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Hessian orders and multinormal distributions

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ABSTRACT

Several well known integral stochastic orders (like the convex order, the supermodular order, etc.) can be defined in terms of the Hessian matrix of a class of functions. Here we consider a generic Hessian order, i.e., an integral stochastic order defined through a convex cone $\mathcal{H}$ of Hessian matrices, and we prove that if two random vectors are ordered by the Hessian order, then their means are equal and the difference of their covariance matrices belongs to the dual of $\mathcal{H}$. Then we show that the same conditions are also sufficient for multinormal random vectors. We study several particular cases of this general result.

1. Introduction

Stochastic orders have been extensively used in statistics, operations research, actuarial sciences, economic theory, queuing theory, etc. The reader is referred to the monographs by Müller and Stoyan [1] and Shaked and Shanthikumar [2] for a detailed exposition of the field.

An important family of orders is obtained by comparing expectations of functions in a certain class. A general treatment for these orders, called "integral orders", has been provided by Müller [3]. The class of functions used to define an integral order can be chosen, for instance, by specifying some properties of its derivatives. In the univariate case, if we choose functions whose first derivative is nonnegative we get the usual stochastic order, if we choose functions whose second derivative is nonnegative we get the convex order, etc. Similarly, in the multivariate case several orders are obtained through properties of the first or second derivatives of the functions used for the comparison. In this paper we focus on functions whose Hessian matrix has some properties. In this way we obtain orders for convex, supermodular, directionally convex, and many other functions.

We consider a convex cone of squared matrices $\mathcal{H}$ and we call $\mathcal{F}_{\mathcal{H}}$ the class of functions whose Hessian matrix is in $\mathcal{H}$. We show that if a random vector $\boldsymbol{X}$ is dominated by $\boldsymbol{Y}$ with respect to the integral stochastic order generated by $\mathcal{F}_{\mathcal{H}}$ then their means are equal, and the difference of the covariance matrices of $\boldsymbol{Y}$ and $\boldsymbol{X}$ belongs to the dual cone $\mathcal{H}^*$. We then prove that these necessary conditions are sufficient for multinormal random vectors.

Houdré et al. [4] and Müller [5] studied integral convex orders of multinormal vectors. Block and Sampson [6], Bäuerle [7], Scarsini [8] and Müller and Scarsini [9] studied orders generated by particular classes $\mathcal{F}$ and they gave results for the

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multinormal case. The main theorem of this paper generalizes and extends several of these results. Various new stochastic orders are studied in this framework.

Finally, we consider functions that are defined in terms of their first and second derivatives and the integral stochastic orders that they generate. We obtain sufficient conditions for the orders generated by these functions when the random vectors have multinormal distributions, and we show that in some cases these conditions are also necessary.

2. Notation and auxiliary results

In this section we recall some well known results about orders of random vectors and we fix the notation that will be used in the sequel.

Definition 2.1. A subset $\mathcal{C}$ of a vector space V is a cone if, for every $x \in \mathcal{C}$ and for every $\lambda \geq 0$, $\lambda x \in \mathcal{C}$. The cone $\mathcal{C}$ is convex if and only if $\alpha x + \beta y \in \mathcal{C}$ for all $\alpha, \beta \geq 0$ and for all $x, y \in \mathcal{C}$.

Definition 2.2. Let $\mathcal{C}$ be a closed convex cone in a vector space V endowed with an inner product $\langle \cdot, \cdot \rangle$. Then

$$\mathcal{C}^* = \{y \in V : \langle x, y \rangle \geq 0 \text{ for all } x \in \mathcal{C}\}$$

is called the dual of $\mathcal{C}$. The set $\mathcal{C}^*$ is a closed convex cone.

If $V = \mathcal{S}$, the space of symmetric $n \times n$ matrices, then define, for $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{S}$,

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \mathrm{Tr}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}) = \mathrm{Tr}(\boldsymbol{AB}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij},$$

where $\boldsymbol{A} = [a_{ij}]_{i,j=1}^n$ and $\boldsymbol{B} = [b_{ij}]_{i,j=1}^n$.

Given a twice-differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, call $\boldsymbol{H}_f$ its Hessian matrix:

$$\boldsymbol{H}_f(\boldsymbol{x}) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{x}) \right]_{i,j \in \{1,\dots,n\}}.$$

For $\mathcal{H} \in \mathcal{S}$ call $\mathcal{F}_{\mathcal{H}}$ the class of twice-differentiable functions f such that $\boldsymbol{H}_f(\boldsymbol{x}) \in \mathcal{H}$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

Definition 2.3. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors. Given a class $\mathcal{F}$ of functions $f : \mathbb{R}^n \to \mathbb{R}$, we say that $\boldsymbol{X}$ is smaller than $\boldsymbol{Y}$ with respect to $\mathcal{F}$ (written $\boldsymbol{X} \leq_{\mathcal{F}} \boldsymbol{Y}$) if

$$\mathbb{E}[f(\boldsymbol{X})] \leq \mathbb{E}[f(\boldsymbol{Y})] \tag{2.1}$$

for all $f \in \mathcal{F}$, such that the expectations exist. An order defined this way is called *integral stochastic order*. If $\mathcal{F} = \mathcal{F}_{\mathcal{H}}$ for some class $\mathcal{H} \subset \mathcal{S}$, then the order is called *Hessian order*.

