See [2, Section 13.2] for details. We notice that in (1.2) the degree of *fractional* differencing is given by d .

If $d \in (-1/2, 1/2) \setminus \{0\}$, then the fractional ARIMA(p, d, q) process $\{X_n\}$ is a long-memory process in the sense that the autocovariance $\gamma(n)$ decays slowly as

$$\gamma(n) \sim Cn^{2d-1} \quad (n \rightarrow \infty), \quad (1.3)$$

where the constant C is given by

$$C := \frac{\Gamma(1-2d) \sin(\pi d)}{\pi} \left\{ \frac{\Theta(1)}{\Phi(1)} \right\}^2 \quad (1.4)$$

(see, e.g., [8]). Notice that the case $d = 0$ corresponds to the ordinary ARMA(p, q) process, for which the autocovariance $\gamma(n)$ decays exponentially as $n \rightarrow \infty$ (see [2, Chapter 3]). From (1.3), we see that the degree d is closely related to the long-range dependence of $\{X_n\}$.

The partial autocorrelation $\alpha(n)$ of a stationary process $\{X_n\}$ is the correlation coefficient of the two residuals obtained after regressing X_0 and X_n on the

intermediate observations X_1, \dots, X_{n-1} . To be more precise, we denote by H the closed real linear hull of $\{X_k : k \in \mathbf{Z}\}$ in $L^2(\Omega, \mathcal{F}, P)$. Then H is a real Hilbert space with inner product $(Y_1, Y_2) := E[Y_1 Y_2]$ and norm $\|Y\| := (Y, Y)^{1/2}$. For $n \in \mathbf{N}$, we write $H_{[1,n]}$ for the subspace of H spanned by $\{X_1, \dots, X_n\}$, and $H_{[1,n]}^\perp$ for its orthogonal complement in H . We denote by $P_{[1,n]}^\perp$ the orthogonal projection operator of H onto $H_{[1,n]}^\perp$. The *partial autocorrelation function* $\alpha(\cdot)$ of $\{X_n\}$ is defined by

$$\alpha(n) := \frac{(P_{[1,n-1]}^\perp X_n, P_{[1,n-1]}^\perp X_0)}{\|P_{[1,n-1]}^\perp X_n\| \cdot \|P_{[1,n-1]}^\perp X_0\|} \quad (n = 2, 3, \dots).$$

Furthermore, $\alpha(1)$ is defined by $\alpha(1) := \gamma(1)/\gamma(0)$. The partial autocorrelation function plays an important role in time-series analysis. Its importance is illustrated in the fact that it appears in the Durbin–Levinson algorithm (see [2, Proposition 5.2.1]).

In [8, Theorem 1.1], the first author has proved that if $0 < d < 1/2$ the partial autocorrelation function $\alpha(\cdot)$ of the fractional ARIMA(p, d, q) process satisfies

$$|\alpha(n)| \sim \frac{d}{n} \quad (n \rightarrow \infty). \quad (1.5)$$

Notice that the degree d , which is important in the fractional ARIMA process, appears explicitly in (1.5). We wish to extend this asymptotic formula to cover the case $-1/2 < d < 0$.

Here is the main theorem.

Theorem 1.1. *Let $p, q \in \mathbf{N} \cup \{0\}$ and $-1/2 < d < 0$, and let $\{X_n\}$ be a fractional ARIMA(p, d, q) process with partial autocorrelation function $\alpha(\cdot)$. Then $\alpha(\cdot)$ satisfies*

$$|\alpha(n)| \sim \frac{|d|}{n} \quad (n \rightarrow \infty). \quad (1.6)$$

See Section 5 for numerical calculation of $n\alpha(n)$.

