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Multivariate Liouville Distributions, III

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We present a panoply of results on partial orderings for the Liouville distributions, including sufficient conditions for two Liouville vectors to be comparable under the stochastic, convex, concave, and Laplace transform orderings. Further, we derive partial orderings for the order statistics and spacings from certain exchangeable Liouville distributions. As applications to reliability theory, we obtain stochastic orderings for $N(t)$ and bounds for $R_k(t)$, the number of components working at time $t \geqslant 0$ and the reliability function, respectively, for a "k-out-of-n" system consisting of components whose lifetimes have a joint Liouville distribution. When the component lifetimes are distributed as a mixture of independent, identically distributed exponential random variables, we derive some results for a conjecture of Lefevre and Malice (*J. Appl. Prob.* **26** (1989), 202–208) on variation comparisons for $R_k(t)$ as the mixing distribution is varied. Following a suggestion and using the methods of Diaconis and Perlman (in *Topics in Statistical Dependence*, IMS Lecture Notes, 1991), we compare the cumulative distribution functions of two linear combinations of an exchangeable Liouville vector when the first vector of coefficients majorizes the second vector of coefficients. We derive sufficient conditions under which the two distribution functions cross exactly once, and obtain bounds for the location of the unique crossing point.

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1. Introduction

An absolutely continuous random vector $(X_1, ..., X_n)$, with positive components, has a multivariate Liouville distribution if its joint density function is proportional to

$$f\left(\sum_{i=1}^{n} x_i\right) \prod_{i=1}^{n} x_i^{a_i-1}, \tag{1.1}$$

where $a_i > 0$, $i = 1, ..., n$, and $f: \mathbf{R}_+ \to \mathbf{R}_+$ is such that (1.1), after normalization, is a density function. Whenever $(X_1, ..., X_n)$ has the density function (1.1), we denote this by $(X_1, ..., X_n) \sim L_n[f; a_1, ..., a_n]$.

The Liouville distributions arise in a variety of statistical and probabilistic contexts. These include multivariate majorization (Diaconis and Perlman [3], Marshall and Olkin [12, p. 308]); generalizations of the Dirichlet and inverted Dirichlet distributions, total positivity and correlation inequalities (Gupta and Richards [5, 6]); and in statistical reliability theory (Gupta and Richards [7]). A comprehensive account of some applications and other aspects of these distributions is provided in the recent monograph by Fang *et al.* [4].

In this article, the main goal is to develop stochastic partial orderings for the Liouville distributions. Let us comment on several motivations for these results. In applications of the Liouville distributions to reliability theory [7], there arises the need for stochastic comparisons between two Liouville vectors $(X_1, ..., X_n) \sim L_n[f; a_1, ..., a_n]$ and $(Y_1, ..., Y_n) \sim L_n[g; b_1, ..., b_n]$. Generally, the exact evaluation of probabilities for $(X_1, ..., X_n)$ is difficult, especially when some of the a_i are non-integral. This problem is the main reason why, in [7], only the case of integral a_i could be treated in depth. However, if it can be proved that $(X_1, ..., X_n)$ is bounded stochastically by $(Y_1, ..., Y_n)$, where the function g is simpler than f, then it may be possible to derive numerical bounds for the probabilities involving $(X_1, ..., X_n)$.

Related problems arise in the work of Lefevre and Malice [10]. Suppose that $X_1, ..., X_n$ represent the random lifetimes of n non-renewable components in a "k-out-of-n" system, where k is fixed, $1 \leqslant k \leqslant n$. Denote by $N(t)$ the number of components working at time $t \geqslant 0$; then system reliability is measured by the function $R_k(t) = P[N(t) \geqslant k]$. If $(X_1, ..., X_n)$ is distributed as a mixture of independent exponential variables, then the results in [10] pertain to stochastic orderings of $(X_1, ..., X_n)$ through the mixing distribution. In the case when the exponential variables are also identically distributed, then $(X_1, ..., X_n)$ follows a Liouville distribution. As a consequence of our results we partially extend the results in [10], deriving stochastic orderings for $N(t)$ and bounds for $R_k(t)$, when

$X_1, \ldots, X_n$ is distributed as a mixture of independent, identically distributed (i.i.d.) gamma variables.

Further motivation for our work arises from the results of Shaked and Tong [15, 16]. In a study of models for dependent data in reliability theory and other applications, those authors introduce several partial orderings for dependent random variables and for order statistics. Here, we will determine conditions under which the Liouville distributions can be ordered using some of the partial orderings given in [15, 16].

We close the introduction by describing the layout of our results. In Section 2 we obtain sufficient conditions such that $(X_1, \ldots, X_n) \geqslant^{\text{st}} (Y_1, \ldots, Y_n)$; that is, for $(X_1, \ldots, X_n)$ to be stochastically greater than $(Y_1, \ldots, Y_n)$. In Section 3, we apply the orderings in Section 2 to derive comparisons for $N(t)$ and $R_k(t)$ in the Liouville case and to partially extend some results of [10] to the case when $(X_1, \ldots, X_n)$ is a mixture of i.i.d. gamma variables. Using the theory of variation diminishing transformations (Karlin [8]), we will also obtain results regarding a conjecture of Lefevre and Malice [10] on comparisons for the system reliability function, $R_k(t)$, as the mixing distribution is varied. In Section 4 we order the Liouville distributions using the partial orders of Shaked and Tong [15, 16]. Further, we obtain orderings for the spacings from Liouville distributions. Finally, in Section 5, we obtain some results for a conjecture of Diaconis and Perlman [3] as applied to the Liouville distributions. Suppose $(X_1, \ldots, X_n) \sim L_n[f; a, \ldots, a]$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ and $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$ are vectors with positive components. When $\boldsymbol{\theta}$ majorizes $\boldsymbol{\phi}$ [12], Diaconis and Perlman [3] suggested that, for certain functions f, the tail probabilities $F_{\boldsymbol{\theta}}(t) = P\,[\sum_{j=1}^{n} \theta_j X_j > t]$ possibly satisfy the *unique crossing conjecture*: The function $F_{\boldsymbol{\theta}}(t) - F_{\boldsymbol{\phi}}(t)$ changes sign exactly once on $\mathbf{R}_+$. We will show that the unique crossing conjecture is valid in some situations if the function $f(e^t)$ is log-concave on $\mathbf{R}$. Further, we will obtain bounds for the location of the unique crossing point.

2. Stochastic Orderings for Liouville Distributions

Throughout, all random vectors are assumed to be absolutely continuous with continuous density functions. We also use the monograph by Marshall and Olkin [12] as a general reference for the concepts and properties of partial orderings of random variables. To avoid pathologies, we assume throughout that all expectations exist and that all density functions are sufficiently smooth. The precise differentiability conditions required on the function f will always be evident from the context.

Suppose $(X_1, \ldots, X_n) \sim L_n[f; a_1, \ldots, a_n]$ and $(Y_1, \ldots, Y_n) \sim L_n[g; b_1, \ldots, b_n]$. In this section we derive conditions on the functions f, g and the parameters $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ such that $(X_1, \ldots, X_n) \geqslant^{\text{st}} (Y_1, \ldots, Y_n)$; that is,

$$E\psi(X_1, \ldots, X_n) \geqslant E\psi(Y_1, \ldots, Y_n) \tag{2.1}$$

for any function $\psi: \mathbf{R}_+^n \to \mathbf{R}$ such that ψ is monotone increasing (in each component), and for which the expectations exist. Our main result in this section is the following.

2.1. THEOREM. *Suppose* $(X_1, \ldots, X_n) \sim L_n[f; a_1, \ldots, a_n]$ *and* $(Y_1, \ldots, Y_n) \sim L_n[g; b_1, \ldots, b_n]$. *If*

(i) $a_j \geqslant b_j$, $j = 1, \ldots, n$, *and* $b_n \geqslant 1$; *and*

(ii) *the function* $f(x+t)/g(x)$ *is monotone increasing in* $x > 0$ *for any* $t \geqslant 0$; *then* $(X_1, \ldots, X_n) \geqslant^{\text{st}} (Y_1, \ldots, Y_n)$.

Further, if $a_j = b_j$, $j = 1, \ldots, n$, *then the same conclusion holds with* (ii) *replaced by*

(iii) *the function* f/g *is monotone increasing on* $\mathbf{R}_+$.

2.2. EXAMPLE. Suppose $f(t) = t^a e^{-\alpha t}$, $g(t) = e^{-\beta t}$, $t > 0$, where $a \geqslant 0$ and α, $\beta > 0$. Then $X_1, \ldots, X_n$ are correlated if $a > 0$ (cf. [12, p. 308]) and $Y_1, \ldots, Y_n$ are independent gamma variables. If condition (i) above holds and $\beta \geqslant \alpha$, then $(X_1, \ldots, X_n) \geqslant^{\text{st}} (Y_1, \ldots, Y_n)$. When $n = 1$ (in this case the assumption $b_n \geqslant 1$ is not necessary) and $a = 0$, we recover a well-known result [17, p. 7] on the stochastic ordering of gamma distributions.

2.3. EXAMPLE. Suppose $f(t) = (1+t)^{-a_{n+1}}$, $g(t) = (1+t)^{-b_{n+1}}$, $t > 0$, where a_{n+1}, $b_{n+1} > 0$. Then $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ follow inverted Dirichlet distributions $\mathrm{ID}(a_1, \ldots, a_n; a_{n+1})$ and $\mathrm{ID}(b_1, \ldots, b_n; b_{n+1})$, respectively. For x, $t \geqslant 0$,

$$\frac{\partial}{\partial x} \log \frac{f(x+t)}{g(x)} = \frac{(b_{n+1} - a_{n+1})(1+x) + b_{n+1} t}{(1+x)(1+x+t)}.$$

Hence if $a_j \geqslant b_j$, $j = 1, \ldots, n$, and $a_{n+1} \leqslant b_{n+1}$ then $f(x+t)/g(x)$ is monotone increasing in $x > 0$ for any $t \geqslant 0$. Then $(X_1, \ldots, X_n) \geqslant^{\text{st}} (Y_1, \ldots, Y_n)$.

The proof of Theorem 2.1 will be based on the following result [12, p. 485]: If the random vectors $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ satisfy

(i) $X_1 \geqslant^{\text{st}} Y_1$;

(ii) $X_j \mid \{X_1 = x_1, \ldots, X_{j-1} = x_{j-1}\} \geqslant^{\text{st}} Y_j \mid \{Y_1, \ldots, Y_{j-1} = y_{j-1}\}$, for all $x_1 \geqslant y_1, \ldots, x_{j-1} \geqslant y_{j-1}$, and for all $j = 2, \ldots, n$; then $(X_1, \ldots, X_n) \geqslant^{\text{st}} (Y_1, \ldots, Y_n)$.

In preparation for the proof of Theorem 2.1 we will need some preliminary notation and results. Recall [5, Proposition 4.1] that if $(X_1, ..., X_n) \sim L_n[f; a_1, ..., a_n]$ then $X_1 \sim L_n[W^a f; a_1]$, where $a = a_2 + \cdots + a_n$ and

$$W^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)\, dt, \qquad x > 0, \tag{2.2}$$

is the fractional derivative of f of order $\alpha > 0$. Further, the conditional random variable $X_j \mid \{X_1 = x_1, ..., X_{j-1} = x_{j-1}\} \sim L_1[f_j; a_j]$, where

$$f_j(t) = \frac{W^{a_{j+1} + \cdots + a_n} f(t + x_1 + \cdots + x_{j-1})}{W^{a_j + \cdots + a_n} f(x_1 + \cdots + x_{j-1})}, \qquad t > 0. \tag{2.3}$$

In particular, the conditional distribution (2.3) depends only on $x_1 + \cdots + x_{j-1}$.

