



# A ridge regression estimation approach to the measurement error model<sup>☆</sup>



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## ABSTRACT

This paper considers the estimation of the parameters of measurement error models where the estimated covariance matrix of the regression parameters is ill conditioned. We consider the Hoerl and Kennard type (1970) ridge regression (RR) modifications of the five quasi-empirical Bayes estimators of the regression parameters of a measurement error model when it is suspected that the parameters may belong to a linear subspace. The modifications are based on the estimated covariance matrix of the estimators of regression parameters. The estimators are compared and the dominance conditions as well as the regions of optimality of the proposed estimators are determined based on quadratic risks.

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## 1. Introduction

The standard assumption in the linear regression analysis is that all the explanatory variables are linearly independent. When this assumption is violated, the problem of multicollinearity enters into the data and it inflates the variance of an ordinary least squares estimator of the regression coefficient, see [28] for more details. Obtaining the estimators for multicollinear data is an important problem in the literature. The ridge regression estimation due to Hoerl and Kennard [13] works well in multicollinear data. The ridge estimators under the normally distributed random errors in a regression model have been studied by e.g., [31,18,19,6,12,3] etc. The details of development of other approaches and the literature related to the ridge regressions are not within the scope of this paper.

Another fundamental assumption in all statistical analyses is that all the observations are correctly observed. When this assumption is violated, the measurement errors creep into the data. Then the usual statistical tools tend to lose their validity, see [8,7] for more details. An important issue in the area of measurement errors is to find the consistent estimators of the parameters which can be accomplished by utilizing some additional information from outside the sample. In the context of multiple linear regression models, the use of additional information in the form of a known covariance matrix of

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measurement errors and a known matrix of reliability ratios, both associated with explanatory variables, has been studied, see e.g., [9,21–26,30,34,16,37] etc.

When the problem of multicollinearity is present in the measurement error ridden data, then an important issue is how to obtain the consistent estimators of regression coefficients. One simple idea is to use the ridge regression estimation over the measurement error ridden data. An obvious question that crops up is what happens then? In this paper, we attempt to answer such questions.

It is well known that Stein [38,14] initially proposed the Stein estimator and positive-rule estimators. The preliminary test estimators were proposed by Bancroft [4]. On the other hand, ridge regression estimators were proposed by Hoerl and Kennard [13] and they combat the problem of multicollinearity for the estimation of regression parameters. Saleh [29, Chapter 4] proposed “quasi-empirical Bayes estimators”. So we have considered five quasi-empirical Bayes estimators by weighing the unrestricted, restricted, preliminary test and Stein-type estimators by the ridge “weight function”. The resulting estimators are studied in measurement error models. The quadratic risks of these estimators have been obtained and optimal regions of superiority of the estimators are determined.

The plan of the paper is as follows. We describe the model set up in Section 2. The details and development of the estimators are presented in Section 3. The comparison of estimators over each other is studied and their dominance conditions are reported in Section 4. The summary and conclusions are placed in Section 5 followed by the references.

## 2. The model description

Consider the multiple regression model with measurement errors

$$Y_t = \beta_0 + \mathbf{x}_t' \boldsymbol{\beta} + e_t, \quad \mathbf{X}_t = \mathbf{x}_t + \mathbf{u}_t, \quad t = 1, 2, \dots, n \quad (2.1)$$

where  $\beta_0$  is the intercept term and  $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_p)'$  is the  $p \times 1$  vector of regression coefficients,  $\mathbf{x}_t = (x_{1t}, x_{2t}, \dots, x_{pt})'$  is the  $p \times 1$  vector of set  $t$ th observations on true but unobservable  $p$  explanatory variables that are observed as  $\mathbf{X}_t = (X_{1t}, X_{2t}, \dots, X_{pt})'$  with  $p \times 1$  measurement error vector  $\mathbf{u}_t = (u_{1t}, u_{2t}, \dots, u_{pt})'$ ,  $u_{it}$  being the measurement error in the  $i$ th explanatory variable  $x_{it}$  and  $e_t$  is the response error in the observed response variable  $Y_t$ . We assume that

$$(\mathbf{x}_t', e_t, \mathbf{u}_t) \sim N_{2p+1} \{ (\boldsymbol{\mu}_x', 0, \mathbf{0}')', \text{BlockDiag}(\Sigma_{xx}, \sigma_{ee}, \Sigma_{uu}) \} \quad (2.2)$$

with  $\boldsymbol{\mu}_x = (\mu_{x1}, \mu_{x2}, \dots, \mu_{xp})'$ ,  $\sigma_{ee}$  is the variance of  $e_t$ 's whereas  $\Sigma_{xx}$  and  $\Sigma_{uu}$  are the covariance matrices of  $\mathbf{x}_t$ 's and  $\mathbf{u}_t$ 's respectively. Clearly,  $(Y_t, \mathbf{X}_t)'$  follows a  $(p+1)$ -variate normal distribution with mean vector  $(\beta_0 + \boldsymbol{\beta}' \boldsymbol{\mu}_x, \boldsymbol{\mu}_x')'$  and covariance matrix

$$\begin{pmatrix} \sigma_{ee} + \boldsymbol{\beta}' \Sigma_{xx} \boldsymbol{\beta} & \boldsymbol{\beta}' \Sigma_{xx} \\ \Sigma_{xx} \boldsymbol{\beta} & \Sigma_{xx} + \Sigma_{uu} \end{pmatrix}. \quad (2.3)$$

Then the conditional expectation of  $Y_t$  given  $\mathbf{X}_t$  is

$$E(Y_t | \mathbf{X}_t) = \gamma_0 + \boldsymbol{\gamma}' \mathbf{X}_t \quad (2.4)$$

where  $\gamma_0 = \beta_0 + \boldsymbol{\beta}' (I_p - K_{xx}') \boldsymbol{\mu}_x$ ,  $\boldsymbol{\gamma} = K_{xx} \boldsymbol{\beta}$ ,  $\boldsymbol{\beta} = K_{xx}^{-1} \boldsymbol{\gamma}$ , and  $K_{xx} = \Sigma_{xx}^{-1} \Sigma_{xx} = (\Sigma_{xx} + \Sigma_{uu})^{-1} \Sigma_{xx}$  is the  $p \times p$  matrix of reliability ratios of  $X$ , see [9].

Our basic problem is the estimation of  $\boldsymbol{\beta}$  under various situations beginning with the primary estimation of  $\boldsymbol{\beta}$  assuming  $\Sigma_{uu}$  is known.

Let

$$S = \begin{pmatrix} S_{YY} & S_{YX} \\ S_{XY} & S_{XX} \end{pmatrix} \quad (2.5)$$

where

- (i)  $S_{YY} = (\mathbf{Y} - \bar{Y} \mathbf{1}_n)' (\mathbf{Y} - \bar{Y} \mathbf{1}_n)$ ,  $Y = (Y_1, Y_2, \dots, Y_n)'$ ,  $\mathbf{1}_n = (1, 1, \dots, 1)'$ .
- (ii)  $S_{XX} = ((S_{X_i X_i}))$ ,  $S_{X_i X_j} = (\mathbf{x}_i - \bar{X}_i \mathbf{1}_n)' (\mathbf{x}_j - \bar{X}_j \mathbf{1}_n)$
- (iii)  $S_{X_i Y} = (\mathbf{x}_i - \bar{X}_i \mathbf{1}_n)' (\mathbf{Y}_i - \bar{Y}_i \mathbf{1}_n)$ ,  $S_{XY} = (S_{X_1 Y}, S_{X_2 Y}, \dots, S_{X_p Y})'$
- (iv)  $\bar{X}_i = \frac{1}{n} \sum_{t=1}^n X_{it}$ ,  $\bar{Y} = \frac{1}{n} \sum_{t=1}^n Y_t$ .

Gleser [9] showed that the maximum likelihood estimators of  $\gamma_0$ ,  $\boldsymbol{\gamma}$  and  $\sigma_{zz}$  are just the naive least squares estimators, viz.,

$$\tilde{\gamma}_{0n} = \bar{Y} - \tilde{\boldsymbol{\gamma}}_n' \bar{\mathbf{X}}, \quad \tilde{\boldsymbol{\gamma}}_n = S_{XX}^{-1} S_{XY} \quad \text{and} \quad \tilde{\sigma}_{zz} = \frac{1}{n} (\mathbf{Y} - \tilde{\gamma}_{0n} \mathbf{1}_n - \tilde{\boldsymbol{\gamma}}_n' \mathbf{X})' (\mathbf{Y} - \tilde{\gamma}_{0n} \mathbf{1}_n - \tilde{\boldsymbol{\gamma}}_n' \mathbf{X}) \quad (2.6)$$

provided

$$\tilde{\sigma}_{ee} = \tilde{\sigma}_{zz} - \tilde{\boldsymbol{\gamma}}_n' K_{xx}^{-1} \Sigma_{uu} \tilde{\boldsymbol{\gamma}}_n \geq 0. \quad (2.7)$$

When  $\Sigma_{uu}$  is known and  $K_{xx} = \Sigma_{xx}^{-1} \Sigma_{xx} = (\Sigma_{xx} + \Sigma_{uu})^{-1} \Sigma_{xx}$  is unknown, then  $K_{xx}$  is estimated consistently by replacing  $\Sigma_{xx}$  and  $\Sigma_{xx} + \Sigma_{uu}$  by their respective consistent estimators as

$$\hat{K}_{xx} = S_{xx}^{-1} (S_{xx} - n \Sigma_{uu}) \quad (2.8)$$

where  $\frac{1}{n} S_{xx}$  is the maximum likelihood estimate of  $(\Sigma_{xx} + \Sigma_{uu})$ .

Thus, the maximum likelihood estimates of  $\beta_0$ ,  $\beta_1$  and  $\sigma_{ee}$  are given by

$$\tilde{\beta}_{0n} = \tilde{\gamma}_{0n} - \tilde{\beta}'_n (I_p - \hat{K}'_{xx}) \bar{\mathbf{x}}, \quad \tilde{\beta}_n = \hat{K}_{xx}^{-1} \tilde{\gamma}_n \quad \text{and} \quad \tilde{\sigma}_{ee} = \tilde{\sigma}_{zz} - \tilde{\beta}'_n \Sigma_{uu} \hat{K}_{xx} \tilde{\beta}_n \quad (2.9)$$

respectively.

Finally,  $\tilde{\beta}_{0n}$  reduces to  $\bar{Y} - \tilde{\beta}'_n \bar{\mathbf{X}}$  and

$$\tilde{\beta}_n = (S_{xx} - n \Sigma_{uu})^{-1} S_{xy} \quad (2.10)$$

provided  $\tilde{\sigma}_{ee} \geq 0$  as in (2.7). The estimators will be designated as the unrestricted estimators of  $\beta_0$  and  $\beta$ . Then by Theorem 2 of [8] we find the large sample covariance matrix of  $\tilde{\beta}_n$  as  $\sigma_{zz} C^{-1}$  where  $C = K'_{xx} \Sigma_{xx} K_{xx} = \Sigma_{xx} \Sigma_{xx}^{-1} \Sigma_{xx}$ .

Then, a consistent estimator of  $C$  is given by

$$C_n = \hat{K}'_{xx} \hat{\Sigma}_{xx} \hat{K}_{xx} = (S_{xx} - n \Sigma_{uu})' S_{xx}^{-1} (S_{xx} - n \Sigma_{uu}). \quad (2.11)$$

In case,  $\beta$  is suspected to belong to the linear subspace of  $H\beta = \mathbf{h}$  where  $H$  is a  $q \times p$  matrix and  $\mathbf{h}$  is a  $q \times 1$  vector of known numbers respectively, the restricted estimator of  $\beta$  is defined by

$$\hat{\beta}_n = \tilde{\beta}_n - C_n^{-1} H' (H C_n^{-1} H')^{-1} (H \tilde{\beta}_n - \mathbf{h}), \quad (2.12)$$

see [33].