As in [7,8], we deduce (1.6) from the asymptotic behaviour of the mean squared prediction error $\|P_{[1,n-1]}^\perp X_n\|$ as $n \rightarrow \infty$, using a Tauberian argument. However, there is one distinction in the proof. The proofs of [7,8] are based on an explicit representation of $\|P_{[1,n-1]}^\perp X_n\|$ (see [7, Theorems 4.5 and 4.6; 8, Theorem 4.1]) in terms of the AR(∞) coefficients a_k and MA(∞) coefficients c_k of $\{X_n\}$. The same representation is not available in the present case $-1/2 < d < 0$ because the series that appear in the representation do not converge absolutely. It turns out that if $-1/2 < d < 0$ we can use a similar representation of $\|P_{[1,n-1]}^\perp X_n\|$ (Theorems 2.2 and 3.3) which is given in terms of ϕ_k and ψ_k defined by

$$\phi_n := \begin{cases} a_0 & (n = 0), \\ a_n - a_{n-1} & (n = 1, 2, \dots) \end{cases} \quad (1.7)$$

and

$$\psi_n := - \sum_{k=n+1}^{\infty} c_k \quad (n = 0, 1, \dots), \quad (1.8)$$

respectively, rather than given in terms of a_k and c_k themselves. Once the representation is obtained, the proof is parallel to that of [8].

In what follows, we write $\sum_{k=0}^{\infty -}$ for the sums that are not necessarily absolutely convergent:

$$\sum_{k=0}^{\infty -} := \lim_{M \rightarrow \infty} \sum_{k=0}^M.$$

2. Representation of the prediction error (1)

In this section, we assume that $\{X_n\}$ is a purely nondeterministic stationary process (hence not necessarily a fractional ARIMA process). Let $\Delta(\cdot)$ be the spectral density of $\{X_n\}$. We define the outer function $h(\cdot)$ of $\{X_n\}$ by

$$h(z) := \sqrt{2\pi} \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\lambda} + z}{e^{i\lambda} - z} \log \Delta(\lambda) d\lambda \right\} \quad (z \in \mathbb{C}, |z| < 1).$$

The function $h(\cdot)$ is actually an outer function which is in the Hardy space H^{2+} of class 2 over the unit disk $|z| < 1$. Using $h(\cdot)$, we define the $\text{MA}(\infty)$ coefficients c_n of $\{X_n\}$ by

$$h(z) = \sum_{n=0}^{\infty} c_n z^n \quad (|z| < 1)$$

and the $\text{AR}(\infty)$ coefficients a_n of $\{X_n\}$ by

$$-\frac{1}{h(z)} = \sum_{n=0}^{\infty} a_n z^n \quad (|z| < 1).$$

See, e.g., [7] for background.

As we stated in Section 1, we write H for the real Hilbert space spanned by $\{X_k : k \in \mathbb{Z}\}$ in $L^2(\Omega, \mathcal{F}, P)$, with inner product $(Y_1, Y_2) := E[Y_1 Y_2]$ and norm $\|Y\| := (Y, Y)^{1/2}$. For $I \subset \mathbb{Z}$, denote by H_I the closed real linear hull of $\{X_k : k \in I\}$ in H . In particular, for $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ with $m \leq n$, we write $H_{(-\infty, m]}$, $H_{[m, \infty)}$ and $H_{[m, n]}$ for H_I with $I = \{k \in \mathbb{Z} : -\infty < k \leq m\}$, $\{k \in \mathbb{Z} : m \leq k < \infty\}$, and $\{k \in \mathbb{Z} : m \leq k \leq n\}$, respectively. For $I \subset \mathbb{Z}$, we denote by P_I the orthogonal projection operator of H onto H_I . We write $P_I^\perp := I_H - P_I$, where I_H is the identity map of H . So P_I^\perp is the orthogonal projection operator of H onto H_I^\perp .

We now consider $P_{(-\infty, 0]} X_n$ for $n \geq 1$. We define

$$b_j^m := \sum_{k=0}^m c_k a_{j+m-k} \quad (m, j \in \mathbb{N} \cup \{0\}).$$

Notice that b_j^m for $j \geq 1$ here corresponds to b_{j-1}^{m+1} defined in [7, (4.4)].

Proposition 2.1. *We assume the following conditions:*

$$\sum_{k=0}^{\infty} |c_k| < \infty, \quad (2.1)$$

$$\sum_{k=0}^{\infty} |a_k|^2 < \infty. \quad (2.2)$$

Then, for $n \in \mathbf{N}$,

$$P_{(-\infty, 0]} X_n = \sum_{j=0}^{\infty} b_{j+1}^{n-1} X_{-j}, \quad (2.3)$$

the sum converging in H (not necessarily absolutely).

