



Nonparametric density estimation for spatial data with wavelets

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ABSTRACT

Nonparametric density estimators are studied for d -dimensional, strongly spatial mixing data which are defined on a general N -dimensional lattice structure. We consider linear and nonlinear hard thresholded wavelet estimators derived from a d -dimensional multi-resolution analysis. We give sufficient criteria for the consistency of these estimators and derive rates of convergence in $L^{p'}$ for $p' \in [1, \infty)$. For this reason, we study density functions which are elements of a d -dimensional Besov space $B_{p,q}^s(\mathbb{R}^d)$. We also verify the analytic correctness of our results in numerical simulations.

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1. Introduction

This article considers methods of nonparametric density estimation for spatially dependent data with wavelets. There is an extensive literature on the density estimation problem for iid data or time series. Recently, inference techniques for spatial data have gained importance because of their relevance in modern applications such as image analysis, forestry, epidemiology or geophysics. See the monographs of [11] and [23] for a systematic introduction on spatial data and random fields.

So far when working with random fields, the kernel method has been a popular tool both in regression and density estimation; see, e.g., [3,8,9,27,28]. More recently, Dabo-Niang and Yao [13] extended the kernel method to functional stationary random fields and estimated the spatial density with respect to a reference measure. Dabo-Niang et al. [12] proposed a kernel method in spatial density estimation which also allows for spatial clustering. Amiri et al. [1] studied asymptotic properties of a recursive version of the Parzen–Rozenblatt estimator.

While the kernel method is efficient if the density has unbounded support, it can sometimes suffer from boundary issues for densities with compact support. Furthermore, the kernel method typically requires the density to satisfy certain smoothness conditions like two-times continuous differentiability. In situations where the density function does not meet these requirements, the wavelet method is an alternative which often performs relatively well because it adapts

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automatically to the regularity of the curve to be estimated. Wavelet estimators assume that the underlying curve belongs to a function space with certain degrees of smoothness. These smoothness parameters can be more subtle than the differentiability criterion from above and will be clarified later. The wavelet estimators do not depend on the smoothness parameters; nevertheless, they behave as if the true curve is known in advance and attain the optimal rates of convergence. This is in particular true for the hard thresholding estimator of Donoho et al. [19].

However, estimating the density of spatial data with wavelets has received little attention. Only the special case of time series has been thoroughly investigated: the wavelet method for density and regression estimators for multivariate and stationary time series is studied in [4,5,37,38]. In a recent article, Li [36] studied wavelet estimators for compactly supported one-dimensional Besov densities on stationary and strongly mixing random fields.

In the present article, we continue with these considerations for d -dimensional densities and study the linear and the hard thresholding estimator based on wavelets without assuming that they are isotropic. It is well-known that the hard thresholding estimator performs better than its linear analogue for certain densities in the one-dimensional setting. We will show a similar behavior for multivariate density functions.

The hard thresholding estimator has a linear basic component with respect to a coarse level j_0 . Additionally, nonlinear details are added for higher levels $j_0 \leq j \leq j_1$ if their contribution is significant in the statistical sense. This implies that this estimator can converge faster than the linear estimator in certain parameter settings.

The generalization to arbitrary dimensions is non-trivial, in particular because the definitions of the underlying Besov space $B_{p,q}^s(\mathbb{R}^d)$ have to be generalized to the d -dimensional case. For isotropic wavelets there already exist such generalizations; see, e.g., [30,39]. However, as we also allow for density estimators with nonisotropic wavelets, we need a more general definition. This is one of the unique features of the present work. Moreover, we allow for density functions on \mathbb{R}^d which do not necessarily have compact support.

We assume that $Z = \{Z(s) : s \in \mathbb{Z}^N\}$ is a random field with equal marginal laws on \mathbb{R}^d which admit a square integrable density f with respect to the d -dimensional Lebesgue measure λ^d . Then for an orthonormal basis $\{b_u : u \in \mathbb{N}_+\}$ of $L^2(\lambda^d)$, we have the representation $f = \sum_{u \in \mathbb{N}_+} \langle f, b_u \rangle b_u$, where $\langle \cdot, \cdot \rangle$ is the inner product on the function space $L^2(\lambda^d)$. Since f is a density, we have the fundamental relationship between an observed sample $\{Z(s) : s \in I\}$ with $I \subseteq \mathbb{Z}^N$ and a coefficient $\langle f, b_u \rangle$ from this representation, viz.

$$\langle f, b_u \rangle = \mathbb{E} [b_u \{Z(s)\}] \approx \frac{1}{|I|} \sum_{s \in I} b_u \{Z(s)\}.$$

It is well-known that replacing the true coefficient with the empirical approximation yields a consistent density estimate for an iid sample of one-dimensional data under certain conditions, see, e.g., [18,29]. In the particular case of wavelets, Kerkycharian and Picard [33] derived rates of convergence for the linear wavelet estimator.

In contrast to linear wavelet estimators, nonlinear wavelet estimators are particularly useful if the density curve features high-frequency oscillations or exhibits an erratic behavior. Rates of convergence of the hard thresholded wavelet estimator were studied in [19] and [25]. Since then the wavelet method for the density problem has been studied in various special settings: Hall et al. [24], Cai [7] and Chicken and Cai [10] considered rates of convergence for wavelet block thresholding. Giné and Nickl [22] gave several uniform limit theorems for wavelet density estimators for a compactly supported density and iid sample data. Xue [44] studied wavelet-based density estimation under censorship. Giné and Madych [21] investigated wavelet projection kernels in the density estimation problem. In this article, we continue the analysis for multivariate sample data which are spatially dependent.

This manuscript is organized as follows. We give the fundamental definitions and summarize the main facts of Besov spaces in d dimensions in Section 2. In Section 3 we study in detail the wavelet density estimators. We give criteria which are sufficient for the consistency of the nonparametric estimators and establish rates of convergence. Section 4 is devoted to numerical applications. We use an algorithm proposed by Kaiser et al. [31] for the simulation of the random field and estimate its marginal density with the linear and the hard thresholded wavelet estimator. Section 5 contains the proofs of the results from Section 3. Appendix A contains useful inequalities for dependent sums. As the wavelet estimators are a priori not necessarily a density, we consider in Appendix B the question under which circumstances a normalized estimator is consistent.

2. Notation and definitions

This section is divided in four parts. First, we introduce the concepts for multidimensional wavelets. Second, we define the multidimensional Besov spaces. Third, we explain the data generating process. Finally, we define the wavelet density estimator. In the following, we write $L^2(\lambda^d)$ for $L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda^d)$, where λ^d is the d -dimensional Lebesgue measure and we write

$$\|f\|_{L^p(\lambda^d)} = \left(\int_{\mathbb{R}^d} |f|^p d\lambda^d \right)^{1/p}$$

for the L^p -norm of a function f on \mathbb{R}^d .

We begin with well-known results on wavelets in d dimensions; see, e.g., the monograph of [2].

Definition 1. Let $\Gamma \subseteq \mathbb{R}^d$ be a lattice, this is a discrete subgroup given by $(\Gamma, +) = (\{\sum_{i=1}^d a_i v_i : a_i \in \mathbb{Z}\}, +)$ for certain $v_1, \dots, v_d \in \mathbb{R}^d$. Furthermore, let $M \in \mathbb{R}^{d \times d}$ be a matrix which preserves the lattice Γ , i.e., $M\Gamma \subseteq \Gamma$ and which is strictly expanding, i.e., all eigenvalues ζ of M satisfy $|\zeta| > 1$. Denote for such a matrix M the absolute value of its determinant by $|M|$. A multi-resolution analysis (MRA) of $L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda^d)$, $d \in \mathbb{N}_+$, with a scaling function $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is an increasing sequence of subspaces of $L^2(\lambda^d)$ given by $\dots \subseteq U_{-1} \subseteq U_0 \subseteq U_1 \subseteq \dots$ such that the following four conditions are satisfied:

1. (Denseness) $\bigcup_{j \in \mathbb{Z}} U_j$ is dense in $L^2(\lambda^d)$.
2. (Separation) $\bigcap_{j \in \mathbb{Z}} U_j = \{0\}$.
3. (Scaling) $f \in U_j$ if and only if $f(M^{-j} \cdot) \in U_0$.
4. (Orthonormality) $\{\Phi(\cdot - \gamma) : \gamma \in \Gamma\}$ is an orthonormal basis of U_0 .

The relation between an MRA and an orthonormal basis of $L^2(\lambda^d)$ is summarized as follows.

Theorem 1 ([41]). Suppose Φ generates a multi-resolution analysis and the $a_k(\gamma)$, satisfy for all integers $j, k \in \{0, \dots, |M| - 1\}$ and $\gamma \in \Gamma$, the equations

$$\sum_{\gamma' \in \Gamma} a_j(\gamma') a_k(M\gamma + \gamma') = |M| \delta(j, k) \delta(\gamma, 0), \quad \sum_{\gamma \in \Gamma} a_0(\gamma) = |M|,$$

where δ is Kronecker's delta. Furthermore, let the functions Ψ_k be defined by $\sum_{\gamma \in \Gamma} a_k(\gamma) \Phi(M \cdot - \gamma)$ for all $k \in \{1, \dots, |M| - 1\}$. Then the set of functions $\{|M|^{j/2} \Psi_k(M^j \cdot - \gamma) : j \in \mathbb{Z}, k \in \{1, \dots, |M| - 1\}, \gamma \in \Gamma\}$ forms an orthonormal basis of $L^2(\lambda^d)$, viz.

$$L^2(\lambda^d) = U_0 \oplus (\oplus_{j \in \mathbb{N}} W_j) = \oplus_{j \in \mathbb{Z}} W_j,$$

where $W_j = \langle |M|^{j/2} \Psi_k(M^j \cdot - \gamma) : k \in \{1, \dots, |M| - 1\}, \gamma \in \Gamma \rangle$.

We also call the scaling function $\Psi_0 = \Phi$ the father wavelet. Moreover, we assume throughout the rest of this article that the MRA is given by compactly supported and bounded wavelets $\Psi_0, \dots, \Psi_{|M|-1}$ unless otherwise stated. Additionally, we assume that the lattice Γ is \mathbb{Z}^d . One could also use different lattices which would have a finer grid than \mathbb{Z}^d ; however, this would also result in more technical complexities and provide little additional insight. Note that the last assumption also implies the eigenvalues of the matrix M to be integers.

Without loss of generality, the support of the wavelets Ψ_k is in $[0, L]^d$ for some $L \in \mathbb{N}_+$; we write $\text{supp } \Psi_k \subseteq [0, L]^d$. The mother wavelets satisfy the balancing condition $\int_{\mathbb{R}^d} \Psi_k d\lambda^d = 0$ for all $k \in \{1, \dots, |M| - 1\}$.

One can derive a d -dimensional, isotropic MRA from a father wavelet φ and a mother wavelet ψ which are defined on the real line. Assume that φ and ψ fulfill the scaling equations

$$\varphi \equiv \sqrt{2} \sum_{\gamma \in \mathbb{Z}} h_\gamma \varphi(2 \cdot - \gamma), \quad \psi \equiv \sqrt{2} \sum_{\gamma \in \mathbb{Z}} g_\gamma \varphi(2 \cdot - \gamma),$$

for real sequences $(h_\gamma : \gamma \in \mathbb{Z})$ and $(g_\gamma : \gamma \in \mathbb{Z})$. Let φ generate an MRA of $L^2(\lambda)$ with the corresponding spaces $U'_j, j \in \mathbb{Z}$. The d -dimensional wavelets are derived as follows: set $\Gamma = \mathbb{Z}^d$ and define the diagonal matrix M as $2I$, where I is the identity matrix. Denote the mother wavelets as pure tensors by $\Psi_k = \xi_{k_1} \otimes \dots \otimes \xi_{k_d}$ for $k \in \{0, 1\}^d \setminus \{0\}$, where $\xi_0 = \varphi$ and $\xi_1 = \psi$. The scaling function is $\Phi = \Psi_0 = \varphi \otimes \dots \otimes \varphi$. Then Φ and the linear spaces $U_j = U'_1 \otimes \dots \otimes U'_d$ form an MRA of $L^2(\lambda^d)$ and the functions $\Psi_k, k \neq 0$, generate an orthonormal basis in that

$$L^2(\lambda^d) = U_0 \oplus (\oplus_{j \in \mathbb{N}} W_j) = \oplus_{j \in \mathbb{Z}} W_j,$$

where $W_j = \langle |M|^{j/2} \Psi_k(M^j \cdot - \gamma) : \gamma \in \mathbb{Z}^d, k \in \{0, 1\}^d \setminus \{0\} \rangle$.

Moreover, we need the characterization of the multivariate Besov space; see [39] or [42]. For that reason we generalize the well-known multivariate notion of the Besov space for isotropic wavelets to nonisotropic wavelets.

Definition 2. Let $s > 0, p, q \in [1, \infty]$ and let $\{\Psi_0, \dots, \Psi_{|M|-1}\}$ be a wavelet basis. Set $\Phi_{j,\gamma} = \Psi_{0,j,\gamma} = |M|^{j/2} \Phi(M^j \cdot - \gamma)$ for the father wavelets and write $\Psi_{k,j,\gamma} = |M|^{j/2} \Psi_k(M^j \cdot - \gamma)$ for the mother wavelets for $k \in \{1, \dots, |M| - 1\}, j \in \mathbb{Z}$ and $\gamma \in \mathbb{Z}^d$. The Besov space $B_{p,q}^s(\mathbb{R}^d)$ is defined as

$$B_{p,q}^s(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \rightarrow \mathbb{R} : \text{there is a wavelet representation} \right. \\ \left. f = \sum_{\gamma \in \mathbb{Z}^d} \theta_{0,\gamma} \Phi_{0,\gamma} + \sum_{k=1}^{|M|-1} \sum_{j \geq 0} \sum_{\gamma \in \mathbb{Z}^d} v_{k,j,\gamma} \Psi_{k,j,\gamma} \text{ such that } \|f\|_{B_{p,q}^s} < \infty \right\},$$

where the Besov norm of f (with the usual modification if $p = \infty$ or $q = \infty$) is

$$\|f\|_{B_{p,q}^s} = \left\| \sum_{\gamma \in \mathbb{Z}^d} \theta_{0,\gamma} \Phi_{0,\gamma} \right\|_{L^p(\lambda^d)} + \left(\sum_{k=1}^{|M|-1} \sum_{j \geq 0} |M|^{jsq} \left\| \sum_{\gamma \in \mathbb{Z}^d} v_{k,j,\gamma} \Psi_{k,j,\gamma} \right\|_{L^p(\lambda^d)}^q \right)^{1/q}. \quad (1)$$

Furthermore, denote the ℓ^p -sequence norm by $\|\cdot\|_{\ell^p}$ and define the equivalent norms (see Lemma 1)

$$\|f\|_{s,p,q} = \|\theta_{0,\cdot}\|_{\ell^p} + \left(\sum_{k=1}^{|M|-1} \sum_{j \geq 0} |M|^{j(s+1/2-1/p)q} \|v_{k,j,\cdot}\|_{\ell^p}^q \right)^{1/q}. \quad (2)$$

In the following, M is a diagonalizable matrix, $M = S^{-1}DS$, where D is a diagonal matrix containing the eigenvalues of M . Denote the maximum of the absolute values of the eigenvalues by $\zeta_{\max} = \max(|\zeta_1|, \dots, |\zeta_d|)$ and the corresponding minimum by $\zeta_{\min} = \min(|\zeta_1|, \dots, |\zeta_d|)$.