Definition 2.4. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *supermodular* if

$$f(\boldsymbol{x}) + f(\boldsymbol{y}) \leq f(\boldsymbol{x} \wedge \boldsymbol{y}) + f(\boldsymbol{x} \vee \boldsymbol{y}) \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n.$$

(See, e.g., [10].)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *directionally convex* if its increments are increasing, i.e.,

$$f(\boldsymbol{x} + \boldsymbol{h}) - f(\boldsymbol{x}) \leq f(\boldsymbol{y} + \boldsymbol{h}) - f(\boldsymbol{y}) \quad \text{for all } \boldsymbol{x} \leq \boldsymbol{y} \text{ and } \boldsymbol{h} \geq \boldsymbol{0}.$$

Directionally convex functions are sometimes called ultramodular (see, e.g., [11]).

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *linear-convex* if

$$f(\boldsymbol{x}) = \psi(\langle \boldsymbol{a}, \boldsymbol{x} \rangle), \tag{2.2}$$

with $\boldsymbol{a} \in \mathbb{R}^n$, and $\psi : \mathbb{R} \to \mathbb{R}$ convex.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *positive-linear-convex* if (2.2) holds with $\boldsymbol{a} \in \mathbb{R}^n_+$, and $\psi : \mathbb{R} \to \mathbb{R}$ convex.

The following well known integral orders will be considered in the sequel. The reader is referred to [1,2] for their properties.

(a) $\boldsymbol{X} \leq_{\mathrm{cx}} \boldsymbol{Y}$ if (2.1) holds for all convex functions f.
(b) $\boldsymbol{X} \leq_{\mathrm{lcx}} \boldsymbol{Y}$ if (2.1) holds for all linear-convex functions f.
(c) $\boldsymbol{X} \leq_{\mathrm{ccx}} \boldsymbol{Y}$ if (2.1) holds for all componentwise convex functions f.
(d) $\boldsymbol{X} \leq_{\mathrm{sm}} \boldsymbol{Y}$ if (2.1) holds for all supermodular functions f.
(e) $\boldsymbol{X} \leq_{\mathrm{dcx}} \boldsymbol{Y}$ if (2.1) holds for all directionally convex functions f.
(f) $\boldsymbol{X} \leq_{\mathrm{plcx}} \boldsymbol{Y}$ if (2.1) holds for all positive-linear-convex functions f.
(g) $\boldsymbol{X} \leq_{\mathrm{st}} \boldsymbol{Y}$ if (2.1) holds for all increasing functions f.
(h) $\boldsymbol{X} \leq_{\mathrm{iplcx}} \boldsymbol{Y}$ if (2.1) holds for all increasing positive-linear-convex functions f.
(i) $\boldsymbol{X} \leq_{\mathrm{ism}} \boldsymbol{Y}$ if (2.1) holds for all increasing supermodular functions f.

The notation $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu_X}, \boldsymbol{\Sigma_X})$ means that $\boldsymbol{X}$ has a multinormal distribution with mean $\boldsymbol{\mu_X}$ and covariance matrix $\boldsymbol{\Sigma_X}$.

The proofs of the following two results can be found in [4,5]; see also [12].

Theorem 2.5. *Let $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu_X}, \boldsymbol{\Sigma_X})$ and $\boldsymbol{Y} \sim \mathcal{N}(\boldsymbol{\mu_Y}, \boldsymbol{\Sigma_Y})$, with $\boldsymbol{\Sigma_X}$ and $\boldsymbol{\Sigma_Y}$ positive definite, and let ϕ_λ be the density function of*

$$\mathcal{N}(\lambda\boldsymbol{\mu_Y} + (1-\lambda)\boldsymbol{\mu_X}, \lambda\boldsymbol{\Sigma_Y} + (1-\lambda)\boldsymbol{\Sigma_X}), \quad \lambda \in [0, 1].$$

Moreover, assume that $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable with $f(\boldsymbol{x}) = O(\|\boldsymbol{x}\|)$ and $\nabla f(\boldsymbol{x}) = O(\|\boldsymbol{x}\|)$ at infinity. Then

$$\mathbb{E}[f(\boldsymbol{Y})] - \mathbb{E}[f(\boldsymbol{X})] = \int_0^1 \int_{\mathbb{R}^n} \left((\boldsymbol{\mu_Y} - \boldsymbol{\mu_X})^{\mathrm{T}} \nabla f(\boldsymbol{x}) + \frac{1}{2}\mathrm{Tr}((\boldsymbol{\Sigma_Y} - \boldsymbol{\Sigma_X})\boldsymbol{H}_f(\boldsymbol{x})) \right) \phi_\lambda(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\lambda.$$