2.4. LEMMA. *If $\alpha \geqslant \beta \geqslant 1$ and $f(x+t)/g(x)$ is monotone increasing in $x > 0$ for all $t \geqslant 0$, then the function $W^\alpha f / W^\beta g$ is monotone increasing on $\mathbf{R}_+$.*

Proof. Throughout, we let $t_+^\alpha = t^\alpha$ or 0 according as $t > 0$ or $t \leqslant 0$, respectively. First, assume that $\alpha > \beta$ and let $h(t) = W^{\alpha-\beta} f(t)$. Then by the semigroup property, $W^{\alpha+\beta} = W^\alpha W^\beta$, of the fractional integral operators, we have $W^\beta h(t) = W^\alpha f(t)$, $t > 0$. By applying the definition of the W^α operators and using the basic composition formula [8, p. 17], we have for $x_1 > x_2 > 0$,

$$\begin{aligned} \begin{vmatrix} W^\alpha f(x_1) & W^\alpha f(x_2) \\ W^\beta g(x_1) & W^\beta g(x_2) \end{vmatrix} &= \begin{vmatrix} W^\beta h(x_1) & W^\beta h(x_2) \\ W^\beta g(x_1) & W^\beta g(x_2) \end{vmatrix} \\ &= \int_0^\infty \int_0^{t_1} \det\left(\frac{(t_i - x_j)_+^{\beta-1}}{\Gamma(\beta)} \right) \\ &\quad \times \begin{vmatrix} h(t_1) & h(t_2) \\ g(t_1) & g(t_2) \end{vmatrix} dt_2\, dt_1. \end{aligned} \tag{2.4}$$

For $\beta \geqslant 1$, the function $t \mapsto t^{\beta-1}/\Gamma(\beta)$, $t > 0$, is PF_2 (a Pólya frequency function of order 2) [8, p. 332]; hence $\det((t_i - x_j)_+^{\beta-1}/\Gamma(\beta)) \geqslant 0$ for $\beta \geqslant 1$ whenever $x_1 > x_2 > 0$, $t_1 > t_2 > 0$. Next, we note that

$$\begin{aligned} \begin{vmatrix} h(t_1) & h(t_2) \\ g(t_1) & g(t_2) \end{vmatrix} &= g(t_2)\, W^{\alpha-\beta} f(t_1) - g(t_1)\, W^{\alpha-\beta} f(t_2) \\ &= \frac{1}{\Gamma(\alpha-\beta)} \int_0^\infty t^{\alpha-\beta-1} \begin{vmatrix} f(t+t_1) & f(t+t_2) \\ g(t_1) & g(t_2) \end{vmatrix} dt. \end{aligned} \tag{2.5}$$

Since $f(x+t)/g(x)$ is monotone increasing in $x>0$ for all $t\geqslant 0$, then $t_1>t_2$ implies $f(t_1+t)/g(t_1)\geqslant f(t_2+t)/g(t_2)$; hence the determinant in (2.5) is nonnegative.

When $\alpha=\beta$ we proceed as before. Then for $x_1>x_2$,

$$\begin{vmatrix} W^\beta f(x_1) & W^\beta f(x_2) \\ W^\beta g(x_1) & W^\beta g(x_2) \end{vmatrix}$$
$$= \int_0^\infty \int_0^{t_1} \det\left(\frac{(t_i-x_j)_+^{\beta-1}}{\Gamma(\beta)}\right) \begin{vmatrix} f(t_1) & f(t_2) \\ g(t_1) & g(t_2) \end{vmatrix}\, dt_2\, dt_1. \tag{2.6}$$

Since f/g is monotone increasing, the second determinant in (2.6) is nonnegative for $t_1>t_2$; and the first determinant has already been seen to be nonnegative. Therefore, $W^\alpha f(x)/W^\alpha g(x)$ is monotone increasing on $\mathbf{R}_+$. ∎

2.5. *Proof of Theorem* 2.1. Since $X_1 \sim L_1[W^{a_2+\cdots+a_n}f;\, a_1]$ and similarly for Y_1, then, ignoring any normalizing constants, the ratio of the marginal densities of X_1 and Y_1 is

$$\frac{t^{a_1-1}W^{a_2+\cdots+a_n}f(t)}{t^{b_1-1}W^{b_2+\cdots+b_n}g(t)} = t^{a_1-b_1}\,\frac{W^{a_2+\cdots+a_n}f(t)}{W^{b_2+\cdots+b_n}g(t)}.$$

Since $a_j\geqslant b_j$, $1, ..., n$, then $t\mapsto t^{a_1-b_1}$, $t>0$, is increasing. By Lemma 2.4, $W^{a_2+\cdots+a_n}f(t)/W^{b_2+\cdots+b_n}g(t)$ is monotone increasing, since $a_2+\cdots+a_n\geqslant b_2+\cdots+b_n$ and $b_n\geqslant 1$. Therefore the ratio of densities is monotone increasing on $\mathbf{R}_+$. By Lehmann [11, p. 74, Lemma 2], it follows that $X_1 \geqslant^{\text{st}} Y_1$.

Next suppose $2\leqslant j\leqslant n$, and $x_1, ..., x_{j-1}$ and $y_1, ..., y_{j-1}$ are fixed, where $x_1\geqslant y_1, ..., x_{j-1}\geqslant y_{j-1}$. Again ignoring normalizing constants, it follows from (2.3) that the ratio of the conditional densities of $X_j \mid \{X_1=x_1, ..., X_{j-1}=x_{j-1}\}$ and $Y_j\mid\{Y_1=y_1, ..., Y_{j-1}=y_{j-1}\}$ is

$$\frac{t^{a_j-1}W^{a_{j+1}+\cdots+a_n}f(t+\sum_{i=1}^{j-1}x_i)}{t^{b_j-1}W^{b_{j+1}+\cdots+b_n}g(t+\sum_{i=1}^{j-1}y_i)} = t^{a_j-b_j}\,\frac{W^{a_{j+1}+\cdots+a_n}f(t+u)}{W^{b_{j+1}+\cdots+b_n}g(t+v)}, \tag{2.7}$$

where $u=x_1+\cdots+x_{j-1}$ and $v=y_1+\cdots+y_{j-1}$. Since $u\geqslant v$ then, applying Lemma 2.4, we see that the ratio (2.7) is monotone increasing in $t>0$. Hence again by [11, p. 74, Lemma 2], we have $X_j\mid\{X_1=x_1, ..., X_{j-1}=x_{j-1}\} \geqslant^{\text{st}} Y_j\mid\{Y_1=y_1, ..., Y_{j-1}=y_{j-1}\}$, $j=2, ..., n$. Therefore $(X_1, ..., X_n)\geqslant^{\text{st}}(Y_1, ..., Y_n)$.

Finally, it remains to consider the case when $a_j=b_j$, $j=1, ..., n$. Here it follows from [5, Theorem 3.2(i)] that

$$(X_1, ..., X_n) =^{\text{st}} V_1(Z_1, ..., Z_n), \tag{2.8}$$

where $(Z_1, ..., Z_n)$ and V_1 are independent; $(Z_1, ..., Z_n)$ follows a (singular) Dirichlet distribution $D(a_1, ..., a_n)$ with density proportional to $\prod_{i=1}^n w_i^{a_i}$, on the simplex

$$\mathscr{S}_n = \{(w_1, ..., w_n) : w_i > 0,\ 1 \leqslant i \leqslant n,\ w_1 + \cdots + w_n = 1\},$$

and $V_1 \sim L_1[f; a_1 + \cdots + a_n]$. Similarly, $(Y_1, ..., Y_n) =^{\mathrm{st}} V_2(Z_1, ..., Z_n)$, where $(Z_1, ..., Z_n)$ is independent of $V_2 \sim L_1[g; b_1 + \cdots + b_n]$. If f/g is monotone increasing on $\mathbf{R}_+$ then the ratio of the densities of V_1 and V_2 is monotone increasing; hence, by [11, p. 74, Lemma 2], $V_1 \geqslant^{\mathrm{st}} V_2$. By [12, p. 484, Proposition 17.B.3], we have $(X_1, ..., X_n) =^{\mathrm{st}} V_1(Z_1, ..., Z_n) \geqslant^{\mathrm{st}} V_2(Z_1, ..., Z_n) =^{\mathrm{st}} (Y_1, ..., Y_n)$. In particular, we do not require the assumption $b_n \geqslant 1$. ∎

There are other partial orderings for which it is of interest to order the Liouville distributions; we refer to Stoyan [17, Section 1.10] for definitions of some of these orderings. In the case of the convex ($\leqslant_{\mathrm{c}}$), concave ($\leqslant_{\mathrm{cv}}$), $\leqslant_{\mathrm{D}}$, $\leqslant_{\mathrm{K}}$, and Laplace transform ($\leqslant_{\mathrm{L}}$) orderings (we use Stoyan's notations), we have the following consequence of Theorem 2.1 for the Liouville distributions.

2.6. Corollary. *Under the hypotheses of Theorem* 2.1,

(i) $(Y_1, ..., Y_n) \leqslant (X_1, ..., X_n)$, *where* $\leqslant$ *denotes* $\leqslant_{\mathrm{c}}$ *or* $\leqslant_{\mathrm{D}}$;

(ii) $(X_1, ..., X_n) \leqslant (Y_1, ..., Y_n)$, *where* $\leqslant$ *denotes* $\leqslant_{\mathrm{cv}}$, $\leqslant_{\mathrm{K}}$, *or* $\leqslant_{\mathrm{L}}$.

The proof of this result follows from the fact that $(X_1, ..., X_n) \geqslant^{\mathrm{st}} (Y_1, ..., Y_n)$ implies $(Y_1, ..., Y_n) \leqslant (X_1, ..., X_n)$, where $\leqslant$ denotes $\leqslant_{\mathrm{c}}$ or $\leqslant_{\mathrm{D}}$. For the orderings $\leqslant_{\mathrm{cv}}$, $\leqslant_{\mathrm{K}}$, or $\leqslant_{\mathrm{L}}$, the inequalities are reversed.

In closing this section, we obtain a criterion which is both necessary and sufficient such that $(X_1, ..., X_n) \geqslant_{\mathrm{L}} (Y_1, ..., Y_n)$; that is, $E(\exp(-\sum_{i=1}^n t_i X_i)) \leqslant E(\exp(-\sum_{i=1}^n t_i Y_i))$ for all $t_i > 0$, $i = 1, ..., n$. In stating this result, we use the notation

$$\hat{f}(t) = \int_{\mathbf{R}_+} e^{-tx} x^{n-1} f(x)\, dx, \qquad t > 0, \tag{2.9}$$

and for $n \geqslant 2$,

$$\hat{f}(t_1, ..., t_n) = \frac{\begin{vmatrix} t_1^{n-2} & t_2^{n-2} & \cdots & t_n^{n-2} \\ t_1^{n-3} & t_2^{n-3} & \cdots & t_n^{n-3} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \\ \hat{f}(t_1) & \hat{f}(t_2) & \cdots & \hat{f}(t_n) \end{vmatrix}}{\prod_{1 \leqslant i < j \leqslant n} (t_i - t_j)}, \tag{2.10}$$

where (2.10) is to be interpreted using l'Hôpital's rule in the case of any equalities among the t_i's. Also, we denote by W_t^{-a} the inverse of W_t^a, the fractional derivative of order a (in the variable t); if a is a nonnegative integer then $W_t^{-a} \equiv (-1)^a (\partial/\partial t)^a$.