Similar to the case in one dimension, we have the following relations between different Besov spaces for multivariate functions; cf. [19].

- (i) If either $s' > s$ and $q = q'$ or if $s' = s$ and $q' \leq q$, then $B_{p,q'}^{s'} \subseteq B_{p,q}^s$. Moreover, if $p' \leq p$ and $s' = s - 1/p + (p')^{-1}$, then $B_{p',q}^{s'} \subseteq B_{p,q}^s$.
- (ii) If $s' = s - 1/p > 0$, then $B_{p,q}^s \subseteq B_{p,\infty}^s \subseteq B_{\infty,\infty}^{s'}$.
- (iii) Furthermore, if a function is Hölder continuous with exponent $r \in (0, 1]$, we see in the following that this function belongs to the Besov space $B_{\infty,\infty}^s$, where the regularity parameter s is given by $r \ln \zeta_{\min}/(d \ln \zeta_{\max})$. In particular, in the one-dimensional case s equals r , otherwise it is strictly smaller.

Moreover, a wavelet is r -regular if every derivative up to order $r \in \mathbb{N}_+$ is rapidly decreasing. In the one-dimensional case, this regularity ensures that the characterization of the Besov norms via the wavelet coefficients as in (1) and (2) is equivalent to the characterization via the modulus of smoothness; compare [35] and [20]. In the one-dimensional case and for r -regular wavelets, the Besov spaces also include the Sobolev spaces $H^s = B_{2,2}^s$. Similar considerations remain true (at least) in the special case of isotropic wavelets. For more details, we refer the reader to [30,39]. We do not consider such equivalent characterizations for a general matrix M in the following but leave this issue up to further research.

For the density estimation problem, we define for $K \in \mathbb{R}_+$, $A \in \mathcal{B}(\mathbb{R}^d)$ and $d \in \mathbb{N}_+$ subsets of $B_{p,q}^s$ as follows:

$$F_{s,p,q}(K, A) = \left\{ f : \mathbb{R}^d \rightarrow \mathbb{R}_{\geq 0} : f \in B_{p,q}^s(\mathbb{R}^d), \int_{\mathbb{R}^d} f \, d\lambda^d = 1, \|f\|_{s,p,q} \leq K, \text{supp}(f) \subseteq A \right\}.$$

If the wavelets ψ_k have compact support and if $s - 1/p > 0$, then it is straightforward to show that finiteness of f with respect to the Besov norm implies that the function is essentially bounded by $\|f\|_{s,p,q}$ times a constant. In particular, if f is a density such that $\|f\|_{s,p,q} < \infty$ and $s > 1/p$, then f is square integrable.

In the statements below, the notation $|M|^j \simeq g(n)$ means that the integer j is chosen as a function of n such that $|M|^j \leq g(n) < |M|^{j+1}$.

We denote the p -norm on \mathbb{R}^N (resp. \mathbb{R}^d) by $\|\cdot\|_p$ and the corresponding metric by d_p for $p \in [1, \infty]$ with the extension $d_p(I, J) = \inf\{d_p(s, t) : s \in I, t \in J\}$ to subsets I, J of \mathbb{R}^N (resp. \mathbb{R}^d). Write $s \leq t$ for $s, t \in \mathbb{R}^N$ if and only if for each $i \in \{1, \dots, N\}$, the single coordinates satisfy $s_i \leq t_i$. We denote the indicator function of a set A by $\mathbf{1}(A)$. For $a \in \mathbb{R}$ we write $a^+ = \max(a, 0)$ for the positive and $a^- = \max(-a, 0)$ for the negative part. Furthermore, we write $e_N = (1, \dots, 1) \in \mathbb{Z}^N$ for the vector whose elements are all equal to 1. If $a, b \in \mathbb{R}^d$ are such that $a \leq b$, then we denote the cube $\{x \in \mathbb{R}^d : a \leq x \leq b\}$ by $[a, b]$.

We call a function $h : \mathbb{R}^d \rightarrow \mathbb{R}$ radial if $h(x) = h(y)$ whenever $\|x\|_2 = \|y\|_2$. A radial function h is non-increasing if $h(x) \leq h(y)$ whenever $\|x\|_2 \geq \|y\|_2$.

In the next step, we describe the data generating process which is given by a d -dimensional random field Z . This random field is defined on an N -dimensional lattice structure, i.e., $Z = \{Z(s) : s \in \mathbb{Z}^N\}$ for some integer $N \geq 1$. The random variables $Z(s)$ are identically distributed on \mathbb{R}^d and their distribution admits a density f .

Denote for a subset I the σ -algebra generated by the $Z(s)$ in I by $\mathcal{F}(I) = \sigma(Z(s) : s \in I)$. The α -mixing coefficient is introduced in [40]; in the present context it is defined for $k \in \mathbb{N}$ as

$$\alpha(k) = \sup_{\substack{I, J \subseteq \mathbb{Z}^N \\ d_\infty(I, J) \geq k}} \sup_{\substack{A \in \mathcal{F}(I), \\ B \in \mathcal{F}(J)}} |\Pr(A \cap B) - \Pr(A)\Pr(B)|.$$

We say that the random field is strongly spatial mixing if $\alpha(k) \rightarrow 0$ for $k \rightarrow \infty$. Bradley [6] gives an introduction to dependence measures for random variables. We require the following condition throughout the article.

Condition 1. Assume that $N \in \mathbb{N}_+$ and let $Z = \{Z(s) : s \in \mathbb{Z}^N\}$ be an \mathbb{R}^d -valued random field. The random variables $Z(s)$ are identically distributed and admit a bounded density f with respect to the Lebesgue measure. Furthermore,

- (A) Z is strongly spatial mixing with exponentially decreasing mixing coefficients, i.e., $\alpha(k) \leq c_0 \exp(-c_1 k)$ for all $k \in \mathbb{N}_+$ for certain $c_0, c_1 \in \mathbb{R}_+$.
- (B) Define the index sets by $I_n = \{s \in \mathbb{Z}^N : e_N \leq s \leq n\} \subseteq \mathbb{Z}_+^N$ for $n \in \mathbb{N}^N$. All index sets considered in the following satisfy $\min(n_1, \dots, n_N) \geq C' \max(n_1, \dots, n_N)$ for a fixed constant $C' \in \mathbb{R}_+$.

(C) Denote the joint density of $Z(s)$ and $Z(t)$ by $f_{Z(s), Z(t)}$. There are two bounded and non-increasing radial functions $h, \tilde{h} : \mathbb{R}^d \rightarrow \mathbb{R}_{\geq 0}$ and an integer $a \in \mathbb{N}_+$ such that $f \leq h$ and $|f_{Z(s), Z(t)}(z_1, z_2) - f(z_1)f(z_2)| \leq \tilde{h}(z_1)\tilde{h}(z_2)$ for all $Z(s), Z(t), s, t \in \mathbb{Z}^N$. Moreover,

$$\|h^{1/(2a)}\|_{L^1(\lambda^d)} < \infty \quad \text{and} \quad \|\tilde{h}^{1/a}\|_{L^2(\lambda^d)} < \infty.$$

The assumption of exponentially decreasing α -mixing coefficients is common [36]. One can show that such a rate is guaranteed for time series under mild conditions [16,43].

The requirement on the constant C' is technical. If we consider a sequence $\{n(k) : k \in \mathbb{N}\} \subseteq \mathbb{N}^N$, where one coordinate $n_i(k)$ tends to infinity, then all other coordinates tend to infinity as well. This will also prove helpful in the following results, where we express the rates of convergence of the estimators in terms of the cardinality of the index set I_n , which is $|I_n| = n_1 \times \cdots \times n_N$. For instance, if we obtain a rate of convergence in $\mathcal{O}(|I_n|^{-\rho})$ for a certain $\rho > 0$, then this also means in terms of a single coordinate i that the rate is in $\mathcal{O}(n_i^{-N\rho})$. This reminds more of the case of iid or time series data, where we usually have observations Z_1, \dots, Z_n . We do not require for our results an asymptotic on the index sets of the kind $I_{n(k)} \subseteq I_{n(k+1)}$ for a sequence $\{n(k) : k \in \mathbb{N}\}$ in \mathbb{N}^N ; we only require that all ratios n_i/n_j are at least C' .

The condition on the function and the joint densities of the variables $Z(s)$ and $Z(t)$ is technical. The fact that the density f is dominated by a radial function h , which satisfies certain integrability conditions, ensures that the tail of the density is well behaved. This is necessary for density functions with an unbounded support. If the density function is bounded and has bounded support, this condition is trivially satisfied.

The second requirement on the joint distribution of the random variables $Z(s)$ and $Z(t)$ restricts the mutual dependence, i.e., $f_{Z(s), Z(t)}(z_1, z_2) \leq f(z_1)f(z_2) + \tilde{h}(z_1)\tilde{h}(z_2)$ for another radial function \tilde{h} which satisfies certain integrability conditions. If Z is strictly stationary this condition reduces to the joint densities of the pairs $(Z(0), Z(s))$. Moreover, as the mixing coefficients vanish with increasing distance, we expect the dominating function \tilde{h} to be determined by the pairs $(Z(s), Z(t))$ where $\|s - t\|_\infty$ is small.

We can now define the density estimators. The density f has the representation

$$f = \sum_{\gamma \in \mathbb{Z}^d} \theta_{0,\gamma} \Phi_{0,\gamma} + \sum_{k=1}^{|M|-1} \sum_{j=0}^{\infty} \sum_{\gamma \in \mathbb{Z}^d} v_{k,j,\gamma} \Psi_{k,j,\gamma},$$

where $\theta_{j,\gamma} = \langle f, \Phi_{j,\gamma} \rangle$ and $v_{k,j,\gamma} = \langle f, \Psi_{k,j,\gamma} \rangle$.

Define the j th approximation of f by $P_j f = \sum_{\gamma \in \mathbb{Z}^d} \theta_{j,\gamma} \Phi_{j,\gamma}$ for every integer $j \geq 0$. Denote the linear estimator of f given the sample $\{Z(s) : s \in I_n\}$ by

$$\tilde{P}_j f = \sum_{\gamma \in \mathbb{Z}^d} \hat{\theta}_{j,\gamma} \Phi_{j,\gamma}, \quad \text{where } \hat{\theta}_{j,\gamma} = \frac{1}{|I_n|} \sum_{s \in I_n} \Phi_{j,\gamma}(Z(s)). \quad (3)$$

The hard thresholding estimator is defined for two levels $0 \leq j_0 \leq j_1$ and a thresholding sequence $(\bar{\lambda}_j : j \in \mathbb{N}) \subseteq \mathbb{R}_+$ as follows

$$\tilde{Q}_{j_0 j_1} f = \sum_{\gamma \in \mathbb{Z}^d} \hat{\theta}_{j_0,\gamma} \Phi_{j_0,\gamma} + \sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} \sum_{\gamma \in \mathbb{Z}^d} \hat{v}_{k,j,\gamma} \mathbf{1}(|\hat{v}_{k,j,\gamma}| > \bar{\lambda}_j) \Psi_{k,j,\gamma}, \quad (4)$$

where $\hat{v}_{k,j,\gamma} = \sum_{s \in I_n} \Psi_{k,j,\gamma}(Z(s)) / |I_n|$. Hence, $\tilde{Q}_{j_0 j_1} f$ consists of a linear estimator with respect to the coarse level j_0 and nonlinear terms of higher frequencies which are added to allow for more details if these are significantly different from zero. This also allows the approximation error and the estimation error of the estimator to vanish at higher rates than the linear estimator for certain parameter constellations; we encounter this below when presenting the results.

As $\tilde{P}_j f$ and $\tilde{Q}_{j_0 j_1} f$ are not necessarily a probability density, one can additionally consider the normalized estimator. We refer to Appendix B for this question.

3. Linear and hard thresholded wavelet density estimation

In this section we study wavelet density estimators for d -dimensional data. Kelly et al. [32] show that for isotropic wavelets and $f \in L^{p'}(\lambda^d)$ with $1 \leq p' < \infty$, the approximation bias vanishes, i.e., $\|f - P_j f\|_{L^{p'}(\lambda^d)} \rightarrow 0$ as $j \rightarrow \infty$. In the case $p' = \infty$ it is not guaranteed that the approximation error vanishes for general elements from $L^{p'}$. For instance, consider the one-dimensional Haar mother wavelet $\psi = \mathbf{1}\{[0, 1/2)\} - \mathbf{1}\{[1/2, 1)\}$ and construct with it the density $f = \mathbf{1}\{[0, 1)\} + \sum_{j=0}^{\infty} \psi(2^{j+1}x - (2^{j+1} - 2))$ on the unit interval $[0, 1]$. Then f jumps between 0 and 1 and these jumps become quite erratic as $x \rightarrow 1$. In particular, the projection $P_j f$ onto U_j cannot capture all jumps. Hence, we have $\liminf_{j \rightarrow \infty} \|f - P_j f\|_\infty \geq 1/2 > 0$ and the approximation property fails in this case. However, if f is a Besov density in $B_{p,q}^s(\mathbb{R}^d)$, we can derive for general admissible matrices M a rate of convergence.

We begin with the linear estimator. The technique of the proof is based on the ideas of Kerkycharian and Picard [33] who consider the case for one-dimensional iid data.