Corollary 2.6. *Let $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu_X}, \boldsymbol{\Sigma_X})$ and $\boldsymbol{Y} \sim \mathcal{N}(\boldsymbol{\mu_Y}, \boldsymbol{\Sigma_Y})$. Assume that $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the conditions of Theorem 2.5. Then $\mathbb{E}[f(\boldsymbol{X})] \le \mathbb{E}[f(\boldsymbol{Y})]$ if the following two conditions hold:*

$$\sum_{i=1}^n (\mu_{X_i} - \mu_{Y_i}) \frac{\partial f(\boldsymbol{x})}{\partial x_i} \ge 0 \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^n, \tag{2.3}$$

and

$$\sum_{i=1}^n \sum_{j=1}^n (\sigma_{X_iX_j} - \sigma_{Y_iY_j}) \frac{\partial^2 f(\boldsymbol{x})}{\partial x_i \partial x_j} \ge 0 \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^n. \tag{2.4}$$

Remark 2.7. The orders (a)–(i) defined above impose no differentiability assumptions on the functions f. Theorem 2.5 and Corollary 2.6 assume that f is twice differentiable. Analogously, in Definition 2.3, Hessian orders are defined only for twice-differentiable functions f. Denuit and Müller [12] proved that all the orders (a)–(i) can be defined considering only infinitely differentiable function. For this reason all the orders (a)–(e) are Hessian orders and it will be possible to apply Theorem 2.5 and Corollary 2.6 to all the orders (a)–(i).

3. Main results

The first result that we present in this section gives necessary conditions for the order of two generic random vectors. Theorem 3.2 will show that these conditions are also sufficient in the multinormal case.

Theorem 3.1. *Let $\mathcal{H} \in \mathcal{S}$ and let $\mathcal{C}_{\mathcal{H}}$ be the closed convex cone generated by $\mathcal{H}$. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors with finite second moments. If $\boldsymbol{X} \le_{\mathcal{F}_{\mathcal{H}}} \boldsymbol{Y}$, then*

(a) $\boldsymbol{\mu_X} = \boldsymbol{\mu_Y}$;
(b) $\boldsymbol{\Sigma_Y} - \boldsymbol{\Sigma_X} \in \mathcal{C}^*_{\mathcal{H}}$.

Proof. (a) Denote as $\boldsymbol{O}$ the $n \times n$ matrix having all components equal to 0. Then $\boldsymbol{O} \in \mathcal{C}_{\mathcal{H}}$.

For $i \in \{1, \dots, n\}$ consider the functions $f_i(\boldsymbol{x}) = x_i$ and $g_i(\boldsymbol{x}) = -x_i$. Observe that $\boldsymbol{H}_{f_i}(\boldsymbol{x}) = \boldsymbol{H}_{g_i}(\boldsymbol{x}) = \boldsymbol{O}$. Hence $f_i, g_i \in \mathcal{F}_{\mathcal{H}}$. Therefore $\boldsymbol{X} \le_{\mathcal{F}_{\mathcal{H}}} \boldsymbol{Y}$ implies

$$\mu_{X_i} = \mathbb{E}[f_i(\boldsymbol{X})] \le \mathbb{E}[f_i(\boldsymbol{Y})] = \mu_{Y_i},$$

and

$$-\mu_{X_i} = \mathbb{E}[g_i(\boldsymbol{X})] \le \mathbb{E}[g_i(\boldsymbol{Y})] = -\mu_{Y_i},$$

which implies $\boldsymbol{\mu_X} = \boldsymbol{\mu_Y}$.
(b) Given (a), let $\mathbb{E}[\boldsymbol{X}] = \mathbb{E}[\boldsymbol{Y}] = \boldsymbol{\mu}$. Choose a matrix $\boldsymbol{A} \in \mathcal{C}_{\mathcal{H}}$ and define a function f as

$$f(\boldsymbol{x}) = \frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{A} (\boldsymbol{x} - \boldsymbol{\mu}).$$

Observe that $f \in \mathcal{F}_{\mathcal{H}}$ since $\boldsymbol{H}_f(\boldsymbol{x}) = \boldsymbol{A}$ for all $\boldsymbol{x}$. Since $\boldsymbol{X} \le_{\mathcal{F}_{\mathcal{H}}} \boldsymbol{Y}$, we have

$$\mathbb{E}\left[\frac{1}{2}(\boldsymbol{X} - \boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{A} (\boldsymbol{X} - \boldsymbol{\mu})\right] = \mathbb{E}[f(\boldsymbol{X})] \le \mathbb{E}[f(\boldsymbol{Y})] = \mathbb{E}\left[\frac{1}{2}(\boldsymbol{Y} - \boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{A} (\boldsymbol{Y} - \boldsymbol{\mu})\right].$$