Since it is suspected that the restrictions  $H\beta = \mathbf{h}$  may hold, we remove the suspicion by testing hypothesis  $H_0$  based on the Wald-type statistic

$$\mathcal{L}_n^* = n(H \tilde{\beta}_n - \mathbf{h})' (H C_n^{-1} H')^{-1} (H \tilde{\beta}_n - \mathbf{h}). \quad (2.13)$$

Thus under  $H_0$ , as  $n \rightarrow \infty$ ,  $\mathcal{L}_n^* \xrightarrow{D} \chi_q^2$ , the chi-square variable with  $q$  degrees of freedom where  $\xrightarrow{D}$  denotes the convergence in distribution.

### 3. Ridge regression estimators of $\beta$

In this section, we introduce the ridge regression estimators of  $\beta$ . For this, we first consider the conditional setup of the least squares method with known reliability matrix  $K_{xx}$  and minimize the quadratic form with Lagrangian multiplier

$$(XK\beta + \gamma_0 \mathbf{1}_p - Y)' (XK\beta + \gamma_0 \mathbf{1}_p - Y) + k\beta' \beta.$$

This minimization yields the normal equation for  $\beta$  as

$$[K'_{xx} S_{xx} K_{xx} + k I_p] \beta = K'_{xx} S_{xy}.$$

Thus, the ridge regression estimator for  $\beta$  is given by

$$\tilde{\beta}_n(k) = \left[ I_p + \left( \hat{K}'_{xx} S_{xx} \hat{K}_{xx} \right)^{-1} \right]^{-1} \tilde{\beta}_n, \quad (3.1)$$

substituting the consistent estimator of  $K_{xx}$  given by (2.8) with

$$\tilde{\beta}_n = (S_{xx} - n \Sigma_{uu})^{-1} S_{xy}.$$

Here, the ridge factor of the ridge estimator is given by

$$R_n(k) = [I_p + k C_n^{-1}]^{-1}, \quad C_n = \hat{K}'_{xx} S_{xx} \hat{K}_{xx} \quad (3.2)$$

which is a consistent estimator of

$$R(k) = [I_p + k C^{-1}]^{-1}, \quad C = K'_{xx} \Sigma_{xx} K_{xx}.$$

Hence, the unrestricted ridge regression estimator  $\tilde{\beta}_n(k)$  is defined by

$$\tilde{\beta}_n(k) = R_n(k) \tilde{\beta}_n.$$

It is easy to verify that as  $n \rightarrow \infty$ , the bias, MSE and trace of MSE expressions for  $\tilde{\beta}_n(k)$  are given by

$$\begin{aligned} b_1(\tilde{\beta}_n(k)) &= -kC^{-1}(k)\beta; \quad C^{-1}(k) = (C + kI_p)^{-1} \\ M_1(\tilde{\beta}_n(k)) &= \sigma_{zz}[R(k)]'C^{-1}[R(k)] + k^2C^{-1}(k)\beta\beta'C^{-1}(k) \\ \text{tr}(M_1(\tilde{\beta}_n(k))) &= \sigma_{zz}\text{tr}\{[R(k)]'C^{-1}[R(k)]\} + k^2\beta'C^{-1}(k)\beta. \end{aligned}$$

Further, since  $\beta$  is suspected to belong to the subspace  $H\beta = h$ , we shall consider four more estimators, namely, the  
(i) *Restricted estimator of  $\beta$*  given by

$$\hat{\beta}_n(k) = R_n(k)\hat{\beta}_n. \quad (3.3)$$

(ii) *Preliminary test estimator (PTE) of  $\beta$*  given by

$$\hat{\beta}_n^{PT}(k) = R_n(k)\hat{\beta}_n^{PT} \quad (3.4)$$

where  $\hat{\beta}_n^{PT} = \tilde{\beta}_n - (\tilde{\beta}_n - \hat{\beta}_n)I(\mathcal{L}_n^* < \chi_q^2(\alpha))$  and  $\chi_q^2(\alpha)$  denotes the  $\alpha$ -level critical value of a Chi-square distribution with  $q$  degrees of freedom.

The preliminary test estimation under the assumption of normally distributed random errors has been pioneered by Bancroft [4] and considered later by Bancroft [5], Han and Bancroft [11], Judge and Bock [15], Kibria and Saleh [20], Saleh [29] and Arashi et al. [3] among others. In the setup of measurement errors models, Kim and Saleh [21–26] have considered the preliminary test and Stein-type estimation.

(iii) The *James–Stein type shrinkage estimator (SE) of  $\beta$*  due to James and Stein [14] is given by

$$\hat{\beta}_n^S(k) = R_n(k)\hat{\beta}_n^S \quad (3.5)$$

where  $\hat{\beta}_n^S = \tilde{\beta}_n - (q-2)(\tilde{\beta}_n - \hat{\beta}_n)\mathcal{L}_n^{*-1}$ .

The Stein-rule estimation technique in various models has been considered by several researchers, see e.g., [27,10,29,35,36,2,6,1] among many others.

(iv) The *Positive rule Stein estimator (PRSE) of  $\beta$*  is given by

$$\hat{\beta}_n^{S+}(k) = R_n(k)\hat{\beta}_n^{S+} \quad (3.6)$$

where  $\hat{\beta}_n^{S+} = \hat{\beta}_n I(\mathcal{L}_n^* < q-2) + \hat{\beta}_n^S I(\mathcal{L}_n^* \geq q-2)$ .

Now, we present the asymptotic distributional properties of the five ridge regression estimators. It may be verified that the test  $\mathcal{L}_n^*$  for the test of  $H\beta = h$  is consistent as  $n \rightarrow \infty$ . Thus all the quasi-empirical Bayes estimators  $\hat{\beta}_n^{PT}$ ,  $\hat{\beta}_n^S$  and  $\hat{\beta}_n^{S+}$  are asymptotically equivalent to  $\tilde{\beta}_n$  under a fixed alternative, while the asymptotic distribution of  $\hat{\beta}_n$  degenerates as  $n \rightarrow \infty$ . To by pass this problem, we consider the asymptotic distribution under the sequence of local alternatives

$$K_{(n)} : H\beta = h + n^{-\frac{1}{2}}\xi; \quad \xi \in R^q. \quad (3.7)$$

The dominance properties of  $\tilde{\beta}_n$ ,  $\hat{\beta}_n$ ,  $\hat{\beta}_n^{PT}$ ,  $\hat{\beta}_n^S$  and  $\hat{\beta}_n^{S+}$  are given in Saleh [29, Chapter. 7.8.2] under (3.7). Let  $\Delta^2$  denotes the departure parameter that indicates the departure of the alternative hypothesis from the null hypothesis. Here, [Theorem 1](#) gives the properties of the ridge regression estimators using the following result:

**Result.** Under  $\{K_{(n)}\}$  and the basic assumptions of the measurement error model, the following holds:

(a)

$$\begin{pmatrix} \sqrt{n}(\tilde{\beta}_n - \beta) \\ \sqrt{n}(\hat{\beta}_n - \beta) \\ \sqrt{n}(\tilde{\beta}_n - \hat{\beta}_n) \end{pmatrix} \xrightarrow{D} N_{3p} \left\{ \begin{pmatrix} 0 \\ -\delta \\ \delta \end{pmatrix}, \sigma_{zz} \begin{pmatrix} C^{-1} & C^{-1} - A & A \\ C^{-1} - A & C^{-1} - A & \mathbf{0} \\ A & \mathbf{0} & A \end{pmatrix} \right\} \quad (3.8)$$

where  $C = K'_{xx}\Sigma_{XX}K_{xx}$ .

(b)

$$\lim_{n \rightarrow \infty} P\{\mathcal{L}_n \leq x \mid K_{(n)}\} = \mathcal{H}_q(x, \Delta^2), \quad \Delta^2 = \frac{1}{\sigma_{zz}}\delta'C\delta, \quad \delta = C^{-1}H'(HC^{-1}H')^{-1}\xi$$

where  $\mathcal{H}_q(x; \Delta^2)$  is the c.d.f. of a noncentral Chi-square distribution with  $q$  degrees of freedom and noncentrality parameter  $\Delta^2$ .

**Theorem 1.** Under  $\{K_{(n)}\}$ , the bias, MSE matrices and risk expressions of the five ridge regression estimators are given by