Theorem 2 (Rate of Convergence of the Linear Estimator). Let $p' \in [1, \infty)$, $p, q \in [1, \infty]$ and $s > 1/p$. Define $s' = s + (1/p' - 1/p) \wedge 0$. Let $A \in \mathcal{B}(\mathbb{R}^d)$ and if $p' < p$, further assume that A is bounded. Let $f \in F_{s,p,q}(K, A)$ for some $K \in \mathbb{R}_+$. If $p' \in [1, 2]$, assume that [Condition 1](#) (C) is satisfied with $a = 1$; if $p' \in (2, \infty)$, assume furthermore that [Condition 1](#) (C) is satisfied with $a = 2$. Let the level j grow at a rate of $|M|^j \simeq |I_n|^{1/(2s'+1)}$. Then

$$\mathbb{E} \left\{ \int_{\mathbb{R}^d} |f - \tilde{P}_j f|^{p'} \right\}^{1/p'} \leq C_1 |I_n|^{-s'/(2s'+1)}$$

for suitable constant $C_1 \in \mathbb{R}_+$. Moreover, if the domain A is bounded and $K > 0$ is a fixed constant, then

$$\sup_{f \in F_{s,p,q}(K,A)} \mathbb{E} \left\{ \int_{\mathbb{R}^d} |f - \tilde{P}_j f|^{p'} \right\}^{1/p'} \leq C_2 |I_n|^{-s'/(2s'+1)} \quad (5)$$

for all dependence structures of the random field Z , which satisfy $|f_{Z(s),Z(t)}(z_1, z_2) - f(z_1)f(z_2)| \leq h_0$ for all $s, t \in \mathbb{Z}^N$ for some fixed $h_0 \in \mathbb{R}_+$. The constants C_1, C_2 depend on the wavelets $\Psi_0, \dots, \Psi_{|M|}$, the matrix M , the bound on the mixing rates, the domain A , the index p' . Furthermore, C_1 also depends on the functions h and \bar{h} , and C_2 on K .

With similar requirements for a real-valued sample iid sample Z_1, \dots, Z_n , Kerkycharian and Picard [33] obtained a rate of convergence which is in $\mathcal{O}(n^{-s'/(2s'+1)})$.

Bouzebda et al. [5] studied the linear wavelet estimator for multivariate time series Z_1, \dots, Z_n in the supremum norm and in the case $M = 2I$. They obtain a bound of $\mathcal{O}\{2^{jd/2}(\ln n)^{1/2}/n^{1/2} + 2^{-jd\rho}\}$ for a level j which depends on n . The first term is the estimation error and the second term is the approximation error (bias); this error depends on a certain smoothness parameter ρ . Thus, their rates are quite similar to our result in particular if we compare it to the intermediate result from [Theorem 5](#) in Section 5 which considers the estimation error of the linear estimator.

Hence, when compared to the one-dimensional iid situation, we see that the estimate with a strongly mixing d -dimensional sample achieves the same rate, namely $|I_n|^{-s'/(2s'+1)}$.

The data dimension d is relevant for the rate of convergence. However, this is not shown in the previous theorem. We highlight this fact in the next two results which show that the dimension d has a negative impact on the Besov parameter s which controls the decay of the coefficients $v_{k,j,\gamma}$ as $j \rightarrow \infty$. We demonstrate that the classical inclusions shift slightly when moving from the one-dimensional to the d -dimensional Besov space: consider an (A, r) -Hölder continuous function with respect to the 2-norm, i.e., $|f(x) - f(y)| \leq A\|x - y\|_2^r$ for all $x, y \in \mathbb{R}^d$ for some $0 < A < \infty$. In the one-dimensional case we have that f belongs to the space $B_{\infty,\infty}^r$, i.e., that $\|f\|_{r,\infty,\infty}$ is finite; see, e.g., [19].

However, in the multivariate case, we find that such a function can be embedded only in a Besov space $B_{\infty,\infty}^s$ for an $s < r$ which yields a slower rate of convergence. Consider a wavelet coefficient of f , viz.

$$\begin{aligned} |v_{k,j,\gamma}| &\leq \left| \int_{\mathbb{R}^d} \{f(x) - f(x_0)\} \Psi_{k,j,\gamma}(x) \, dx \right| + |f(x_0)| \left| \int_{\mathbb{R}^d} \Psi_{k,j,\gamma}(x) \, dx \right| \\ &\leq \sup\{|f(x) - f(x_0)| : x \in \text{supp}(\Psi_{k,j,\gamma})\} |M|^{-j/2} \|\Psi_k\|_1 \\ &\leq A(L\sqrt{d}\|M^{-j}\|_2)^r |M|^{-j/2} \|\Psi_k\|_1 \leq C(\zeta_{\min})^{-jr} |M|^{-j/2}, \end{aligned}$$

where $\text{supp}(\Psi_k) \subseteq [0, L]^d$ and the point $x_0 \in \text{supp}(\Psi_{k,j,\gamma})$ is in the support of $\Psi_{k,j,\gamma}$ and $C \in \mathbb{R}_+$ is a suitable constant. Hence, we have for the $\|\cdot\|_{s,\infty,\infty}$ -norm of f if $p = q = \infty$:

$$\sup_{k,j,\gamma} |M|^{j(s+1/2)} |v_{k,j,\gamma}| \leq C \sup_j (\zeta_{\max})^{jsd} (\zeta_{\min})^{-jr} < \infty$$

if $s \leq (r/d) \times (\ln \zeta_{\min}/\ln \zeta_{\max})$. Hence, if f is an (A, r) -Hölder density and $s = r \ln \zeta_{\min}/(d \ln \zeta_{\max}) \leq r$, then $\|f\|_{s,\infty,\infty} < \infty$. We see that the difference in the eigenvalues ζ_{\min} and ζ_{\max} has little impact as it only enters with the logarithm. However, far more relevant is the data dimension d which scales the regularity parameter with its inverse d^{-1} .

One finds in simple examples that the bound of the first inequality is sharp. Indeed, consider the Lipschitz function which is non-constant in only one coordinate, $f(x) = x_1$ and use an MRA given by isotropic Haar wavelets. In this case, one has

$$\sup_{k,j,\gamma} |M|^{j(s+1/2)} |v_{k,j,\gamma}| = \sup_j 2^{j(ds-1)/4} < \infty \Leftrightarrow s \leq 1/d.$$

Using this insight, we can formulate two statements which also reveal that with increasing data dimension d the rate of convergence deteriorates.

Corollary 1 (Hölderian Densities). Let f be a compactly supported d -dimensional (A, r) -Hölderian density. The linear estimator attains the rate given in Eq. (5) for the parameter choice $s' = s = r \ln \zeta_{\min}/(d \ln \zeta_{\max})$.

The next result also applies to density functions with unbounded support.

Theorem 3 (Differentiable Densities). Let $p' \in [1, \infty)$. If $p' \leq 2$ (resp. $p' > 2$), assume moreover that [Condition 1](#) (C) is satisfied with $a = 1$ (resp. $a = 2$). Additionally, the gradient of f is bounded by a non-increasing radial function $\bar{h} \in L^p(\lambda^d)$, i.e., $\|Df\|_2 \leq \bar{h}$. Choose $j = j_0 + \lfloor (2 \ln \zeta_{\min} + d \ln \zeta_{\max})^{-1} \ln |I_n| \rfloor$, for a $j_0 \in \mathbb{N}$. Then the linear estimator attains the rate $\mathcal{O}\{|I_n|^{-\ln \zeta_{\min}/(2 \ln \zeta_{\min} + d \ln \zeta_{\max})}\}$.

Next, we study the nonlinear hard thresholding estimator of Donoho et al. [19], who consider this estimator for one-dimensional iid data. It is well-known that hard thresholding preserves the visual appearance of jumps and peaks of the density. A short and heuristic motivation for this estimator is as follows: the proof of [Theorem 2](#) (in particular, [Theorem 5](#) and Eq. (11)) reveals that the bias (approximation error) of the linear estimator is of order $\mathcal{O}(|M|^{-js'})$ while the stochastic term (estimation error) is of order $|M|^{j/2}|I_n|^{1/2}$. This cannot be optimal for a density $f \in B_{p,q}^s$ if $p' > p$ because in this case the bias is of the wrong order as $s' < s$; see Donoho et al. [19] for a deeper discussion and more details. However, if the density f belongs to a Besov space $B_{p,q}^s$, then this restriction entails that many coefficients $v_{k,j,\gamma}$ are forced to decay at a high rate: in particular, the decay in the ℓ^p -sequence norms $\|v_{k,j,\cdot}\|_{\ell^p}$ has to overcompensate the exponential growth of $|M|^{j(s+1/2-1/p)}$. Consequently, it makes sense to add finer levels of the density (more details) such that the bias is again of the right order and to set insignificant estimates of these details, $\hat{v}_{k,j,\gamma}$, to zero.

Theorem 4 (Rate of Convergence of the Hard Thresholding Estimator). *Let $p' \in [1, \infty)$, $p, q \in [1, \infty]$ and $s > 1/p$. Let $A \in \mathcal{B}(\mathbb{R}^d)$ and if $p' < p$, further assume that A is bounded. Let [Condition 1](#) (C) be satisfied with $a = 4$. Let $f \in F_{s,p,q}(K, A)$ for some $K \in \mathbb{R}_+$. Set $\tilde{\lambda}_j = K_0 \sqrt{j/|I_n|}$ for a constant K_0 specified in (19). Define $\varepsilon = sp - (p' - p)/2$ and $s' = s + (1/p' - 1/p) \wedge 0$. Moreover, define the levels j_0 and j_1 as*

$$|M|^{j_0} \simeq |I_n|^{1-2\alpha} \text{ and } |M|^{j_1} \simeq |I_n|^{\alpha/s'}, \quad \text{where } \alpha = \begin{cases} s/(2s+1) & \text{if } \varepsilon \geq 0, \\ s'/(2s+1-2/p) & \text{if } \varepsilon < 0. \end{cases}$$

Then

$$\mathbb{E} \left\{ \int_{\mathbb{R}^d} |f - \tilde{Q}_{j_0,j_1} f|^{p'} d\lambda^d \right\}^{1/p'} \leq C_1 |I_n|^{-\alpha} (\ln |I_n|)^{(3p'-p)/2p' \times \mathbf{1}(p' > p)}.$$

Moreover, if the domain A is bounded and $K > 0$ is a fixed constant, then

$$\sup_{f \in F_{s,p,q}(K,A)} \mathbb{E} \left[\int_{\mathbb{R}^d} |f - \tilde{Q}_{j_0,j_1} f|^{p'} d\lambda^d \right]^{1/p'} \leq C_2 |I_n|^{-\alpha} (\ln |I_n|)^{(3p'-p)/2p' \times \mathbf{1}(p' > p)}.$$

for all dependence structures of the random field Z , which satisfy $|f_{Z(s),Z(t)}(z_1, z_2) - f(z_1)f(z_2)| \leq h_0$ for all $s, t \in \mathbb{Z}^N$ for some fixed $h_0 \in \mathbb{R}_+$. The constants C_1, C_2 depend on the wavelets Ψ_0, \dots, Ψ_M , the matrix M , the bound on the mixing rates, the domain A , the index p' . Furthermore, C_1 depends also on the functions h and \tilde{h} , and C_2 on K .

Note that the exponent α is smooth in the parameters s, p, p' , i.e., for the case $\varepsilon = 0$, we could also use the definition of the case where $\varepsilon < 0$. This also means that the rate of convergence is smooth in the parameter p' .

We see that the rates have the identical structure as the rates of [19] who consider the classical case for a one-dimensional density and iid data. If $p' \leq p$, then $\varepsilon > 0$ and we obtain that j_1 and j_0 grow at the same rate. So the linear estimator is the preferred choice. If $p' > p$, however, one computes that in each case the exponent α is strictly greater than $s'/(2s+1)$, the latter is the exponent which determines the rate of the linear estimator. Consequently, the nonlinear estimator performs better in this case.

Li [36] studies the hard thresholding estimator in the special case $p' = 2$ for random fields similar as we do, however, the data are one-dimensional. He obtains for a one-dimensional density $f \in F_{s,p,q}(K, [-A, A])$ a rate in L^2 of $\mathcal{O}((\ln |I_n|/|I_n|)^{s/(2s+1)})$. So our results are a generalization as we do not only consider general p' but also allow for multivariate data and nonisotropic wavelets. Moreover, if $p' \geq p$, the density function can have an unbounded support.

An alternative to hard thresholding is soft thresholding of the coefficients. Here the absolute value of the estimated coefficients $v_{k,j,\gamma}$ undergoes the nonlinear shrinkage process $x \mapsto \text{sgn } x \times (x - \delta)_+$ for a certain $\delta > 0$.

This procedure can be interpreted as suppressing the noise in the estimated coefficients. Hence, one could also investigate the soft thresholding density estimator in the present setting. Fundamental properties of the soft thresholding method have been investigated by Donoho et al. [20]. Delouille et al. [17] studied the soft and hard thresholding estimator for design-adapted wavelets in nonparametric regression for one-dimensional iid data. They obtained a rate of convergence in L^2 which is in $\mathcal{O}((\ln n/n)^{r/(2r+1)})$ if the regression function is Hölder continuous with exponent $r \in (1/2, 1)$. This corresponds to our findings for Hölder continuous densities.

4. Numerical results

We give an example for the estimation of a two-dimensional density with strongly spatial mixing sample data on a regular two-dimensional lattice. Hence, concerning the parameter choice from the previous sections, we have a lattice dimension $N = 2$ and two-dimensional data, i.e., $d = 2$. The section is divided into three parts. First, we describe our decision rule how to choose the tuning parameters λ, j_0 and j_1 . Second, we sketch the process which generates the sample data on the lattice. Third, we present some numerical results for the estimation of a selected density function.

We follow a simple validation approach in order to choose the tuning parameters. We do not use leave-one out cross-validation because we face a dependent sample and cross-validation could corrupt the inner stochastic structure. In what

follows I_n is a finite and rectangular subset of \mathbb{N}_+^2 . We can construct a graph G from this set if we use the four-nearest-neighborhood structure on \mathbb{Z}^2 to construct the edge set. This neighborhood structure will define the dependence between the random variables $\{Z(s) : s \in I_n\}$ which all have the same marginal density f . The index set I_n is partitioned into two sets $I_{n,1}$ and $I_{n,2}$. Here we assume that each index set is a connected set with respect to the four-nearest neighborhood structure.

The density is estimated from the sample data which belongs to the index set $I_{n,1}$; we denote the estimate by \hat{f}_n . As in the present example $I_n = I_{(n_1, n_2)} = \{s : 1 \leq s_i \leq n_i, i \in \{1, 2\}\}$, we choose $I_{n,1}$ as $\{s : 1 \leq s_i \leq \lfloor 0.9n_i \rfloor, i \in \{1, 2\}\}$. So $I_{n,1}$ is also a rectangular set and the estimator is computed with approximately 80% of the data.