Proof (Compare the proof of Inoue [7, Theorem 4.4]). Consider the spectral representation of $\{X_n\}$ written as

$$X(n) = \int_{-\pi}^{\pi} e^{in\lambda} Z(d\lambda) \quad (n \in \mathbf{Z}),$$

where Z is the spectral measure such that

$$E[Z(A)\overline{Z(B)}] = \int_{A \cap B} \Delta(\lambda) d\lambda$$

(see [2, Section 4.8]). We put

$$\zeta_n := \int_{-\pi}^{\pi} e^{in\lambda} \{\overline{h(e^{i\lambda})}\}^{-1} Z(d\lambda) \quad (n \in \mathbf{Z}).$$

Then it follows that

$$\left\| \sum_{k=0}^m a_k X_{n-k} + \zeta_n \right\|^2 = \int_{-\pi}^{\pi} |f_m(\lambda)|^2 \Delta(\lambda) d\lambda \quad (m \in \mathbf{N}), \quad (2.4)$$

where

$$f_m(\lambda) := \frac{1}{h(e^{i\lambda})} + \sum_{k=0}^m a_k e^{ik\lambda} \quad (-\pi < \lambda < \pi).$$

Since (2.2) implies $h^{-1} \in H^{2+}$ (cf. [7, Proposition 4.2]), we have the Fourier expansion $1/h(e^{i\lambda}) = -\sum_{k=0}^{\infty} a_k e^{ik\lambda}$ in $L^2((-\pi, \pi), d\lambda)$, which yields $f_m(\lambda) = -\sum_{m+1}^{\infty} a_k e^{ik\lambda}$. Thus it follows from (2.2) that $f_m(\cdot)$ converges to zero as $m \rightarrow \infty$ in $L^2((-\pi, \pi), d\lambda)$. On the other hand, (2.1) implies that

$$2\pi\Delta(\lambda) = \left| \sum_{n=0}^{\infty} c_n e^{in\lambda} \right|^2 \leq \left(\sum_{n=0}^{\infty} |c_n| \right)^2 \quad \text{a.e. on } (-\pi, \pi),$$

hence the spectral density $\Delta(\cdot)$ is essentially bounded on $(-\pi, \pi)$. Thus the integral on the right-hand side of (2.4) tends to zero as $m \rightarrow \infty$, and so we obtain the $\text{AR}(\infty)$

representation of $\{X_n\}$ of the form

$$\sum_{j=0}^{\infty-} a_j X_{n-j} + \xi_n = 0 \quad (n \in \mathbf{Z}).$$

As in the proof of [7, Theorem 4.4], this allows us to obtain (2.3). \square

We shall see in Section 3 that the fractional ARIMA(p, d, q) process with $-1/2 < d < 0$ satisfies (2.1) and (2.2).

We put

$$\varepsilon(n) := \frac{\|P_{[-n+2,0]}^\perp X_1\|^2 - \|P_{(-\infty,0]}^\perp X_1\|^2}{\|P_{(-\infty,0]}^\perp X_1\|^2} \quad (n = 2, 3, \dots). \quad (2.5)$$

Notice that this definition is slightly different from that in [7, (4.11)]. In fact, $\varepsilon(n)$ here corresponds to $\varepsilon(n-2)$ in [7]. The next theorem is an analogue of [7, Theorem 4.5].

Theorem 2.2. *We assume (2.1) and (2.2). Then, for $n = 2, 3, \dots$,*

$$\varepsilon(n) = \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} d_k(n, p)^2, \quad (2.6)$$

where $d_1(n, p) := \sum_{v=0}^{\infty-} a_{v+n+p} c_v$ and, for $k = 2, 3, \dots$,

$$d_k(n, p) := \sum_{m_{k-1}=0}^{\infty-} a_{n+m_{k-1}} \sum_{m_{k-2}=0}^{\infty-} b_{n+m_{k-2}}^{m_{k-1}} \cdots \sum_{m_1=0}^{\infty-} b_{n+m_1}^{m_2} \sum_{v=0}^{\infty-} b_{n+p+v}^{m_1} c_v.$$

Notice that $d_k(n, p)$ above corresponds to $d_k(n-2, p)$ defined in [7, Theorem 4.5]. We can prove Theorem 2.2 using [7, Theorem 3.1] and Proposition 2.1, in the same way as the proof of [7, Theorem 4.5]. We omit the details.