$$\begin{aligned}
 \text{(i) } \mathbf{b}_1(\tilde{\beta}_n(k)) &= -kC^{-1}(k)\beta, \quad C(k) = C + kI_p \\
 M_1(\tilde{\beta}_n(k)) &= \sigma_{zz}R(k)C^{-1}R(k)' + k^2C^{-1}(k)\beta\beta'C^{-1}(k), \\
 R_1(\tilde{\beta}_n(k); W) &= \sigma_{zz}\text{tr}[WR(k)C^{-1}R(k)'] + k^2\beta'C^{-1}(k)\beta. \\
 \text{(ii) } \mathbf{b}_2(\hat{\beta}_n(k)) &= -kC^{-1}(k)\beta - R(k)\delta, \\
 M_2(\hat{\beta}_n(k)) &= \sigma_{zz}R(k)[C^{-1} - A]R'(k) + [kC^{-1}(k)\beta + R(k)\delta][kC^{-1}(k)\beta + R(k)\delta]', \\
 R_2(\hat{\beta}_n(k); W) &= \sigma_{zz}\text{tr}[W(R(k)(C^{-1} - A)R(k)')] + [kC^{-1}(k)\beta + R(k)\delta]'W[kC^{-1}(k)\beta + R(k)\delta]. \\
 \text{(iii) } \mathbf{b}_3(\hat{\beta}_n^{PT}(k)) &= -kC^{-1}(k)\beta - R(k)\delta\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2], \\
 M_3(\hat{\beta}_n^{PT}(k)) &= \sigma_{zz}[R(k)C^{-1}R(k)'] - \sigma_{zz}[R(k)AR(k)']\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] \\
 &\quad [R(k)\delta\delta'R(k)']\{2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]\} + k^2[C^{-1}(k)\beta\beta'C^{-1}(k)] \\
 &\quad + k[R(k)\delta\beta'C^{-1}(k) + C^{-1}(k)\beta\delta'R(k)]\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2], \\
 R_3(\hat{\beta}_n^{PT}(k); W) &= \sigma_{zz}\text{tr}[W(R(k)C^{-1}R(k)')] - \sigma_{zz}\text{tr}[W(R(k)AR(k)')]\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] \\
 &\quad + [\delta'R(k)WR(k)\delta][2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]] + k^2[\beta'R^{-1}(k)WR^{-1}(k)\beta] \\
 &\quad + 2k[\delta'R(k)'WR^{-1}(k)\beta]\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]. \\
 \text{(iv) } \mathbf{b}_4(\hat{\beta}_n^S(k)) &= -kC^{-1}(k)\beta + (q-2)R(k)\delta E[\chi_{q+2}^{-2}(\Delta^2)], \\
 M_4(\hat{\beta}_n^S(k)) &= \sigma_{zz}[R(k)\Sigma_{XX}^{-1}R(k)'] - \sigma_{zz}(q-2)[R(k)AR(k)']\{2E[\chi_{q+2}^{-2}(\Delta^2)] \\
 &\quad - (q-2)E[\chi_{q+2}^{-4}(\Delta^2)]\} + (q^2-4)[R(k)\delta\delta'R(k)']E[\chi_{q+4}^{-4}(\Delta^2)] \\
 &\quad + k^2C^{-1}(k)\beta\beta'C^{-1}(k) + k[R(k)\delta\beta'C^{-1}(k) + C^{-1}(k)\beta\delta'R(k)']E[\chi_{q+2}^{-2}(\Delta^2)], \\
 R_4(\hat{\beta}_n^S(k); W) &= \sigma_{zz}\text{tr}(W[R(k)\Sigma_{XX}^{-1}R(k)']) \\
 &\quad - \sigma_{zz}(q-2)\text{tr}(W[R(k)AR(k)'])\{2E[\chi_{q+2}^{-2}(\Delta^2)] - (q-2)E[\chi_{q+2}^{-4}(\Delta^2)]\} \\
 &\quad + (q^2-4)\text{tr}[W(R(k)\delta\delta'R(k)')]E[\chi_{q+4}^{-4}(\Delta^2)] + k^2\beta'C^{-1}(k)WC^{-1}(k)\beta \\
 &\quad + k\text{tr}\{W[R(k)\delta\beta'C^{-1}(k) + C^{-1}(k)\beta\delta'R(k)']\}E[\chi_{q+2}^{-2}(\Delta^2)]. \\
 \text{(v) } \mathbf{b}_5(\hat{\beta}_n^{S+}(k)) &= \mathbf{b}_4(\hat{\beta}_n^S(k)) - R(k)\delta\{\mathcal{H}_{q+2} - (q-2)E[\chi_{q+2}^{-2}(\Delta^2)]I(\chi_{q+2}^{-2}(\Delta^2) < (q-2))\}, \\
 M_5(\hat{\beta}_n^{S+}(k)) &= M_4(\hat{\beta}_n^S(k)) - \sigma_{zz}[R(k)AR(k)']E\left[(1 - (q-2)\chi_{q+2}^{-2}(\Delta^2))^2 I(\chi_{q+2}^2(\Delta^2) < (q-2))\right] \\
 &\quad - [R(k)\delta\delta'R(k)']\left\{2E[(1 - (q-2)\chi_{q+2}^{-2}(\Delta^2))I(\chi_{q+2}^{-2}(\Delta^2) < (q-2))]\right. \\
 &\quad \left.- E[(1 - (q-2)\chi_{q+4}^{-2}(\Delta^2))^2 I(\chi_{q+4}^{-2}(\Delta^2) < (q-2))]\right\} \\
 &\quad \times \{R(k)\delta\beta'C^{-1}(k) + C^{-1}(k)\beta\delta'R(k)'\} \\
 &\quad \times E[(1 - (q-2)\chi_{q+2}^{-2}(\Delta^2))I(\chi_{q+4}^2(\Delta^2) < (q-2))], \\
 R_5(\hat{\beta}_n^{S+}(k); W) &= R_4(\hat{\beta}_n^S(k); W) \\
 &\quad - \left\{\sigma_{zz}\text{tr}(R(k)AR(k)')E\left[(1 - (q-2)\chi_{q+2}^{-2}(\Delta^2))^2 I(\chi_{q+2}^2(\Delta^2) < (q-2))\right]\right. \\
 &\quad \left.+ (\delta'W^2\delta)\left\{2E[(1 - (q-2)\chi_{q+2}^{-2}(\Delta^2))I(\chi_{q+2}^2(\Delta^2) < (q-2))]\right.\right. \\
 &\quad \left.- E[(1 - (q-2)\chi_{q+4}^{-2}(\Delta^2))^2 I(\chi_{q+4}^2(\Delta^2) < (q-2))]\right\} \\
 &\quad \left.- 2k\delta'R(k)'WC^{-1}(k)\beta E[(1 - (q-2)\chi_{q+2}^{-2}(\Delta^2))I(\chi_{q+2}^2(\Delta^2) < (q-2))]\right\}.
 \end{aligned}$$

#### 4. Comparison of estimators of $\beta$

We compare the five ridge regression estimators of  $\beta$  based on the risk criterion as a function of the departure parameter,  $\Delta^2$ , as a function of ridge constant  $k$  and as a function of both  $(\Delta^2, k)$ . Comparison among the ridge regression estimators needs the study of the risk-difference of the estimators in comparing. On the other hand, comparison of the ridge regression estimators and the corresponding estimators (say  $\tilde{\beta}_n$  and  $R(k)\tilde{\beta}_n$  etc.) needs the study of the derivatives of the ridge estimators with respect to  $k$ . These procedures are adopted throughout in Section 4.1 onwards.

#### 4.1. Comparison of $\tilde{\beta}_n(k)$ , $\hat{\beta}_n^S(k)$ and $\hat{\beta}_n^{S+}(k)$ based on risks as a function of the departure parameter $\Delta^2$

First we compare  $\tilde{\beta}_n(k)$  and  $\hat{\beta}_n^S(k)$ . Here the risk difference may be written as

$$\begin{aligned} R_1(\tilde{\beta}_n(k); I_p) - R_4(\hat{\beta}_n^S(k); I_p) \\ = (q-2)\sigma_{zz}\text{tr}[R^2(k)A] \left\{ (q-2)E[\chi_{q+2}^{-4}(\Delta^2)] \left( 1 - \frac{(q+2)\delta R^2(k)\delta}{2\Delta^2\sigma_{zz}\text{tr}[R^2(k)A]} \right) 2\Delta^2E[\chi_{q+4}^{-4}(\Delta^2)] \right\} \\ + 2(q-2)k\delta'R(k)C^{-1}(k)\beta E[\chi_{q+2}^{-2}(\Delta^2)]. \end{aligned} \quad (4.1)$$

The right hand side of (4.1) is non-negative if and only if

$$\frac{Ch_{\max}(R^2(k)C^{-1})}{\text{tr}[R^2(k)A]} \geq \frac{q+2}{2}. \quad (4.2)$$

Hence,  $\hat{\beta}_n^S(k)$  dominates  $\tilde{\beta}_n(k)$  uniformly in  $\Delta^2$  for all  $k \in (0, \infty)$

Next, we consider the risk difference

$$\begin{aligned} R_4(\hat{\beta}_n^S(k); I_p) - R_5(\hat{\beta}_n^{S+}(k); I_p) &= \sigma_{zz}\text{tr}[R^2(k)A] E \left[ (1 - (q-2)\chi_{q+2}^{-2}(\Delta^2))^2 I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right] \\ &\quad - \delta R^2(k)\delta \left\{ 2E \left[ (1 - (q-2)\chi_{q+2}^{-2}(\Delta^2)) I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right] \right. \\ &\quad \left. - E \left[ (1 - (q-2)\chi_{q+2}^{-2}(\Delta^2))^2 I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right] \right\} \\ &\quad - 2k\delta'R(k)C^{-1}(k)\beta E[(1 - (q-2)\chi_{q+2}^{-2}(\Delta^2)) I(\chi_{q+2}^2(\Delta^2) < (q-2))]. \end{aligned} \quad (4.3)$$

Since  $\chi_{q+2}^2(\Delta^2) - (q-2) < 0$ , so the right hand side of (4.3) is non-negative. Hence  $\hat{\beta}_n^{S+}(k)$  dominates  $\hat{\beta}_n^S(k)$  uniformly in  $\Delta^2 \in (0, \infty)$ . As a result, the dominance picture for the above three estimators is given by

$$R_5(\hat{\beta}_n^{S+}(k); I_p) \leq R(\hat{\beta}_n^S(k); I_p) \leq R_1(\tilde{\beta}_n(k); I_p),$$

for all  $\Delta^2 \in (0, \infty)$ . Hence  $\hat{\beta}_n^{S+}(k)$  is preferable to either  $\hat{\beta}_n^S(k)$  or  $\tilde{\beta}_n(k)$ . These results are similar to  $\hat{\beta}_n^{S+}$  compared to  $\hat{\beta}_n^S$  and  $\tilde{\beta}_n$  (see [29, Chapter 7]).

#### 4.2. Comparison of $\tilde{\beta}_n^{PT}(k)$ , $\tilde{\beta}_n(k)$ and $\hat{\beta}_n(k)$ based on risks as a function of the departure parameter $\Delta^2$

First, we consider the comparison between the risks of  $\hat{\beta}_n^{PT}(k)$  and  $\tilde{\beta}_n(k)$  as follows:

$$\begin{aligned} R_3(\tilde{\beta}_n(k); I_p) - R_1(\hat{\beta}_n^{PT}(k); I_p) &= \sigma_{zz}\text{tr}[R(k)AR(k)'] \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] \\ &\quad - [\delta'R(k)R(k)\delta] [2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]] \\ &\quad - 2k[\delta'R(k)C^{-1}(k)\beta] \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]. \end{aligned} \quad (4.4)$$

The expression on the right hand side is non-negative whenever

$$\delta'R(k)R(k)\delta \leq \frac{\{\sigma_{zz}\text{tr}[R(k)AR(k)'] - 2k\delta'R(k)C^{-1}(k)\beta\} \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]}{[2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]]}. \quad (4.5)$$

The use of the Courant–Fisher theorem once again yields that (4.5) is non-negative whenever  $\Delta^2 \in (0, \Delta_1^2(\alpha, k))$  where

$$\Delta_1^2(\alpha, k) = \frac{\{\sigma_{zz}\text{tr}[R(k)AR(k)'] - 2k\delta'R(k)C^{-1}(k)\beta\} \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]}{Ch_{\max}(R(k)R(k)C^{-1}) [2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]]}. \quad (4.6)$$

Thus  $\tilde{\beta}_n(k)$  is dominated by  $\hat{\beta}_n^{PT}(k)$  whenever  $\Delta^2 \in (0, \Delta_1^2(\alpha, k))$  and  $\tilde{\beta}_n(k)$  dominates  $\hat{\beta}_n^{PT}(k)$  whenever  $\Delta^2 \in (\Delta_2^2(\alpha, k), \infty)$  where

$$\Delta_2^2(\alpha, k) = \frac{\{\sigma_{zz}\text{tr}[R(k)AR(k)'] - 2k\delta'R(k)C^{-1}(k)\beta\} \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]}{Ch_{\min}(R(k)R(k)C^{-1}) [2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]]}. \quad (4.7)$$

Now, if  $\alpha = 0$ , then  $\chi_q^2(0) = \infty$  implies  $\mathcal{H}$  functions are unity and we get risk comparison of  $\tilde{\beta}_n(k)$  and  $\hat{\beta}_n(k)$ . Thus

$$R_1(\tilde{\beta}_n(k); I_p) - R_2(\hat{\beta}_n(k); I_p) = \sigma_{zz} \text{tr}[R(k)AR(k)'] - \delta' R^2(k) \delta - 2k \delta' R(k)' C^{-1}(k) \beta. \quad (4.8)$$

The right hand side is non-negative whenever  $\Delta^2 \in (0, \Delta_1(0, k))$  where

$$\Delta_1(0, k) = \frac{\sigma_{zz} \text{tr}[R(k)AR(k)'] - 2k \delta' R(k)' C^{-1}(k) \beta}{Ch_{\max}[R'(k)R(k)C^{-1}]}. \quad (4.9)$$