In general, the integrated squared error can be decomposed as

$$\text{ISE}(f, \hat{f}_n) = \int_{\mathbb{R}^d} (\hat{f}_n - f)^2 d\lambda^d = \left\{ \int_{\mathbb{R}^d} \hat{f}_n^2 d\lambda^d - 2 \int_{\mathbb{R}^d} \hat{f}_n f d\lambda^d \right\} + \int_{\mathbb{R}^d} f^2 d\lambda^d = \text{ver}(\hat{f}_n, f) + \|f\|_{L^2(\lambda^d)}^2. \quad (6)$$

Since in practice the true density function is unknown, it is sufficient for a comparison of density estimates to compute the full validation criterion with the subsample $I_{n,2}$, which is L-shaped in the present example. We define

$$\widehat{\text{ver}}(\hat{f}_n, f, I_{n,2}) = \int_{\mathbb{R}^d} \hat{f}_n^2 d\lambda^d - 2 \frac{1}{|I_{n,2}|} \sum_{s \in I_{n,2}} \hat{f}_n\{Z(s)\}, \quad (7)$$

which is the empirical analogue of $\text{ver}(\hat{f}_n, f)$ as on average we expect only a small dependence between \hat{f}_n and the random variables $Z(s)$ for $s \in I_{n,2}$.

For hard thresholding, we use an approach similar to an algorithm which has been proposed by [26] for the choice of the primary level j_0 in the context of cross-validation. The idea is to define a suitable partition $R_1 \cup \dots \cup R_S$ of the domain of definition of \hat{f}_n (resp. of f), where each R_k collects regions of relatively homogeneous roughness. These regions can be determined with a pilot estimator. We compute the validation criterion for each R_k for the levels $j \in \{j_0, \dots, j_1\}$ with $j_0 \leq j_1$ and with the purely linear wavelet estimator $\hat{P}_j f$ from Eq. (3) restricted to R_k . Abbreviate the level which minimizes (7) for region R_k by j_k . Then choose $j^* = \min(j_1, \dots, j_S)$ as the primary level.

Moreover, we follow an approach used in [29] for the hard thresholding estimator from (4) and set each threshold as a multiple of $\max\{|\hat{v}_{k,j,\gamma}| : k \in \{1, \dots, |M| - 1\}, \gamma \in \mathbb{Z}^d\}$ for $j \in \{j^*, \dots, j_1\}$. This multiple is the same for all $j \in \{j^*, \dots, j_1\}$. In the following, we refer to these multiples as the threshold, e.g., a threshold 0.1 means that $\bar{\lambda}_j$ is equal to 0.1 times $\max\{|\hat{v}_{k,j,\gamma}| : k \in \{1, \dots, |M| - 1\}, \gamma \in \mathbb{Z}^d\}$.

Next, we sketch the data generating process. We use the algorithm of Kaiser et al. [31] to simulate five random vectors Z_1, \dots, Z_5 . The marginals of each vector are standard normally distributed. The underlying graph is the same for each random vector and is the regular two-dimensional lattice with the four-nearest-neighborhood structure and edge lengths $n_1 = n_2$. We perform the simulation for four different values of n_1 , namely 20, 35, 50 and 65. So that $|I_n|$ is 400, 1225, 2500 and 4225.

The dependence within a random vector Z_i is generated as follows. Write Z for this vector for simplicity. If $Z = \{Z(s) : s \in I_n\}$ is multivariate Gaussian with expectation $\alpha \in \mathbb{R}^{|I_n|}$ and covariance $\Sigma \in \mathbb{R}^{|I_n| \times |I_n|}$, then Z has the density

$$f_Z(z) = (2\pi)^{-|I_n|/2} \det(\Sigma)^{-1/2} \exp \left\{ -\frac{1}{2} (z - \alpha)^\top \Sigma^{-1} (z - \alpha) \right\}.$$

Using the notation P for the precision matrix Σ^{-1} and $-s = I_n \setminus \{s\}$, we have

$$Z(s) | Z(-s) \sim \mathcal{N} \left[\alpha(s) - \{P(s, s)\}^{-1} \sum_{t \neq s} P(s, t) \{Z(t) - \alpha(t)\}, \{P(s, s)\}^{-1} \right].$$

Write $\text{Ne}(s)$ for the neighbors of s in I_n with respect to the four-nearest-neighborhood structure. Since $P = \Sigma^{-1}$ is symmetric and since we can assume that $\{P(s, s)\}^{-1} > 0$, Z is a Markov random field if and only if for all nodes $s \in I_n$,

$$\forall t \in \text{Ne}(s) \quad P(s, t) \neq 0 \quad \text{and} \quad \forall t \in I_n \setminus \text{Ne}(s) \quad P(s, t) = 0.$$

Cressie [11] investigates the conditional specification

$$Z(s) | Z(-s) \sim \mathcal{N} \left[\alpha(s) + \sum_{t \in \text{Ne}(s)} c(s, t) \{Z(t) - \alpha(t)\}, \varsigma^2(s) \right], \quad (8)$$

where $C = (c(s, t))_{s,t}$ is a $|I_n| \times |I_n|$ matrix and $D = \text{diag}(\varsigma^2(s) : s \in I_n)$ is a diagonal matrix such that the coefficients satisfy the necessary condition $\varsigma^2(s) c(t, s) = \varsigma^2(t) c(s, t)$ for $s \neq t$ and $c(s, s) = 0$ as well as $c(s, t) = 0 = c(t, s)$ if s, t are not neighbors. This means $P(s, t) = -c(s, t) P(s, s)$, i.e., $\Sigma^{-1} = P = D^{-1}(I - C)$. If $I - C$ is invertible and $(I - C)^{-1}D$ is symmetric and positive definite, then the entire random field is multivariate normal with $Z \sim \mathcal{N}[\alpha, (I - C)^{-1}D]$.

In particular, it is plausible in many applications to use equal weights $c(s, t)$: we can write the matrix C as ηH , where H is the adjacency matrix of G , i.e., $H(s, t)$ is 1 if s, t are neighbors, and otherwise it is 0. Denote the minimal eigenvalue of H by h_0 and the corresponding maximal eigenvalue by h_m . We know from the properties of the Neumann series that $I - C$ is invertible if $(h_0)^{-1} < \eta < (h_m)^{-1}$ when $h_0 < 0 < h_m$; the latter condition is often satisfied in applications. We choose

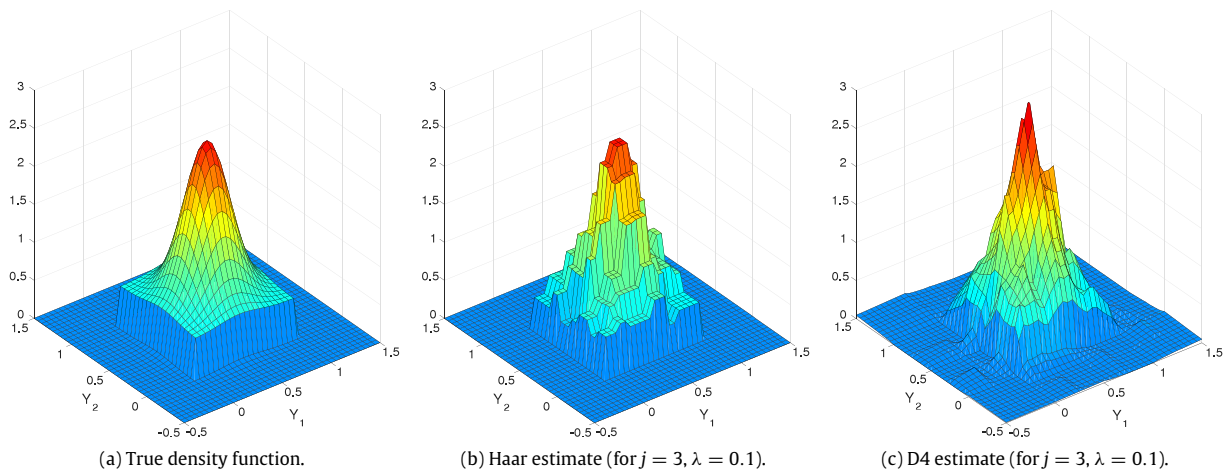


Fig. 1. Estimation of a mixture density with a sample from the two-dimensional lattice of size 4225.

the conditional variance $\varsigma^2(s)$ such that the diagonal matrix D consists of the inverse elements of the diagonal of the matrix $(I - C)^{-1}$. Hence, the marginals of the $Z(s)$ are $\mathcal{N}(0, 1)$.

The graph structure of the index set I_n allows us to partition I_n into two sets C_1 and C_2 , which are disjoint with respect to the edges from the four-nearest neighborhood structure such that within each set C_i any two points s, t are no neighbors. The sets C_i are termed *concliques*; see [31]. An important property is now the following: if $Z(s), s \in I_n$ is a Markov random field, then the conditional distribution of the $Z(s)$ with $s \in C_1$ given the $Z(s)$ with $s \in C_2$ factorizes as a product due to the conditional independence. The same is true if we change the roles of C_1 and C_2 .

This insight yields the following MCMC algorithm: All $Z(s), s \in I_n$ are initialized according to a certain distribution. Then with the help of Eq. (8) conditional on the conclique C_2 the random variables $Z(s)$ with $s \in C_1$ are updated. Afterwards, the random variables which belong to C_2 are updated with (8) based on the (new) realizations of C_1 . The last two steps are repeated many times until the random field approximately reaches its stationary distribution. Hence, if we compare this method to the Gibbs sampler, we see that a complete update of the random field can be performed in two steps. More details on this procedure, in particular its asymptotic properties, can be found, e.g., in [34].

As we do not simulate one random vector but instead five, we use a Gaussian copula in the update steps such that also four of the random vectors are dependent, namely Z_1, Z_2, Z_3 and Z_4 . However, Z_5 is independent of the first four. So we have a dependence within each component Z_i and the first four components are also dependent among themselves.

We run 1000 iterations of the MCMC-algorithm for the simulation of (Z_1, \dots, Z_5) . The parametrization of the multivariate Gaussian distribution is chosen as follows: $\alpha_i(s) \equiv 0$ and $\sigma_i = 1$ for all $s \in I_n$ and $i \in \{1, \dots, 5\}$. The dependence parameters η_i that determine the interaction within a distribution Z_i are chosen as follows: 0.2, -0.1, -0.22, 0.2 and 0.22; note that $|\eta_i| = 0.22$ constitutes a strong dependence, whereas $\eta_i = 0$ indicates independence. In this case the admissible range of η is very close to $(-0.25, 0.25)$ which is the parameter space of η for a lattice wrapped on a torus. The approximate correlations of the first four Z_i are given by $\rho_{1,2} \approx 0.1$, $\rho_{1,3} \approx 0$, $\rho_{1,4} \approx 0$, $\rho_{2,3} \approx 0$, $\rho_{2,4} \approx 0$ and $\rho_{3,4} \approx 0.1$.

With these distributions we define a random variable Y with a non-continuous density as follows. First, we retransform Z_5 to a discrete random variable S which takes the states 0 and 1 with probability 1/2. Second, we transform Z_1 and Z_2 to random variables U_1 and U_2 which are both uniformly distributed on $[0, 1]$. Third, we define X_1 and X_2 as rescaled and shifted Z_3 and Z_4 such that they are normally distributed with parameters $\mu = 0.5$ and $\sigma^2 = 0.2$. Set now $Y = \mathbf{1}(S = 0) \times (U_1, U_2) + \mathbf{1}(S = 1) \times (X_1, X_2)$; then Y admits the density

$$f_{(Y_1, Y_2)} = \frac{1}{2} \mathbf{1}_{[0,1]^2} + \frac{1}{2} \mathcal{N} \left[\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, 0.2^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right],$$

where $\rho \approx 0.1$. A density plot is given in Fig. 1(a). We estimate the density $f_{(Y_1, Y_2)}$ with the linear and the nonlinear wavelet estimators based on isotropic Haar wavelets and Daubechies 4-wavelets and the sample $Z(s), s \in I_{n,1}$. We abbreviate the Daubechies wavelet by D4; see [14].

Then we compute for several levels the verification criterion from Eq. (7) with the random variables $Z(s), s \in I_{n,2}$. We perform this whole procedure 1000 times in total for each sample size. This means that we compute for each of the 1000 simulations the verification criterion for each estimator at each level j . Afterwards we can compute the average and the empirical standard deviation of this statistic for each scenario. The numerical results are given in Table 1. Here the verification criterion is computed for different levels j reaching from 0 to 4. This means for the linear estimator that the coefficients $\theta_{j,\gamma}$ are computed for each of these levels. For the nonlinear estimator the coefficients $\theta_{j,\gamma}$ are computed only for the level $j = 0$.

Table 1

Approximate validation criterion from (7) computed for the density estimation problem with the Haar wavelet and the D4-wavelet. The hard thresholding estimator is computed with respect to the levels $j_0 = 0$ and $j_1 \in \{1, \dots, 4\}$. The thresholds 0.1, 0.2, 0.3 are relative thresholds as explained in the text.