2.7. PROPOSITION. *Suppose that* $(X_1, ..., X_n) \sim L_n[f; a_1, ..., a_n]$ *and* $(Y_1, ..., Y_n) \sim L_n[g; b_1, ..., b_n]$. *Then* $(X_1, ..., X_n) \geqslant_{\mathrm{L}} (Y_1, ..., Y_n)$ *if and only if*

$$\frac{1}{W^a f(0)} \prod_{i=1}^{n} W_{t_i}^{-(a_i-1)} \hat{f}(t_1, ..., t_n) - \frac{1}{W^b g(0)} \prod_{i=1}^{n} W_{t_i}^{-(b_i-1)} \hat{g}(t_1, ..., t_n) \leqslant 0, \tag{2.11}$$

for all $t_i > 0$, $i = 1, ..., n$, *where* $a = a_1 + \cdots + a_n$ *and* $b = b_1 + \cdots + b_n$.

Proof. By a straightforward calculation, we have $W_t^a e^{-tx} = x^{-a} e^{-tx}$; equivalently, $W_t^{-a} e^{-tx} = x^a e^{-tx}$, $t > 0$, $x > 0$. With $a = a_1 + \cdots + a_n$,

$$\begin{aligned} E \exp\left(-\sum_{i=1}^{n} t_i X_i\right) &= \frac{1}{W^a f(0)} \int_{\mathbf{R}_+^n} f\left(\sum_{i=1}^{n} x_i\right) \prod_{i=1}^{n} x_i^{a_i-1} e^{-t_i x_i} \, dx_i \\ &= \frac{1}{W^a f(0)} \prod_{i=1}^{n} W_{t_i}^{-(a_i-1)} \int_{\mathbf{R}_+^n} f\left(\sum_{i=1}^{n} x_i\right) \prod_{i=1}^{n} e^{-t_i x_i} \, dx_i \\ &= \frac{W^n f(0)}{W^a f(0)} \prod_{i=1}^{n} W_{t_i}^{-(a_i-1)} E \exp\left(-\sum_{i=1}^{n} t_i U_i\right), \end{aligned}$$

where $(U_1, ..., U_n) \sim L_n[f; 1, ..., 1]$. Applying (2.8), we obtain $E \exp(-\sum_{i=1}^n t_i U_i) = E \exp(-V_1 \sum_{i=1}^n t_i Z_i)$, where $V_1 \sim L_1[f; n]$ independently of $(Z_1, ..., Z_n) \sim D(1, ..., 1)$. However, by (2.9), (2.10), and the classical Hermite–Genocchi formula [9, Eq. (2.4)],

$$\begin{aligned} E \exp\left(-V_1 \sum_{i=1}^{n} t_i Z_i\right) &= E_{V_1} E_{Z_1, ..., Z_n} \exp\left(-V_1 \sum_{i=1}^{n} t_i Z_i\right) \\ &= (-1)^{n-1} E_{V_1} \sum_{i=1}^{n} \frac{e^{-t_i V_1}}{\prod_{j \neq i} (t_i - t_j)} \\ &= \frac{(-1)^{n-1}}{W^n f(0)} \sum_{i=1}^{n} \frac{\hat{f}(t_i)}{\prod_{j \neq i} (t_i - t_j)} \\ &= \frac{1}{W^n f(0)} \hat{f}(t_1, ..., t_n), \end{aligned}$$

where the last equality follows by a simple determinantal calculation. Putting all these facts together, we obtain

$$E\exp\left(-\sum_{i=1}^{n} t_i X_i\right)=\frac{1}{W^a f(0)}\prod_{i=1}^{n} W_{t_i}^{-(a_i-1)}\hat{f}(t_1, ..., t_n),$$

which leads immediately to (2.11). ∎

It is easy to see that two gamma random variables with common scale parameter are comparable under the Laplace transform ordering. As the following example shows, this result does not seem to extend to general Liouville distributions.

2.8. Example. Suppose that $(X_1, X_2) \sim L_2[f; 1, 1]$, $(Y_1, Y_2) \sim L_2[g; 1, 1]$, where $f(t)=t^{\alpha}e^{-t}$, $g(t)=t^{\beta}e^{-t}$, $t>0$, and $\alpha\neq\beta$. Then by Proposition 2.7, $(X_1, X_2)\geqslant_{\mathrm{L}}(Y_1, Y_2)$ if and only if

$$\frac{(1+t_2)^{-(\alpha+2)}-(1+t_1)^{-(\alpha+2)}}{t_1-t_2}\leqslant\frac{(1+t_2)^{-(\beta+2)}-(1+t_1)^{-(\beta+2)}}{t_1-t_2} \tag{2.12}$$

for all t_1, $t_2> -1$. Replacing t_i by t_i-1, $i=1, 2$, then (2.12) becomes

$$\frac{t_1^{\alpha+2}-t_2^{\alpha+2}}{(t_1 t_2)^{\alpha}(t_1-t_2)}\leqslant\frac{t_1^{\beta+2}-t_2^{\beta+2}}{(t_1 t_2)^{\beta}(t_1-t_2)}. \tag{2.13}$$

Evaluating both sides of (2.13) as t_1, $t_2\to t$ we find that $t^{\alpha}/(\alpha+2)\geqslant t^{\beta}/(\beta+2)$ for all $t>0$, which implies that $\alpha>\beta$. But when we set $t=1$, we obtain $\alpha\leqslant\beta$, which is a contradiction. Therefore (X_1, X_2) and (Y_1, Y_2) are not comparable under $\geqslant_{\mathrm{L}}$.

3. Reliability of Systems with Dependent Components

Suppose $X_1, ..., X_n$ are the random lifetimes of n non-renewable components. Assume that there is a common environment affecting the n components, so that $X_1, ..., X_n$ will be correlated. We refer to Shaked [14] and Lefevre and Malice [10] as two of many articles which study models for the reliability of systems with correlated components.

Here we consider the situation when the joint distribution of $(X_1, ..., X_n)$ is multivariate Liouville. This generalizes the classical situation when $X_1, ..., X_n$ are i.i.d. gamma variables or even when $X_1, ..., X_n$ is a mixture of i.i.d. gamma variables [13]. We will study the reliability of the system by analysing the distribution of $N(t; X_1, ..., X_n)$, the number of components operating at time t. Also we study the system reliability function

$R_k(t; X_1, ..., X_n) = P[N(t; X_1, ..., X_n) \geqslant k]$, $k = 1, ..., n$. These results represent a partial extension of the results of [10], where it was assumed that $X_1, ..., X_n$ was distributed as a mixture of independent exponential variables.

3.1. THEOREM. *Suppose the Liouville vectors* $(X_1, ..., X_n)$ *and* $(Y_1, ..., Y_n)$ *satisfy the hypotheses of Theorem* 2.1. *Then* $N(t; X_1, ..., X_n) \geqslant^{\text{st}} N(t; Y_1, ..., Y_n)$, *and* $R_k(t; X_1, ..., X_n) \geqslant R_k(t; Y_1, ..., Y_n)$, $1 \leqslant k \leqslant n$, $t > 0$.

Proof. Let $\chi_t(x) = 1$ or 0 according as $x > t$ or $x \leqslant t$, respectively. Then

$$N(t; X_1, ..., X_n) = \sum_{i=1}^{n} \chi_t(X_i),$$

and similarly for $N(t; Y_1, ..., Y_n)$. Since χ_t is monotone increasing on $\mathbf{R}_+$, then $N(t; x_1, ..., x_n)$ is monotone increasing in $(x_1, ..., x_n)$ for any $t > 0$. Since $(X_1, ..., X_n) \geqslant^{\text{st}} (Y_1, ..., Y_n)$ then, by (2.1), we have $N(t; X_1, ..., X_n) \geqslant^{\text{st}} N(t; Y_1, ..., Y_n)$.

In turn, $N(t; X_1, ..., X_n) \geqslant^{\text{st}} N(t; Y_1, ..., Y_n)$ implies that $P[N(t; X_1, ..., X_n) \geqslant k] \geqslant P[N(t; Y_1, ..., Y_n) \geqslant k]$, $1 \leqslant k \leqslant n$; equivalently, that $R_k(t; X_1, ..., X_n) \geqslant R_k(t; Y_1, ..., Y_n)$. ∎

3.2. COROLLARY. *Suppose the Liouville vectors* $(X_1, ..., X_n)$ *and* $(Y_1, ..., Y_n)$ *are positive mixtures of independent gamma variables with common scale parameter; say,* $(X_1, ..., X_n) =^{\text{st}} (Z_1, ..., Z_n)/V_1$, $(Y_1, ..., Y_n) =^{\text{st}} (Z_1, ..., Z_n)/V_2$, *where the* Z_i *are mutually independent and* $Z_i \sim G(a_i, b)$. *If* $V_2 \geqslant^{\text{st}} V_1$ *then* $N(t; X_1, ..., X_n) \geqslant^{\text{st}} N(t; Y_1, ..., Y_n)$.

Proof. Since $V_2 \geqslant^{\text{st}} V_1$ then $1/V_1 \geqslant^{\text{st}} 1/V_2$, hence $(X_1, ..., X_n) \geqslant^{\text{st}} (Y_1, ..., Y_n)$. The result now follows immediately from Theorem 3.1. ∎

In the case when the Z_i are exponentially distributed, Corollary 3.2 reduces to a special case of Theorem 1 of Lefevre and Malice [10].

In the last part of this section we consider a problem raised by Lefevre and Malice [10, p. 207]. Suppose n random variables $X_1, ..., X_n$ are distributed as a mixture of independent exponential variables, so that the joint distribution function of $X_1, ..., X_n$ is

$$F(x_1, ..., x_n) = \int_{\mathbf{R}_+} \prod_{j=1}^{n} [1 - \exp(-\eta \lambda_j x_j)] \, dG_H(\eta), \tag{3.1}$$

$x_j > 0$, $j = 1, ..., n$. In (3.1), H denotes the effect of the common environment with distribution function $G_H(\eta)$, $\eta > 0$; and $\lambda_j > 0$, $j = 1, ..., n$, are the parameters of the underlying exponential distributions. More precisely, λ_j represents the failure rate of the jth component under a control environment corresponding to $\eta = 1$. If G_H is supported on $(1, \infty)$ then the components are operating in a harsh environment; and if G_H is supported on $(0, 1)$ then the components are operating in a kinder, gentler environment.

Corresponding to the distribution function $F(x_1, ..., x_n)$, we denote by $N(t, H)$ the number of components operating at time $t > 0$ under the environment H. Further, for $k = 1, ..., n$ the function

$$R_k(t, H) := P[N(t, H) \geqslant k] = \int_{\mathbf{R}_+} R_k(t, \eta)\, dG_H(\eta)$$

is the reliability of a k-out-of-n system at time $t > 0$ under the environment H, where $R_k(t, \eta)$ is the system reliability function corresponding to the distribution function

$$F_\eta(x_1, ..., x_n) = \prod_{j=1}^{n} [1 - \exp(-\eta \lambda_j x_j)], \qquad x_j > 0,\ 1 \leqslant j \leqslant n. \tag{3.2}$$

The conjecture of Lefevre and Malice [10, p. 207] may be stated as follows:

3.3. *Conjecture* [10]. Let H and $\hat{H}$ be two random environments. For $k = 1, ..., n$, the number of sign changes of $R_k(t, H) - R_k(t, \hat{H})$ on $\mathbf{R}_+$ is less than or equal to the number of sign changes of $G_H(\eta) - G_{\hat{H}}(\eta)$ as η varies on $\mathbf{R}_+$.