This is equivalent to

$$\mathbb{E}[\mathrm{Tr}((\boldsymbol{X} - \boldsymbol{\mu})^{\mathrm{T}} (\boldsymbol{X} - \boldsymbol{\mu}) \boldsymbol{A})] \le \mathbb{E}[\mathrm{Tr}((\boldsymbol{Y} - \boldsymbol{\mu})^{\mathrm{T}} (\boldsymbol{Y} - \boldsymbol{\mu}) \boldsymbol{A})]$$

and therefore

$$\mathrm{Tr}(\mathbb{E}[(\boldsymbol{Y}-\boldsymbol{\mu})(\boldsymbol{Y}-\boldsymbol{\mu})^{\mathrm{T}}-(\boldsymbol{X}-\boldsymbol{\mu})(\boldsymbol{X}-\boldsymbol{\mu})^{\mathrm{T}}]\boldsymbol{A})\geq 0$$

which corresponds to

$$\mathrm{Tr}((\boldsymbol{\Sigma}_{\boldsymbol{Y}}-\boldsymbol{\Sigma}_{\boldsymbol{X}})\boldsymbol{A})\geq 0.$$

This holds for any $\boldsymbol{A}\in\mathcal{C}_{\mathcal{H}}$ if and only if $\boldsymbol{\Sigma}_{\boldsymbol{Y}}-\boldsymbol{\Sigma}_{\boldsymbol{X}}\in\mathcal{C}^{*}_{\mathcal{H}}$. □

When the vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ are multinormal, the necessary conditions in Theorem 3.1 are also sufficient.

Theorem 3.2. *Let $\mathcal{H}\in\mathcal{S}$ and let $\mathcal{C}_{\mathcal{H}}$ be the closed convex cone generated by $\mathcal{H}$. Let $\boldsymbol{X}\sim\mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{X}},\boldsymbol{\Sigma}_{\boldsymbol{X}})$ and $\boldsymbol{Y}\sim\mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{Y}},\boldsymbol{\Sigma}_{\boldsymbol{Y}})$. Then*

$$\boldsymbol{X}\leq_{\mathcal{F}_{\mathcal{H}}}\boldsymbol{Y} \tag{3.1}$$

if and only if

$$\boldsymbol{\mu}_{\boldsymbol{X}}=\boldsymbol{\mu}_{\boldsymbol{Y}}, \tag{3.2}$$

and

$$\boldsymbol{\Sigma}_{\boldsymbol{Y}}-\boldsymbol{\Sigma}_{\boldsymbol{X}}\in\mathcal{C}^{*}_{\mathcal{H}}. \tag{3.3}$$

Proof. *If*: If (3.2) holds, then (2.3) is trivially satisfied.

If (3.3) holds for $f\in\mathcal{F}_{\mathcal{H}}$, then

$$\mathrm{Tr}((\boldsymbol{\Sigma}_{\boldsymbol{Y}}-\boldsymbol{\Sigma}_{\boldsymbol{X}})\boldsymbol{H}_f(\boldsymbol{x}))\geq 0,$$

i.e., (2.4) holds.

Corollary 2.6 provides the desired result.

Only if: See Theorem 3.1. □

Theorem 3.2 generalizes a host of known results, as will be shown in Section 4.

4. Examples

In the following we will provide several examples and some useful lemmas. As mentioned before, a key element in the sequel is the result in [12] that shows that most of the commonly used integral stochastic orders have a generator consisting of infinitely differentiable functions.

4.1. Well known Hessian orders

Convex order

Proposition 4.1 *([8,5]). Let $\boldsymbol{X}\sim\mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{X}},\boldsymbol{\Sigma}_{\boldsymbol{X}})$ and $\boldsymbol{Y}\sim\mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{Y}},\boldsymbol{\Sigma}_{\boldsymbol{Y}})$. Then $\boldsymbol{X}\leq_{\mathrm{cx}}\boldsymbol{Y}$ if and only if $\boldsymbol{\mu}_{\boldsymbol{X}}=\boldsymbol{\mu}_{\boldsymbol{Y}}$ and $\boldsymbol{\Sigma}_{\boldsymbol{Y}}-\boldsymbol{\Sigma}_{\boldsymbol{X}}$ is positive semidefinite.*

Denote as $\mathcal{C}_{\mathrm{psd}}\in\mathcal{S}$ the cone of positive semidefinite matrices. A twice-differentiable function f is convex if and only if $\boldsymbol{H}_f\in\mathcal{C}_{\mathrm{psd}}$. The cone $\mathcal{C}_{\mathrm{psd}}$ is self-dual, i.e., $\mathcal{C}^{*}_{\mathrm{psd}}=\mathcal{C}_{\mathrm{psd}}$ (see [13] Lemma 16.2.3). Hence Proposition 4.1 is a corollary of Theorem 3.2.

