Journal of Multivariate Analysis 146 (2016) 119–137

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Detecting and estimating intensity of jumps for discretely observed ARMA$D(1, 1)$ processes

D. Blanke [a,*], D. Bosq [b]

[a] Avignon University, LMA EA2151, 33 rue Louis Pasteur, F-84000 Avignon, France

[b] Sorbonne Universités, UPMC Univ Paris 06, EA 3124, Laboratoire de Statistique Théorique et Appliquée, F-75005, 4 place Jussieu, Paris, France

ARTICLE INFO

Article history:
Received 31 December 2014
Available online 5 September 2015

AMS 2010 subject classifications:
primary 62M
60F17

Keywords:
Functional linear processes
ARMA$D(1, 1)$ processes
Jumps
Estimation of intensity
Discrete data

ABSTRACT

We consider n equidistributed random functions, defined on [0,1], and admitting fixed or random jumps, the context being $D[0, 1]$-valued ARMA(1, 1) processes. We begin with properties of ARMA$D(1, 1)$ processes. Next, different scenarios are considered: fixed instants with a given but unknown probability of jumps (the deterministic case), random instants with ordered intensities (the random case), and random instants with non ordered intensities (the completely random case). By using discrete data and for each scenario, we identify the instants of jumps, whose number is either random or fixed, and then estimate their intensity.

1. Introduction

1.1. There is an abundant literature concerning functional data analysis (FDA) and prediction of stochastic processes in infinite dimensional spaces. In particular, the books by Ramsay and Silverman [50], Ferraty and Vieu [31], Ferraty and Romain [30], Horváth and Kokoszka [35] and the recent book edited by Bongiorno et al. [9] contain interesting theoretical and practical results. See also [11,13]. In general, X takes its value in $L^2 = L^2([0, h])$ or in $C = C([0, h])$, but, in some situations, one may consider that a *jump* does exist if there is a large peak: see, for example the annual sediment in [9, p. 8]. Thus, it is perhaps more natural to consider the space $D = D([0, h])$ which is càdlàg and equipped with the Skorohod metric d° (see [6, p. 125]): with that metric, D becomes a separable complete metric space. Note that this metric is not easy to compute. In this paper, we consider càdlàg processes from a functional point of view: by this way, we work in the context of FDA with jumps.

1.2. Works dedicated to *jumps in stochastic processes* appear very often: actually, there are more than 1200 papers concerning them. Thus, we may only give recent and limited references. For example, processes with jumps are widely used in finance: we may refer to [20,54]; [39, part 2], [29, ch. 10]; [49], etc.; but applications can also be found in fields as varied as the environment, medicine, reliability, etc., see e.g. [33,4,16,10]. Many mathematical models have been proposed and studied [25,42,34,24], and statistical estimation appears e.g. in [18,17,26], etc. Note that the pioneer paper concerning jumps appears in Lévy [44]. Other references of interest will appear below.

* Corresponding author.
E-mail addresses: delphine.blanke@univ-avignon.fr (D. Blanke), denis.bosq@upmc.fr (D. Bosq).

http://dx.doi.org/10.1016/j.jmva.2015.08.014
0047-259X/

1.3. Now, here and in the books quoted in Section 1.1, our purpose is somewhat different since we want to observe a process over a *sequence of time intervals*. More precisely, let $(\xi_t,\ t \in \mathbb{R})$ be a real measurable continuous time process. We put

$$X_n(t) = \xi_{(n-1)h+t}, \quad 0 \le t \le h,\ n \in \mathbb{Z} \tag{1.1}$$

where $h > 0$ is a time interval. The process may contain some jumps and we envisage to *detect* them and to estimate *intensity of jumps*, given the data $X_1, \ldots, X_n$.

Another motivation should be *prediction of* X_{n+1} *over the time interval* $[nh, (n+1)h]$. One way to predict X_{n+1} would be to treat *continuous time and jumps separately* (see [51,1,52], etc.). As an example, consider the functional autoregressive process of order 1 (ARD(1)):

$$X_{n+1}(t) = \rho(X_n)(t) + Z_{n+1}(t), \quad 0 \le t \le h,\ n \in \mathbb{Z},$$

where ρ is a continuous linear operator with respect to the sup-norm. Then, in order to separate the continuous part from the jump's part, we may suppose that $\rho(D) \subset C$. That condition is satisfied by the *Ornstein–Uhlenbeck process driven by a Levy process*, cf Example 2.1. Another classical example is given by: $\rho_r(x)(t) = \int_0^h r(s,t)x(s)\,ds$, $0 \le t \le h, x \in D$ where r satisfies Example 2.2, see also [21,35], etc. Thus, $\rho_r(x) \in C$. Finally, the condition $\rho(D) \subset C$ seems quite standard and characterizes the unpredictability of jumps by confining them in the innovation process. Now, the best probabilistic predictor of X_{n+1} is $\rho(X_n)$ and it can be approximated by using an estimator of ρ. An exponential rate is obtained in [11, p. 222–235], when the detector and intensity of jumps appear in the current paper. One direction (currently under development) will consist in combining the two approaches to improve the prediction.