Thus  $\hat{\beta}_n(k)$  dominates  $\tilde{\beta}_n(k)$  whenever  $\Delta^2 \in (0, \Delta_1(0, k))$  and  $\tilde{\beta}_n(k)$  dominates  $\hat{\beta}_n(k)$  whenever  $\Delta^2 \in (\Delta_2(0, k), \infty)$  where

$$\Delta_2^2(0, k) = \frac{\sigma_{zz} \text{tr}[R(k)AR(k)'] - 2k \delta' R(k)' C^{-1}(k) \beta}{Ch_{\min}[R'(k)R(k)C^{-1}]}. \quad (4.10)$$

Now we consider the relative efficiency (RE) of  $\hat{\beta}_n^{PT}(k)$  compared with  $\tilde{\beta}_n(k)$  similar to  $\hat{\beta}_n^{PT}$  compared with  $\tilde{\beta}_n$ . Accordingly, we provide a maximum and minimum (Max and Min) rule for the optimum choice of the level of significance of the  $\hat{\beta}_n^{PT}(k)$  for testing the null hypothesis  $H_0 : H\beta = h$ . For fixed value of  $k(k > 0)$ , this RE is a function of  $\alpha$  and  $\Delta^2$ . Let us denote this by

$$E(\alpha, \Delta, k) = \frac{R(\tilde{\beta}_n(k); I_p)}{R(\hat{\beta}_n^{PT}(k); I_p)} = \left[ 1 - \frac{f_{14}(\alpha, k, \Delta^2)}{\sigma_{zz} \text{tr}[R(k)AR(k)'] + k^2 \beta' C^{-2}(k) \beta} \right]^{-1} \quad (4.11)$$

where

$$f_{14}(\alpha, k, \Delta^2) = \sigma_{zz} \text{tr}[R(k)AR(k)'] \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \delta' R(k)' R(k) \delta \{ 2 \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2] \} - 2k \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] \delta' R(k)' C^{-1}(k) \beta. \quad (4.12)$$

For a given  $k$ , the function  $E(\alpha, \Delta^2, k)$ , is a function of  $\alpha$  and  $\Delta^2$ . This function for  $\alpha \neq 0$  has its maximum under the null hypothesis with following value,

$$E_{\max}(\alpha, 0, k) = \left[ 1 - \frac{\text{tr}[R(k)AR(k)'] \mathcal{H}_{q+2}[\chi_q^2(\alpha); 0]}{\text{tr}[R(k)AR(k)'] + k^2 \beta' C^{-2}(k) \beta} \right]^{-1}. \quad (4.13)$$

For given  $k$ ,  $E_{\max}(\alpha, 0, k)$  is a decreasing function of  $\alpha$ . While, the minimum efficiency  $E_{\min}$  is an increasing function of  $\alpha$ . For  $\alpha \neq 0$ , as  $\Delta^2$  varies the graphs of  $E(0, \Delta, k)$  and  $E(1, \Delta, k)$  intersect in the range  $0 < \Delta^2 < \Delta_1^2(\alpha, k)$ , which is given in (4.13). Therefore, in order to choose an estimator with optimum relative efficiency, we adopt the following rule for fixed values of  $k$ . If  $0 < \Delta^2 < \Delta_1^2(\alpha, k)$ , we choose  $\tilde{\beta}_n(k)$  since  $E(0, \Delta, k)$  is the largest in this interval. However,  $\Delta^2$  is unknown and there is no way of choosing a uniformly best estimator. Therefore, following Saleh [29], we will use the following criterion for selecting the significance level of the preliminary test.

Suppose the experimenter does not know the size of  $\alpha$  and wants an estimator which has relative efficiency not less than  $E_{\min}$ . Then among the set of estimators with  $\alpha \in A$ , where  $A = \{\alpha : E(\alpha, \Delta, k) \geq E_{\min} \text{ for all } \Delta\}$ , the estimator is chosen to maximize  $E(\alpha, \Delta, k)$  over all  $\alpha \in A$  and all  $\Delta^2$ . Thus we solve for  $\alpha$  from the following equation.

$$\max_{0 \leq \alpha \leq 1} \min_{\Delta^2} E(\alpha, \Delta, k) = E_{\min}. \quad (4.14)$$

Readers are referred to Saleh and Kibria [32] for tabular values of maximum and minimum guaranteed efficiencies ARE's for various values of  $\alpha, k, \sigma^2$ .

#### 4.3. Comparison of the risk of estimators of $\beta$ as a function of ridge constant $k$

First note that the asymptotic covariance matrix of the unrestricted estimator of  $\beta$  is  $\sigma_{zz} C^{-1}$  where  $\Leftrightarrow C = K'_{xx} \Sigma_{xx} K_{xx}$  is a positive definite matrix. Thus we can find an orthogonal matrix  $\Gamma$  such that

$$\Gamma' (K'_{xx} \Sigma_{xx} K_{xx}) \Gamma = \Gamma' C \Gamma = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_p), \quad (4.15)$$

where  $\lambda_1 \geq \lambda_2 \geq \dots, \lambda_p > 0$  are the characteristic roots of the matrix  $(K'_{xx} \Sigma_{xx} K_{xx})$ . It is easy to see that the characteristic roots of  $[I_p + k (K'_{xx} \Sigma_{xx} K_{xx})^{-1}]^{-1} = R(k)$  and of  $[(K'_{xx} \Sigma_{xx} K_{xx}) + k I_p] = R^{-1}(k)$  are

$$\left( \frac{\lambda_1}{(\lambda_1 + k)^2}, \frac{\lambda_2}{(\lambda_2 + k)^2}, \dots, \frac{\lambda_p}{(\lambda_p + k)^2} \right) \quad \text{and} \quad (\lambda_1 + k, \lambda_2 + k, \dots, \lambda_p + k) \quad (4.16)$$

respectively. Hence we obtain the following identities:

$$\text{tr}[R(k)C^{-1}R(k)'] = \text{tr}\left[R(k)\left(K'_{xx}\Sigma_{xx}K_{xx}\right)^{-1}R(k)'\right] = \sum_{i=1}^p \frac{\lambda_i}{(\lambda_i + k)}, \quad (4.17)$$

$$\boldsymbol{\beta}'R^{-2}(k)\boldsymbol{\beta} = \sum_{i=1}^p \frac{\theta_i^2}{(\lambda_i + k)^2}, \quad \boldsymbol{\theta} = \boldsymbol{\Gamma}'\boldsymbol{\beta} = (\theta_1, \theta_2, \dots, \theta_p)', \quad (4.18)$$

$$\text{tr}[R(k)AR(k)'] = \sum_{i=1}^p \frac{a_{ii}^*\lambda_i^2}{(\lambda_i + k)^2} \quad (4.19)$$

where  $a_{ii}^* \geq 0$  is the diagonal matrix of  $A^* = \boldsymbol{\Gamma}'A\boldsymbol{\Gamma}$  and

$$\text{tr}[\boldsymbol{\delta}'R(k)R(k)\boldsymbol{\delta}] = \sum_{i=1}^p \frac{\lambda_i\delta_i^{*2}}{(\lambda_i + k)^2} \quad (4.20)$$

where  $\delta_i^*$  is the  $i$ th element of  $\boldsymbol{\delta}^* = \boldsymbol{\delta}'\boldsymbol{\Gamma}$ . Similarly

$$\text{tr}\left[\boldsymbol{\delta}'R(k)\left(K'_{xx}\Sigma_{xx}K_{xx} + kI_p\right)^{-1}R(k)\boldsymbol{\delta}\right] = \sum_{i=1}^p \frac{\theta_i\lambda_i\delta_i^*}{(\lambda_i + k)^2}. \quad (4.21)$$

#### 4.3.1. Comparison of $\tilde{\boldsymbol{\beta}}_n(k)$ and $\tilde{\boldsymbol{\beta}}_n$

In this case we have the risk of  $\tilde{\boldsymbol{\beta}}_n(k)$  as

$$R(\tilde{\boldsymbol{\beta}}_n(k); I_p) = \sigma_{zz} \sum_{i=1}^p \frac{\lambda_i}{(\lambda_i + k)^2} + k^2 \sum_{i=1}^p \frac{\theta_i^2}{(\lambda_i + k)^2}. \quad (4.22)$$

Clearly for  $k = 0$ , the risk equals that the risk of  $\tilde{\boldsymbol{\beta}}_n$ . Note that the first term of (4.22) is a continuous, monotonically decreasing function of  $k$  and its derivative with respect to  $k$  approaches  $-\infty$  as  $k \rightarrow 0^+$  and  $\lambda_p \rightarrow 0$ . The second term is also a continuous monotonically increasing function of  $k$  and its derivative with respect to  $k$  tends to zero as  $k \rightarrow 0^+$  and the second term approaches  $\boldsymbol{\beta}\boldsymbol{\beta}'$  as  $k \rightarrow \infty$ . Differentiating with respect to  $k$ , we get

$$\frac{\partial R(\tilde{\boldsymbol{\beta}}_n(k); I_p)}{\partial k} = 2 \sum_{i=1}^p \frac{\lambda_i}{(\lambda_i + k)^3} (k\theta_i^2 - \sigma_{zz}). \quad (4.23)$$

Thus a sufficient condition for (4.23) to be negative is that  $0 < k < k_0^*$  where

$$k_0^* = \frac{\sigma_{zz}}{\theta_{\max}}, \quad (4.24)$$

where  $\theta_{\max}$  = Largest element of  $\boldsymbol{\theta}$  and  $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_p)'$ .

Thus, we have the following theorem.

**Theorem 2.** There always exists a  $k \in (0, k_0^*)$  such that  $R_1(\tilde{\boldsymbol{\beta}}_n; I_p) \geq R_1(\tilde{\boldsymbol{\beta}}_n(k); I_p)$ .

#### 4.3.2. Comparison of $\hat{\boldsymbol{\beta}}_n(k)$ and $\hat{\boldsymbol{\beta}}_n$

The risk function under the hypothesis  $H\boldsymbol{\beta} \neq \mathbf{h}$  is given by

$$R_2(\hat{\boldsymbol{\beta}}_n(k); I_p) = \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} [\sigma_{zz}(\lambda_i - a_{ii}^*) + k^2\theta_i^2 + \lambda_i^2\delta_i^{*2} + 2k\theta_i\delta_i^*\lambda_i]. \quad (4.25)$$

Thus differentiating (4.25) with respect to  $k$ , we obtain a sufficient condition for  $\partial R_2(\hat{\boldsymbol{\beta}}_n(k); I_p)/\partial k$  to be negative as  $k \in (0, k_2^*)$  where

$$k_2^* = \frac{\min_{1 \leq i \leq p} [(\lambda_i - a_{ii}^*) - \lambda_i^2\delta_i^*(\theta_i - \delta_i^*)]}{\max_{1 \leq i \leq p} \lambda_i\theta_i(\theta_i - \delta_i^*)}. \quad (4.26)$$

Thus a sufficient condition for the restricted ridge regression estimator to have smaller risk value than the unrestricted ridge regression estimator is that there exists a value of  $k$  such that  $0 < k < k_1$  where  $k_1$  is given by

$$k_1 = \frac{\min_{1 \leq i \leq p} [\sigma_{zz}a_{ii}^* - \lambda_i^2\delta_i^{*2}]}{\max_{1 \leq i \leq p} (2\theta_i\delta_i^*\lambda_i)}. \quad (4.27)$$



We conclude that

$$R_1(\tilde{\beta}_n(k); I_p) - R_2(\hat{\beta}_n(k); I_p) \geq 0 \quad \text{for all } k \text{ such that } 0 < k < k_1,$$

since

$$R_1(\tilde{\beta}_n(k); I_p) - R_2(\hat{\beta}_n(k); I_p) = \sum_{i=1}^p \frac{(\sigma_{zz} a_{ii}^* - \lambda_i^2 \delta_i^{*2} - 2k\theta_i \lambda_i \delta_i^*)}{(\lambda_i + k)^2}.$$

**Theorem 3.** There always exist a  $k \in (0, k_2^*)$  such that  $R_2(\hat{\beta}_n; I_p) \geq R_2(\hat{\beta}_n(k); I_p)$ . Under  $H\beta = \mathbf{h}$ ,  $k_2^*$  equals  $k_1$ .