Directionally convex order

Proposition 4.2 *([5]). Let $\boldsymbol{X}\sim\mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{X}},\boldsymbol{\Sigma}_{\boldsymbol{X}})$ and $\boldsymbol{Y}\sim\mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{Y}},\boldsymbol{\Sigma}_{\boldsymbol{Y}})$. Then $\boldsymbol{X}\leq_{\mathrm{dcx}}\boldsymbol{Y}$ if and only if $\boldsymbol{\mu}_{\boldsymbol{X}}=\boldsymbol{\mu}_{\boldsymbol{Y}}$ and $\sigma_{X_iX_j}\leq\sigma_{Y_iY_j}$ for all $i,j\in\{1,\ldots,n\}$.*

Denote as $\mathcal{C}_{+}\in\mathcal{S}$ the cone of nonnegative matrices. A twice-differentiable function f is directionally convex if and only if $\boldsymbol{H}_f\in\mathcal{C}_{+}$. The cone $\mathcal{C}_{+}$ is self-dual, i.e., $\mathcal{C}^{*}_{+}=\mathcal{C}_{+}$ (see [13] Lemma 16.2.3). Hence Proposition 4.2 is a corollary of Theorem 3.2.

Properties of the directionally convex order have been studied, e.g., by Shaked and Shanthikumar [14], Meester and Shanthikumar [15,16], Bäuerle and Rolski [17] and Müller and Scarsini [18].

Supermodular order

Proposition 4.3 *([7,9]). Let $\boldsymbol{X}\sim\mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{X}},\boldsymbol{\Sigma}_{\boldsymbol{X}})$ and $\boldsymbol{Y}\sim\mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{Y}},\boldsymbol{\Sigma}_{\boldsymbol{Y}})$. Then $\boldsymbol{X}\leq_{\mathrm{sm}}\boldsymbol{Y}$ if and only if $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the same marginals and $\sigma_{X_iX_j}\leq\sigma_{Y_iY_j}$ for all $i,j\in\{1,\ldots,n\}$, $i\neq j$.*

Denote as $\mathcal{C}_{+\,\mathrm{off}}\in\mathcal{S}$ the cone of matrices with nonnegative off-diagonal elements. A twice-differentiable function f is supermodular if and only if $\boldsymbol{H}_f\in\mathcal{C}_{+\,\mathrm{off}}$. We have

$$\mathcal{C}^{*}_{+\,\mathrm{off}}=\{\boldsymbol{B}\in\mathcal{S}:b_{ij}\geq 0\text{ and }b_{ii}=0\}.$$

Hence Proposition 4.3 is a corollary of Theorem 3.2.

Results concerning the submodular order can be obtained by considering the cone of Z-matrices (see, e.g., [19]), i.e., matrices with nonpositive off-diagonal elements.

Componentwise convex order

Proposition 4.4 *([1, Theorem 3.6.5]). Let $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu_X}, \boldsymbol{\Sigma_X})$ and $\boldsymbol{Y} \sim \mathcal{N}(\boldsymbol{\mu_Y}, \boldsymbol{\Sigma_Y})$. Then $\boldsymbol{X} \leq_{\mathrm{ccx}} \boldsymbol{Y}$ if and only if $\boldsymbol{\mu_X} = \boldsymbol{\mu_Y}$ and $\sigma^2_{X_i} \leq \sigma^2_{Y_i}$ for all $i \in \{1, \ldots, n\}$, and $\sigma_{X_iX_j} = \sigma_{Y_iY_j}$ for all $i \neq j \in \{1, \ldots, n\}$.*

Denote as $\mathcal{C}_{+\mathrm{diag}} \in \mathcal{S}$ the cone of matrices with nonnegative elements on the main diagonal. A twice-differentiable function f is componentwise convex if and only if $\boldsymbol{H}_f \in \mathcal{C}_{+\mathrm{diag}}$. It is easy to see that

$$\mathcal{C}^*_{+\mathrm{diag}} = \{\boldsymbol{B} \in \mathcal{S} : b_{ii} \geq 0 \text{ and } b_{ij} = 0 \text{ for } i \neq j\}.$$

Hence Proposition 4.4 is a corollary of Theorem 3.2.

4.2. Other orders

Copositive and completely positive orders

Definition 4.5. An $n \times n$ matrix $\boldsymbol{A}$ is copositive if the quadratic form $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \geq \boldsymbol{0}$.