1.4. A more general model should be the ARMA$D(p,p)$ process defined by

$$X_n - \rho_1(X_{n-1}) - \cdots - \rho_p(X_{n-p}) = Z_n - \rho_1'(Z_{n-1}) - \cdots - \rho_p'(Z_{n-p}), \quad n \in \mathbb{Z},$$

where X_n and Z_n are D-valued and where $\rho_j, \rho_j', j, j' = 1, \ldots, p$ are continuous linear operators with respect to the sup-norm. In order to study this process, it should be possible to work in the space $D([0,h]^p)$ (cf [43]). Note that if $\rho_j, \rho_j', j, j' = 1, \ldots, p$ are C-valued, X_n and Z_n have again the same jumps.

Now, since this model is difficult to handle, and in order to simplify the exposition, we take $p = 1$ and write

$$X_n - \rho(X_{n-1}) = Z_n - \rho'(Z_{n-1}), \quad n \in \mathbb{Z},$$

note that, Z_{n-1} may be replaced with an exogenous variable (see for example [32]).

1.5. We now give some practical examples of *jumps over time intervals*:

- a patient's *electrocardiogram* at each minute [46,48,45];
- the *temperature* day by day [56];
- *El Niño* southern oscillation (ENSO): a prediction over one year shows a jump in May [5];
- *wave amplitude* [55];
- *pollution* day by day [35];
- *credit cards* transaction and its prediction [35];
- another example is *electricity consumption*: it admits a jump early in the morning and in the evening (see [3,27,28]);
- administration of a *drug treatment*: each day produces a shock at time intervals (see [40]);
- *astronomical time series* with 100000 data (see [48]);
- *earthquake* and *explosion*: [46];
- predicting *ozone* [36,22,15,23];
- predicting the *euro–dollar rate* [41];
- finally, the *mistral gust* during one day or one week is one of our objective for prediction: 240000 data are at our disposal. Predicting the greatest jump should be of interest, see [37].

1.6. In our considered framework, preliminary results were first obtained by Bosq [12] and, the case of observations in continuous time also appears in [7]. Here, we use *high frequency data* (HFD); this scheme appears in many situations (see [8,19,2] among others). Concerning prediction with HFD, practical results will be studied later with combined predictors. In particular, we will apply the results to the mistral gusts with big data.

1.7. In Section 2, we introduce the ARMA$D(1,1)$ model which is connected with FDA:

$$X_n - m - \rho(X_{n-1} - m) = Z_n - \rho'(Z_{n-1}), \quad n \in \mathbb{Z} \tag{1.2}$$

where m is a trend and $\rho(D) \subset C$, $\rho'(D) \subset C$ so that (X_n) and (Z_n) have the same jumps. We give several properties of (1.2) as well as examples. In the following, we study various types of jumps.

In Section 3, we consider data of the form $X_i(\frac{\ell}{q_n})$, $\ell = 0, \ldots, q_n$, $q_n \ge 1$, $i = 1, \ldots, n$; where ℓ and q_n are integers and $(X_1, \ldots, X_n)$ are D-valued realizations of (1.2). We consider the case of fixed but unknown instants of jumps $t_1, \ldots, t_k$, where t_j, denotes the jth jump, $j = 1, \ldots, k$ and k is unknown too. In this part, each jump may occur randomly at time t_j with unknown probability $p_j \in]0,1]$, $j = 1, \ldots, k$, so the number of jumps is a random variable depending on $i = 1, \ldots, n$.

We propose and study detectors of jumps and next, we derive estimators of each intensity of jumps by estimating p_j and plug-in the detectors.

Section 4 is devoted to the case of random instants of jumps: $0 < T_1 < T_2 < \cdots < T_{K_i} < 1$ with K_i a $\mathbb{N}$-valued random variable. We consider the case where intensities of jumps have the same ordering in each X_i. To estimate these intensities, we detect the k, $k \geq 1$, first jumps by considering separately each X_i. Here, as K_i is random, the difficulty is to select the sample paths with at least k jumps. In this section, we also derive results for estimating the maximal jump.