Sample size	j	Haar				D4			
		Linear	Nonlinear: hard threshold			Linear	Nonlinear: hard threshold		
			0.1	0.2	0.3		0.1	0.2	0.3
400	0	−0.922 (0.040)	—	—	—	−0.583 (0.054)	—	—	—
400	1	−0.871 (0.054)	−0.922 (0.050)	−0.922 (0.050)	−0.922 (0.050)	−1.090 (0.138)	−1.094 (0.141)	−1.083 (0.139)	−1.058 (0.133)
400	2	−1.026 (0.139)	−1.067 (0.137)	−1.064 (0.136)	−1.055 (0.134)	−1.161 (0.167)	−1.166 (0.170)	−1.154 (0.169)	−1.128 (0.161)
400	3	−0.928 (0.187)	−0.980 (0.180)	−0.981 (0.178)	−0.982 (0.174)	−1.045 (0.202)	−1.052 (0.205)	−1.046 (0.202)	−1.029 (0.192)
400	4	−0.384 (0.266)	−0.494 (0.253)	−0.538 (0.246)	−0.606 (0.239)	−0.506 (0.276)	−0.520 (0.278)	−0.565 (0.275)	−0.622 (0.261)
1225	0	−0.922 (0.021)	—	—	—	−0.584 (0.032)	—	—	—
1225	1	−0.878 (0.025)	−0.928 (0.025)	−0.928 (0.025)	−0.928 (0.025)	−1.089 (0.078)	−1.093 (0.080)	−1.083 (0.079)	−1.058 (0.075)
1225	2	−1.052 (0.074)	−1.090 (0.074)	−1.089 (0.074)	−1.086 (0.074)	−1.189 (0.094)	−1.194 (0.096)	−1.180 (0.094)	−1.153 (0.089)
1225	3	−1.043 (0.092)	−1.087 (0.091)	−1.087 (0.091)	−1.084 (0.090)	−1.163 (0.102)	−1.168 (0.104)	−1.157 (0.102)	−1.131 (0.097)
1225	4	−0.867 (0.114)	−0.932 (0.112)	−0.942 (0.110)	−0.961 (0.107)	−0.977 (0.120)	−0.985 (0.122)	−0.988 (0.119)	−0.987 (0.113)
2500	0	−0.923 (0.016)	—	—	—	−0.583 (0.024)	—	—	—
2500	1	−0.881 (0.019)	−0.931 (0.018)	−0.931 (0.018)	−0.930 (0.018)	−1.091 (0.059)	−1.095 (0.060)	−1.084 (0.059)	−1.059 (0.056)
2500	2	−1.063 (0.056)	−1.101 (0.055)	−1.101 (0.055)	−1.099 (0.055)	−1.196 (0.069)	−1.201 (0.071)	−1.186 (0.070)	−1.158 (0.066)
2500	3	−1.079 (0.065)	−1.119 (0.064)	−1.119 (0.064)	−1.116 (0.064)	−1.197 (0.071)	−1.202 (0.072)	−1.188 (0.071)	−1.161 (0.067)
2500	4	−1.000 (0.073)	−1.051 (0.072)	−1.056 (0.072)	−1.063 (0.071)	−1.111 (0.078)	−1.117 (0.080)	−1.108 (0.078)	−1.092 (0.074)
4225	0	−0.922 (0.012)	—	—	—	−0.584 (0.018)	—	—	—
4225	1	−0.881 (0.014)	−0.930 (0.013)	−0.930 (0.013)	−0.930 (0.013)	−1.089 (0.045)	−1.092 (0.046)	−1.081 (0.045)	−1.057 (0.043)
4225	2	−1.063 (0.041)	−1.101 (0.040)	−1.101 (0.040)	−1.100 (0.040)	−1.197 (0.053)	−1.201 (0.054)	−1.186 (0.053)	−1.158 (0.050)
4225	3	−1.087 (0.047)	−1.128 (0.047)	−1.127 (0.047)	−1.125 (0.046)	−1.207 (0.054)	−1.212 (0.055)	−1.196 (0.054)	−1.169 (0.051)
4225	4	−1.044 (0.051)	−1.091 (0.050)	−1.094 (0.050)	−1.097 (0.050)	−1.158 (0.056)	−1.163 (0.057)	−1.152 (0.056)	−1.130 (0.054)

Then the nonlinear details $v_{k,j,\gamma}$ are added successively for $j \in \{1, \dots, 4\}$. We also show in the table which estimator, which levels j and which threshold are the most suitable for each sample size.

For a better comparison, we also give in Table 2 the results, which are derived with an independent reference sample $\tilde{Z} = (\tilde{Z}_1, \dots, \tilde{Z}_5)$. This means that the random variables within a component \tilde{Z}_i are iid, i.e., $\tilde{Z}_i(s)$ are iid for $s \in I_n$ and for fixed $i \in \{1, \dots, 5\}$. The correlations between the vectors \tilde{Z}_i correspond to those of the Z_i . Note that we use for hard thresholding several multiples of $\max\{|\hat{v}_{k,\ell,\gamma}| : k \in \{1, \dots, |M| - 1\}, \gamma \in \mathbb{Z}^2\}$, however, the multiple is the same for all levels j^*, \dots, j_1 and only varies for the entire estimator. Examples of density estimates are given in Figs. 1(b) and 1(c). The estimators have been corrected for possible negative regions; we refer to Appendix B for details.

First, we see that the estimator from the dependent sample performs as well as the estimator from the independent sample, as suggested by the theoretical results. This is true for each sample size and for each wavelet type. Moreover, we find that the optimal multiple of the threshold $\tilde{\lambda}$ is the same for all sample sizes. In particular it is different from zero in each case.

We note that the values of the verification criterion can be compared across the different wavelets, this follows from Eqs. (6) and (7). Hence in terms of the validation criterion, we find that the Daubechies wavelet performs better than the

Table 2

Approximate validation criterion from Eq. (7) with independent reference samples. The hard thresholding estimator is computed with respect to the levels $j_0 = 0$ and $j_1 \in \{1, \dots, 4\}$. The thresholds 0.1, 0.2, 0.3 are relative thresholds as explained in the text.

Sample size	j	Haar				D4			
		Linear	Nonlinear: hard threshold			Linear	Nonlinear: hard threshold		
			0.1	0.2	0.3		0.1	0.2	0.3
400	0	-0.923 (0.036)	—	—	—	-0.587 (0.051)	—	—	—
400	1	-0.875 (0.049)	-0.927 (0.046)	-0.926 (0.046)	-0.926 (0.046)	-1.087 (0.130)	-1.091 (0.133)	-1.082 (0.132)	-1.056 (0.125)
400	2	-1.035 (0.132)	-1.073 (0.127)	-1.071 (0.126)	-1.063 (0.126)	-1.167 (0.151)	-1.173 (0.154)	-1.162 (0.153)	-1.134 (0.146)
400	3	-0.933 (0.178)	-0.985 (0.167)	-0.987 (0.165)	-0.988 (0.162)	-1.048 (0.187)	-1.055 (0.190)	-1.049 (0.188)	-1.031 (0.180)
400	4	-0.388 (0.263)	-0.497 (0.247)	-0.541 (0.242)	-0.614 (0.236)	-0.505 (0.264)	-0.519 (0.265)	-0.560 (0.259)	-0.620 (0.246)
1225	0	-0.922 (0.019)	—	—	—	-0.584 (0.027)	—	—	—
1225	1	-0.879 (0.027)	-0.929 (0.024)	-0.929 (0.024)	-0.929 (0.024)	-1.093 (0.068)	-1.097 (0.069)	-1.087 (0.069)	-1.060 (0.065)
1225	2	-1.053 (0.066)	-1.092 (0.062)	-1.091 (0.062)	-1.088 (0.062)	-1.190 (0.077)	-1.195 (0.079)	-1.183 (0.078)	-1.153 (0.074)
1225	3	-1.047 (0.081)	-1.090 (0.078)	-1.090 (0.077)	-1.088 (0.076)	-1.165 (0.086)	-1.170 (0.088)	-1.160 (0.087)	-1.132 (0.083)
1225	4	-0.870 (0.102)	-0.933 (0.097)	-0.945 (0.096)	-0.965 (0.093)	-0.978 (0.107)	-0.986 (0.108)	-0.990 (0.105)	-0.987 (0.100)
2500	0	-0.923 (0.015)	—	—	—	-0.585 (0.021)	—	—	—
2500	1	-0.881 (0.018)	-0.932 (0.018)	-0.932 (0.018)	-0.931 (0.017)	-1.094 (0.051)	-1.097 (0.052)	-1.088 (0.051)	-1.061 (0.048)
2500	2	-1.062 (0.050)	-1.101 (0.047)	-1.101 (0.047)	-1.099 (0.047)	-1.199 (0.057)	-1.203 (0.058)	-1.190 (0.057)	-1.159 (0.054)
2500	3	-1.079 (0.056)	-1.121 (0.055)	-1.120 (0.055)	-1.117 (0.055)	-1.199 (0.061)	-1.204 (0.062)	-1.192 (0.061)	-1.162 (0.058)
2500	4	-1.002 (0.064)	-1.052 (0.064)	-1.057 (0.064)	-1.063 (0.063)	-1.114 (0.069)	-1.120 (0.070)	-1.113 (0.068)	-1.094 (0.065)
4225	0	-0.922 (0.011)	—	—	—	-0.586 (0.015)	—	—	—
4225	1	-0.882 (0.014)	-0.932 (0.013)	-0.932 (0.013)	-0.932 (0.013)	-1.095 (0.038)	-1.098 (0.039)	-1.089 (0.039)	-1.062 (0.036)
4225	2	-1.067 (0.035)	-1.104 (0.033)	-1.104 (0.033)	-1.104 (0.033)	-1.202 (0.042)	-1.207 (0.043)	-1.193 (0.043)	-1.162 (0.040)
4225	3	-1.093 (0.039)	-1.132 (0.038)	-1.132 (0.038)	-1.130 (0.038)	-1.212 (0.043)	-1.217 (0.044)	-1.204 (0.044)	-1.174 (0.041)
4225	4	-1.049 (0.044)	-1.095 (0.043)	-1.097 (0.042)	-1.101 (0.041)	-1.163 (0.047)	-1.169 (0.048)	-1.159 (0.048)	-1.134 (0.045)

Haar wavelet for each sample size in this example. However, they have in common that the optimal level j is the same for each sample size: the level $j = 2$ minimizes the criterion for the sample sizes 400 and 1225. It is level $j = 3$ for the larger sample sizes 2500 and 4225.

5. Proofs of the theorems in Section 3

Throughout this section, we use the common convention to abbreviate constants in \mathbb{R} by A_i or A or likewise by C_i or C . Furthermore, we use the convention to write $\|\cdot\|_p$ for the norm of $L^p(\lambda^d)$, $p \in [1, \infty]$. The idea of the first lemma dates back at least to Meyer [39]. It applies in particular to wavelets ψ_k which have compact support.

Lemma 1 (Norm Equivalence on Besov Spaces). *The norms in (1) and in (2) are equivalent provided that the wavelets ψ_k are integrable and $\sup_{x \in \mathbb{R}^d} \sum_{\gamma \in \mathbb{Z}^d} |\psi_k(x - \gamma)| < \infty$ for all $k \in \{0, \dots, |M| - 1\}$.*

Proof. We show that there are $0 < C_1, C_2 < \infty$ depending on s, p, q such that $C_1 \|f\|_{s,p,q} \leq \|f\|_{B_{p,q}^s} \leq C_2 \|f\|_{s,p,q}$. First we consider the left inequality. Define for $j \geq j_0$ the functions $g_j^{(k)} = \sum_{\gamma \in \mathbb{Z}^d} v_{k,j,\gamma} \psi_{k,j,\gamma}$ for $k \in \{1, \dots, |M| - 1\}$ and $g_j^{(0)} = \sum_{\gamma \in \mathbb{Z}^d} \theta_{j_0,\gamma} \phi_{j_0,\gamma}$. Denote the Hölder conjugate of p by u , then by the property of an orthonormal basis and Hölder's inequality applied to the measure $|\psi_{k,j,\gamma}| d\lambda^d$, we have

$$|v_{k,j,\gamma}| \leq \left(\int_{\mathbb{R}^d} |g_j^{(k)}|^p |\psi_{k,j,\gamma}| d\lambda^d \right)^{1/p} \left(\int_{\mathbb{R}^d} |\psi_{k,j,\gamma}| d\lambda^d \right)^{1/u},$$

and thus

$$\|v_{k,j,\cdot}\|_{\ell^p} \leq |M|^{j(1/p-1/2)} \|\psi_k\|_1^{1/u} \|g_j^{(k)}\|_p \left\| \sum_{\gamma \in \mathbb{Z}^d} |\psi_k(\cdot - \gamma)| \right\|_{\infty}^{1/p}$$

with the usual modification if $p = 1$ or $p = \infty$; the same reasoning is true for the vector $\theta_{j_0,\cdot}$. Then, $\|f\|_{B_{p,q}^s} \geq C_1 \|f\|_{s,p,q}$, where

$$C_1 = \min_{k \in \{0, \dots, |M|-1\}} \left\{ \|\psi_k\|_1^{-1/u} \left\| \sum_{\gamma \in \mathbb{Z}^d} |\psi_k(\cdot - \gamma)| \right\|_{\infty}^{-1/p} \right\} < \infty.$$

For the right inequality, consider the point-wise inequality

$$|g_j^{(k)}| \leq \sum_{\gamma \in \mathbb{Z}^d} |v_{k,j,\gamma}| |\psi_{k,j,\gamma}|^{1/p} |\psi_{k,j,\gamma}|^{1/u} \leq \left(\sum_{\gamma \in \mathbb{Z}^d} |v_{k,j,\gamma}|^p |\psi_{k,j,\gamma}| \right)^{1/p} \left(\sum_{\gamma \in \mathbb{Z}^d} |\psi_{k,\ell,\gamma}| \right)^{1/u}$$

for all $k \in \{1, \dots, |M| - 1\}$; it is also true if $k = 0$. Thus,

$$\|g_j^{(k)}\|_p \leq \left\| \sum_{\gamma \in \mathbb{Z}^d} |\psi_k(\cdot - \gamma)| \right\|_{\infty}^{1/u} \|\psi_k\|_1^{1/p} |M|^{j(1/2-1/p)} \|v_{k,j,\cdot}\|_{\ell^p}.$$

Hence, $\|f\|_{B_{p,q}^s} \leq C_2 \|f\|_{s,p,q}$ with $C_2 = \max_{k \in \{0, \dots, |M|-1\}} \left\| \sum_{\gamma \in \mathbb{Z}^d} |\psi_k(\cdot - \gamma)| \right\|_{\infty}^{1/u} \|\psi_k\|_1^{1/p} < \infty$. \square

We are now prepared to give bounds on the estimation error of the linear estimator.

Theorem 5. Let $p' \in [1, \infty)$ and assume the density f to be in $L^{p'}(\lambda^d) \cap L^\infty(\lambda^d)$.