In general, this conjecture appears to be formidable. The difficulty lies in the complicated nature of the reliability function $R_k(t, H)$. When $k = n$ or when $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, however, we will obtain results for this problem. These results will be derived using the theory of variation diminishing transformations [8].

3.4. THEOREM. *Conjecture* 3.3 *is valid in the following two cases*:

(i) $k = n$;

(ii) $1 \leqslant k < n$, $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, *and* $G_H(\eta) - G_{\hat{H}}(\eta)$ *has at most one sign change as* η *varies on* $\mathbf{R}_+$.

Proof. By integration-by-parts, we have

$$\begin{aligned} R_k(t, H) - R_k(t, \hat{H}) &= \int_{\mathbf{R}_+} R_k(t, \eta)\, d[G_H(\eta) - G_{\hat{H}}(\eta)] \\ &= \int_{\mathbf{R}_+} [G_H(\eta) - G_{\hat{H}}(\eta)]\, K(t, \eta)\, d\eta, \end{aligned} \tag{3.3}$$

where

$$K(t, \eta) = -\frac{\partial}{\partial \eta} R_k(t, \eta).$$

When $k = n$ we have $R_n(t, \eta) = \exp[-(\sum_{j=1}^{n} \lambda_j) t\eta]$, and then

$$K(t, \eta) = \left(\sum_{j=1}^{n} \lambda_j\right) t \exp\left[-\left(\sum_{j=1}^{n} \lambda_j\right) t\eta\right].$$

By [8, p. 18] the kernel $L(t, \eta) = \exp(-t\eta)$ is SRR_∞ (strictly reverse rule of order $r = \infty$) on $\mathbf{R}_+^2$, hence so is $K(t, \eta)$. Therefore it follows from the fundamental theorem of variation diminishing transformations [8, p. 233, Theorem 5.3.1], applied to the integral equation (3.3), that the number of sign changes of $R_k(t, H) - R_k(t, \hat{H})$ as t increases on $\mathbf{R}_+$ is less than or equal to the number of sign changes of $G_H(\eta) - G_{\hat{H}}(\eta)$ as η increases on $\mathbf{R}_+$.

To prove (ii) we assume, without further loss of generality, that $\lambda_j = 1$ for all $j = 1, ..., n$. If $N(t, \eta)$ denotes the number of working components at time t, relative to the distribution function (3.2), then $N(t, \eta) \sim b(n, \exp(-t\eta))$, a binomial distribution on n trials with $\exp(-t\eta)$ as probability of success. By a well-known relationship between the binomial and the beta distributions [14, p. 506], we have

$$R_k(t, \eta) = P[N(t, \eta) \geqslant k] = k \binom{n}{k} \int_0^{\exp(-t\eta)} u^{k-1}(1-u)^{n-k} \, du. \quad (3.4)$$

Differentiating (3.4) we obtain

$$K(t, \eta) = k \binom{n}{k} t e^{-nt\eta}(e^{t\eta} - 1)^{n-k}. \quad (3.5)$$

We claim that $K(t, \eta)$ is SRR_2 on $\mathbf{R}_+^2$; equivalent, if $\psi(t) = e^{-nt}(e^t - 1)^{n-k}$ then the kernel $L_1(t, \eta) = \psi(t\eta)$ is SRR_2 on $\mathbf{R}_+^2$. To see this, note that L_1 is SRR_2 on $\mathbf{R}_+$ if and only if the translation kernel $L_2(t, \eta) = L_1(e^t, e^\eta) = \psi(e^{t+\eta})$ is SRR_2 on $\mathbf{R}^2$. Since L_2 is a translation kernel then it is SRR_2 on $\mathbf{R}^2$ if and only if the function $\psi_1(t) = \psi(e^t)$, $t \in \mathbf{R}$, is strictly log-concave [8, p. 159]. By a direct calculation, $(\log \psi_1)''(t) = -e^t(\exp(e^t) - 1)^{-2} \psi_2(e^t)$, where $\psi_2(t) = ke^{2t} - (n+k) e^t + (n-k) te^t + n$. Since $\psi_2(0) = 0$ and $\psi_2'(t) = [2k(e^t - 1) + (n-k)t] e^t > 0$ for all $t > 0$, then $\psi_2(t) > 0$ for all $t > 0$. Therefore $(\log \psi_1)''(t) < 0$, $t \in \mathbf{R}$, so that ψ_1 is strictly log-concave.

Having proved that $K(t, \eta)$ is SRR_2 on $\mathbf{R}_+^2$, the conclusion (ii) again follows from [8, p. 233, Theorem 5.3.1]. ▮

3.5. COROLLARY. *Theorem* 3.4 *remains valid if* $X_1, ..., X_n$ *are distributed as a mixture of independent Weibull variables, with distribution function*

$$F(x_1, ..., x_n) = \int_{\mathbf{R}_+} \prod_{j=1}^{n} [1 - \exp(-\eta\lambda_j x_j^{\beta})] \, dG_H(\eta), \qquad x_j > 0, \ 1 \leqslant j \leqslant n,$$

where $\beta > 0$.

Proof. Let $K(t, \eta)$ be the kernel defined in the proof of Theorem 3.4. Then, proceeding as before, we need to show that the kernel $\tilde{K}(t, \eta) = K(t^{\beta}, \eta)$ is SRR_r on $\mathbf{R}_+^2$ with $r = \infty$ or $r = 2$ according to whether K corresponds to case (i) or (ii), respectively. Since the function $t \mapsto t^{\beta}$, $t > 0$, is strictly increasing then, by [8], $\tilde{K}$ is SRR_r whenever K is SRR_r. Then the result follows from Theorem 3.4. ∎

3.6. *Remark.* (i) Since the kernel $K(t, \eta)$ is *strictly* RR_r (with $r = 2$ if $1 \leqslant k \leqslant n-1$ and $r = \infty$ if $k = n$), then by [8, p. 237, Corollary 5.3.1], the sign changes of $R_k(t, H) - R_k(t, \hat{H})$ occurs at isolated crossing points, and these crossing points are the only zeros of $R_k(t, H) - R_k(t, \hat{H})$ on $\mathbf{R}_+$.

(ii) For $k < n$ the kernel $K(t, \eta)$ in (3.5) is not RR_{∞}. On reviewing the proof of Theorem 3.4, it is evident that $K(t, \eta)$ fails to be RR_{∞} if and only if the function ψ_1 is not PF_{∞} on $\mathbf{R}$. To show this, we use I. J. Schoenberg's representation formula [8, p. 345, Theorem 7.3.2] for the Laplace transforms of PF_{∞} functions on $\mathbf{R}$.

First, we note that ψ_1 is integrable on $\mathbf{R}$. Next, the Laplace transform of ψ_1 is

$$\begin{aligned}\int_{\mathbf{R}} e^{-st} \psi_1(t) \, dt &= \int_{\mathbf{R}} e^{-st} \exp(-ne^t)(\exp(e^t) - 1)^{n-k} \, dt \\ &= \int_{\mathbf{R}_+} u^{-s-1} e^{-nu} (e^u - 1)^{n-k} \, du. \end{aligned} \tag{3.6}$$

This transform exists in the left half-plane $\mathrm{Re}(s) < 0$. It can be computed explicitly by expanding the term $(e^u - 1)^{n-k}$ through the binomial theorem and then integrating term-by-term. Then it becomes clear that (3.6) is not of the form required by [8, p. 345, Theorem 7.3.2]. Hence, ψ_1 is not PF_{∞} on $\mathbf{R}$.

It remains an open problem to find the largest r such that $K(t, \eta)$ is RR_r on $\mathbf{R}_+^2$. There also remains the question of whether Theorem 3.4 can be extended to the case when the λ_j are not all equal. As the following example shows, this is a difficult question even when $n = 2$.

3.7. Example. Suppose $n=2$. Since $N(t,\eta)=U_1+U_2$, where U_1 and U_2 are independent binomial variables with $U_j \sim b(1, \exp(-\lambda_j t\eta))$, $j=1,2$, then

$$R_1(t,\eta)=1-P[N(t,\eta)=0]=e^{-\lambda_1 t\eta}+e^{-\lambda_2 t\eta}-e^{-(\lambda_1+\lambda_2)t\eta}.$$

Therefore $K(t,\eta)=\psi(t\eta)$, where

$$\psi(t)=\lambda_1 e^{-\lambda_1 t}+\lambda_2 e^{-\lambda_2 t}-(\lambda_1+\lambda_2)\, e^{-(\lambda_1+\lambda_2)t}, \qquad t>0.$$

To see whether K is, say, SRR_2, we need to determine if the function $\psi(e^t)$, $t \in \mathbf{R}$, is log-concave. Since K is SRR_2 if and only if the kernel $L(t,\eta)=\lambda_1^{-1}K(\lambda_1^{-1}t,\eta)$ is SRR_2 then we can assume, without loss of generality, that $\lambda_1=1$.

Denoting λ_2 by λ, and writing $\psi(e^t)$ in the form

$$\log \psi(e^t)= -(1+\lambda)\, e^t+\log[\exp(\lambda e^t)+\lambda \exp(e^t)-1-\lambda],$$

we obtain

$$[\log \psi(e^t)]''= -[\exp(\lambda e^t)+\lambda \exp(e^t)-1-\lambda]^{-2}\, e^t \Psi(\exp(e^t)),$$

where

$$\begin{aligned}\Psi(u)&=(1+\lambda)[u^\lambda+\lambda(u-1)-1]^2+\lambda^2(u^\lambda+u)^2 \log u\\ &\quad -\lambda[u^\lambda+\lambda(u-1)-1][u^\lambda(1+\lambda \log u)+u(1+\log u)], \qquad u>1.\end{aligned}$$

It is clear that $\Psi(u)>0$ for sufficiently large u. Extensive numerical work supports the conjecture that $\Psi(u)>0$ for all $u>1$. If valid, it implies that $\psi(e^t)$ is log-concave on $\mathbf{R}$, or K is SRR_2. Then the sign change properties of $R_k(t,H)-R_k(t,\hat{H})$ are as stated in Theorem 3.4(ii).

In the spirit of Theorem 3.4 the following result compares $R_k(t,H)$ and $R_k(t,\hat{H})$ when $H \leqslant^* \hat{H}$, where $\leqslant^*$ denotes the star-shaped ordering [2, p. 217].