In Section 5, we consider a final scheme, the completely random one where the ordering of jumps varies from each sample X_i. Similarly as in the previous section, we detect the jumps with each trajectories considered separately. To estimate their intensities, their random ordering makes the problem intricate but we propose a method in the case where the number of jumps is fixed. It is based on a trick, derived from Viète's formula, that allows us to provide estimations (based on numerical approximation for a number of jumps greater than 4).

2. ARMAD(1, 1) processes

2.1. Model and properties

In order to study the jumps of the real continuous time process $X = (X_t,\ 0 \leq t \leq h)$, $h > 0$, we consider the space $D = D([0, h])$ of càdlàg real functions defined over $[0, h]$. The sup-norm $\|x\| = \sup_{0\leq t\leq h} |x(t)|$ entails non-separability of D. Thus, it is more convenient to use the modified Skorohod metric d° (cf [6, p. 125]); with that metric, D becomes a complete separable space.

The process X being defined on the probability space $(\Omega, \mathcal{A}, P)$, we suppose that it is $\mathcal{A} - \mathcal{D}$ measurable where $\mathcal{D}$ is the σ-algebra generated by d°. Concerning measurability we refer to [38].

Now, if ρ is a bounded linear operator, i.e. $\|\rho\|_{\mathcal{L}} = \sup_{x\in D, \|x\|\leq h} \|\rho(x)\| < \infty$, then, it is $\mathcal{D}$–$\mathcal{D}$ measurable. Also if there is a jump at t_0, $x \mapsto x(t_0) - x(t_0^-)$ is a continuous linear form on $(D, \|\cdot\|)$, see [47].

We consider the ARMAD(1, 1) process defined as

$$X_n - m - \rho(X_{n-1} - m) = Z_n - \rho'(Z_{n-1}), \quad n \in \mathbb{Z}, \tag{2.1}$$

where ρ and ρ' are bounded linear operators, $m = \mathbb{E}\,(X_n)$, and (Z_n) is a strong white noise, i.e. the sequence (Z_n) is i.i.d., and such that $\mathbb{E}\,\|Z_n\|^2 < \infty$, $\mathbb{E}\,(Z_n) = 0$. Note also the presence of the trend m.

In order to show existence of the ARMAD process we make the following assumption, weaker than those considered in [7]:

Assumption 2.1 (A2.1). $\exists j_0 \geq 1\ :\ \left\|\rho^{j_0}\right\|_{\mathcal{L}} < 1$ and $\exists j_1 \geq 1\ :\ \left\|\rho'^{j_1}\right\|_{\mathcal{L}} < 1$.

Lemma 2.1. *If Assumption 2.1 holds, we have*

$$X_n - m =_{L^2_D} \sum_{j=0}^{\infty} \rho^j\Big(Z_{n-j} - \rho'(Z_{n-1-j})\Big), \quad n \in \mathbb{Z}, \tag{2.2}$$

so the process $(X_n - m,\ n \in \mathbb{Z})$ is stationary and $(Z_n,\ n \in \mathbb{Z})$ is the innovation of $(X_n,\ n \in \mathbb{Z})$.

Proof. To simplify the exposition, let us assume that $m = 0$. We may write $Y_n = Z_n - \rho'(Z_{n-1})$, $n \in \mathbb{Z}$ then, (Y_n) is an equidistributed sequence and $\|Y_n\| \leq \|Z_n\| + \left\|\rho'\right\|_{\mathcal{L}} \|Z_{n-1}\|$, thus

$$\mathbb{E}\,\|Y_n\|^2 \leq 2\mathbb{E}\,\|Z_n\|^2 + 2\left\|\rho'\right\|^2_{\mathcal{L}} \mathbb{E}\,\|Z_{n-1}\|^2 \leq 2(1 + \left\|\rho'\right\|^2_{\mathcal{L}})\mathbb{E}\,\|Z_0\|^2 < \infty.$$

Now, we study

$$\mathbb{E}\left\|\sum_{j\geq k+1} \rho^j(Y_{n-j})\right\|^2 \leq \sum_{j,j'\geq k+1} \|\rho\|^j_{\mathcal{L}} \|\rho\|^{j'}_{\mathcal{L}}\, \mathbb{E}\,\left(\left\|Y_{n-j}\right\| \left\|Y_{n-j'}\right\|\right) \leq \mathbb{E}\,\left(\|Y_0\|^2\right)\left(\sum_{j\geq k+1} \|\rho\|^j_{\mathcal{L}}\right)^2$$

and A2.1 implies $\sum_{j\geq k+1} \|\rho\|^j_{\mathcal{L}} \xrightarrow[k\to\infty]{} 0$ which gives (2.2). Finally, the condition $\exists j_1 \geq 1$: $\left\|\rho'^{j_1}\right\|_{\mathcal{L}} < 1$ gives invertibility and consequently, (Z_n) is the innovation of (X_n). ■

We consider the following assumption:

Assumption 2.2 (A2.2). $\rho(D) \subset C$, $\rho'(D) \subset C$; also, $m \in C$.