1. If $p' \in [1, 2]$ and if [Condition 1](#) (C) is satisfied with $a = 1$, then

$$\mathbb{E} \left\{ \int_{\mathbb{R}^d} |\tilde{P}_j f - P_j f|^{p'} d\lambda^d \right\}^{1/p'} \leq C_{p'} (2L+1)^d \|\Phi\|_{L^\infty(\lambda^d)} \|\Phi\|_{L^2(\lambda^d)} \times \left\{ \|h^{1/2}\|_{L^1(\lambda^d)}^{1/2} + \|\tilde{h}\|_{L^2(\lambda^d)} \right\} \frac{|M|^{j/2}}{|I_n|^{1/2}}.$$

2. If $p' \in (2, \infty)$ and [Condition 1](#) (C) is satisfied with $a = 2$, then

$$\begin{aligned} \mathbb{E} \left\{ \int_{\mathbb{R}^d} |\tilde{P}_j f - P_j f|^{p'} d\lambda^d \right\}^{1/p'} &\leq C_{p'} (2L+1)^d \|\Phi\|_{L^\infty(\lambda^d)}^{1/p'} \|\Phi\|_{L^{p'}(\lambda^d)} \left\{ \|h^{1/4}\|_{L^1(\lambda^d)}^{1/p'} + \|\tilde{h}\|_{L^1(\lambda^d)}^{1/p'} \right\} \\ &\quad \times \left[\frac{|M|^{j(1-1/(2p'))} q_n^{N(1-1/p')}}{|I_n|^{1-1/(2p')}} + \frac{|M|^{j/2}}{|I_n|^{1/2}} \right], \end{aligned}$$

where $q_n = 2(p' - 1)/c_1 \ln |I_n|$ depends on p' and the bound of the mixing coefficients.

The constant $C_{p'}$ depends on p' , the bound of the mixing coefficients which is given by the numbers $c_0, c_1 \in \mathbb{R}_+$.

Proof of Theorem 5. We write \tilde{f}_j (resp. f_j) instead of $\tilde{P}_j f$ (resp. $P_j f$) to keep the notation simple. Without loss of generality, the support of Φ is contained in $[0, L]^d$, $L \in \mathbb{N}_+$. Hence, a fixed $x \in \mathbb{R}^d$ is contained in the support of at most $(2L+1)^d$ translations of Φ . Consequently, if we apply the Hölder inequality to the counting measure over the index γ , the estimation error is at most

$$\mathbb{E} \left\{ \int_{\mathbb{R}^d} |f_j - \tilde{f}_j|^{p'} d\lambda^d \right\} \leq (2L+1)^{d(p'-1)} \|\Phi\|_{p'}^{p'} |M|^{j(p'/2-1)} \sum_{\gamma \in \mathbb{Z}^d} \mathbb{E} \{ |\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}|^{p'} \}. \quad (9)$$

We investigate the sum in (9). First, let $p' \in [1, 2]$, then we obtain with Proposition 1 that

$$\left[\sum_{\gamma \in \mathbb{Z}^d} \mathbb{E} \{ |\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}|^2 \} \right]^{1/2} \leq C_2 L^{d/2} 2^{d/2} \|\Phi\|_\infty \{ \|h^{1/2}\|_1^{1/2} + \|\tilde{h}\|_2 \} \frac{|M|^{j/2}}{|I_n|^{1/2}}.$$

This yields the claim for $p' \leq 2$, if we use that $(2L + 1)^{d/2} L^{d/2} 2^{d/2} \leq (2L + 1)^d$ and the Hölder inequality to bound $\mathbb{E}(\int_{\mathbb{R}^d} |f_j - \tilde{f}_j|^{p'} d\lambda^d)^{1/p'}$ by $\mathbb{E}(\int_{\mathbb{R}^d} |f_j - \tilde{f}_j|^2 d\lambda^d)^{1/2}$. Second, if $p' > 2$, we use the decomposition

$$\sum_{\gamma \in \mathbb{Z}^N} \mathbb{E} \{ |\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}|^{p'} \} \leq \sum_{\gamma \in \mathbb{Z}^N} \mathbb{E} \{ |\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}|^2 \}^{1/2} \mathbb{E} \{ |\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}|^{2(p'-1)} \}^{1/2}. \quad (10)$$

We bound the last factor inside the sum with Lemma 3, noting that $2(p' - 1) > 2$, viz.

$$\mathbb{E} \{ |\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}|^{2(p'-1)} \}^{1/2} \leq C_{2(p'-1)} \left[|I_n|^{-(p'-1)} + \left\{ \frac{|M|^{j/2} q_n^N}{|I_n|} \right\}^{2(p'-1)} + \frac{|M|^{j(p'-1)}}{|I_n|^{c_1 B}} \right]^{1/2},$$

where the constant $C_{2(p'-1)}$ depends on B . If we choose $B = 2(p' - 1)/c_1$, then the last term inside the parentheses is negligible. Moreover, it follows with Proposition 1 that

$$\sum_{\gamma \in \mathbb{Z}^N} \mathbb{E} \{ |\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}|^2 \}^{1/2} \leq C_2 L^{d/2} 2^{d/2} \|\Phi\|_\infty \{ \|h^{1/4}\|_1 + \|\tilde{h}\|_1 \} |M|^j |I_n|^{-1/2}.$$

We combine (9) and (10) to obtain the result. Note that we have $(2L + 1)^{d(p'-1)/p'} L^{d/2} 2^{d/p'} \leq (2L + 1)^d$. \square

Next comes the proof of Theorem 2 which quantifies the rate of convergence of the linear estimator.

Proof of Theorem 2. Denote the Hölder conjugate of p' by u , i.e., $(p')^{-1} + u^{-1} = 1$. We show that the approximation error $\|f - P_j f\|_{p'}$ can be bounded above as follows:

$$\|f - P_j f\|_{p'} \leq C_A \max_{1 \leq k \leq |M|-1} \|\Psi_k\|_1 \max_{1 \leq k \leq |M|-1} \left\| \sum_{\gamma \in \mathbb{Z}^d} |\Psi_k(\cdot - \gamma)| \right\|_\infty^{1/u} \times \|f\|_{s,p,\infty} |M|^{1-j s'} / (1 - |M|^{-s'}), \quad (11)$$

where the constant C_A only differs from 1 if $p > p'$, in which case it depends on the domain A .

We have to distinguish the cases $p \leq p'$ and $p > p'$ but can treat this in one formula. We proceed as in the proof of Lemma 1, viz.

$$\left\| \sum_{\gamma \in \mathbb{Z}^d} v_{k,j,\gamma} \Psi_{k,j,\gamma} \right\|_{p'} \leq \max_{1 \leq k \leq |M|-1} \left\| \sum_{\gamma \in \mathbb{Z}^d} |\Psi_k(\cdot - \gamma)| \right\|_\infty^{1/u} \|\Psi_k\|_1^{1/p'} |M|^{j(1/2-1/p')} \|v_{k,j,\cdot}\|_{\ell^{p'}},$$

with the notation that u is the Hölder conjugate to p' . In the case $p > p'$, the number of nonzero coefficients on the j th level (for the k th mother wavelet) is bounded above by $C_A |M|^j$, where C_A depends on the domain of f . This follows from the dilatation rules of volumes under linear transformations and from the fact that the domain A is bounded. Consequently, we have in both cases $p \geq p'$ and $p < p'$ the inequalities for the ℓ^p -sequence norms,

$$\|v_{k,j,\cdot}\|_{\ell^{p'}} \leq C_A |M|^{j(1/p'-1/p)^+} \|v_{k,j,\cdot}\|_{\ell^p}$$

where $C_A = 1$ if $p' \geq p$. Then with Hölder's inequality and the Besov property of f ,

$$\|f - P_j f\|_{p'} \leq C_A \max_{1 \leq k \leq |M|-1} \|\Psi_k\|_1^{1/p'} \max_{1 \leq k \leq |M|-1} \left\| \sum_{\gamma \in \mathbb{Z}^d} |\Psi_k(\cdot - \gamma)| \right\|_\infty^{1/u} \times \|f\|_{s,p,\infty} |M|^{1-j s'} / (1 - |M|^{-s'}) \leq C |M|^{-j s'} \quad (12)$$

with the definition $s' = s + (1/p' - 1/p) \wedge 0$. Note that $s' > 0$ as $s > 1/p$. The constant C depends on the matrix M , the wavelets, f and, if $p < p'$, also on the domain A . The estimation error is given in Theorem 5. The growth rate of j equalizes the rates of the terms $|M|^{-j s'}$ and $|M|^{j/2} |I_n|^{1/2}$; both behave as $|I_n|^{-s'/(2s'+1)}$. This implies in particular that the term $|M|^{j(1-1/(2p'))} q_n^{N(1-1/p')} / |I_n|^{1-1/(2p')}$, which appears in the case $p' > 2$, is negligible. This proves the first statement of this theorem.

The amendment concerning the rate of convergence of $\sup_{f \in F_{s,p,q}(K,A)} \mathbb{E}(\int_{\mathbb{R}^d} |f - \tilde{P}_j f|^{p'})^{1/p'}$ can be easily verified now. Since the support of f is contained in a bounded set, the integrability requirement for the dominating function h is satisfied because the requirement $\|f\|_{s,p,q} \leq K$ implies a uniform bound on the maximum norm of f . We only need that the mutual dependence between the variables $Z(s)$ and $Z(t)$ is as required in Condition 1 (C). This, however, follows by the assumptions of the amendment. \square

Proof of Theorem 3. We prove that the approximation error is in $\mathcal{O}\{(\zeta_{\min})^{-j}\}$; the claim follows then with an application of Theorem 5. Since the father and mother wavelets ψ_k are compactly supported on $[0, L]^d$, there are at most $(2L+1)^d$ wavelets not equal to zero for fixed $x \in \mathbb{R}^d$. Hence, for all $j \in \mathbb{Z}$ and $k \in \{1, \dots, |M|-1\}$

$$\int_{\mathbb{R}^d} \left| \sum_{\gamma \in \mathbb{Z}^d} v_{k,j,\gamma} \psi_{k,j,\gamma} \right|^{p'} d\lambda^d \leq (2L+1)^{dp'} \|\psi_k\|_{p'}^{p'} |M|^{j(p'/2-1)} \sum_{\gamma \in \mathbb{Z}^d} |v_{k,j,\gamma}|^{p'} = \mathcal{O}\{(\zeta_{\min})^{-jp'}\}.$$

Here we use the following bound on the wavelet coefficients $v_{k,\ell,\gamma}$:

$$\begin{aligned} |v_{k,j,\gamma}|^{p'} &\leq |M|^{-jp/2} \|\psi_k\|_1^{p'} \sup\{|f(x) - f(y)| : x, y \in \text{supp}(\psi_{k,j,\gamma})\}^{p'} \\ &\leq |M|^{-jp/2} \|\psi_k\|_1^{p'} \|\sup\{\tilde{h}\{M^{-j}(u + \gamma)\} : u \in [0, L]^d\}\| M^{-j} \|_2 \sqrt{dL}^{p'}. \end{aligned}$$

Thus, the approximation error is bounded above as follows:

$$\|f - P_j f\|_{p'} \leq \sum_{k=1}^{|M|-1} \sum_{\ell=j}^{\infty} \left\| \sum_{\gamma \in \mathbb{Z}^d} v_{k,\ell,\gamma} \psi_{k,\ell,\gamma} \right\|_{p'} = \mathcal{O}\{(\zeta_{\min})^{-j}\}.$$

This concludes the argument. \square

We prove now the statement concerning the rate of convergence of the hard thresholding density estimator.

Proof of Theorem 4. Write the approximation with respect to the j_1 th and j_0 th level as

$$Q_{j_0,j_1} f = P_{j_1} f = \sum_{\gamma \in \mathbb{Z}^d} \theta_{j_0,\gamma} \Phi_{j_0,\gamma} + \sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} v_{k,j,\gamma} \psi_{k,j,\gamma}.$$

We decompose the error as follows

$$\begin{aligned} \mathbb{E} \left\{ \|f - \tilde{Q}_{j_0,j_1} f\|_{p'}^{p'} \right\}^{1/p'} &\leq \|f - Q_{j_0,j_1} f\|_{p'} + \mathbb{E} \left\{ \left\| \sum_{\gamma \in \mathbb{Z}^d} (\hat{\theta}_{j_0,\gamma} - \theta_{j_0,\gamma}) \Phi_{j_0,\gamma} \right\|_{p'}^{p'} \right\}^{1/p'} \\ &\quad + \sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} \mathbb{E} \left\{ \left\| \sum_{\gamma \in \mathbb{Z}^d} \{\hat{v}_{k,j,\gamma} \mathbf{1}(|\hat{v}_{k,j,\gamma}| > \bar{\lambda}_j) - v_{k,j,\gamma}\} \psi_{k,j,\gamma} \right\|_{p'}^{p'} \right\}^{1/p'} \\ &\equiv J_1 + J_2 + J_3 \end{aligned} \quad (13)$$

and consider these three terms separately. We infer from Theorem 2 that the approximation error J_1 is at most $|M|^{-j_1 s'} \simeq |I_n|^{-\alpha}$ times a constant which only depends on the domain A , the parameters of the Besov space, the wavelets ψ_k and the Besov norm $\|f\|_{s,p,\infty} \leq \|f\|_{s,p,q}$; for its exact value see (12).

For linear estimation error J_2 , we use Theorem 5. So, $J_2 \leq C |M|^{j_0/2} / |I_n|^{1/2} \simeq |I_n|^{-1/2}$. We consider the nonlinear details term in the estimation error which is the third term on the RHS of (13) and which constitutes the main error. It can be decomposed and bounded above as follows

$$\begin{aligned} J_3 &\leq (2L+1)^{d(p'-1)/p'} \sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} |M|^{j(1/2-1/p')} \|\psi_k\|_{p'}^{p'} \\ &\quad \times \left[\left[\sum_{\gamma \in \mathbb{Z}^d} |v_{k,j,\gamma}|^{p'} \mathbf{1}(|v_{k,j,\gamma}| \leq 2\bar{\lambda}_j) + \left\{ \sum_{\gamma \in \mathbb{Z}^d} \Pr(|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}| > \bar{\lambda}_j) |v_{k,j,\gamma}|^{p'} \right\} \right]^{1/p'} \right]^{1/p'} \\ &\quad + \left[\sum_{\gamma \in \mathbb{Z}^d} \mathbb{E} \left\{ |\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}|^{p'} \mathbf{1}(|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}| > \bar{\lambda}_j/2) \right\} \right]^{1/p'} \\ &\quad + \left[\sum_{\gamma \in \mathbb{Z}^d} \mathbb{E} \left\{ |\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}|^{p'} \mathbf{1}\{|v_{k,j,\gamma}| > \bar{\lambda}_j/2\} \right\} \right]^{1/p'} \right]^{1/p'}. \end{aligned} \quad (14)$$

We derive the rates of convergence for each term in (14) separately; many techniques are quite similar to the classical proof given by [19]. The first error in (14) is the dominating error. If $p' > p$, it can be bounded above as

$$\begin{aligned} \sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} |M|^{j(1/2-1/p')} \left\{ \sum_{\gamma \in \mathbb{Z}^d} |v_{k,j,\gamma}|^p (2\bar{\lambda}_j)^{p'-p} \mathbf{1}(|v_{k,j,\gamma}| \leq 2\bar{\lambda}_j) \right\}^{1/p'} \\ \leq \sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} |M|^{j(1/2-1/p')} (2\bar{\lambda}_j)^{(p'-p)/p'} |M|^{-j(s+1/2-1/p)p/p'} \|f\|_{s,p,\infty}^{p/p'} \\ \leq C \|f\|_{s,p,\infty}^{p/p'} |I_n|^{-(p'-p)/(2p')} \sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} j^{(p'-p)/(2p')} |M|^{-je/p'}. \end{aligned} \quad (15)$$

If $\varepsilon \neq 0$, (15) is bounded above by

$$|I_n|^{-(p'-p)/(2p')} |M|^{\max(-j_0\varepsilon/p', -j_1\varepsilon/p')} j_1^{(p'-p)/(2p')} \simeq |I_n|^{-\alpha} (\ln|I_n|)^{(p'-p)/2p'}.$$

If $\varepsilon = 0$, (15) is bounded above by $|I_n|^{-(p'-p)/(2p')} (j_1 - j_0) j_1^{(p'-p)/(2p')} \simeq |I_n|^{-\alpha} (\ln|I_n|)^{(3p'-p)/2p'}$.