3.8. Proposition. *Let H and $\hat{H}$ be random environments. If $H \leqslant^* \hat{H}$ then, for any $\alpha>0$, $R_k(t,H)-R_k(\alpha t,\hat{H})$ has at most one sign change on $\mathbf{R}_+$.*

Proof. For any $\alpha>0$, it is clear that $R_k(t,\eta)=R_k(\alpha t,\eta/\alpha)$. Therefore,

$$\begin{aligned}R_k(\alpha t,\hat{H})&=\int_{\mathbf{R}_+} R_k(\alpha t,\eta)\, dG_{\hat{H}}(\eta)\\ &=\int_{\mathbf{R}_+} R_k(\alpha t,\eta/\alpha)\, dG_{\hat{H}}(\eta/\alpha).\end{aligned}$$

By integration-by-parts, we have

$$R_k(t, H) - R_k(\alpha t, \hat{H}) = \int_{\mathbf{R}_+} R_k(t, \eta)\, d[G_H(\eta) - G_{\hat{H}}(\eta/\alpha)]$$

$$= -\int_{\mathbf{R}_+} [\bar{G}_H(\eta) - \bar{G}_{\hat{H}}(\eta/\alpha)]\, K(t, \eta)\, d\eta,$$

where $\bar{G}_H(\eta) \equiv 1 - G_H(\eta)$. Since $H \leqslant^* \hat{H}$ then $\bar{G}_H(\eta) - \bar{G}_{\hat{H}}(\eta/\alpha)$ has at most one sign change on $\mathbf{R}_+$ [2, p. 218, Theorem 9.2]. Since $K(t, \eta)$ is SRR_2 then by the variation diminishing property of RR_2 kernels it follows that $R_k(t, H) - R_k(\alpha t, \hat{H})$ has at most the same number of sign changes as $\bar{G}_H(\eta) - \bar{G}_{\hat{H}}(\eta/\alpha)$. ∎

For general $\lambda_1, ..., \lambda_n$ the intractability of $R_k(t, \eta)$ makes it difficult to extend the analysis given in Theorem 3.4. This leads us to consider the behavior of $R_k(t, H)$ as a function of $(\lambda_1, ..., \lambda_n)$.

For fixed t and η, some results are known about the behavior of $N(t, \eta)$ and $R_k(t, \eta)$ as a function of $(\lambda_1, ..., \lambda_n)$. For example, as a direct application of [12, p. 360, 12.F.1], we have the following result: If $\psi: \{0, 1, ..., n\} \to \mathbf{R}$ is convex then $E\psi(N(t, \eta))$ is Schur-convex in $(\lambda_1, ..., \lambda_n)$. If also $\psi(0) \leqslant \psi(1) \leqslant \cdots \leqslant \psi(n-1)$ then, in addition, $E\psi(N(t, \eta))$ is monotone decreasing in $(\lambda_1, ..., \lambda_n)$.

In the case of $R_k(t, H)$, few results are available (since $R_n(t, H)$ is a function of $\lambda_1 + \cdots + \lambda_n$, we need only consider the case $k < n$). We have only the following result.

3.9. PROPOSITION. *Fix $t \in \mathbf{R}_+$. Then*

(i) *If $1 \leqslant k < n$ and G_H is supported on the interval $\{\eta : \eta > 0, \sum_{j=1}^n \exp(-\lambda_j t\eta) \geqslant k+1\}$ then $R_k(t, H)$ is Schur-concave in $(\lambda_1, ..., \lambda_n)$.*

(ii) *If $3 \leqslant k < n$ and G_H is supported on the interval $\{\eta : \eta > 0, \sum_{j=1}^n \exp(-\lambda_j t\eta) \leqslant k-2\}$ then $R_k(t, H)$ is Schur-convex in $(\lambda_1, ..., \lambda_n)$.*

Proof. By [12, p. 376, Theorem 12.K.1], $R_k(t, \eta)$ is Schur-concave or Schur-convex in $(\lambda_1, ..., \lambda_n)$ according as $k \leqslant -1 + \sum_{j=1}^n \exp(-\lambda_j t\eta)$ or $k \geqslant 2 + \sum_{j=1}^n \exp(-\lambda_j t\eta)$, respectively. Since a mixture of Schur-concave (resp. Schur-convex) functions is again Schur-concave (resp. Schur-convex), the result follows from the fact that $R_k(t, H)$ is a mixture of $R_k(t, \eta)$. ∎

Proposition 3.9 should be compared with Theorem 2(b) in [10].

4. Partial Orderings through Positive Dependence Properties

In this section, we order Liouville vectors $(X_1, ..., X_n)$ and $(Y_1, ..., Y_n)$ through positive dependence properties and through dependence of the corresponding order statistics $(X_{(1)}, ..., X_{(n)})$ and $(Y_{(1)}, ..., Y_{(n)})$, respectively. We will determine conditions under which $(X_1, ..., X_n)$ and $(Y_1, ..., Y_n)$ may be compared using the orderings of [15, 16]. In particular, we use those results to obtain stochastic orderings for the order statistics $(X_{(1)}, ..., X_{(n)})$ and $(Y_{(1)}, ..., Y_{(n)})$, and for the spacings $(X_{(2)} - X_{(1)}, X_{(3)} - X_{(2)}, ..., X_{(n)} - X_{(n-1)})$ and $(Y_{(2)} - Y_{(1)}, Y_{(3)} - Y_{(2)}, ..., Y_{(n)} - Y_{(n-1)})$.

When $(X_1, ..., X_n)$ is *positive dependent by mixture* (PDM); that is, $(X_1, ..., X_n)$ is stochastically equal to a mixture of i.i.d. random variables; many results on partial orderings for the order statistics have been obtained, cf. [14]. Thus, it is natural to determine conditions when a Liouville vector is also PDM. As the following results shows, in the interesting cases, a Liouville vector is PDM if and only if it is a mixture of i.i.d. gamma variables.

4.1. Proposition. *Suppose the Liouville vector $(X_1, ..., X_n)$ is PDM, and the mixing measure is complete. Then $(X_1, ..., X_n)$ is a mixture of i.i.d. gamma variables.*

Proof. Since $X_1, ..., X_n$ are exchangeable, then we have $a_1 = a_2 = \cdots = a_n = a$, say. By equating the Liouville and PDM densities, we see that $(X_1, ..., X_n)$ is PDM if and only if

$$c_n f\left(\sum_{i=1}^n x_i\right) \prod_{i=1}^n x_i^{a-1} = \int_\Omega \prod_{i=1}^n f^{(\omega)}(x_i)\, d\tau(\omega), \tag{4.1}$$

where c_n is a normalizing constant; $\Omega \subseteq \mathbf{R}^d$, $d < \infty$; $\{f^{(\omega)} : \omega \in \Omega\}$ is a family of density functions; and τ is a distribution function on Ω. By assumption, τ is also complete. If we set $f^{(\omega)}(t) = t^{a-1} g^{(\omega)}(t)$, then (4.1) becomes

$$c_n f\left(\sum_{i=1}^n x_i\right) = \int_\Omega \prod_{i=1}^n g^{(\omega)}(x_i)\, d\tau(\omega). \tag{4.2}$$

Substituting $x_1 = t$ and $x_i = 0$ $(i = 2, ..., n)$ in (4.2), we obtain

$$c_n f(t) = \int_\Omega [g^{(\omega)}(0)]^{n-1} g^{(\omega)}(t)\, d\tau(\omega). \tag{4.3}$$

In particular, we can assume, without loss of generality, that $g^{(\omega)}(t) \neq 0$ for all $t \geq 0$ and $\omega \in \Omega$. Next, setting $x_i = 0$ $(i = 3, ..., n)$ in (4.2), we obtain

$$c_n f(x_1 + x_2) = \int_\Omega [g^{(\omega)}(0)]^{n-2} g^{(\omega)}(x_1) g^{(\omega)}(x_2) \, d\tau(\omega). \tag{4.4}$$

Therefore by (4.3) and (4.4),

$$\begin{aligned} \int_\Omega [g^{(\omega)}(0)]^{n-1} g^{(\omega)}(x_1 + x_2) \, d\tau(\omega) \\ &= c_n f(x_1 + x_2) \\ &= \int_\Omega [g^{(\omega)}(0)]^{n-2} g^{(\omega)}(x_1) g^{(\omega)}(x_2) \, d\tau(\omega). \end{aligned} \tag{4.5}$$

Since τ is complete and $g^{(\omega)}(0) \neq 0$ for all ω, then it follows from (4.5) that

$$g^{(\omega)}(0) \, g^{(\omega)}(x_1 + x_2) = g^{(\omega)}(x_1) \, g^{(\omega)}(x_2) \tag{4.6}$$

for all $\omega \in \Omega$ and $x_1, x_2 \in \mathbf{R}_+$. It is well known that (4.6), Cauchy's equation, has the general solution $g^{(\omega)}(t) = g^{(\omega)}(0) \exp(-\lambda_\omega t)$, $\lambda_\omega > 0$. If we change notation and set $\lambda_\omega \to \lambda\omega$, then $\Omega \subseteq \mathbf{R}_+$ and $g^{(\omega)}(t) = g^{(\omega)}(0) \exp(-\lambda\omega t)$. Further, the joint density function of $(X_1, ..., X_n)$ is

$$c_n \left(\prod_{i=1}^{n} x_i^{a-1} \right) \int_\Omega \exp\left(-\lambda\omega \sum_{i=1}^{n} x_i \right) [g^{(\omega)}(0)]^n \, d\tau(\omega);$$

hence $(X_1, ..., X_n)$ is a mixture of i.i.d. gamma variables. ∎

If we try to apply the results of [14] on PDM distributions to obtain partial orderings for the Liouville distributions, then it follows from Proposition 4.1 that we often can obtain results only for the case of mixtures of i.i.d. gamma variables. Even more, it turns out that some results of [14] do not appear to extend to the Liouville distributions. As an example it is proved in [14] that if $X_1, ..., X_n$ are i.i.d., $(Y_1, ..., Y_n)$ are PDM, and $X_1 =^{\text{st}} Y_1$, then

$$(F_{X_{(1)}}(t), ..., F_{X_{(n)}}(t)) \succ (F_{Y_{(1)}}(t), ..., F_{Y_{(n)}}(t)), \tag{4.7}$$

where $\succ$ denotes the usual majorization order [12]. As the following example shows, these results do not appear to extend to the Liouville distributions.

4.2. Example. Let $X_1, ..., X_n$ be i.i.d. random variables, $(Y_1, ..., Y_n) \sim$

$L_n[g;\, 1, \ldots, 1]$, where $g(t) = te^{-t}$, $t > 0$, and $X_1 =^{\text{st}} Y_1$. By a straightforward calculation,

$$h(t) := \frac{W^n g(t)}{W^n g(0)} = \left(1 + \frac{t}{n}\right) e^{-t}, \qquad t > 0.$$

For $1 \leqslant k \leqslant n$, the joint density function of $(Y_{(1)}, \ldots, Y_{(k)})$ is [7, Eq. (3.12)]

$$\frac{n!}{(n-k)!\; W^n g(0)}\; W^{n-k} g\left(\sum_{i=1}^{k} y_{(i)} + (n-k)\, y_{(k)}\right),$$

for $0 < y_{(1)} < \cdots < y_{(k)} < \infty$. Using the inclusion–exclusion principle, it can then be proved that the marginal density function of $Y_{(k)}$ is

$$\frac{n!}{(n-k)!\,(k-1)!\; W^n g(0)} \times \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} W^{n-1} g((n-k+1+j)\, y_{(k)}), \qquad y_{(k)} > 0,$$

and in turn that

$$\begin{aligned} P(Y_{(k)} > t) &= \frac{n!}{(n-k)!\,(k-1)!\; W^n g(0)} \times \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \frac{W^n g((n-k+1+j)t)}{n-k+1+j} \\ &= k \binom{n}{k} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \times (n-k+1+j)^{-1}\, h((n-k+1+j)t), \qquad t > 0. \end{aligned}$$

Similarly, using standard results on the order statistics of i.i.d. variables, it can be shown that

$$P(X_{(k)} > t) = k \binom{n}{k} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \times (n-k+1+j)^{-1}\, h(t)^{n-k+1+j}, \qquad t > 0.$$