#### 4.3.3. Comparison of $\hat{\beta}_n^{PT}(k)$ and $\hat{\beta}_n^{PT}$

We consider the  $R_3(\hat{\beta}_n^{PT}(k); I_p)$  under  $H\beta \neq \mathbf{h}$  which is a function of eigenvalues and  $k$  is given as follows:

$$R_3(\hat{\beta}_n^{PT}(k); I_p) = \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} [\sigma_{zz} \{\lambda_i - a_{ii}^* \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]\} + k^2 \theta_i^2 + 2k\theta_i \lambda_i \delta_i^* \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] + \lambda_i^2 \delta_i^{*2} \{2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]\}]. \quad (4.28)$$

Differentiation of  $R_3(\hat{\beta}_n^{PT}(k); I_p)$  with respect to  $k$ , we obtain a sufficient condition for  $\hat{\beta}_n^{PT}(k)$  is superior to  $\hat{\beta}_n^{PT}$  is that there exists a  $k \in (0, k_2(\alpha, \Delta^2))$ . The derivative  $\partial R_3(\hat{\beta}_n^{PT}(k); I_p) / \partial k$  is negative where  $k_2(\alpha, \Delta^2)$  is defined by

$$k_2(\alpha, \Delta^2) = \frac{f_2(\alpha, \Delta^2)}{g_1(\alpha, \Delta^2)} \quad (4.29)$$

where

$$f_2(\alpha, \Delta^2) = \min_{1 \leq i \leq p} [\sigma_{zz} (\lambda_i - a_{ii}^* \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]) + \lambda_i^2 \delta_i^{*2} \{2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]\} - \theta_i \lambda_i^2 \delta_i^* \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]], \quad (4.30)$$

$$g_1(\alpha, \Delta^2) = \frac{\max_{1 \leq i \leq p} [\lambda_i \theta_i \{\theta_i - \delta_i^* \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]\}]}{\max_{1 \leq i \leq p} (\theta_i^2, \lambda_i)}. \quad (4.31)$$

**Theorem 4.** There always exist a  $k \in (0, k_2(\alpha, \Delta^2))$  such that  $R_3(\hat{\beta}_n^{PT}; I_p) \geq R_3(\hat{\beta}_n^{PT}(k); I_p)$ . Under  $H\beta = \mathbf{h}$ ,  $k_2(\alpha, \Delta^2)$  equals  $k_3(\alpha, \Delta^2) = \min_{1 \leq i \leq p} [\lambda_i - a_{ii}^* \mathcal{H}_{q+2}(\chi_q^2(\alpha); 0)]$ .

**Remark.** Suppose  $k > 0$ , then the following statements hold true following Kaciranlar et al. [17]:

1. If  $g_1(\alpha, \Delta^2) > 0$ , it follows that for each  $k > 0$  with  $k < k_2(\alpha, \Delta^2)$ ,  $\hat{\beta}_n^{PT}(k)$  has smaller risk than that of  $\hat{\beta}_n^{PT}$ .
2. If  $g_1(\alpha, \Delta^2) < 0$ , it follows that for each  $k > 0$  with  $k > k_2(\alpha, \Delta^2)$ ,  $\hat{\beta}_n^{PT}(k)$  has smaller risk than that of  $\hat{\beta}_n^{PT}$ .
3. If  $\alpha = 0$ , we obtain the comparison conditions for  $\hat{\beta}_n(k)$  and  $\hat{\beta}_n$  and if  $\alpha = 1$ , we obtain the comparison conditions for  $\tilde{\beta}_n(k)$  and  $\tilde{\beta}_n$ .

#### 4.3.4. Comparison of $\hat{\beta}_n^{PT}(k)$ , $\tilde{\beta}_n(k)$ and $\hat{\beta}_n(k)$

Consider first the comparison of  $\hat{\beta}_n^{PT}(k)$  and  $\hat{\beta}_n(k)$ .

Under the alternative hypothesis  $H\beta \neq \mathbf{h}$ , the difference between the risks of  $\hat{\beta}_n^{PT}(k)$  and  $\hat{\beta}_n(k)$  in terms of eigenvalues and  $k$  is given by

$$R_3(\hat{\beta}_n^{PT}(k); I_p) - R_2(\hat{\beta}_n(k); I_p) = \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} [\sigma_{zz} a_{ii}^* [1 - \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]] - \lambda_i^2 \delta_i^{*2} [1 - 2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] + \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]] - 2k\theta_i \lambda_i \delta_i^* [1 - \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]]] \quad (4.32)$$

and the right hand side is negative whenever

$$k_3(\Delta^2, \alpha) = \frac{\max_{1 \leq i \leq p} \{ [\sigma_{zz} a_{ii}^* [1 - \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]] - \lambda_i^2 \delta_i^{*2} [1 - 2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] + \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]] \}}{\min_{1 \leq i \leq p} \{ 2\theta_i \lambda_i \delta_i^* [1 - \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2]] \}}. \quad (4.33)$$

Thus  $\hat{\beta}_n^{PT}(k)$  dominates  $\hat{\beta}_n(k)$  whenever  $k_3(\Delta^2, \alpha) < k$ , otherwise the reverse holds true. For  $\alpha = 1$ , we find that  $\hat{\beta}_n^{PT}(k)$  is dominated by  $\hat{\beta}_n(k)$  when

$$k_3(\Delta^2, 1) = \frac{\max_{1 \leq i \leq p} [\sigma_{zz} a_{ii}^* - \lambda_i^2 \delta_i^{*2}]}{\min_{1 \leq i \leq p} (2\theta_i \lambda_i \delta_i^*)}. \quad (4.34)$$

Under the null hypothesis  $H_0 : H\beta = \mathbf{h}$ ,  $\hat{\beta}_n(k)$  is superior to  $\hat{\beta}_n^{PT}(k)$  since the risk difference equals

$$\sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} [\sigma_{zz} a_{ii}^* \{1 - \mathcal{H}_{q+2}[\chi_q^2(\alpha); 0]\}] \geq 0.$$

Again the risk difference of  $\hat{\beta}_n^{PT}(k)$  and  $\tilde{\beta}_n(k)$  in terms of eigenvalues and  $k$  points out to the fact that  $\hat{\beta}_n^{PT}(k)$  dominates  $\tilde{\beta}_n(k)$  when  $k \in k_4(\Delta^2, \alpha)$ , where

$$k_4(\Delta^2, \alpha) = \frac{\max_{1 \leq i \leq p} \{ [\sigma_{zz} \lambda_i^2 a_{ii}^* \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \lambda_i^2 \delta_i^{*2} [2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]] \}}{\min_{1 \leq i \leq p} \{ 2\theta_i \lambda_i \delta_i^* \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] \}} \quad (4.35)$$

and  $\tilde{\beta}_n(k)$  dominates  $\hat{\beta}_n^{PT}(k)$  whenever  $k_4(\Delta^2, \alpha) > k$ .

#### 4.3.5. Comparison of $\hat{\beta}_n^S(k)$ and $\hat{\beta}_n^S$

In this section, we compare  $\hat{\beta}_n^S(k)$  and  $\hat{\beta}_n^S$  when the risk is a function of  $(\alpha, \Delta^2)$ . First we consider the risk as a function of  $k$  and then as a function of  $\Delta^2$ . We now consider the risk function of  $\hat{\beta}_n^S(k)$ . Then, a sufficient condition for  $\partial R_4(\hat{\beta}_n^S(k); I_p)/\partial k$  to be negative is that  $k < k_5(\Delta^2)$  where

$$k_5(\Delta^2) = \frac{f_3(\Delta^2)}{g_2(\Delta^2)}, \quad (4.36)$$

where

$$f_3(\Delta^2) = \min_{1 \leq i \leq p} \left\{ \sigma_{zz} \left\{ \lambda_i - (q-2)\sigma_{zz} a_{ii}^* \left[ (q-2)E[\chi_{q+2}^{-4}(\Delta^2)] \right. \right. \right. \\ \left. \left. \left. + \left( 1 - \frac{(q+2)\delta_i^{*2}\lambda_i^2}{2\Delta^2\sigma_{zz}a_{ii}^*} \right) (2\Delta^2)E[\chi_{q+4}^{-4}(\Delta^2)] \right] \right\} + (q-2)\theta_i \lambda_i^2 \delta_i^* E[\chi_{q+2}^{-2}(\Delta^2)] \right\} \quad (4.37)$$

$$g_2(\Delta^2) = \max_{1 \leq i \leq p} [\lambda_i \theta_i \{ \theta_i - (q-2)\delta_i^* E[\chi_{q+2}^{-4}(\Delta^2)] \}]. \quad (4.38)$$

Suppose  $k > 0$ , then the following statements hold true following Kaciranlar et al. [17]:

1. If  $g_2(\Delta^2) > 0$ , it follows that for each  $k > 0$  with  $k < k_5(\Delta^2)$ ,  $\hat{\beta}_n^S(k)$  has smaller risk than that of  $\hat{\beta}_n^S$ .
2. If  $g_2(\Delta) < 0$ , it follows that for each  $k > 0$  with  $k > k_5(\Delta)$ ,  $\hat{\beta}_n^S(k)$  has smaller risk than that of  $\hat{\beta}_n^S$ .

Now we consider the risk as a function of  $\Delta^2$ .

To find a sufficient condition on  $\Delta^2$ , the difference in the risks of  $\hat{\beta}_n^S(k)$  and  $\hat{\beta}_n^S$  will be non-positive when

$$\delta'[I_p - R(k)'R(k)]\delta \geq \frac{f_4(\Delta^2)}{(q^2 - 4)E[\chi_{q+4}^{-4}(\Delta^2)]}, \quad (4.39)$$

where

$$f_4(\Delta^2) = \sigma_{zz} [\text{tr}(R(k)C^{-1}R(k)') - \text{tr}(C^{-1})] + (q-2)\sigma_{zz} [\text{tr}(A) - \text{tr}(R(k)AR(k)')] \\ \times \{ 2E[\chi_{q+2}^{-2}(\Delta^2)] - (q-2)E[\chi_{q+2}^{-4}(\Delta^2)] \} \\ + k^2 \beta' C^{-2}(k) \beta + 2(q-2)k \delta' R(k)' C^{-1}(k) \beta E[\chi_{q+2}^{-2}(\Delta^2)]. \quad (4.40)$$

Since  $\Delta^2 > 0$ , we assume that the numerator of (4.39) is positive. Then the  $\hat{\beta}_n^S$  is superior to  $\hat{\beta}_n^S(k)$  when

$$\Delta^2 \geq \frac{f_4(\Delta^2)}{(q^2 - 4)Ch_{\max}[(I_p - R(k)'R(k))C^{-1}]E[\chi_{q+4}^{-4}(\Delta^2)]} = \Delta_{\max}^2(k), \quad \text{say} \quad (4.41)$$

where  $Ch_{\max}(M)$  is the maximum characteristic root of the matrix  $(M)$ . However,  $\hat{\beta}_n^S$  is inferior to  $\hat{\beta}_n^S(k)$  when

$$\Delta^2 < \frac{f_4(\Delta^2)}{(q^2 - 4)Ch_{\min}[(I_p - R(k)'R(k))C^{-1}]E[\chi_{q+4}^{-4}(\Delta^2)]} = \Delta_{\min}^2(k), \quad (4.42)$$

where  $Ch_{\min}(M)$  is the minimum characteristic root of the matrix  $(M)$ .