We treat the first error term in (14) in the case $p \geq p'$. We have $\varepsilon > 0$ and $s = s'$. Moreover, the density has bounded support. We find in this case

$$\sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} |M|^{j(1/2-1/p')} \|v_{k,j,\cdot}\|_{\ell^{p'}} \leq C_A \|f\|_{s,p,\infty} \sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} |M|^{-js}, \quad (16)$$

where C_A is the constant which depends on the support of f and which is introduced in the proof of Theorem 2. Consequently, this last inequality behaves as $|M|^{-j_0s} = |I_n|^{-s/(1+2s)} = |I_n|^{-\alpha}$.

For the remaining three errors from (14), we need two bounds. First, we prove that given the growth rate of j_1

$$\sup_{\gamma \in \mathbb{Z}^d} E\{|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}|^{p'}\}^{1/p'} \leq \sup_{\gamma \in \mathbb{Z}^d} E\{|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}|^{2p'}\}^{1/2p'} \leq C |I_n|^{-1/2}. \quad (17)$$

We know from Lemma 3 that the leftmost expectation of (17) is bounded above by

$$C \left(|I_n|^{-1/2} + \frac{|M|^{j_1/2} q_n^N}{|I_n|} + \frac{|M|^{j_1/2}}{|I_n|^{c_1 B/(2p')}} \right)$$

where $q_n = B \ln|I_n|$ for a $B > 0$ arbitrary but fixed and where the constant C depends on B . Hence, if we choose B sufficiently large, namely, $B = \{2p' + 1 + \alpha/(2s')\}/c_1$, it only remains to show that the term in the middle is negligible. It corresponds to $|I_n|^{\alpha/(2s')-1} (\ln|I_n|)^N$.

Consequently, we only need that $1 - \alpha/s' > 0$. If $\varepsilon < 0$, then $1 - \alpha/s' = 2(sp - 1)/(p + 2(sp - 1)) > 0$ because by assumption $sp > 1$. Moreover, if $\varepsilon \geq 0$, we use that $1/p' \geq 1/(2sp + p)$. The relation $1 - \alpha/s' > 0$ is equivalent to $s'(1 + 2s) > s$. Now $s' \geq s + 1/(2sp + p) - 1/p$. Thus, $s'(1 + 2s) \geq s(2s + 1 - 2/p) > s$, where we use again that $s > 1/p$. All in all, we find that (17) is true.

Second, we show that $\Pr(|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}| > \bar{\lambda}_j)^{1/(2p')}$ vanishes at a rate $|I_n|^{-C_p K_0^2}$ which is negligible given the choice of K_0 , which is defined below in (19). We infer from Lemma 2 that this probability can be bounded above by

$$2 \exp \left\{ -\frac{|I_n| \bar{\lambda}_j^2 / (2p')}{A_2 + A_1 q_n^N |M|^{j/2} \bar{\lambda}_j} \right\} + A_3 \left(\frac{|M|^{j/2}}{\bar{\lambda}_j |I_n|^{c_1 B - 1/2}} \right)^{1/(2p')}$$

for certain constants A_1, \dots, A_3 independent of I_n and j . So it remains to compute the asymptotics of the following three terms:

$$\begin{aligned} \frac{|I_n| \bar{\lambda}_j^2}{2p' A_2} &\geq \frac{K_0^2 j_0}{2p' A_2} \simeq \frac{K_0^2 (1 - 2\alpha)}{(2p' \ln|M|)} \ln|I_n| \\ \frac{|I_n| \bar{\lambda}_j^2}{q_n^N |M|^{j/2} \bar{\lambda}_j} &\geq CK_0 |I_n|^{(1-\alpha/s')/2} \ln|I_n|^{1/2-N} \\ \frac{|M|^{j/(4p')}}{\bar{\lambda}_j^{1/(2p')} |I_n|^{(c_1 B - 1/2)/(2p')}} &\leq \left[K_0^{-1} j_1^{-1/2} |I_n|^{-\{c_1 B - 1 - \alpha/(2s')\}} \right]^{1/(2p')}. \end{aligned} \quad (18)$$

We see that the error on the second line of (18) is negligible because $1 - \alpha/s' > 0$. The error on the third line vanishes at a rate greater than $|I_n|^{-1}$ and is negligible as well. Hence, the choice

$$K_0^2 \simeq p' \ln|M|/(1 - 2\alpha) \quad (19)$$

implies that the probability in question decays at a rate of at least $|I_n|^{-1}$, i.e.,

$$\Pr(|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}| > \bar{\lambda}_j)^{1/(2p')} \leq C|I_n|^{-1}. \quad (20)$$

We use the norm inequalities in the $\ell^{p'}$ -spaces in both cases $p' \geq p$ and $p' < p$ to bound the second error in (14):

$$\begin{aligned} & \sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} |M|^{j(1/2-1/p')} \left\{ \sum_{\gamma \in \mathbb{Z}^d} \Pr(|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}| > \bar{\lambda}_j) |v_{k,j,\gamma}|^{p'} \right\}^{1/p'} \\ & \leq CC_A \sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} |M|^{j(1/2-1/p')} |M|^{j(1/p'-1/p)^+} \|v_{k,j,\cdot}\|_{\ell^p} \Pr(|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}| > \bar{\lambda}_j)^{1/p'} \\ & \leq CC_A \|f\|_{s,p,\infty} \sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} |M|^{-js'} \Pr(|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}| > \bar{\lambda}_j)^{1/p'}. \end{aligned}$$

Consequently, this second error is negligible if we use the result from (20).

For the third error in (14) we need the estimate from Proposition 1, viz.

$$\begin{aligned} \sum_{\gamma \in \mathbb{Z}^d} E(|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}|^{2p'})^{1/2} & \leq \sum_{\gamma \in \mathbb{Z}^d} E(|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}|^2)^{1/4} E(|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}|^{2(2p'-1)})^{1/4} \\ & \leq C_p \{(\|h^{1/8}\|_1 + \|\tilde{h}^{1/2}\|_1) \|\psi_k\|_\infty^{1/2} |M|^j |I_n|^{-1/4}\} |I_n|^{-(2p'-1)/4}. \end{aligned}$$

Now we obtain, using Hölder's inequality in both cases $p' \geq p$ and $p' < p$:

$$\sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} |M|^{j(1/2-1/p')} \left(\sum_{\gamma \in \mathbb{Z}^d} E(|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}|^{2p'})^{1/2} \Pr(|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}| > \bar{\lambda}_j/2)^{1/2} \right)^{1/p'} \leq C(j_1 - j_0) |I_n|^{-(1-\alpha/s')-1/2}.$$

Consequently, this error is negligible.

The fourth error in (14) can be treated similar. We use that $E(|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}|^{p'})^{1/p'} \leq C|I_n|^{-1/2}$ from Eq. (17). If $p' > p$,

$$\begin{aligned} & \sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} |M|^{j(1/2-1/p')} \left[\sum_{\gamma \in \mathbb{Z}^d} E\{|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}|^{p'} \mathbf{1}(|v_{k,j,\gamma}| > \bar{\lambda}_j/2)\} \right]^{1/p'} \\ & \leq C \sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} |M|^{j(1/2-1/p')} |I_n|^{-1/2} \|v_{k,j,\cdot}\|_{\ell^p}^{p/p'} (\bar{\lambda}_j/2)^{-p/p'} \\ & \leq C \|f\|_{s,p,\infty}^{p/p'} |I_n|^{-(p'-p)/(2p')} \sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} |M|^{-js/p'} j^{-p/(2p')}. \end{aligned} \quad (21)$$

Note that (21) is asymptotically less than the first nonlinear details term given in (15) and can be neglected. In the case $p' \leq p$, this error term can be bounded above by

$$\|f\|_{s,p,\infty} \sum_{k=1}^{|M|-1} \sum_{j=j_0}^{j_1-1} |M|^{-js} j^{-1/2}$$

times a constant. This follows similarly as the derivation of (21). In particular, this error is negligible too, when compared to the first details term in the case $p' \leq p$, see (16).

The amendment concerning the uniform convergence follows along the same lines as in the case for the linear estimator, see the proof of Theorem 2. This finishes the proof. \square

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Appendix A. Exponential and moment inequalities for dependent sums

Proposition 1. Assume the random field Z to satisfy Condition 1.

1. If [Condition 1](#) (C) is satisfied with $a = 1$, then for all $j \in \mathbb{Z}$

$$\left[\sum_{\gamma \in \mathbb{Z}^N} \mathbb{E} \{ |\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}|^2 \} \right]^{1/2} \leq C_1 L^{d/2} 2^{d/2} \|\Phi\|_\infty \{ \|h^{1/2}\|_1^{1/2} + \|\tilde{h}\|_2 \} \frac{|M|^{j/2}}{|I_n|^{1/2}}.$$

2. If [Condition 1](#) (C) is satisfied with $a = 2$, then for all $j \in \mathbb{Z}$

$$\sum_{\gamma \in \mathbb{Z}^N} \mathbb{E} \{ |\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}|^2 \}^{1/2} \leq C_2 L^{d/2} 2^d \|\Phi\|_\infty \{ \|h^{1/4}\|_1 + \|\tilde{h}\|_1 \} \frac{|M|^j}{|I_n|^{1/2}}.$$

3. If [Condition 1](#) (C) is satisfied with $a = 4$, then for all $j \in \mathbb{Z}$

$$\sum_{\gamma \in \mathbb{Z}^N} \mathbb{E} \{ |\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}|^2 \}^{1/4} \leq C_4 L^{d/4} 2^d \|\Phi\|_\infty^{1/2} \{ \|h^{1/8}\|_1 + \|\tilde{h}^{1/2}\|_1 \} \frac{|M|^j}{|I_n|^{1/4}}.$$

In all cases the constants $C_1, C_2, C_4 \in \mathbb{R}_+$ do not depend on $n \in \mathbb{N}_+^N$. They depend on the bound of the mixing coefficients determined by the numbers c_0 and c_1 and on the data dimension d . Moreover, the same results are true if we replace $\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}$ by $\hat{\nu}_{k,j,\gamma} - \nu_{k,j,\gamma}$ for all $k \in \{1, \dots, |M|\}, j \in \mathbb{Z}$.

Proof of Proposition 1. We only prove the statement concerning the coefficients $\theta_{j,\gamma}$ and assume without loss of generality that $j > 0$. We begin with the following decomposition of the variance:

$$\begin{aligned} \mathbb{E} \{ |\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}|^2 \} &\leq |I_n|^{-2} \sum_{s \in I_n} \Phi_{j,\gamma}^2 \{Z(s)\} + |I_n|^{-2} \sum_{\substack{s, t \in I_n, \\ \|s-t\| > |M|^{j/N}}} \text{cov}[\Phi_{j,\gamma} \{Z(s)\}, \Phi_{j,\gamma} \{Z(t)\}] \\ &\quad + |I_n|^{-2} \sum_{\substack{s, t \in I_n, \\ \|s-t\| \leq |M|^{j/N}}} \text{cov}[\Phi_{j,\gamma} \{Z(s)\}, \Phi_{j,\gamma} \{Z(t)\}]. \end{aligned} \quad (\text{A.1})$$

We easily find that the first summand in (A.1) is at most

$$\frac{\|\Phi\|_\infty^2}{|I_n|} \int_{\mathbb{R}^d} \mathbf{1}(\text{supp } \Phi(\cdot - \gamma)) f(M^{-j}y) dy \leq \frac{\|\Phi\|_\infty^2}{|I_n|} \int_{\mathbb{R}^d} \mathbf{1}(\text{supp } \Phi(\cdot - \gamma)) h(M^{-j}y) dy. \quad (\text{A.2})$$

Consider the second summand, here we apply the inequality of [15] to bound the covariance by the fourth moments times the mixing coefficient. We obtain the upper bound

$$\begin{aligned} |I_n|^{-2} \sum_{\substack{s, t \in I_n, \\ \|s-t\| > |M|^{j/N}}} \mathbb{E}[\Phi_{j,\gamma}^4 \{Z(s)\}]^{1/2} \alpha(\|s-t\|)^{1/2} \\ \leq \frac{|M|^{j/2} \|\Phi\|_\infty^2}{|I_n|^2} \sum_{\substack{s, t \in I_n, \\ \|s-t\| > |M|^{j/N}}} \left[\int_{\mathbb{R}^d} \mathbf{1}(\text{supp } \Phi(\cdot - \gamma)) f(M^{-j}y) dy \right]^{1/2} \alpha(\|s-t\|)^{1/2} \\ \leq C \frac{\|\Phi\|_\infty^2}{|I_n|} \left[\int_{\mathbb{R}^d} \mathbf{1}(\text{supp } \Phi(\cdot - \gamma)) h(M^{-j}y) dy \right]^{1/2} \sum_{k > |M|^{j/N}} k^{2N-1} \alpha(k)^{1/2}, \end{aligned} \quad (\text{A.3})$$

where we use in the last inequality that $\sum_{s, t \in I_n, \|s-t\| > |M|^{j/N}} \alpha(\|s-t\|)^{1/2} \leq C |I_n| \sum_{k > |M|^{j/N}} k^{N-1} \alpha(k)^{1/2}$ for a constant C and the fact that in this case $|M|^{j/2}$ is less than $k^{N/2} \leq k^N$.