Definition 4.6. An $n \times n$ matrix $\boldsymbol{A}$ is completely positive if there exists a nonnegative $m \times n$ matrix $\boldsymbol{B}$ such that $\boldsymbol{A} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}$.

The reader is referred to [13, Section 16.2], for an extensive treatment of copositive and completely positive matrices.

Denote as $\mathcal{C}_{\mathrm{cop}}$ the cone of copositive matrices and as $\mathcal{C}_{\mathrm{cp}}$ the cone of completely positive matrices. The following lemma can be found in [13, Lemma 16.2.2].

Lemma 4.7. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two closed convex cones, then $\mathcal{C}_1 \cap \mathcal{C}_2$ and $\mathcal{C}_1 + \mathcal{C}_2$ are closed convex cones. In addition, $(\mathcal{C}_1 + \mathcal{C}_2)^* = \mathcal{C}_1^* \cap \mathcal{C}_2^*$ and $(\mathcal{C}_1 \cap \mathcal{C}_2)^* = \mathcal{C}_1^* + \mathcal{C}_2^*$.*

The cones $\mathcal{C}_{\mathrm{cop}}$ and $\mathcal{C}_{\mathrm{cp}}$ are both closed and convex. In addition (see [13] Theorem 16.2.1),

$$\mathcal{C}^*_{\mathrm{cop}} = \mathcal{C}_{\mathrm{cp}}, \quad \text{and} \quad \mathcal{C}^*_{\mathrm{cp}} = \mathcal{C}_{\mathrm{cop}}.$$

It is easy to see that every elementwise nonnegative matrix is copositive and every positive semidefinite matrix is copositive. Therefore,

$$\mathcal{C}_{\mathrm{cop}} \supseteq \mathcal{C}_+ + \mathcal{C}_{\mathrm{psd}} \tag{4.1}$$

and

$$\mathcal{C}_{\mathrm{cp}} = \mathcal{C}^*_{\mathrm{cop}} \subseteq (\mathcal{C}_+ + \mathcal{C}_{\mathrm{psd}})^* = \mathcal{C}^*_+ \cap \mathcal{C}^*_{\mathrm{psd}} = \mathcal{C}_+ \cap \mathcal{C}_{\mathrm{psd}}. \tag{4.2}$$

Diananda [20] shows that the inclusions in (4.1) and (4.2) are equalities for $n \leq 4$. Hall [13, p. 349] provides a counterexample for $n > 4$.

The following Hessian orders can be defined.

(a) $\boldsymbol{X} \leq_{\mathrm{cp}} \boldsymbol{Y}$ if (2.1) holds for all functions f such that $\mathcal{H}_{\mathcal{F}} \in \mathcal{C}_{\mathrm{cp}}$.
(b) $\boldsymbol{X} \leq_{\mathrm{cop}} \boldsymbol{Y}$ if (2.1) holds for all functions f such that $\mathcal{H}_{\mathcal{F}} \in \mathcal{C}_{\mathrm{cop}}$.

Corollary 4.8. *Let $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu_X}, \boldsymbol{\Sigma_X})$ and $\boldsymbol{Y} \sim \mathcal{N}(\boldsymbol{\mu_Y}, \boldsymbol{\Sigma_Y})$.*

(a) *$\boldsymbol{X} \leq_{\mathrm{cp}} \boldsymbol{Y}$ if and only if $\boldsymbol{\mu_X} = \boldsymbol{\mu_Y}$ and $\boldsymbol{\Sigma_Y} - \boldsymbol{\Sigma_X}$ is copositive.*
(b) *$\boldsymbol{X} \leq_{\mathrm{cop}} \boldsymbol{Y}$ if and only if $\boldsymbol{\mu_X} = \boldsymbol{\mu_Y}$ and $\boldsymbol{\Sigma_Y} - \boldsymbol{\Sigma_X}$ is completely positive.*
(c) *For $n \leq 4$, $\mathbb{E}[f(\boldsymbol{X})] \leq \mathbb{E}[f(\boldsymbol{Y})]$ holds for all functions f that are both convex and directionally convex, if and only if $\boldsymbol{\mu_X} = \boldsymbol{\mu_Y}$ and $\boldsymbol{\Sigma_Y} - \boldsymbol{\Sigma_X} = \boldsymbol{T} + \boldsymbol{W}$, where $\boldsymbol{T}$ is positive semidefinite, and $\boldsymbol{W}$ has nonnegative elements.*
(d) *For $n \leq 4$, $\mathbb{E}[f(\boldsymbol{X})] \leq \mathbb{E}[f(\boldsymbol{Y})]$ holds for all functions f such that*

$$f(\boldsymbol{x}) = \alpha g(\boldsymbol{x}) + \beta h(\boldsymbol{x}),$$

with $\alpha, \beta \geq 0$, g convex, and h directionally convex, if and only if $\boldsymbol{\mu_X} = \boldsymbol{\mu_Y}$ and $\boldsymbol{\Sigma_Y} - \boldsymbol{\Sigma_X}$ is positive semidefinite and has nonnegative elements.