3. Representation of the prediction error (2)

In this section, we assume that $\{X_n\}$ is a fractional ARIMA(p, d, q) with $p, q \in \mathbf{N} \cup \{0\}$ and

$$-1/2 < d < 0. \quad (3.1)$$

Let a_n and c_n be as in Section 2. We define ϕ_n and ψ_n by (1.7) and (1.8), respectively. By (3.5), the sequence (c_n) satisfies (2.1), and so ψ_n are well defined. In the arguments below, we write C for positive constants which are not necessarily the same.

Since the spectral density $\Delta(\cdot)$ of $\{X_n\}$ is given by (1.1), we can write the outer function $h(\cdot)$ of $\{X_n\}$ explicitly as

$$h(z) = \frac{\Theta(z)}{\Phi(z)} (1-z)^{-d} \quad (|z| < 1) \quad (3.2)$$

(cf. [8, (2.1)]). Since we have assumed (3.1), it follows from (3.2) that

$$\sum_{k=0}^{\infty} c_k = 0. \quad (3.3)$$

This is one of the key features of the fractional ARIMA process $\{X_n\}$ with (3.1).

Now we have

$$|a_n| \leq \frac{C}{(n+1)^{1+d}} \quad (n = 0, 1, \dots), \quad (3.4)$$

$$|c_n| \leq \frac{C}{(n+1)^{1-d}} \quad (n = 0, 1, \dots), \quad (3.5)$$

$$|\phi_n| \leq \frac{C}{(n+1)^{2+d}} \quad (n = 0, 1, \dots). \quad (3.6)$$

See [12, Section 3; 8, Lemma 2.2]. From (3.5) and (3.4), we find that c_n and a_n satisfy (2.1) and (2.2), respectively. It also follows from (3.5) that

$$|\psi_n| \leq \frac{C}{(1+n)^{-d}} \quad (n = 0, 1, \dots). \quad (3.7)$$

We define

$$\beta(n) := \sum_{v=0}^{\infty} \psi_v \phi_{v+n+1} \quad (n = 0, 1, \dots)$$

and

$$B(n) := \sum_{v=0}^{\infty} |\psi_v| \cdot |\phi_{v+n+1}| \quad (n = 0, 1, \dots).$$

Since

$$\int_0^{\infty} \frac{dv}{v^{-d}(v+x)^{2+d}} = \frac{1}{x(1+d)} \quad (0 < x < \infty),$$

we can apply (3.6) and (3.7) to obtain

$$|\beta(n)| \leq B(n) \leq \frac{C}{(n+1)} \quad (n = 0, 1, \dots). \quad (3.8)$$

For $k \in \mathbf{N}$ and $n, p, m \in \mathbf{N} \cup \{0\}$, we define $D_k(n, p, m)$ inductively by

$$D_1(n, p, m) := B(n+p+m),$$

$$D_{k+1}(n, p, m) := \sum_{m_k=0}^{\infty} B(m+m_k+n) D_k(n, p, m_k) \quad (k = 1, 2, \dots).$$

Lemma 3.1. For $k \in \mathbf{N}$, we have

$$\sum_{m=0}^{\infty} D_k(n, p, m)^2 < \infty \quad (n, p \in \mathbf{N} \cup \{0\}). \quad (3.9)$$

Proof. We use induction. By (3.8), we find that (3.9) holds for $k = 1$. We assume that (3.9) holds for $k \geq 1$. Then by (3.8), we have

$$D_{k+1}(n, p, m) \leq C \sum_{m_k=0}^{\infty} \frac{1}{m + m_k + 1} D_k(n, p, m_k).$$

Since the operator T defined by

$$(Tu)_m := \sum_{i=0}^{\infty} \frac{u_i}{m + i + 1} \quad (u = (u_i) \in l^2)$$

is a bounded linear operator from l^2 to l^2 (see [5, Chapter IX]), the inequality above implies (3.9) for $k + 1$. Thus the lemma follows by induction on k . \square