#### 4.3.6. Comparison of $\hat{\beta}_n^S(k)$ and $\tilde{\beta}_n(k)$ as a function of $k$

Consider the difference in the risks of  $\hat{\beta}_n^S(k)$  and  $\tilde{\beta}_n(k)$ . Then, a sufficient condition for the risk difference to be non-negative is whenever  $0 < k < k_6(\Delta^2)$  where

$$k_6(\Delta^2) = \frac{f_4(\Delta^2)}{g_2(\Delta^2)}, \quad (4.43)$$

where

$$f_4(\Delta^2) = \min_{1 \leq i \leq p} \left\{ \sigma_{zz} a_{ii}^* \lambda_i^2 [(q-2)E[\chi_{q+2}^{-4}(\Delta^2)]] + \left( 1 - \frac{(q+2)\delta_i^{*2}}{2\Delta^2 \sigma_{zz} a_{ii}^*} \right) (2\Delta^2)E[\chi_{q+4}^{-4}(\Delta^2)] \right\} \quad (4.44)$$

$$g_2(\Delta^2) = \max_{1 \leq i \leq p} [2\theta_i \lambda_i \delta_i^* E[\chi_{q+2}^{-2}(\Delta^2)]] . \quad (4.45)$$

Suppose  $k > 0$ , then the following statements hold true following Kaciranlar et al. [17]:

1. If  $g_2(\Delta^2) > 0$ , it follows that for each  $k > 0$  with  $k < k_6(\Delta^2)$ ,  $\hat{\beta}_n^S(k)$  has smaller risk than that of  $\tilde{\beta}_n(k)$ .
2. If  $g_2(\Delta^2) < 0$ , it follows that for each  $k > 0$  with  $k > k_6(\Delta^2)$ ,  $\hat{\beta}_n^S(k)$  has smaller risk than that of  $\tilde{\beta}_n(k)$ .

Note that this risk difference under  $H_0 : H\beta = h$  is  $\sum_{i=1}^p \frac{(q-2)\sigma_{zz}\lambda_i^2 a_{ii}^*}{(\lambda_i+k)^2} \geq 0$ . Therefore  $\hat{\beta}_n^S(k)$  always dominates  $\tilde{\beta}_n(k)$  under the null hypothesis for  $q \geq 3$ .

#### 4.3.7. Comparison of $\hat{\beta}_n^S(k)$ and $\hat{\beta}_n(k)$

Consider the difference in the risks of  $\hat{\beta}_n^S(k)$  and  $\hat{\beta}_n(k)$ . Then, a sufficient condition for the difference to be non-negative is that  $0 < k < k_7(\Delta^2)$  where

$$k_7(\Delta^2) = \frac{f_5(\Delta^2)}{g_3(\Delta^2)}, \quad (4.46)$$

where

$$f_5(\Delta^2) = \max_{1 \leq i \leq p} \left\{ \sigma_{zz} a_{ii}^* \lambda_i^2 \left\{ 1 - (q-2) [(q-2)E[\chi_{q+2}^{-4}(\Delta^2)]] \right. \right. \\ \left. \left. + \left( 1 - \frac{(q+2)\lambda_i^2 \delta_i^{*2}}{2\Delta^2 \sigma_{zz} a_{ii}^*} \right) (2\Delta^2)E[\chi_{q+4}^{-4}(\Delta^2)] \right\} - \lambda_i^2 \delta_i^{*2} \right\} \quad (4.47)$$

$$g_3(\Delta^2) = \min_{1 \leq i \leq p} [2\theta_i \lambda_i \delta_i^* [1 - (q-2)E[\chi_{q+2}^{-2}(\Delta^2)]]].$$

Suppose  $k > 0$ , then the following statements hold true:

1. If  $g_3(\Delta^2) > 0$ , it follows that for each  $k > 0$  with  $k < k_7(\Delta^2)$ ,  $\hat{\beta}_n^S(k)$  has smaller risk than that of  $\hat{\beta}_n(k)$ .
2. If  $g_3(\Delta^2) < 0$ , it follows that for each  $k > 0$  with  $k > k_7(\Delta^2)$ ,  $\hat{\beta}_n(k)$  has smaller risk than that of  $\hat{\beta}_n^S(k)$ .

Note that this risk difference under  $H_0 : H\beta = h$  is  $2\sigma_{zz} \sum_{i=1}^p \frac{a_{ii}^*}{(\lambda_i+k)^2} \geq 0$ . Thus  $\hat{\beta}_n(k)$  is superior to  $\hat{\beta}_n^S(k)$  under  $H_0 : H\beta = h$ .

Next, consider the difference in the risks of  $\hat{\beta}_n^S(k)$  and  $\tilde{\beta}_n^{PT}(k)$ . We define

$$k_8(\alpha, \Delta^2) = \frac{f_6(\alpha, \Delta^2)}{g_4(\alpha, \Delta^2)}, \quad (4.48)$$

where

$$f_6(\alpha, \Delta^2) = \max_{1 \leq i \leq p} \left\{ \sigma_{zz} a_{ii}^* \lambda_i^2 \left\{ \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - (q-2) [(q-2)E[\chi_{q+2}^{-4}(\Delta^2)]] \right. \right. \\ \left. \left. + \left( 1 - \frac{(q+2)\delta_i^{*2}}{2\Delta^2 \sigma_{zz} a_{ii}^*} \right) (2\Delta^2)E[\chi_{q+4}^{-4}(\Delta^2)] \right\} - \lambda_i^2 \delta_i^{*2} [2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]] \right\}. \quad (4.49)$$

$$g_4(\Delta^2) = \min_{1 \leq i \leq p} [2\theta_i \lambda_i \delta_i^{*2} [\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - (q-2)E[\chi_{q+2}^{-4}(\Delta^2)]]]. \quad (4.50)$$

Suppose  $k > 0$ , then the following statements hold true:

1. If  $g_4(\Delta^2) > 0$ , it follows that for each  $k > 0$  with  $k > k_8(\alpha, \Delta^2)$ ,  $\hat{\beta}_n^S(k)$  has smaller risk than that of  $\hat{\beta}_n^{PT}(k)$ .
2. If  $g_4(\alpha, \Delta^2) < 0$ , it follows that for each  $k > 0$  with  $k < k_8(\alpha, \Delta^2)$ ,  $\hat{\beta}_n^S(k)$  has smaller risk than that of  $\hat{\beta}_n^{PT}(k)$ .

The risk difference under  $H_0 : H\beta = \mathbf{h}$  reduces to

$$\sum_{i=1}^p \frac{\left[ 2\mathcal{H}_{q+2}[\chi_q^2(\alpha); 0] - \left( \frac{q-2}{q} \right) \right] \sigma_{zz} a_{ii}^*}{(\lambda_i + k)^2}.$$

Therefore the risk of  $\hat{\beta}_n^S(k)$  is smaller than the risk of  $\hat{\beta}_n^{PT}(k)$  when

$$\chi_q^2(\alpha) \leq \mathcal{H}_{q+2}^{-1}[(q-2), 0] \quad (4.51)$$

where  $\chi_q^2(\alpha)$  is the upper  $\alpha$ -level critical value from the chi-square distribution with  $q$  degrees of freedom. Otherwise the risk of  $\hat{\beta}_n^{PT}(k)$  is smaller than the risk of  $\hat{\beta}_n^S(k)$ .

#### 4.3.8. Comparison of $\hat{\beta}_n^{S+}(k)$ and $\hat{\beta}_n^S(k)$ as a function of $k$

We consider the risk of  $\hat{\beta}_n^{S+}(k)$  under  $H_1 : H\beta \neq \mathbf{h}$ .

A sufficient condition for the risk difference between  $\hat{\beta}_n^S(k)$  and  $\hat{\beta}_n^{S+}(k)$  to be non-negative is whenever  $0 < k < k_9(\Delta^2)$  which is obtained by differentiating the risk of  $\hat{\beta}_n^{S+}(k)$  with respect to  $k$  to obtain

$$k_9(\Delta^2) = \frac{f_7(\Delta^2)}{g_5(\Delta^2)}, \quad (4.52)$$

where

$$f_7(\alpha, \Delta^2) = \min_{1 \leq i \leq p} \left\{ \sigma_{zz} \left\{ \lambda_i - (q-2)a_{ii}^* [(q-2)E[\chi_{q+2}^{-2}(\Delta^2)]] \right. \right. \\ \left. \left. + \left( 1 - \frac{(q+2)\lambda_i \delta_i^{*2}}{2\sigma_{zz} \Delta^2 a_{ii}^*} \right) (2\Delta^2)E[\chi_{q+4}^{-4}(\Delta^2)] \right. \right. \\ \left. \left. - a_{ii}^* E \left[ (1 - (q-2)\chi_{q+2}^{-2}(\Delta^2))^2 I(\chi_{q+2}^2(\Delta^2) \leq (q-2)) \right] \right. \right. \\ \left. \left. - \lambda_i^2 \delta_i^{*2} E \left[ (1 - (q-2)\chi_{q+4}^{-2}(\Delta^2))^2 I(\chi_{q+4}^2(\Delta^2) \leq (q-2)) \right] \right. \right. \\ \left. \left. + (\theta_i - 2\delta_i^*) \lambda_i^2 \delta_i^{*2} E \left[ ((q-2)\chi_{q+2}^{-2}(\Delta^2) - 1) I(\chi_{q+2}^2(\Delta^2) \leq (q-2)) \right] \right. \right. \\ \left. \left. + dq\theta_i \delta_i^* \lambda_i^2 E[\chi_{q+2}^{-2}(\Delta^2)] \right\} \right\} \quad (4.53)$$

$$g_5(\alpha, \Delta^2) = \max_{1 \leq i \leq p} \left[ \lambda_i \theta_i \left\{ \theta_i + \delta_i^* E((q-2)\chi_{q+2}^{-2}(\Delta^2) - 1) I(\chi_{q+2}^2(\Delta^2) \leq (q-2)) \right. \right. \\ \left. \left. - (q-2)\delta_i^* E[\chi_{q+2}^{-2}(\Delta^2)] \right\} \right]. \quad (4.54)$$

Differentiating the risk of  $\hat{\beta}_n^{S+}(k)$  with respect to  $k$  gives a sufficient condition for  $\partial R(\hat{\beta}_n^{S+}(k); I_p)/\partial k$  to be negative under  $H_0 : H\beta = \mathbf{h}$  that  $k \in (0, k_{10}(\alpha))$  where

$$k_{10}(\alpha) = \frac{\min_{1 \leq i \leq p} \sigma_{zz} \left\{ \lambda_i - (q-2)a_{ii}^* - a_{ii}^* E \left[ 1 - ((q-2)\chi_{q+2}^{-2}(0))^2 I[\chi_{q+2}^2(0) < (q-2)] \right] \right\}}{\max_{1 \leq i \leq p} (\theta_i^2 \lambda_i)}. \quad (4.55)$$

Suppose the numerator of (4.55) is positive, then  $\hat{\beta}_n^{S+}(k)$  dominates  $\hat{\beta}_n^{S+}$  when  $k > 0$  belongs to the region  $k \in (0, k_9(\alpha))$ . Suppose  $k > 0$ , then the following statements hold true following Kaciranlar et al. [17]:

1. If  $g_5(\Delta^2) > 0$ , it follows that for each  $k > 0$  with  $k < k_9(\Delta^2)$ ,  $\hat{\beta}_n^{S+}(k)$  has smaller risk than that of  $\hat{\beta}_n^{S+}$ .
2. If  $g_5(\Delta^2) < 0$ , it follows that for each  $k > 0$  with  $k > k_9(\Delta^2)$ ,  $\hat{\beta}_n^{S+}$  has smaller risk than that of  $\hat{\beta}_n^{S+}(k)$ .