Therefore, for $1 \leqslant r \leqslant n$,

$$\begin{aligned} \sum_{k=1}^{r} [F_{Y_{(k)}}(t) - F_{X_{(k)}}(t)] &= \sum_{k=1}^{r} [P(X_{(k)} > t) - P(Y_{(k)} > t)] \\ &= \sum_{k=1}^{r} k \binom{n}{k} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} (n-k+1+j)^{-1} \\ &\quad \times [h(t)^{n-k+1+j} - h((n-k+1+j)t)]. \end{aligned}$$

Replacing j by $k-1-j$ and reversing the order of summation, we obtain

$$\sum_{k=1}^{r} [F_{Y_{(k)}}(t) - F_{X_{(k)}}(t)] = \sum_{j=0}^{r-1} c_{r,j}[h(t)^{n-j} - h((n-j)t)], \tag{4.8}$$

where

$$\begin{aligned} c_{r,j} &= (n-j)^{-1} \sum_{k=j+1}^{r} (-1)^{k-j-1} k \binom{n}{k}\binom{k-1}{j} \\ &= \binom{n}{j} \sum_{k=0}^{r-j-1} (-1)^k \binom{n-j-1}{k}. \end{aligned}$$

Substituting $r=1$ in (4.8), we obtain

$$\begin{aligned} F_{Y_{(1)}}(t) - F_{X_{(1)}}(t) &= h(t)^n - h(nt) \\ &= \left[\left(1+\frac{t}{n}\right)^n - (1+t)\right] e^{-nt} > 0, \qquad t>0. \end{aligned} \tag{4.9}$$

When $r=2$, then (4.8) becomes

$$\begin{aligned} &\sum_{k=1}^{2} [F_{Y_{(k)}}(t) - F_{X_{(k)}}(t)] \\ &\quad = (n-2)[h(nt) - h(t)^n] - n[h((n-1)t) - h(t)^{n-1}]. \end{aligned} \tag{4.10}$$

If $n=2$ then (4.10) is identically zero. For $n \geqslant 3$, we can write (4.10) as

$$\begin{aligned} \frac{\sum_{k=1}^{2} [F_{Y_{(k)}}(t) - F_{X_{(k)}}(t)]}{(n-2)\, e^{-nt}} &= \frac{n}{n-2}\left[\left(1+\frac{t}{n}\right)^{n-1} - 1 - \frac{n-1}{n}\, t\right] e^t \\ &\quad - \left[\left(1+\frac{t}{n}\right)^n - 1 - t\right]. \end{aligned} \tag{4.11}$$

To show that (4.11) is positive for all $t>0$, replace the term e^t by the smaller quantity $1+(t/n)$; then the right-hand side becomes a nontrivial polynomial with nonnegative coefficients; hence it is positive for all $t>0$.

The positivity of (4.9) and (4.11) are partial reverses to (4.7), suggesting that, perhaps, the full reverse of (4.7) holds. Note also that $c_{n,j}=0$ (by the binomial expansion of $[1+(-1)]^{n-j-1}$); therefore

$$\sum_{k=1}^{n} [F_{Y_{(k)}}(t) - F_{X_{(k)}}(t)] \equiv 0.$$

So at least for $n=2$, 3, the full reverse of (4.7) is valid. However, for $n=4$, (4.8) is negative for all $t>0$ when $r=3$; we will postpone the proof of this result to an appendix, since the details are somewhat involved.

Next, we obtain the stochastic orderings for the order statistics. As before, we assume that $(X_1, ..., X_n) \sim L_n(f; a_1, ..., a_n)$ and $(Y_1, ..., Y_n) \sim L_n(g; b_1, ..., b_n)$.

4.3. PROPOSITION. *Suppose that* $a_1 = \cdots = a_n = b_1 = \cdots = b_n$ *and the function* f/g *is monotone increasing. If the constants* c_{ij} $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant k)$ *are such that* $\sum_{i=1}^n c_{ij} = 0$, $1 \leqslant j \leqslant k$, *then*

$$\left(\left|\sum_{i=1}^n c_{i1} X_{(i)}\right|, ..., \left|\sum_{i=1}^n c_{ik} X_{(i)}\right|\right) \geqslant^{\mathrm{st}} \left(\left|\sum_{i=1}^n c_{i1} Y_{(i)}\right|, ..., \left|\sum_{i=1}^n c_{ik} Y_{(i)}\right|\right).$$

In particular, we have the ordering $(X_{(2)} - X_{(1)}, ..., X_{(n)} - X_{(n-1)}) \geqslant^{\mathrm{st}} (Y_{(2)} - Y_{(1)}, ..., Y_{(n)} - Y_{(n-1)})$ *for the spacings.*

Proof. Let a be the common value of the a_j and b_j. As in the proof of Theorem 2.1, we have

$$(X_1, ..., X_n) =^{\mathrm{st}} V_1(Z_1, ..., Z_n), \qquad (Y_1, ..., Y_n) =^{\mathrm{st}} V_2(Z_1, ..., Z_n), \tag{4.12}$$

where $(Z_1, ..., Z_n)$ has a (singular) Dirichlet distribution, $D(a, ..., a)$, on the simplex $\mathscr{S}_n$, $V_1 =^{\mathrm{st}} \sum_{i=1}^n X_i \sim L_1[f; na]$, $V_2 =^{\mathrm{st}} \sum_{i=1}^n Y_i \sim L_1[g; na]$, and $(Z_1, ..., Z_n)$ is independent of both V_1 and V_2. Since f/g is monotone increasing then, by Theorem 2.1, $V_1 \geqslant^{\mathrm{st}} V_2$. Then, the stochastic representations (4.12) are subsumed by the multiplicative model of Shaked and Tong [15, Theorem 1, Model B]. A careful scrutiny of Shaked and Tong's result reveals that their proof remains valid if $Z_1, ..., Z_n$ are exchangeable (instead of being only i.i.d.). Since the distribution $D(a, ..., a)$ is exchangeable, then the result follows. ∎

4.4. COROLLARY. *If* $a_1 = \cdots = a_n = b_1 = \cdots = b_n$, f/g *is monotone increasing, and* $\sum_{i=1}^n c_i = 0$, *then*

(i) $|\sum_{i=1}^n c_i X_{(i)}| \geqslant^{\mathrm{st}} |\sum_{i=1}^n c_i Y_{(i)}|$;

(ii) $|\sum_{i=1}^n c_i X_i| \geqslant^{\mathrm{st}} |\sum_{i=1}^n c_i Y_{(i)}|$;

(iii) $E(X_{(1)}, ..., X_{(n)}) \succ E(Y_{(1)}, ..., Y_{(n)})$.

Proof. The inequality (i) follows from Proposition 4.3 with $k=1$. In the terminology of [16], (i) means that $(X_1, ..., X_n)$ is more dispersed than $(Y_1, ..., Y_n)$, written $(X_1, ..., X_n) \succ_{\mathrm{A}} (Y_1, ..., Y_n)$.

Shaked and Tong [16] also proved that (i) implies (ii), and (ii) implies (iii). (They denote (ii) and (iii) by the orderings $\succ_{\mathrm{B}}$ and $\succ_{\mathrm{D}}$, respectively.) ∎

4.5. *Remark.* Shaked and Tong [16] also considered a fourth partial order: $(X_1, ..., X_n) \succ_C (Y_1, ..., Y_n)$ if $(F_{X_{(1)}}(t), ..., F_{X_{(n)}}(t)) \succ (F_{Y_{(1)}}(t), ..., F_{Y_{(n)}}(t))$ for all $t > 0$. By choosing $f(t) = e^{-t}$ and $g(t) = te^{-t}$, we find that even for the Liouville distributions, $(X_1, ..., X_n) \succ_A (Y_1, ..., Y_n)$ does not imply $(X_1, ..., X_n) \succ_C (Y_1, ..., Y_n)$. This result was shown for other distributions in [16]. It remains an open problem to characterize the Liouville vectors which can be ordered under $\succ_C$.

5. The Unique Crossing Conjecture

Suppose that $(X_1, ..., X_n) \sim L_n[f; a, ..., a]$ is an exchangeable Liouville random vector, and $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ is a vector of nonnegative weights. Following Diaconis and Perlman [3], we investigate the behavior of the distribution function

$$F_{\boldsymbol{\theta}}(t) = P\left[\sum_{i=1}^{n} \theta_i X_i \leqslant t\right], \qquad t > 0,$$

as $\boldsymbol{\theta}$ varies. Given two weight vectors $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and $\boldsymbol{\phi} = (\phi_1, ..., \phi_n)$, we want to compare $F_{\boldsymbol{\theta}}(t)$ and $F_{\boldsymbol{\phi}}(t)$ whenever $\boldsymbol{\theta}$ majorizes $\boldsymbol{\phi}$, $\boldsymbol{\theta} \succ \boldsymbol{\phi}$, and $\boldsymbol{\phi}$ is not a permutation of $\boldsymbol{\theta}$. The problem considered by Diaconis and Perlman [3], for the case when $X_1, ..., X_n$ are i.i.d. gamma variables, is the

Unique Crossing Conjecture. If $\boldsymbol{\theta} \succ \boldsymbol{\phi}$ then $F_{\boldsymbol{\theta}}(t) - F_{\boldsymbol{\phi}}(t)$ changes sign exactly once on $\mathbf{R}_+$. This crossing occurs at a unique point t^*, the only zero of $F_{\boldsymbol{\theta}}(t) - F_{\boldsymbol{\phi}}(t)$ on $\mathbf{R}_+$.

Here, we find conditions on f and a so that the unique crossing conjecture (UCC) is valid for the Liouville distributions $L_n[f; a, ..., a]$.

5.1. Lemma. *If $\boldsymbol{\theta} \succ \boldsymbol{\phi}$, then $F_{\boldsymbol{\theta}} - F_{\boldsymbol{\phi}}$ changes sign at least once on $\mathbf{R}_+$. Further, if t^* exists then*

$$F_{\boldsymbol{\theta}}(t) - F_{\boldsymbol{\phi}}(t) \begin{cases} > 0, & \text{if } \ 0 < t < t^*, \\ < 0, & \text{if } \ t^* < t < \infty. \end{cases} \tag{5.1}$$

Proof. Since $\boldsymbol{\theta} \succ \boldsymbol{\phi}$ then $\sum_{i=1}^{n} \theta_i = \sum_{i=1}^{n} \phi_i$. Therefore,

$$\sum_{i \neq j} \phi_i \phi_j - \sum_{i \neq j} \theta_i \theta_j = \left[\left(\sum_{i=1}^{n} \phi_i\right)^2 - \sum_{i=1}^{n} \phi_i^2\right] - \left[\left(\sum_{i=1}^{n} \theta_i\right)^2 - \sum_{i=1}^{n} \theta_i^2\right]$$

$$= \sum_{i=1}^{n} \theta_i^2 - \sum_{i=1}^{n} \phi_i^2 > 0, \tag{5.2}$$

where positivity follows from the fact that $\sum_{i=1}^{n} \theta_i^2$ is a strictly Schur-convex function of θ.