For $k \in \mathbf{N}$ and $n, p, m \in \mathbf{N} \cup \{0\}$, we define $\delta_k(n, p, m)$ inductively by

$$\delta_1(n, p, m) := -\beta(m + n + p),$$

$$\delta_{k+1}(n, p, m) := - \sum_{m_k=0}^{\infty} \beta(m + m_k + n) \delta_k(n, p, m_k) \quad (k = 1, 2, \dots).$$

By (3.8) and Lemma 3.1, $\delta_k(n, p, m)$ are well-defined and the following inequality holds:

$$|\delta_k(n, p, m)| \leq D_k(n, p, m) \quad (k \in \mathbf{N}, n, p, m \in \mathbf{N} \cup \{0\}).$$

For $k \in \mathbf{N}$ and $n, p, m \in \mathbf{N} \cup \{0\}$, we define $d_k(n, p, m)$ also inductively by

$$d_1(n, p, m) := \sum_{v_1=0}^{\infty-} b_{v_1+n+p}^m c_{v_1},$$

$$d_{k+1}(n, p, m) := \sum_{m_k=0}^{\infty-} b_{m_k+n}^m d_k(n, p, m_k) \quad (k = 1, 2, \dots).$$

We notice that Theorem 2.2 includes the assertion that these sums converge.

Proposition 3.2. For $k \in \mathbf{N}$, we have

$$d_k(n, p, m) = \sum_{v=0}^m c_v \delta_k(n, p, m - v) \quad (p \geq 0, m \geq 0, n \geq 2). \quad (3.10)$$

Proof. We use induction. By (3.3) and summation by parts, we have

$$\begin{aligned} d_1(n, p, m) &= \sum_{v_1=0}^{\infty-} \left(\sum_{v=0}^m c_v a_{n+p+v_1+m-v} \right) c_{v_1} \\ &= - \sum_{v_1=0}^{\infty-} \left(\sum_{v=0}^m c_v \phi_{n+1+p+v_1+m-v} \right) \psi_{v_1}, \end{aligned}$$

which, by (3.8) and Fubini's theorem, implies (3.10) with $k = 1$.

Now we assume (3.10) for $k \geq 1$. From (3.4), (3.5), and Lemma 3.1, we find that, for $m \geq 0$,

$$\begin{aligned} &\sum_{v_{k+1}=0}^{\infty} |c_{v_{k+1}}| \sum_{m_k=0}^{\infty} |a_{n+m_k+v_{k+1}+m}| \cdot |\delta_k(n, p, m_k)| \\ &\leq C \left(\sum_{v_{k+1}=0}^{\infty} |c_{v_{k+1}}| \right) \sum_{m_k=0}^{\infty} \frac{1}{(m_k + 1)^{1+d}} D_k(n, p, m_k) < \infty. \end{aligned}$$

Hence, by Fubini's theorem, we obtain

$$\begin{aligned} d_{k+1}(n, p, m) &= \sum_{m_k=0}^{\infty-} \left(\sum_{v=0}^m c_v a_{n+m_k+m-v} \right) \sum_{v_{k+1}=0}^{m_k} c_{v_{k+1}} \delta_k(n, p, m_k - v_{k+1}) \\ &= \sum_{v=0}^m c_v \sum_{v_{k+1}=0}^{\infty} c_{v_{k+1}} \sum_{m_k=v_{k+1}}^{\infty} a_{n+m_k+m-v} \delta_k(n, p, m_k - v_{k+1}) \\ &= \sum_{v=0}^m c_v \sum_{v_{k+1}=0}^{\infty} c_{v_{k+1}} \sum_{m_k=0}^{\infty} a_{n+m_k+v_{k+1}+m-v} \delta_k(n, p, m_k). \end{aligned}$$

Applying summation by parts to this, we get

$$d_{k+1}(n, p, m) = - \sum_{v=0}^m c_v \sum_{v_{k+1}=0}^{\infty-} \psi_{v_{k+1}} \sum_{m_k=0}^{\infty} \phi_{n+1+m_k+v_{k+1}+m-v} \delta_k(n, p, m_k),$$

which, together with Lemma 3.1 and Fubini's theorem, implies (3.10) for $k + 1$. Thus the proposition follows by induction on k . \square

Recall $d_k(n, p)$ from Section 2. The next theorem, combined with Theorem 2.2, gives the desired representation of $\varepsilon(n)$ in terms of ϕ_k and ψ_k .