To obtain a condition on  $\Delta^2$ , we consider the risk difference between  $\hat{\beta}_n^{S+}(k)$  and  $\hat{\beta}_n^{S+}$ . It may be shown that the risk-difference is non-positive when

$$\delta'[I_p - R(k)'R(k)]\delta \geq \frac{f_8(\Delta^2)}{E^*(\Delta^2)} \quad (4.56)$$

where

$$\begin{aligned} E^*(\Delta^2) &= (q-2) - E \left[ (1 - (q-2)\chi_{q+2}^{-2}(\Delta^2))^2 I(\chi_{q+4}^2(\Delta^2) < (q-2)) \right] \\ &\quad - 2((q-2)\chi_{q+2}^{-2}(\Delta^2) - 1) I(\chi_{q+2}^2(\Delta^2) < (q-2)) \\ f_8(\Delta^2, k) &= \sigma_{zz} [\text{tr}(R(k)'C^{-1}R(k)) - \text{tr}(C^{-1})] \\ &\quad + \sigma_{zz} [\text{tr}(A) - \text{tr}(R(k)'AR(k))] \{2E[\chi_{q+2}^{-2}(\Delta^2)] - (q-2)E[\chi_{q+2}^{-4}(\Delta^2)]\} \\ &\quad + E \left[ (1 - (q-2)\chi_{q+2}^{-2}(\Delta^2))^2 I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right] + 2k\delta'R(k)'C^{-1}(k)\beta \\ &\quad \times \{(q-2) - E[(q-2)\chi_{q+2}^{-2}(\Delta^2) - 1] I(\chi_{q+2}^2(\Delta^2) < (q-2))\}. \end{aligned} \quad (4.57)$$

Since  $\Delta^2 > 0$ , assume that both the numerator and the denominator of (4.57) are positive or negative respectively. Then  $\hat{\beta}_n^{S+}(k)$  dominates  $\hat{\beta}_n^{S+}$  when

$$\Delta^2 > \Delta_3^2(k) = \frac{f_8(\Delta^2, k)}{\text{Ch}_{\max}[I_p - R(k)'R(k)C^{-1}]E^*(\Delta^2)} \quad (4.58)$$

and  $\hat{\beta}_n^{S+}$  dominates  $\hat{\beta}_n^{S+}(k)$  when

$$\Delta^2 < \Delta_4^2(k) = \frac{f_8(\Delta^2, k)}{\text{Ch}_{\min}[I_p - R(k)'R(k)C^{-1}]E^*(\Delta^2)}. \quad (4.59)$$

#### 4.3.9. Comparison of $\hat{\beta}_n^{S+}(k)$ with $\hat{\beta}_n^{PT}(k)$ and $\hat{\beta}_n^S(k)$

Since  $\hat{\beta}_n^{S+}(k)$  and  $\hat{\beta}_n^S(k)$  are the particular cases of  $\hat{\beta}_n^{PT}(k)$ , the comparison between  $\hat{\beta}_n^S(k)$  and  $\hat{\beta}_n^{S+}(k)$  as well as between  $\hat{\beta}_n^S(k)$  and  $\hat{\beta}_n^{S+}(k)$  can be skipped.

#### 4.3.10. Comparison between $\hat{\beta}_n^{S+}(k)$ and $\hat{\beta}_n^{PT}(k)$

Case 1: Under the null hypothesis  $H_0 : H\beta = \mathbf{h}$

The risk difference is

$$\begin{aligned} R(\hat{\beta}_n^{S+}(k); I_p) - R(\hat{\beta}_n^{PT}(k); I_p) &= \sum_{i=1}^p \frac{\sigma_{zz} a_{ii}^* \lambda_i^2}{(\lambda_i + k)^2} \left\{ \left[ \mathcal{H}_{q+2}[\chi_q^2(\alpha); 0] - (q-2) \right. \right. \\ &\quad \left. \left. - E \left[ (1 - (q-2)\chi_{q+2}^{-2}(0))^2 \right] I(\chi_{q+2}^2 \leq (q-2)) \right] \right\} \geq 0, \end{aligned}$$

for all  $\alpha$  satisfying the condition

$$\left\{ \alpha : \chi_q^2(\alpha) \geq \mathcal{H}_{q+2}^{-1} \left( q-2 + E \left[ (1 - (q-2)\chi_{q+2}^{-2}(0))^2 I(\chi_{q+2}^2 \leq (q-2)) \right] \right) \right\}. \quad (4.60)$$

Thus the risk of  $\hat{\beta}_n^{PT}(k)$  is smaller than the risk of  $\hat{\beta}_n^{S+}(k)$  when the critical value  $\chi_q^2(\alpha)$  satisfies the condition (4.60). However the risk of  $\hat{\beta}_n^{S+}(k)$  is smaller than the risk of  $\hat{\beta}_n^{PT}(k)$  when the critical value  $\chi_q^2(\alpha)$  satisfies (4.60) with reverse inequality sign.

Case 2: Under the alternative hypothesis  $H_1 : H\beta \neq \mathbf{h}$

Under the alternative hypothesis, the difference in the risks of  $\hat{\beta}_n^{S+}(k)$  and  $\hat{\beta}_n^{PT}(k)$  is

$$\begin{aligned} R(\hat{\beta}_n^{S+}(k); I_p) - R(\hat{\beta}_n^{PT}(k); I_p) &= \sigma_{zz} \text{tr}[R(k)AR(k)'] \left\{ \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - (q-2)(2E[\chi_{q+2}^{-2}(\Delta^2)] - (q-2)E[\chi_{q+2}^{-4}(\Delta^2)]) \right. \\ &\quad - E \left[ \left( 1 - (q-2)\chi_{q+2}^{-2}(\Delta^2) \right)^2 I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right] \left. \right\} - \delta'R(k)'R(k)\delta \left\{ 2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] \right. \\ &\quad - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2] - (q-2)(2E[\chi_{q+2}^{-2}(\Delta^2)] - 2E[\chi_{q+4}^{-4}(\Delta^2)] + (q-2)E[\chi_{q+2}^{-4}(\Delta^2)]) \\ &\quad + E \left[ \left( 1 - (q-2)\chi_{q+2}^{-2}(\Delta^2) \right)^2 I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right] \\ &\quad + 2E \left[ ((q-2)\chi_{q+2}^{-2}(\Delta^2) - 1) I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right] \left. \right\} \\ &\quad + 2k\delta'R(k)'C(k)'\beta \left\{ (q-2)E[\chi_{q+2}^{-2}(\Delta^2)] \right. \\ &\quad \left. - \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - E \left[ ((q-2)\chi_{q+2}^{-2}(\Delta^2) - 1) I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right] \right\}. \end{aligned} \quad (4.61)$$

The right hand side of (4.61) will be non-positive when

$$\delta'R(k)'R(k)\delta \geq \frac{f_9(\alpha, \Delta^2)}{g_6(\alpha, \Delta^2)}, \quad (4.62)$$

where

$$\begin{aligned} f_9(\alpha, \Delta^2) &= \sigma_{zz} \text{tr}[R(k)AR(k)'] \left\{ \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - (q-2)(2E[\chi_{q+2}^{-2}(\Delta^2)] - (q-2)E[\chi_{q+2}^{-4}(\Delta^2)]) \right. \\ &\quad - E \left[ \left( 1 - (q-2)\chi_{q+2}^{-2}(\Delta^2) \right)^2 I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right] \left. \right\} \\ &\quad + 2k\delta'R(k)'C(k)'\beta \left\{ (q-2)E[\chi_{q+2}^{-2}(\Delta^2)] - \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] \right. \\ &\quad \left. - E \left[ ((q-2)\chi_{q+2}^{-2}(\Delta^2) - 1) I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right] \right\} \\ g_6(\alpha, \Delta^2) &= 2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2] - (q^2-2)E[\chi_{q+4}^{-4}(\Delta^2)] \\ &\quad + E \left[ \left( 1 - (q-2)\chi_{q+2}^{-2}(\Delta^2) \right)^2 I(\chi_{q+4}^2(\Delta^2) < (q-2)) \right] \\ &\quad + 2E \left[ ((q-2)\chi_{q+2}^{-2}(\Delta^2) - 1) I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right]. \end{aligned} \quad (4.63)$$

Since  $\Delta^2 > 0$ , assume that both the numerator and the denominator of (4.62) are positive or negative respectively. Then  $\hat{\beta}_n^{S+}(k)$  dominates  $\hat{\beta}_n^{PT}(k)$  when

$$\Delta^2 \geq \Delta_5^2(\alpha, k) = \frac{f_9(\alpha, \Delta^2)}{Ch_{\max}[R(k)'R(k)C^{-1}]g_6(\alpha, \Delta^2)} \quad (4.64)$$

and  $\hat{\beta}_n^{PT}(k)$  dominates  $\hat{\beta}_n^{S+}(k)$  when

$$\Delta^2 < \Delta_6^2(\alpha, k) = \frac{f_9(\Delta^2, k)}{Ch_{\min}[R(k)'R(k)C^{-1}]g_6(\alpha, \Delta^2)}. \quad (4.65)$$

Now consider the difference in the risk functions of  $\hat{\beta}_n^{S+}(k)$  and  $\hat{\beta}_n^{PT}(k)$  as a function of eigenvalues as follows:

$$\begin{aligned} R(\hat{\beta}_n^{S+}(k); I_p) - R(\hat{\beta}_n^{PT}(k); I_p) &= \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{ \sigma_{zz} a_{ii}^* \lambda_i^2 \left\{ \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - (q-2) \left[ (q-2)E[\chi_{q+2}^{-4}(\Delta^2)] \right. \right. \right. \\ &\quad \left. \left. + \left( 1 - \frac{(q+2)\delta_i^{*2}}{2\Delta^2\sigma_{zz}a_{ii}^*} \right) (2\Delta^2)E[\chi_{q+4}^{-4}(\Delta^2)] \right] \right\} - \lambda_i^2 \delta_i^{*2} \left[ 2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2] \right] \right. \\ &\quad - \left\{ \sigma_{zz} a_{ii}^* \lambda_i^2 E \left[ \left( 1 - (q-2)\chi_{q+2}^{-2}(\Delta^2) \right)^2 I(\chi_{q+2}^2(\Delta^2) \leq (q-2)) \right] \right. \\ &\quad \left. + \lambda_i^2 \delta_i^{*2} E \left[ \left( 1 - (q-2)\chi_{q+4}^{-2}(\Delta^2) \right)^2 I(\chi_{q+4}^2(\Delta^2) \leq (q-2)) \right] \right\} \\ &\quad - 2\lambda_i^2 \delta_i^{*2} E \left[ ((q-2)\chi_{q+4}^{-2}(\Delta^2) - 1) I(\chi_{q+4}^2(\Delta^2) \leq (q-2)) \right] \\ &\quad - 2k\theta_i \lambda_i \delta_i^* \left\{ \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - (q-2)E[\chi_{q+2}^{-2}(\Delta^2)] \right. \\ &\quad \left. + E \left[ ((q-2)\chi_{q+2}^{-2}(\Delta^2) - 1) I(\chi_{q+2}^2(\Delta^2) \leq (q-2)) \right] \right\}. \end{aligned} \quad (4.66)$$