Proof. (a) and (b) are just an immediate corollary of Theorem 3.2.

(c) and (d) follow from the characterization of [20], and from Lemma 4.7. □

Positive-linear-convex order

Proposition 4.9 *([1, Theorem 3.5.5]). Let $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu_X}, \boldsymbol{\Sigma_X})$ and $\boldsymbol{Y} \sim \mathcal{N}(\boldsymbol{\mu_Y}, \boldsymbol{\Sigma_Y})$. Then $\boldsymbol{X} \leq_{\mathrm{plcx}} \boldsymbol{Y}$ if and only if $\boldsymbol{\mu_X} = \boldsymbol{\mu_Y}$ and $\boldsymbol{\Sigma_Y} - \boldsymbol{\Sigma_X}$ is copositive.*

Notice that $\leq_{\text{plcx}}$ is an integral stochastic order, but not a Hessian order. Nevertheless, in the multinormal case, it is equivalent to the Hessian order $\leq_{\text{cp}}$. Given two arbitrary random vectors, the completely positive order implies the positive-linear-convex order.

Theorem 4.10. *If $\boldsymbol{X} \leq_{\text{cp}} \boldsymbol{Y}$ then $\boldsymbol{X} \leq_{\text{plcx}} \boldsymbol{Y}$.*

Proof. It suffices to show that the Hessian of a positive-linear-convex function is completely positive. Given $\boldsymbol{a} \geq \boldsymbol{0}$ and a convex function $\psi : \mathbb{R} \to \mathbb{R}$, define f as in (2.2). Then

$$\boldsymbol{H}_f = \boldsymbol{a} \cdot \boldsymbol{a}^{\mathrm{T}} \psi'' \left(\boldsymbol{a}^{\mathrm{T}} \boldsymbol{x}\right).$$

Since ψ is convex, we have $\psi'' \left(\boldsymbol{a}^{\mathrm{T}} \boldsymbol{x}\right) \geq 0$. Hence $\boldsymbol{H}_f$ is completely positive, according to Definition 4.6. □

When $\boldsymbol{X}$ and $\boldsymbol{Y}$ are multinormal, the completely positive and the positive-linear-convex orders are equivalent.

Theorem 4.11. *Let $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu_X}, \boldsymbol{\Sigma_X})$ and $\boldsymbol{Y} \sim \mathcal{N}(\boldsymbol{\mu_Y}, \boldsymbol{\Sigma_Y})$. Then $\boldsymbol{X} \leq_{\text{cp}} \boldsymbol{Y}$ if and only if $\boldsymbol{X} \leq_{\text{plcx}} \boldsymbol{Y}$.*

Proof. The result follows immediately from Corollary 4.8(a) and Proposition 4.9. □

5. Increasing orders

In the univariate case the following result holds (see [5]).

Proposition 5.1. *Let $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$. Then $X \leq_{\text{icx}} Y$ if and only if $\mu_X \leq \mu_Y$ and $\sigma_X^2 \leq \sigma_Y^2$.*

A similar result does not exist in the multivariate case, but trivial sufficient conditions hold.

Denote as $\mathcal{I}$ the class of increasing functions $f : \mathbb{R}^n \to \mathbb{R}$, and for some $\mathcal{H} \subset \mathcal{S}$ define

$$\mathcal{I}_{\mathcal{H}} = \mathcal{I} \cap \mathcal{F}_{\mathcal{H}}.$$

Theorem 5.2. *Let $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu_X}, \boldsymbol{\Sigma_X})$ and $\boldsymbol{Y} \sim \mathcal{N}(\boldsymbol{\mu_Y}, \boldsymbol{\Sigma_Y})$. If $\boldsymbol{\mu_X} \leq \boldsymbol{\mu_Y}$ and $\boldsymbol{\Sigma_Y} - \boldsymbol{\Sigma_X} \in \mathcal{C}^*_{\mathcal{H}}$, then $\boldsymbol{X} \leq_{\mathcal{I}_{\mathcal{H}}} \boldsymbol{Y}$.*

Proof. The two conditions of Corollary 2.6 are trivially satisfied. □

For the orders in the sequel, the conditions in Theorem 5.2 are also necessary.

Increasing completely positive order

The following result is well known.

Theorem 5.3 ([1, Theorem 3.5.5]). *Let $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu_X}, \boldsymbol{\Sigma_X})$ and $\boldsymbol{Y} \sim \mathcal{N}(\boldsymbol{\mu_Y}, \boldsymbol{\Sigma_Y})$. Then $\boldsymbol{X} \leq_{\text{iplcx}} \boldsymbol{Y}$ if and only if $\boldsymbol{\mu_X} \leq \boldsymbol{\mu_Y}$ and $\boldsymbol{\Sigma_Y} - \boldsymbol{\Sigma_X}$ is copositive.*

In the multinormal case the completely positive order and the positive-linear-convex one are equivalent (see Theorem 4.11). The same holds trivially for their increasing versions.