From (2.1) and A2.2, one obtains for each time of jump t_0 (fixed or random)

$$\Delta_n(t_0) := X_n(t_0) - X_n(t_0^-) = Z_n(t_0) - Z_n(t_0^-),\ n \in \mathbb{Z}$$

which shows that X_n and Z_n have the same jumps and that $\big(\Delta_n(t_0)\big)$ is i.i.d. This assumption is reasonable since we have the following examples. Here, we may suppose that $m = 0$ in order to simplify the exposition.

Example 2.1. Consider the *Ornstein–Uhlenbeck driven by a Levy process* given by:

$$\xi_t = \int_{-\infty}^{t} e^{-\theta(t-s)} \,\mathrm{d}L(s), \quad t \in \mathbb{R}\ (\theta > 0)$$

[14,20] and observed over a sequence of time intervals. By using (1.1), we obtain an ARD(1) process that satisfies $X_{n+1}(t) = \rho_\theta(X_n)(t) + Z_{n+1}(t)$, $0 \le t \le h$, $n \in \mathbb{Z}$, where the linear operator ρ_θ has the shape $\rho_\theta(x)(t) = e^{-\theta t}x(h)$, $0 \le t \le h$, $x \in D$ and with

$$Z_{n+1}(t) = \int_{nh}^{nh+t} e^{-\theta(nh+t-s)} \,\mathrm{d}L(s), \quad 0 \le t \le h,\ n \in \mathbb{Z}.$$

Then (Z_n) is a strong white noise which may contain jumps, and since $\rho_\theta(x) \in C$, X_n and Z_n have the same jumps.

Example 2.2. Set $\rho_r(x)(t) = \int_0^h r(s,t)x(s)\,\mathrm{d}s$, $0 \le t \le h$, $x \in D$, with $\left|r(s,t) - r(s,t')\right| \le c\left|t-t'\right|^\alpha$; $0 < \alpha \le 1$, $0 \le t, t' \le h$, $c > 0$, then

$$\left|\rho_r(x)(t') - \rho_r(x)(t)\right| \le c\left|t-t'\right|^\alpha \int_0^h |x(s)|\ \mathrm{d}s \xrightarrow[|t-t'|\to 0]{} 0$$

since x is bounded (see [6], p. 122). Then, a classical example of ARMAD(1, 1) may be derived with $X_n - \rho_r(X_{n-1}) = Z_n - \rho_{r'}(Z_{n-1})$, $n \in \mathbb{Z}$ where $\rho_{r'}$ satisfies a similar condition as ρ_r.

Example 2.3. Put $X_{n+1} = \rho(X_n) + Z_{n+1}$, $n \in \mathbb{Z}$, where $\left\|\rho^{j_0}\right\|_{\mathcal{L}} < 1$ for some $j_0 \ge 1$. Then, it is possible to predict X_{n+1} by considering *continuous time and jumps separately* (see [51,1,52]). Thus, we may suppose that $\rho(D) \subset C$.

2.2. Discrete data

Here the data are supposed to be discrete. They take the form $X_i(\frac{\ell}{q_n})$, $\ell = 0, \dots, q_n$, $q_n \ge 1$, $i = 1, \dots, n$, where ℓ and q_n are integers and $q_n \to \infty$ as $n \to \infty$. Now, in all the following we set $h = 1$ so if $(\xi_t,\ t \in \mathbb{R})$ is the real measurable continuous time process such that $X_i(t) = \xi_{i-1+t}$, one observes ξ_t at $nq_n + 1$ discrete times $0, q_n^{-1}, \dots, n - q_n^{-1}, n$.