Now we define

$$k_{11}(\alpha, \Delta^2) = \frac{f_{10}(\alpha, \Delta^2)}{g_7(\alpha, \Delta^2)}, \quad (4.67)$$

where

$$\begin{aligned} f_{10}(\alpha, \Delta^2) = & \max_{1 \leq i \leq p} \left\{ \sigma_{zz} a_{ii}^* \lambda_i^2 \left\{ \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - (q-2) [(q-2)E[\chi_{q+2}^{-4}(\Delta^2)]] \right. \right. \\ & + \left( 1 - \frac{(q+2)\delta_i^{*2}}{2\Delta^2 \sigma_{zz} a_{ii}^*} \right) (2\Delta^2)E[\chi_{q+4}^{-4}(\Delta^2)] \left. \right\} - \lambda_i^2 \delta_i^{*2} [2\mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - \mathcal{H}_{q+4}[\chi_q^2(\alpha); \Delta^2]] \\ & - \left\{ \sigma_{zz} a_{ii}^* \lambda_i^2 E \left[ (1 - (q-2)\chi_{q+2}^{-2}(\Delta^2))^2 I(\chi_{q+2}^2(\Delta^2) \leq (q-2)) \right] \right. \\ & + \lambda_i^2 \delta_i^{*2} E \left[ (1 - (q-2)\chi_{q+4}^{-2}(\Delta^2))^2 I(\chi_{q+4}^2(\Delta^2) \leq (q-2)) \right] \left. \right\} \\ & - 2\lambda_i^2 \delta_i^{*2} E \left[ ((q-2)\chi_{q+2}^{-2}(\Delta^2) - 1) I(\chi_{q+2}^2(\Delta^2) \leq (q-2)) \right] \left. \right\} \end{aligned} \quad (4.68)$$

$$\begin{aligned} g_7(\alpha, \Delta^2) = & \min_{1 \leq i \leq p} \left[ 2\lambda_i \theta_i \delta_i^* \left\{ \mathcal{H}_{q+2}[\chi_q^2(\alpha); \Delta^2] - (q-2)E[\chi_{q+2}^{-2}(\Delta^2)] \right. \right. \\ & + E \left[ ((q-2)\chi_{q+2}^{-2}(\Delta^2) - 1) I(\chi_{q+2}^2(\Delta^2) \leq (q-2)) \right] \left. \right\} \left. \right]. \end{aligned} \quad (4.69)$$

Suppose  $k > 0$ , then the following statements hold true following Kaciranlar et al. [17]:

1. If  $g_7(\alpha, \Delta^2) > 0$ , it follows that for each  $k > 0$  with  $k < k_{11}(\alpha, \Delta^2)$ ,  $\hat{\beta}_n^{S+}(k)$  has smaller risk than that of  $\hat{\beta}_n^{PT}(k)$ .
2. If  $g_7(\alpha, \Delta^2) < 0$ , it follows that for each  $k > 0$  with  $k > k_{11}(\alpha, \Delta^2)$ ,  $\hat{\beta}_n^{S+}(k)$  has smaller risk than that of  $\hat{\beta}_n^{PT}(k)$ .

**Remark.** For  $\alpha = 0$ , we obtain the condition for the superiority of  $\hat{\beta}_n^{S+}(k)$  over  $\hat{\beta}_n(k)$  and for  $\alpha = 1$ , we obtain the superiority condition of  $\hat{\beta}_n^{S+}(k)$  over  $\tilde{\beta}_n(k)$ .

#### 4.3.11. Comparison of $\hat{\beta}_n^{S+}(k)$ and $\hat{\beta}_n^S(k)$

The risk difference of  $\hat{\beta}_n^{S+}(k)$  and  $\hat{\beta}_n^S(k)$  is

$$\begin{aligned} R(\hat{\beta}_n^{S+}(k); I_p) - R(\hat{\beta}_n^S(k); I_p) = & - \left\{ \sigma_{zz} \text{tr}[R(k)'AR(k)]E \left[ (1 - (q-2)\chi_{q+2}^{-2}(\Delta^2))^2 I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right] \right. \\ & + [\delta'R(k)'R(k)\delta]E \left[ (1 - (q-2)\chi_{q+4}^{-2}(\Delta^2))^2 I(\chi_{q+4}^2(\Delta^2) < (q-2)) \right] \left. \right\} \\ & - 2[\delta'R(k)'R(k)\delta]E \left[ ((q-2)\chi_{q+2}^{-2}(\Delta^2) - 1) I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right] \\ & - 2k\delta'R(k)'C^{-1}(k)\beta E \left[ ((q-2)\chi_{q+2}^{-2}(\Delta^2) - 1) I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right]. \end{aligned} \quad (4.70)$$

Case 1: Suppose  $\delta'R(k)'C^{-1}(k)\beta > 0$ , then the right hand side of (4.70) is negative, since the expectation of a positive random variable is positive. Thus for all  $\Delta^2$  and  $k$ ,

$$R(\hat{\beta}_n^{S+}(k); I_p) \leq R(\hat{\beta}_n^S(k); I_p).$$

Therefore under this condition, the  $\hat{\beta}_n^{S+}(k)$  not only confirms the inadmissibility of  $\hat{\beta}_n^S(k)$  but also provides a simple superior estimator for the ill-conditioned data.

Case 2: Suppose  $\delta'R(k)'C^{-1}(k)\beta < 0$ , then the right hand side of (4.70) is positive when

$$\delta'R(k)'R(k)\delta \geq \frac{f_{11}(\alpha, \Delta^2)}{g_8(\alpha, \Delta^2)}, \quad (4.71)$$

where

$$\begin{aligned} f_{11}(\alpha, \Delta^2) = & 2k\delta'R(k)'C^{-1}(k)\beta E \left[ ((q-2)\chi_{q+2}^{-2}(\Delta^2) - 1) I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right] \\ & - \sigma_{zz} \text{tr}[R(k)'AR(k)]E \left[ (1 - (q-2)\chi_{q+2}^{-2}(\Delta^2))^2 I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right] \\ g_8(\alpha, \Delta^2) = & \left\{ E \left[ (1 - (q-2)\chi_{q+2}^{-2}(\Delta^2))^2 I(\chi_{q+2}^2(\Delta^2) < (q-2)) \right] \right. \\ & \left. - 2E \left[ ((q-2)\chi_{q+2}^{-2}(\Delta^2) - 1) I(\chi_{q+2}^2 < (q-2)) \right] \right\}. \end{aligned}$$

Since  $\Delta^2 > 0$ , assume that both the numerator and the denominator of (4.71) are positive or negative respectively. Then  $\hat{\beta}_n^{S+}(k)$  dominates  $\hat{\beta}_n^S(k)$  when

$$\Delta^2 > \Delta_7^2(\alpha, k) = \frac{f_{11}(\alpha, \Delta^2)}{Ch_{\max}[R(k)'R(k)C^{-1}]g_8(\alpha, \Delta^2)} \quad (4.72)$$

and  $\hat{\beta}_n^S(k)$  dominates  $\hat{\beta}_n^{S+}(k)$  when

$$\Delta^2 < \Delta_8^2(\alpha, k) = \frac{f_{11}(\Delta^2, k)}{Ch_{\min}[R(k)'R(k)C^{-1}]g_8(\alpha, \Delta^2)}. \quad (4.73)$$

Thus, it is observed that the  $\hat{\beta}_n^{S+}(k)$  does not uniformly dominates the  $\tilde{\beta}_n(k)$ ,  $\hat{\beta}_n(k)$ ,  $\hat{\beta}_n^{PT}(k)$  and  $\hat{\beta}_n^S(k)$ .

#### 4.4. Comparison of risks as a function of $(\Delta^2, k) \in (0, \infty) \times (0, 1)$

In this section, we consider the conditions on the parameters  $(\Delta^2, k)$  simultaneously for the comparison of estimators in the following theorems.

**Theorem 5.** Under  $\{K_{(n)}\}$  and assumed regularity conditions,  $R_1(\tilde{\beta}_n(k); I_p) \geq R_4(\hat{\beta}_n^S(k); I_p)$  in the interval  $(\Delta^2, k) \in (0, \infty) \times (0, k_0^*)$  as  $n \rightarrow \infty$ . Otherwise  $R_4(\hat{\beta}_n^S(k); I_p) \geq R_1(\tilde{\beta}_n(k); I_p)$ .

**Theorem 6.** Under  $\{K_{(n)}\}$  and assumed regularity conditions,  $R_4(\hat{\beta}_n^S(k); I_p) \geq R_5(\tilde{\beta}_n^{S+}(k); I_p)$  in the interval  $(\Delta^2, k) \in (0, \infty) \times (0, k_9(\Delta^2))$  as  $n \rightarrow \infty$ .

As a result, the dominance relations hold as

$$R_1(\tilde{\beta}_n(k); I_p) \geq R_4(\hat{\beta}_n^S(k); I_p) \geq R_5(\tilde{\beta}_n^{S+}(k); I_p)$$

in the interval  $(\Delta^2, k) \in (0, \infty) \times (0, k_9(\Delta^2))$  where  $k_9(\Delta^2)$  is given by (4.52) as  $n \rightarrow \infty$ .

Thus, again the estimator  $\hat{\beta}_n^{S+}(k)$  is preferable over others for applied statistics.

## 5. Summary and conclusions

In this paper, we have combined the idea of the preliminary test and the Stein-rule estimator with the RR approach to obtain a better estimator for the regression parameter  $\beta$  in a multiple measurement error model. Accordingly, we considered five RRR-estimators, namely,  $\tilde{\beta}_n(k)$ ,  $\hat{\beta}_n(k)$ ,  $\hat{\beta}_n^{PT}(k)$ ,  $\hat{\beta}_n^S(k)$  and  $\hat{\beta}_n^{S+}(k)$  for estimating the parameters  $(\beta)$  when it is suspected that the parameter  $\beta$  may belong to a linear subspace defined by  $H\beta = h$ . The performances of the estimators are compared based on the quadratic risk function under both null and alternative hypotheses. Under the restriction  $H_0$ , the  $\hat{\beta}_n(k)$  performed the best compared with other estimators, however, it performed the worst even when  $\Delta^2$  moves away from its origin. Note under the risk of  $\tilde{\beta}_n(k)$  is constant while the risk of  $\hat{\beta}_n(k)$  is unbounded as  $\Delta^2$  goes to  $\infty$ . Also under  $H_0$ , the risk of  $\hat{\beta}_n^{PT}(k)$  is smaller than the risks of  $\hat{\beta}_n^S(k)$  and  $\hat{\beta}_n^{S+}(k)$  for satisfying (4.60) for  $q \geq 3$ . Thus, neither  $\hat{\beta}_n^{PT}(k)$  nor  $\hat{\beta}_n^{S+}(k)$  nor  $\hat{\beta}_n^S(k)$  dominate each other uniformly. Note that the application of  $\hat{\beta}_n^{S+}(k)$  and  $\hat{\beta}_n^S(k)$  is constrained by the requirement that  $q \geq 3$ . However, from Section 4.4,  $\hat{\beta}_n^{S+}(k)$  is preferable to  $\hat{\beta}_n^S(k)$  since it dominates uniformly in  $\Delta^2$  for  $k \in (0, \infty)$  while for  $q < 3$ ,  $\hat{\beta}_n^{PT}(k)$  is preferable which depends on the size of test  $\alpha$  which may be determined by the maximin rule given by (4.14).

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