Since $X_1, ..., X_n$ have the same marginal (and nondegenerate) distributions, then

$$\frac{\operatorname{Cov}(X_1, X_2)}{\operatorname{Var}(X_1)} = \frac{\operatorname{Cov}(X_1, X_2)}{\sqrt{\operatorname{Var}(X_1) \operatorname{Var}(X_2)}} < 1,$$

so that $\operatorname{Cov}(X_1, X_2) < \operatorname{Var}(X_1)$. Also

$$\operatorname{Var}\left(\sum_{i=1}^{n} \theta_i X_i\right) = \left(\sum_{i=1}^{n} \theta_i^2\right) \operatorname{Var}(X_1) + \left(\sum_{i \neq j} \theta_i \theta_j\right) \operatorname{Cov}(X_1, X_2).$$

By (5.2),

$$\begin{aligned}
&\operatorname{Var}\left(\sum_{i=1}^{n} \theta_i X_i\right) - \operatorname{Var}\left(\sum_{i=1}^{n} \phi_i X_i\right) \\
&= \left(\sum_{i=1}^{n} \theta_i^2 - \sum_{i=1}^{n} \phi_i^2\right) \operatorname{Var}(X_1) + \left(\sum_{i \neq j} \theta_i \theta_j - \sum_{i \neq j} \phi_i \phi_j\right) \operatorname{Cov}(X_1, X_2) \\
&= \left(\sum_{i=1}^{n} \theta_i^2 - \sum_{i=1}^{n} \phi_i^2\right) (\operatorname{Var}(X_1) - \operatorname{Cov}(X_1, X_2)) > 0.
\end{aligned}$$

Therefore, $\operatorname{Var}(\sum_{i=1}^{n} \theta_i X_i) > \operatorname{Var}(\sum_{i=1}^{n} \phi_i X_i)$, so that $F_{\boldsymbol{\theta}} \not\equiv F_{\boldsymbol{\phi}}$.

Since $\boldsymbol{\theta} \succ \boldsymbol{\phi}$ also implies that $E(\sum_{i=1}^{n} \theta_i X_i) = E(\sum_{i=1}^{n} \phi_i X_i)$, then it follows that $F_{\boldsymbol{\theta}} - F_{\boldsymbol{\phi}}$ changes sign at least once on $\mathbf{R}_+$. Finally, the proof of (5.1) is exactly as in [3]. ∎

5.2. *Remark.* The proof of Lemma 5.1 does not use the fact that $(X_1, ..., X_n)$ are Liouville distributed. So the result holds for all $X_1, ..., X_n$ which are exchangeable and such that the first two moments of X_1 are finite.

Recall from (4.12) the stochastic representation $(X_1, ..., X_n) =^{\text{st}} V(Z_1, ..., Z_n)$, where $V =^{\text{st}} \sum_{i=1}^{n} X_i \sim L_1[f; na]$, $(Z_1, ..., Z_n)$ has a Dirichlet distribution $D(a, ..., a)$ on the simplex $\mathscr{S}_n$, and V is independent of $(Z_1, ..., Z_n)$. Define

$$H_{\boldsymbol{\theta}}(t) = P\left[\sum_{i=1}^{n} \theta_i Z_i \leqslant t\right], \qquad t > 0,$$

and let g_V denote the density of V. Then

$$\begin{aligned} F_{\boldsymbol\theta}(t) &= P\left[V \sum_{i=1}^{n} \theta_i Z_i \leqslant t\right] \\ &= \int_{\mathbf{R}_+} H_{\boldsymbol\theta}(t/v)\, g_V(v)\, dv \\ &= t \int_{\mathbf{R}_+} H_{\boldsymbol\theta}(u)\, g_V(tu^{-1})\, u^{-2}\, du, \end{aligned}$$

and similarly for $F_{\boldsymbol\phi}(t)$. Since $V \sim L_1[f; na]$ then $g_V(t) = c_1 f(t)\, t^{na-1}$, $t > 0$, c_1 constant, and, therefore,

$$F_{\boldsymbol\theta}(t) - F_{\boldsymbol\phi}(t) = c_1 t^{na} \int_{\mathbf{R}_+} [H_{\boldsymbol\theta}(u) - H_{\boldsymbol\phi}(u)]\, f(tu^{-1})\, u^{-(na+1)}\, du. \tag{5.3}$$

If the kernel $K(t, u) = f(tu^{-1})$ is strictly totally positive of order 2 (STP_2) on $\mathbf{R}_+^2$ [8, p. 15] then the number of sign changes of $F_{\boldsymbol\theta} - F_{\boldsymbol\phi}$ on $\mathbf{R}_+$ is less than or equal to the number of sign changes of $H_{\boldsymbol\theta} - H_{\boldsymbol\phi}$ on $\mathbf{R}_+$, provided that the latter is at most one.

For the rest of the paper, we assume that the kernel K is STP_2. The following result summarizes what is known about the UCC for the Liouville distributions. The proofs of these results are essentially the same as those given in the i.i.d. gamma case by Diaconis and Perlman [3].

5.3. Proposition. *The UCC is valid in the following cases:*

(i) $n = 2$;

(ii) $n \geqslant 3$, $a > 1$, *and* $\boldsymbol\theta$ *and* $\boldsymbol\phi$ *differ in exactly two components*;

(iii) $n = 3$, $a = 1$;

(iv) $n \geqslant 2$ *and* $\boldsymbol\phi = (\bar\theta, ..., \bar\theta)$, *where* $\bar\theta = (\theta_1 + \cdots + \theta_n)/n$.

Proof. Since the proofs of (i)–(iv) are much the same as in [3], we will only provide a proof of (i).

To prove (i), note that since K is STP_2 then any sign changes of $F_{\boldsymbol\theta} - F_{\boldsymbol\phi}$ occur at isolated crossing points and these crossing points are the only zeros of $F_{\boldsymbol\theta} - F_{\boldsymbol\phi}$. To check uniqueness of the crossing point, the argument we use is precisely that of Diaconis and Perlman [3, Proposition 2.1]. They use the symmetry of the distribution of $Z_1 - Z_2$, together with the assumption $\boldsymbol\theta \succ \boldsymbol\phi$, to show that $H_{\boldsymbol\theta} - H_{\boldsymbol\phi}$ has exactly one sign change on $\mathbf{R}_+$. Specifically, if without loss of generality, $\boldsymbol{\theta}=(\bar{\theta}+\alpha, \bar{\theta}-\alpha)$ and $\boldsymbol{\phi}=(\bar{\theta}+\beta, \bar{\theta}-\beta)$, where $\bar{\theta}\geqslant\alpha>|\beta|\geqslant 0$, then [3, Eq. (2.9)],

$$H_{\boldsymbol{\theta}}(u)-H_{\boldsymbol{\phi}}(u)\begin{cases}>0, & \text{if } \bar{\theta}-\alpha<u<\bar{\theta}\\ =0, & \text{if } u\leqslant\bar{\theta}-\alpha,\ u=\bar{\theta}, \text{ or } u\geqslant\bar{\theta}+\alpha\\ <0, & \text{if } \bar{\theta}<u<\bar{\theta}+\alpha.\end{cases} \tag{5.4}$$

By Lemma 5.1, it then follows that $F_{\boldsymbol{\theta}}-F_{\boldsymbol{\phi}}$ has exactly one sign change on $\mathbf{R}_+$.

In (ii)–(iv), the proof again follows from (5.3) and the fact [3] that in each case, the function $H_{\boldsymbol{\theta}}-H_{\boldsymbol{\phi}}$ has exactly one sign change on $\mathbf{R}_+$. ∎

5.4. Example. Suppose that $f(t)=(1+t)^{-b}$, so that (X_1, X_2) has an inverted Dirichlet distribution. By checking that for any $u_1>u_2>0$, the function $f(tu_1^{-1})/f(tu_2^{-1})$ is strictly increasing, it follows that $f(tu^{-1})$ is STP_2. Therefore $F_{\boldsymbol{\theta}}-F_{\boldsymbol{\phi}}$ has exactly one sign change on $\mathbf{R}_+$.

5.5. *Remark.* The kernel $K(t,u)=f(tu^{-1})$ is STP_2 on $\mathbf{R}_+^2$ iff the function $g(t):=f(e^t)$, $t\in\mathbf{R}$, strictly log-concave. To see this, note that K is STP_2 iff $K_1(t,u)=K(e^t,e^u)\equiv f(e^{t-u})$ is STP_2 on $\mathbf{R}^2$. That is, the function g is strictly PF_2 on $\mathbf{R}$. By [8, p. 332, Proposition 1.2], g is strictly log-concave.

When $n=2$, we can again extend the analysis in [3, Proposition 2.2] to obtain an upper bound for the crossing point, t^*. This analysis is appealing, since it holds for a large class of Liouville distributions, while providing in the i.i.d. gamma case the same upper bound obtained by Diaconis and Perlman [3]. In what follows, we let $\psi_t(u)=f(tu^{-1})\,u^{-(2a+1)}$, $u>0$, $t>0$. Further, for the rest of the paper, we will assume that f is log-concave and that $f'(0+)/f(0+)$ exists and is nonzero.

5.6. Lemma. *Suppose that $n=2$. Then there exists t_0, depending only on $\bar{\theta}$, such that*

$$\psi_t(u)-\psi_t(2\bar{\theta}-u)<0, \qquad 0<u<\bar{\theta},$$

and for all $t\geqslant t_0$.

Proof. Note that $\psi_t(u)-\psi_t(2\bar{\theta}-u)<0$ if and only if

$$\frac{f(t/(2\bar{\theta}-u))}{f(tu^{-1})}>\frac{(2\bar{\theta}-u)^{2a+1}}{u^{2a+1}}. \tag{5.5}$$

Then with $t = \bar{\theta} t_1$ and $v = (\bar{\theta} - u)/u$, (5.5) is equivalent to positivity of the function

$$g(v) := \log f\left(\frac{(v+1)\,t_1}{2v+1}\right) - \log f((v+1)\,t_1)$$
$$- (2a+1)\log(2v+1), \qquad v > 0.$$

Note that $g(0+) = 0$. Further,

$$g'(v) = -t_1 \left[\frac{f'\left(\frac{(v+1)\,t_1}{2v+1}\right)}{(2v+1)^2 f\left(\frac{(v+1)\,t_1}{2v+1}\right)} + \frac{f'((v+1)\,t_1)}{f((v+1)\,t_1)}\right] - \frac{2(2a+1)}{2v+1}. \tag{5.6}$$

Since $f(tu^{-1})$ is STP_2 in (t, u) then, by Remark 5.5, $[\log f(e^t)]' = e^t f'(e^t)/f(e^t)$ is strictly decreasing on $\mathbf{R}$; equivalently, the function $vf'(v)/f(v)$ is strictly decreasing on $\mathbf{R}_+$. Then for $v \in \mathbf{R}_+$,

$$\frac{vf'(v)}{f(v)} < \lim_{v \to 0+} \frac{vf'(v)}{f(v)} = 0,$$

since $f'(0+)/f(0+)$ exists. Therefore $f'(v) < 0$, or f is strictly decreasing, on $\mathbf{R}_+$. It also follows from these observations that, since $f'(0+)/f(0+)$ is nonzero then it is negative; otherwise, we would have $f'(v) > 0$ for v sufficiently close to zero. We will let $\delta = -f'(0+)/f(0+)$.