Theorem 5.4. *Let $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu_X}, \boldsymbol{\Sigma_X})$ and $\boldsymbol{Y} \sim \mathcal{N}(\boldsymbol{\mu_Y}, \boldsymbol{\Sigma_Y})$. Then $\boldsymbol{X} \leq_{\text{icp}} \boldsymbol{Y}$ if and only if*

$$\boldsymbol{\mu_X} \leq \boldsymbol{\mu_Y} \quad \text{and} \quad \boldsymbol{\Sigma_Y} - \boldsymbol{\Sigma_X} \in \mathcal{C}^*_{\text{cp}}.$$

Proof. *If*: the result follows from Theorem 5.2.
Only if: combine Theorems 4.10 and 5.3. □

Increasing supermodular order

Necessary and sufficient conditions can be obtained for the increasing supermodular order.

Theorem 5.5. *Let $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu_X}, \boldsymbol{\Sigma_X})$ and $\boldsymbol{Y} \sim \mathcal{N}(\boldsymbol{\mu_Y}, \boldsymbol{\Sigma_Y})$. Then $\boldsymbol{X} \leq_{\text{ism}} \boldsymbol{Y}$ if and only if*

$$\boldsymbol{\mu_X} \leq \boldsymbol{\mu_Y} \quad \text{and} \quad \boldsymbol{\Sigma_Y} - \boldsymbol{\Sigma_X} \in \mathcal{C}^*_{+\,\text{off}}.$$

The following two lemmas will be needed for the proof of Theorem 5.5.

Lemma 5.6 ([5]). *Let $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$. Then $X \leq_{\text{st}} Y$ if and only if $\mu_X \leq \mu_Y$ and $\sigma_X^2 = \sigma_Y^2$.*

Lemma 5.7 ([1, Theorem 3.9.6]). *The increasing supermodular order has the following properties:*

(a) *If $\boldsymbol{X} \leq_{\text{ism}} \boldsymbol{Y}$ then $X_i \leq_{\text{st}} Y_i$ for all $i \in \{1, \ldots, n\}$.*
(b) *If $\boldsymbol{X} \leq_{\text{ism}} \boldsymbol{Y}$ then $\boldsymbol{X} \leq_{\text{iplcx}} \boldsymbol{Y}$.*

Proof of Theorem 5.5. *If*: the result follows from Theorem 5.2.
Only if: Considering for $i \in \{1, \ldots, n\}$ the functions $f_i(\boldsymbol{x}) = x_i$, which are trivially increasing and supermodular, we get $\boldsymbol{\mu_X} \leq \boldsymbol{\mu_Y}$.

If, for some $i \in \{1, \ldots, n\}$, we have $\sigma^2_{Y_i} - \sigma^2_{X_i} \neq 0$, then, by Lemma 5.6, $X_i \leq_{\mathrm{st}} Y_i$ does not hold. But, by Lemma 5.7(a), this is a contradiction. Therefore $\sigma^2_{Y_i} = \sigma^2_{X_i}$.

Suppose now that there exist $i \neq j \in \{1, \ldots, n\}$ such that $\sigma_{Y_iY_j} - \sigma_{X_iX_j} < 0$. Then $\boldsymbol{\Sigma_Y} - \boldsymbol{\Sigma_X} \notin \mathcal{C}_{\mathrm{cop}}$. By Theorem 5.3, $\boldsymbol{X} \not\leq_{\mathrm{iplcx}} \boldsymbol{Y}$, which, by Lemma 5.7(b), is a contradiction. □

6. Conclusions

Integral orders are defined in terms of inequalities for the expectation of a class $\mathcal{F}$ of functions f. These functions are often defined in terms of properties of their derivatives. Necessary conditions for some integral orders have been studied in the past (see, e.g., [21]), and they involve conditions on the moments of the compared random vectors.

Here, we studied functions $f \in \mathcal{F}$ that have a Hessian matrix in some convex cone $\mathcal{C}$. We showed that if two random vectors are ordered, then their means are equal and the difference of their covariance matrices is in the dual cone $\mathcal{C}^*$. These necessary conditions are also sufficient for multinormal random vectors. In particular, they are computationally tractable and of immediate use in systems with underlying normal random variables.

An interesting aspect of this approach is that it unifies and generalizes some existing results. Moreover, we think that it shows the direction for a possible generalization along the lines of s-convex orders studied by Denuit et al. [22]. An integral order could be defined in terms of an s-dimensional matrix of sth derivatives, and necessary conditions could be obtained through matrices of sth moments. We plan to pursue this investigation in a future paper.

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