Theorem 3.3. For $n = 2, 3, \dots$ and $p \in \mathbf{N} \cup \{0\}$, we have

$$\begin{aligned} d_1(n, p) &= -\beta(n + p), \\ d_2(n, p) &= \sum_{m_1=0}^{\infty} \beta(m_1 + n) \beta(m_1 + n + p), \end{aligned}$$

and, for $k \geq 3$,

$$\begin{aligned} d_k(n, p) = & (-1)^k \sum_{m_{k-1}=0}^{\infty} \beta(m_{k-1} + n) \sum_{m_{k-2}=0}^{\infty} \beta(m_{k-1} + m_{k-2} + n) \\ & \cdots \sum_{m_2=0}^{\infty} \beta(m_3 + m_2 + n) \sum_{m_1=0}^{\infty} \beta(m_2 + m_1 + n) \beta(m_1 + n + p), \end{aligned}$$

the sums converging absolutely.

Proof. Since $d_k(n, p, 0) = c_0 d_k(n, p)$, the theorem follows immediately from Proposition 3.2. \square

4. Proof of the main theorem

In this section, we prove Theorem 1.1. We assume that $\{X_n\}$ is a fractional ARIMA(p, d, q) process with (3.1). Let $\Phi(z)$, $\Theta(z)$, a_n , c_n , ϕ_n , and ψ_n be as in the previous section.

As in [8, Section 2], for $\delta \in \mathbf{R}$, $n \in \mathbf{N} \cup \{0\}$, and a real sequence $(\lambda_k)_{k=0}^{\infty}$, we define $\lambda_n(\delta)$ by the following equality for formal power series:

$$(1 - z)^{\delta} \sum_{k=0}^{\infty} \lambda_k z^k = \sum_{k=0}^{\infty} \lambda_k(\delta) z^k \quad (|z| < 1).$$

Using the binomial coefficients, we can write $\lambda_n(\delta)$ as

$$\lambda_n(\delta) = \sum_{k=0}^n \lambda_k (-1)^{n-k} \binom{\delta}{n-k} \quad (n = 0, 1, \dots).$$

We notice that

$$\lambda_n(\delta) - \lambda_{n-1}(\delta) = \lambda_n(\delta + 1) \quad (\delta \in \mathbf{R}, n \in \mathbf{N}). \quad (4.1)$$

We define two real sequences $(\lambda_n)_{n=0}^{\infty}$ and $(\mu_n)_{n=0}^{\infty}$ by

$$-\frac{\Phi(z)}{\Theta(z)} = \sum_{n=0}^{\infty} \lambda_n z^n \quad (|z| < 1)$$

and

$$\frac{\Theta(z)}{\Phi(z)} = \sum_{n=0}^{\infty} \mu_n z^n \quad (|z| < 1),$$

respectively. One can easily show that both (λ_n) and (μ_n) decay exponentially.

Lemma 4.1. For $n = 0, 1, \dots$,

$$\phi_n = \lambda_n(d + 1), \quad (4.2)$$

$$\psi_n = \mu_n(-d - 1). \quad (4.3)$$

Proof. Let $|z| < 1$. Then by (3.2), we have

$$\sum_{n=0}^{\infty} a_n z^n = (1-z)^d \sum_{n=0}^{\infty} \lambda_n z^n = \sum_{n=0}^{\infty} \lambda_n(d) z^n,$$

hence $a_n = \lambda_n(d)$ for $n \geq 0$. Therefore, using (4.1), we obtain (4.2).

Similarly, it follows from (3.2) that $\sum_n c_n z^n = (1-z)^{-d} \sum_n \mu_n z^n$. Therefore, using (3.6), we obtain

$$(1-z)^{-d-1} \sum_{n=0}^{\infty} \mu_n z^n = \left(\sum_{n=0}^{\infty} z^n \right) \left(\sum_{n=0}^{\infty} c_n z^n \right) = \sum_{n=0}^{\infty} \psi_n z^n,$$

whence (4.3). \square

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Using Theorems 2.2 and 3.3, and Lemma 4.1, we run through the arguments in [8, Sections 3 and 4] with d , a_n , and c_n replaced by $d+1$, ϕ_n , and ψ_n , respectively. Then, as in [8, Theorem 4.3], we find that