Since f is also log-concave then $f'(v)/f(v)$ is monotone decreasing; hence $f'(v)/f(v) \leqslant f'(0+)/f(0+) = -\delta$ for all $v \in \mathbf{R}_+$. Then by (5.6),

$$g'(v) > \left[\frac{1}{(2v+1)^2} + 1\right] \delta t_1 - \frac{2(2a+1)}{2v+1}.$$

Therefore $g'(v) > 0$, and in turn $g(v) > 0$, for all $v > 0$ if we choose t_1 so that

$$\begin{aligned}\delta t_1 &> \sup_{0<v<\infty} \frac{2(2a+1)}{2v+1}\left[\frac{1}{(2v+1)^2} + 1\right]^{-1} \\ &= 2(2a+1) \sup_{0<v<\infty} \frac{2v+1}{(2v+1)^2+1} \\ &= 2a+1,\end{aligned}$$

the supremum being attained at $v = 0$. This proves that $\psi_t(u) - \psi_t(2\bar{\theta} - u) < 0$ for all $0 < u < \bar{\theta}$, where $t \geqslant t_0 = (2a+1)\bar{\theta}/\delta$. ∎

5.7. COROLLARY. *Suppose that* $n=2$. *Then* $t^* < (2a+1)\bar{\theta}/\delta$, $\delta = -f'(0+)/f(0+)$.

Proof. As before, the proof is exactly as in [3]. By (5.4) and the symmetry about zero of the distribution of $Z_1 - Z_2$, the function $\Lambda(u) := H_{\boldsymbol{\theta}}(u) - H_{\boldsymbol{\phi}}(u)$ is antisymmetric about $\bar{\theta}$, $\Lambda(u) = -\Lambda(2\bar{\theta} - u)$. By (5.3),

$$\begin{aligned} c_1^{-1} t^{-2a}[F_{\boldsymbol{\theta}}(t) - F_{\boldsymbol{\phi}}(t)] &= \int_0^{\bar{\theta}} \Lambda(u)\, \psi_t(u)\, du + \int_{\bar{\theta}}^{2\bar{\theta}} \Lambda(u)\, \psi_t(u)\, du \\ &= \int_0^{\bar{\theta}} \Lambda(u)[\psi_t(u) - \psi_t(2\bar{\theta} - u)]\, du. \end{aligned}$$

By Lemma 5.5, $\psi_t(u) - \psi_t(2\bar{\theta} - u) < 0$ for $0 < u < \bar{\theta}$ when $t \geqslant (2a+1)\bar{\theta}/\delta$; hence $F_{\boldsymbol{\theta}}(t) - F_{\boldsymbol{\phi}}(t) < 0$ if $t \geqslant (2a+1)\bar{\theta}/\delta$. By (5.1), $t^* < (2a+1)\bar{\theta}/\delta$. ∎

For $n \geqslant 3$, we can also obtain bounds for t^* similar to the results of [1, 3]. The following results are proved used the methods of [1].

5.8. PROPOSITION. *Suppose that there exists a positive constant* γ *such that* $f'(t)/f(t) \geqslant -\gamma$ *for all* $t \in \mathbf{R}_+$. *Then*

(i) $F_{\boldsymbol{\theta}}(t)$ *is Schur-concave for* $\boldsymbol{\theta}$ *in the region* $\{\boldsymbol{\theta} : t \geqslant (na+1)\,\delta^{-1} \max_{1 \leqslant i \leqslant n} \theta_i\}$.

(ii) $F_{\boldsymbol{\theta}}(t)$ *is Schur-convex for* $\boldsymbol{\theta}$ *in the region* $\{\boldsymbol{\theta} : t \leqslant (na+1)\,\gamma^{-1} \min_{1 \leqslant i \leqslant n} \theta_i\}$.

(iii) *If* $\boldsymbol{\theta} \succ \boldsymbol{\phi}$ *then*

$$(na+1)\,\gamma^{-1} \min_{1 \leqslant i \leqslant n} \theta_i \leqslant t^* \leqslant (na+1)\,\delta^{-1} \max_{1 \leqslant i \leqslant n} \theta_i.$$

Proof. It follows from (2.8) that

$$F_{\boldsymbol{\theta}}(t) = P\left[V_1 \sum_{i=1}^n \theta_i Z_i \leqslant t\right] = Eg\left(\sum_{i=1}^n \theta_i Z_i\right),$$

where, since $V_1 \sim L_1[f; na]$,

$$g(u) = P(V_1 \leqslant tu^{-1}) = c \int_0^{tu^{-1}} x^{na-1} f(x)\, dx,$$

$c = 1/W^{na} f(0)$. Then $g'(u) = -ct^{na} u^{-(na+1)} f(tu^{-1})$ and

$$\begin{aligned} g''(u) &= ct^{na} u^{-(na+2)} f(tu^{-1}) \left[na + 1 + \frac{tf'(tu^{-1})}{uf(tu^{-1})} \right] \\ &\leqslant ct^{na} u^{-(na+2)} f(tu^{-1})(na + 1 - \delta tu^{-1}), \end{aligned}$$

since $f'(tu^{-1})/f(tu^{-1}) \leqslant -\delta$, as shown earlier. Therefore $g''(u) \leqslant 0$, or g is concave, if $t \geqslant \delta^{-1}(na+1)u$. So if $\boldsymbol{\theta}$ is such that $t \geqslant (na+1)\,\delta^{-1} \max \theta_i$, then $\sum \theta_i Z_i \leqslant (\max \theta_i) \sum Z_i \leqslant \delta t/(na+1)$, a.s. Then $Eg(\sum \theta_i Z_i)$ is symmetric and concave, hence Schur-concave. in $\boldsymbol{\theta}$. This proves (i), and (ii) is proved by a similar argument.

To prove (iii) note that if $\boldsymbol{\theta} \succ \boldsymbol{\phi}$, then $F_{\boldsymbol{\theta}}(t) - F_{\boldsymbol{\phi}}(t)$ is nonnegative or nonpositive according as $F_{\boldsymbol{\theta}}(t)$ is Schur-concave or Schur-convex, respectively, in $\boldsymbol{\theta}$. Then the conclusion follows directly from (i) and (ii) and the uniqueness of t^*. ∎

Appendix

Here, we prove that (4.8) is negative for $n=4$ and $r=3$. Using the notation

$$\Delta_r(t) := \sum_{k=1}^{r} \left[F_{Y_{(k)}}(t) - F_{X_{(k)}}(t)\right],$$

then for any indeterminate x, it follows from (4.8) that

$$\begin{aligned}\sum_{r=1}^{n} x^{r-1} \Delta_r(t) &= \sum_{r=1}^{n} x^{r-1} \sum_{j=0}^{r-1} c_{r,j}\left[h(t)^{n-j} - h((n-j)t)\right] \\ &= \sum_{j=0}^{n-1} \left(\sum_{r=j+1}^{n} x^{r-1} c_{r,j}\right) \left[h(t)^{n-j} - h((n-j)t)\right]. \qquad \text{(A.1)}\end{aligned}$$

However,

$$\begin{aligned}\sum_{r=j+1}^{n} x^{r-1} c_{r,j} &= \binom{n}{j} \sum_{r=j+1}^{n} x^{r-1} \sum_{k=0}^{r-j-1} (-1)^k \binom{n-j-1}{k} \\ &= \binom{n}{j} \sum_{k=0}^{n-j-1} \sum_{r=k+j+1}^{n} (-1)^k \binom{n-j-1}{k} x^{r-1} \\ &= \binom{n}{j} \sum_{k=0}^{n-j-1} (-1)^k \binom{n-j-1}{k} \sum_{r=k+j+1}^{n} x^{r-1} \\ &= \binom{n}{j} (1-x)^{-1} \sum_{k=0}^{n-j-1} (-1)^k \binom{n-j-1}{k} (x^{k+j} - x^n).\end{aligned}$$

Using the binomial to evaluate these two sums, we obtain

$$\sum_{r=j+1}^{n} x^{r-1} c_{r,j} = \binom{n}{j} x^j (1-x)^{n-j},$$

and then, by (A.1),

$$\sum_{r=1}^{n} x^{r-1} \Delta_r(t) = \sum_{j=0}^{n-1} \binom{n}{j} x^j (1-x)^{n-j} \left[h(t)^{n-j} - h((n-j)t)\right]$$

$$= \sum_{j=0}^{n} \binom{n}{j} x^j (1-x)^{n-j} \left[h(t)^{n-j} - h((n-j)t)\right].$$

This latter expression can be used, if necessary, to evaluate (A.1) in closed form. In particular, by substituting $x = 1$, we find that $\sum_{r=1}^{n} \Delta_r(t) \equiv 0$.

In the case when $n = 4$, we have shown that for $t > 0$, $\Delta_1(t) > 0$, $\Delta_2(t) > 0$, $\Delta_4(t) \equiv 0$, and $\sum_{j=1}^{4} \Delta_j(t) \equiv 0$. Therefore $\Delta_3(t) = -[\Delta_1(t) + \Delta_2(t)] < 0$, $t > 0$. ∎

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REFERENCES

[1] BOCK, M. E., DIACONIS, P., HUFFER, F. W., AND PERLMAN, M. D. (1987). Inequalities for linear combinations of gamma random variables. *Canad. J. Statist.* **15** 387–395.

[2] DHARMADHIKARI, S., AND JOAG-DEV, K. (1988). *Unimodality, Convexity, and Applications.* Academic Press, New York.

[3] DIACONIS, P., AND PERLMAN, M. D. (1991). Bounds for tail probabilities of weighted sums of independent gamma random variables. In *Topics in Statistical Dependence* (H. W. Block *et al.*, Eds.). IMS Lecture Notes, Monograph Series, Vol. 16, pp. 147–166. Inst. Math. Statistics, Hayward, CA.

[4] FANG, K.-T., KOTZ, S., AND NG, K. W. (1990). *Symmetric Multivariate and Related Distributions.* Chapman & Hall, London.

[5] GUPTA, R. D., AND RICHARDS, D. ST. P. (1987). Multivariate Liouville distributions. *J. Multivariate Anal.* **23** 233–256.

[6] GUPTA, R. D., AND RICHARDS, D. ST. P. (1990). The Dirichlet distributions and polynomial regression. *J. Multivariate Anal.* **32** 95–102.

[7] GUPTA, R. D., AND RICHARDS, D. ST. P. (1992). Multivariate Liouville distributions, II. *Probab. Math. Statist.* **12** 291–309.

[8] KARLIN, S. (1968). *Total Positivity.* Stanford Univ. Press, Stanford, CA.

[9] KARLIN, S., MICCHELLI, C. A., AND RINOTT, Y. (1986). Multivariate splines: A probabilistic perspective. *J. Multivariate Anal.* **20** 69–90.

[10] LEFEVRE, C., AND MALICE, M.-P. (1989). On a system of components with joint lifetimes distributed as a mixture of independent exponential laws. *J. Appl. Probab.* **26** 202–208.

[11] LEHMANN, E. (1959). *Testing Statistical Hypotheses.* Wiley, New York.

[12] MARSHALL, A. W., AND OLKIN, I. (1979). *Inequalities: Theory of Majorization and Its Applications.* Academic Press, New York.

[13] NAYAK, T. (1987). Multivariate Lomax distributions: Properties and usefulness in reliability theory. *J. Appl. Probab.* **24** 170–177.

[14] SHAKED, M. (1977). A concept of positive dependence for exchangeable random variables. *Ann. Statist.* **5** 505–515.

[15] SHAKED, M., AND TONG, Y. L. (1984). Stochastic orderings of spacings from dependent random variables. In *Inequalities in Statistics and Probability* (Y. L. Tong, Ed.). IMS Lecture Notes, Monograph Series, Vol. 5, pp. 141–149. Inst. Math. Statistics, Hayward, CA.

[16] SHAKED, M., AND TONG, Y. L. (1985). Some partial orderings of exchangeable random variables by positive dependence. *J. Multivariate Anal.* **17** 333–349.

[17] STOYAN, D. (1983). *Comparison Methods for Queues and Other Stochastic Models.* Wiley, New York.