The third summand can be bounded above with the help of the requirement on the joint densities, by assumption we have for all locations $s, t \in \mathbb{Z}^N$ that

$$|f_{Z(s), Z(t)}(z_1, z_2) - f(z_1)f(z_2)| \leq \tilde{h}(z_1)\tilde{h}(z_2)$$

for a non-increasing radial function \tilde{h} . Consequently, we obtain

$$\begin{aligned} |I_n|^{-2} \sum_{\substack{s, t \in I_n, \\ \|s-t\| \leq |M|^{j/N}}} \int_{\mathbb{R}^d \times \mathbb{R}^d} |M|^j |\Phi(M^j z_1 - \gamma)| |\Phi(M^j z_2 - \gamma)| |f_{Z(s), Z(t)}(z_1, z_2) - f(z_1)f(z_2)| dz_1 dz_2 \\ \leq |I_n|^{-1} \|\Phi\|_\infty^2 \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}(\text{supp } \Phi(\cdot - \gamma) \times \text{supp } \Phi(\cdot - \gamma)) (z_1, z_2) \tilde{h}(M^{-j}z_1) \tilde{h}(M^{-j}z_2) dz_1 dz_2. \end{aligned} \quad (\text{A.4})$$

Note that we have used the relation $\sum_{s, t \in I_n, \|s-t\| \leq |M|^{j/N}} |M|^{-j} \leq |I_n|$ in the derivation of the last inequality.

It remains to bound the sum of the variances $E(|\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}|^2)$, respectively the sum of the corresponding standard deviations, respectively the sum of the square root of the standard deviations. We use the following concept on the integrals from (A.4), (A.2) and (A.3): the support of $\Phi(\cdot - \gamma)$ is the cube $[\gamma, \gamma + Le_N]$. Let y_γ^* be among the points y in this cube such that $M^{-j}y$ is nearest to the origin, i.e., y_γ^* satisfies $\|M^{-j}y_\gamma^*\|_\infty = \inf\{\|M^{-j}y\|_\infty : y \in [\gamma, \gamma + Le_N]\}$. Then we have with the properties of the non-increasing radial functions h and \tilde{h} and for $a \in \{1, 2, 4, 8\}$,

$$\sum_{\gamma \in \mathbb{Z}^d} \left[\int_{\mathbb{R}^d} \mathbf{1}_{\{\text{supp } \Phi(\cdot - \gamma)\}} h(M^{-j}z) dz \right]^{1/a} \leq L^{d/a} \sum_{\gamma \in \mathbb{Z}^d} h(M^{-j}y_\gamma^*)^{1/a} \leq C L^{d/a} 2^d \|h^{1/a}\|_1 |M|^j,$$

where the constant C depends on the data dimension d . The factor $|M|^j$ in the last inequality is due to a change of variables. Similarly, we obtain for the integral from (A.4)

$$\sum_{\gamma \in \mathbb{Z}^d} \left[\int_{\mathbb{R}^d} \mathbf{1}_{\{\text{supp } \Phi(\cdot - \gamma)\}}(z) \tilde{h}(M^{-j}z) \right]^{2/a} \leq C L^{2d/a} 2^d \|\tilde{h}^{2/a}\|_1 |M|^j.$$

This finishes the proof. \square

Lemma 2. Assume the random field Z to satisfy Condition 1. Set $q_n = B \ln |I_n|$ for some $B > 0$. Then there are positive constants A_1, A_2, A_3 such that for all $j \in \mathbb{Z}$ and $\gamma \in \mathbb{Z}^N$

$$\Pr(|\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}| \geq x) \leq \begin{cases} 2 \exp\left(-\frac{|I_n|x^2}{A_2 + 2^N A_1 q_n^N |M|^{j/2} x/3}\right) + A_3 \frac{2^N |M|^{j/2}}{x |I_n|^{c_1 B}} & \text{for } x \leq A_1 |M|^{j/2}, \\ 2 \exp\left(-\frac{|I_n|x^2}{A_2 + 2^N A_1 q_n^N |M|^{j/2} x/3}\right) & \text{for } x > A_1 |M|^{j/2}. \end{cases}$$

Here the constant $c_1 > 0$ is due to the bound on the mixing coefficients and guaranteed by Condition 1. The same result is also true for $\Pr(|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}| \geq x)$ for all $k \in \{1, \dots, |M|\}, j \in \mathbb{Z}$ and $\gamma \in \mathbb{Z}^N$.

Proof. We use Lemma 4.6 from [36], we only have to replace the factor $2^{j/2}$ by $|M|^{j/2}$. We use that for a rectangular set $\tilde{I} \subseteq \mathbb{Z}^N$ and in case that Condition 1 is satisfied, it is true that for all k, j, γ

$$E \left[\left[\sum_{s \in \tilde{I}} \Psi_{k,j,\gamma}(Z_s) - E\{\Psi_{k,j,\gamma}(Z_s)\} \right]^2 \right] \leq C \|\Psi_{k,j,\gamma}\|_\infty^2 L^{2d} \left(\|h\|_\infty + \|h\|_\infty^{1/2} + \|\tilde{h}\|_\infty^2 \right) |\tilde{I}|,$$

where the constant C only depends on the lattice dimension N and the mixing coefficients.

Moreover, we bound the probability $P_2(x)$ from Eqs. (A.4) and (A.7) in Lemma 4.6 of [36] with the maximum norm of the functions, which is $|M|^{j/2}$ times a constant. This yields the result. \square

Lemma 3. Let $q \geq 2$ and $B > 0$, set $q_n = B \ln |I_n|$. Then it is true that

$$E(|\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}|^q) \leq C_q \left\{ |I_n|^{-q/2} + \left(\frac{|M|^{j/2} q_n^N}{|I_n|} \right)^q + \frac{|M|^{jq/2}}{|I_n|^{c_1 B}} \right\},$$

where the constant C_q depends on B . The same relation is true for $E[|\hat{v}_{k,j,\gamma} - v_{k,j,\gamma}|^q]$.

Proof. We use Lemma 2. Define $x^* = \frac{3A_2}{2^N A_1} (q_n^N |M|^{j/2})^{-1}$. Then

$$\begin{aligned} E(|\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}|^q) &= q \int_0^\infty x^{q-1} \Pr(|\hat{\theta}_{j,\gamma} - \theta_{j,\gamma}| \geq x) dx \\ &\leq 2q \int_0^\infty x^{q-1} \left\{ \exp\left(-\frac{|I_n|x^2}{A_2 + 2^N A_1 q_n^N |M|^{j/2} x/3}\right) + \mathbf{1}_{\{x \leq A_1 |M|^{j/2}\}} A_3 \frac{|M|^{j/2}}{x |I_n|^{c_1 B}} \right\} dx \\ &\leq 2q \int_0^{x^*} x^{q-1} \exp\left(-\frac{|I_n|x^2}{2A_2}\right) dx + 2q \int_{x^*}^\infty x^{q-1} \exp\left(-\frac{|I_n|x^2}{2^{N+1} A_1 q_n^N |M|^{j/2} x}\right) dx \\ &\quad + 2q A_3 \int_0^{A_1 |M|^{j/2}} x^{q-1} \frac{2^N |M|^{j/2}}{x |I_n|^{c_1 B}} dx. \end{aligned} \tag{A.5}$$

The first integral in (A.5) is bounded above by

$$q \left(\frac{2A_2}{|I_n|} \right)^{q/2} \gamma \left(\frac{q}{2}, C \frac{|I_n|}{q_n^N |M|^{j/2}} \right) \leq C |I_n|^{-q/2},$$

here we (temporarily) denote the lower incomplete gamma function by $\gamma(\cdot, \cdot)$. Likewise, the second integral is bounded above by (modulo a constant)

$$\left(\frac{q_n^N |M|^{j/2}}{|I_n|} \right)^q \Gamma \left(q, C \frac{|I_n|}{q_n^{2N} |M|^j} \right) \leq C \left(\frac{q_n^N |M|^{j/2}}{|I_n|} \right)^q.$$

Here $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function.

The last integral in (A.5) is at most $|M|^{jq/2} |I_n|^{-c_1 B}$ (times a constant). This finishes the proof. \square

Appendix B. The question of normalization

This appendix contains a result on the convergence of the normalized density estimator: let $p \geq 1$ and $(f_k : k \in \mathbb{N}_+)$ be a sequence of density projections onto (increasing) subspaces of $L^p(\lambda^d) \cap L^2(\lambda^d)$. Furthermore, let $(\tilde{f}_k : k \in \mathbb{N}_+) \subseteq L^p(\lambda^d \otimes \text{Pr}) \cap L^2(\lambda^d \otimes \text{Pr})$ be a corresponding sequence of density estimators. Define the normalized density estimator

$$\hat{f}_k = \frac{1}{S_k} \tilde{f}_k^+, \quad \text{where } S_k = \int_{\mathbb{R}^d} \tilde{f}_k^+ d\lambda^d \quad (\text{B.1})$$

is the normalizing constant. We have in this case the general result.

Proposition 2 (*L^p -Convergence of \hat{f}_k*). *Let $p \in [1, \infty)$ and $f \in L^p(\lambda^d)$ be a density. If the estimator \tilde{f}_k converges to f in $L^p(\lambda^d)$ a.s. and in $L^1(\lambda^d)$ a.s., then \hat{f}_k converges to f in $L^p(\lambda^d)$ a.s. Furthermore, let \tilde{f}_k converge to f in $L^p(\lambda^d \otimes \text{Pr})$ and in $L^1(\lambda^d \otimes \text{Pr})$; additionally, if $p > 1$, let $\liminf_{k \rightarrow \infty} \|S_k\|_{L^\infty(\text{Pr})} \geq \delta > 0$. Then the estimator \hat{f}_k converges to f in $L^p(\lambda^d \otimes \text{Pr})$.*

Proof of Proposition 2. It remains to prove the desired convergence for the term $|\hat{f}_k - \tilde{f}_k|^p$:

$$\int_{\mathbb{R}^d} |\hat{f}_k - \tilde{f}_k|^p d\lambda^d \leq 2^p \int_{\mathbb{R}^d} (\tilde{f}_k^-)^p d\lambda^d + 2^p \left| 1 - \frac{1}{S_k} \right|^p \int_{\mathbb{R}^d} (\tilde{f}_k^+)^p d\lambda^d. \quad (\text{B.2})$$

Consider the first term in (B.2),

$$\int_{\mathbb{R}^d} |\tilde{f}_k^-|^p d\lambda^d \leq 2^p \int_{\mathbb{R}^d} |f - \tilde{f}_k|^p d\lambda^d + 2^p \int_{\mathbb{R}^d} f^p \mathbf{1}_{\{f < f - \tilde{f}_k\}} d\lambda^d. \quad (\text{B.3})$$

An application of Lebesgue's dominated convergence theorem shows that the second error in (B.3) converges to zero both in the mean and a.s.: indeed we define, for $\varepsilon_1, \varepsilon_2 \in (0, 1)$,

$$L(\varepsilon_1) = \inf \left\{ a \in \mathbb{R}_+ : \int_{[-a, a]^d} f^p d\lambda^d \geq 1 - \varepsilon_1 \right\} < \infty, \quad K(\varepsilon_1) = [-L(\varepsilon_1), L(\varepsilon_1)]^d \quad \text{and} \quad A(\varepsilon_2) = \{f > \varepsilon_2\}.$$

We get

$$\int_{\{f < f - \tilde{f}_k\}} f^p d\lambda^d \leq \varepsilon_1 + \int_{K(\varepsilon_1)} f^p \mathbf{1}_{\{f < f - \tilde{f}_k\}} d\lambda^d \leq \varepsilon_1 + \int_{K(\varepsilon_1) \cap A(\varepsilon_2)} f^p \mathbf{1}_{\{\varepsilon_2 < |f - \tilde{f}_k|\}} d\lambda^d + \varepsilon_2^p \lambda^d(K(\varepsilon_1)).$$

If $|f - \tilde{f}_k| \rightarrow 0$ in $L^1(\lambda^d \otimes \text{Pr})$ and $f \in L^p(\lambda^d)$, then

$$\limsup_{k \rightarrow \infty} \mathbb{E} \left\{ \int_{K(\varepsilon_1) \cap A(\varepsilon_2)} f^p \mathbf{1}_{\{\varepsilon_2 < |f - \tilde{f}_k|\}} d\lambda^d \right\} = 0$$

with Lebesgue's dominated convergence theorem applied to the product measure $\lambda^d \otimes \text{Pr}$. In the same way, if $|f - \tilde{f}_k| \rightarrow 0$ in $L^1(\lambda^d)$ on a set $\Omega_0 \in \mathcal{A}$ with $\text{Pr}(\Omega_0) = 1$ and $f \in L^p(\lambda^d)$, then

$$\limsup_{k \rightarrow \infty} \int_{K(\varepsilon_1) \cap A(\varepsilon_2)} f^p \mathbf{1}_{\{\varepsilon_2 < |f - \tilde{f}_k|\}} d\lambda^d = 0$$

with Lebesgue's dominated convergence theorem applied to λ^d for each $\omega \in \Omega_0$. In addition, this implies $S_k \rightarrow 1$ in the mean and a.s. This finishes the computations on the first term in (B.2). We can bound the second term in (B.2) as

$$\left| 1 - \frac{1}{S_k} \right|^p \int_{\mathbb{R}^d} (\tilde{f}_k^+)^p d\lambda^d \leq 2^p \left| 1 - \frac{1}{S_k} \right|^p \int_{\mathbb{R}^d} f^p d\lambda^d + 2^p \left| 1 - \frac{1}{S_k} \right|^p \int_{\mathbb{R}^d} |\tilde{f}_k - f|^p d\lambda^d. \quad (\text{B.4})$$

The error $|1 - 1/S_k|$ on the RHS of (B.4) converges to zero almost surely by the continuous mapping theorem. In particular, the RHS of (B.4) converges to zero almost surely. We come to the convergence in mean. Again by the continuous mapping theorem, the first term on the RHS of (B.4) converges to zero in probability. Furthermore, there is a $k^* \in \mathbb{N}_+$ such that for $k \geq k^*$ this term is bounded above by $2^p(1 + 1/\delta)^p \|f\|_p^p$. Hence, the family $\{|1 - 1/S_k|^p : k \geq k^*\}$ is uniformly integrable and this factor converges to zero in the mean. In addition, the first factor in the second term on the RHS of (B.4) is bounded for all $k \geq k^*$ and, thus, the whole term converges to zero in the mean. \square

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