$$\lim_{n \rightarrow \infty} n\varepsilon(n) = \frac{1}{\pi^2} \arcsin^2 \{ \sin((d+1)\pi) \} = d^2$$

or

$$\varepsilon(n) \sim \frac{d^2}{n} \quad (n \rightarrow \infty) \quad (4.4)$$

(this result has its own interest and we refer to [7, Theorem 6.4; 9–11] for relevant work). We also find that, as in [8, Proposition 4.4],

$$\forall \lambda > 1, \quad \limsup_{n \rightarrow \infty} \sup_{n \leq m \leq \lambda n} n^2 \{ \delta(m) - \delta(n) \} \leq 0 \quad (\text{hence} = 0), \quad (4.5)$$

where

$$\delta(n) := \varepsilon(n) - \varepsilon(n+1) \quad (n = 2, 3, \dots).$$

Notice that

$$\sum_{k=n}^{\infty} \delta(k) = \varepsilon(n) \quad (n = 2, 3, \dots). \quad (4.6)$$

By (4.4)–(4.6), the Monotone Density Theorem (see [1, Section 1.7.6]) gives

$$\delta(n) \sim \frac{d^2}{n^2} \quad (n \rightarrow \infty).$$

Since the Durbin–Levinson algorithm implies

$$\alpha(n)^2 \sim \delta(n) \quad (n \rightarrow \infty)$$

(see [7, p. 101]), (1.6) follows. \square

5. Estimation of the differencing parameter

For a fractional ARIMA(p, d, q) process with $d \in (-1/2, 1/2) \setminus \{0\}$, Theorem 1.1 and [8, Theorem 1.1] imply that

$$\lim_{n \rightarrow \infty} n|\alpha(n)| = |d|. \quad (5.1)$$

The question arises if (5.1) gives an efficient method for estimation of the important parameter d (cf. [2, Section 13.2; 3]). We leave this question open here. See Tables 1 and 2 for the values of $n\alpha(n)$ for various n and d . It should be noticed that the values of $\alpha(n)$ there are not statistically estimated ones via, say, computer simulation. They are the exact values (modulo figures of order 10^{-4}) that are calculated from the exact values of the autocovariance function $\gamma(\cdot)$ via the Durbin–Levinson algorithm. The

Table 1

Values of $n\alpha(n)$ for FARIMA(1, d , 0) with $\Theta(z) = 1$ and $\Phi(z) = 1 - 0.3z$

d	$n = 1$	$n = 10$	$n = 20$	$n = 50$	$n = 100$
-0.4	-0.048	-0.355	-0.376	-0.390	-0.395
-0.3	0.025	-0.268	-0.284	-0.293	-0.297
-0.2	0.106	-0.181	-0.190	-0.196	-0.198
-0.1	0.198	-0.091	-0.095	-0.098	-0.099
0.1	0.414	0.093	0.096	0.099	0.099
0.2	0.542	0.187	0.194	0.197	0.199
0.3	0.682	0.284	0.292	0.297	0.298
0.4	0.835	0.382	0.391	0.396	0.398

Table 2

Values of $n\alpha(n)$ for FARIMA(0, d , 1) with $\Theta(z) = 1 - 0.5z$ and $\Phi(z) = 1$

d	$n = 1$	$n = 10$	$n = 20$	$n = 50$	$n = 100$
-0.4	-0.535	-0.488	-0.436	-0.413	-0.407
-0.3	-0.509	-0.373	-0.329	-0.311	-0.305
-0.2	-0.479	-0.255	-0.220	-0.207	-0.204
-0.1	-0.443	-0.133	-0.111	-0.104	-0.102
0.1	-0.345	0.123	0.112	0.104	0.102
0.2	-0.271	0.256	0.225	0.209	0.205
0.3	-0.156	0.394	0.340	0.314	0.307
0.4	0.071	0.536	0.455	0.420	0.410

values of $\gamma(\cdot)$ are, in turn, obtained using the analytic representations of $\gamma(\cdot)$ for the fractional ARIMA(1, d , 0) and (0, d , 1) processes (Lemmas 1 and 2 in [6, Section 5]).

Acknowledgments

We thank two referees for their helpful comments.

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