The instants of jumps associated with X_i are denoted by $T_{i1}, \dots, T_{iK_i}$, they can be fixed or random, as well as K_i, and they satisfy $0 < T_{i1} < \cdots T_{iK_i} < 1$, $i = 1, \dots, n$, almost surely (a.s.). Next, in order to avoid local irregularity we need the following hypothesis:

Assumption 2.3 (A2.3)**.** For $0 < \alpha \le 1$, $(s,t) \in [0,1]^2$:

(i) For $x \in D$, the functions $\rho(x)$, $\rho'(x)$ and m are Hölderian:

$$\begin{aligned}
&|\rho(x)(t) - \rho(x)(s)| \le a(x)\,|t-s|^\alpha \quad (a > 0),\\
&\left|\rho'(x)(t) - \rho'(x)(s)\right| \le b(x)\,|t-s|^\alpha \quad (b > 0),\\
&|m(t) - m(s)| \le c_m\,|t-s|^\alpha \quad (c_m > 0).
\end{aligned}$$

(ii) For i.i.d. and integrable M_i: $|Z_i(t) - Z_i(s)| \le M_i\,|t-s|^\alpha$, $(s,t) \in I_{iK_i}$ where $I_{iK_i} = [0, T_{i1}[^2 \cup \cdots \cup [T_{iK_i}, 1[^2$, $i = 1, \dots, n$.

Note that Example 2.2 satisfies A2.3-(i) with $a(x) = c\int_0^1 |x(s)|\ \mathrm{d}s$ and that the Ornstein–Uhlenbeck process or the fractional Brownian motion with jumps satisfies condition A2.3-(ii).

In the following, we will use repeatedly the following result since it gives a measure of proximity between increments of X and Z.

Lemma 2.2. *Under the condition* A2.3-(i), *we have:*

$$\left| |X_i(s) - X_i(t)| - |Z_i(s) - Z_i(t)| \right| \le \left(a(X_{i-1}) + b(Z_{i-1}) + c_m\,\|I - \rho\|_{\mathcal{L}}\right)|s-t|^\alpha,$$

$i = 1, \dots, n$, $(s,t) \in [0,1]^2$.

Proof. The relation (2.1) gives

$$\begin{aligned}
|X_i(s) - X_i(t)| &\le |Z_i(s) - Z_i(t)| + |\rho(X_{i-1})(s) - \rho(X_{i-1})(t)|\\
&\quad + \left|\rho'(Z_{i-1})(s) - \rho'(Z_{i-1})(t)\right| + \|I - \rho\|_{\mathcal{L}}\,|m(s) - m(t)|
\end{aligned} \tag{2.3}$$

next, A2.3-(i) implies $|X_i(s) - X_i(t)| \le |Z_i(s) - Z_i(t)| + \left(a(X_{i-1}) + b(Z_{i-1}) + c_m\,\|I - \rho\|_{\mathcal{L}}\right)|s-t|^\alpha$, and reversing the inequality, one obtains the result. ■

Note that if m is constant, it disappears in (2.3) and the term $c_m \|I-\rho\|_{\mathcal{L}}$ is no longer relevant. One may handle this case with the choice $c_m = 0$. The next result shows that, excluding the jump's times, X_i satisfies also a Hölder type condition. It is a direct consequence of Lemma 2.2 and condition A2.3-(ii).

Corollary 2.1. *Under Assumption 2.3, we have*

$$|X_i(s) - X_i(t)| \le \left(a(X_{i-1}) + b(Z_{i-1}) + M_i + c_m \|I-\rho\|_{\mathcal{L}}\right) |s-t|^{\alpha}, \quad i = 1, \ldots, n$$

provided $(s,t) \in I_{iK} = [0, T_{i1}[^2 \cup \cdots \cup [T_{iK}, 1[^2$, $i = 1, \ldots, n$.

Now, throughout the paper, we will suppose that Assumptions 2.1, 2.2 and 2.3 hold.

3. Fixed jumps with k unknown

3.1. Framework

In this part, we consider the model (2.1). The (Z_i) are i.i.d. functional random variables such that each Z_i, $i = 1, \ldots, n$, has at most k distinct jumps, with a *fixed* but *unknown* $k \ge 1$. These jumps may occur randomly at fixed times $t_1, \ldots, t_k$ with $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$ such that $t_{j+1} - t_j \ge \delta_0 > 0$ for all $j = 1, \ldots, k-1$. More precisely, we set:

$$\Delta_{ij} = \left|Z_i(t_j) - Z_i(t_j^-)\right| = \left|X_i(t_j) - X_i(t_j^-)\right| = I_{ij} Y_{ij}, \quad i = 1, \ldots, n$$

where $Z_i(t_j^-) = \lim_{\eta \searrow 0} Z_i(t_j - \eta)$ and $(I_{ij},\ j = 1, \ldots, k)$ are positive random variables that describe the jump amplitudes. Here, we suppose that $\mathbb{P}(I_{ij} \ge \delta_1) = 1$ for some positive δ_1 and, that $(Y_{ij},\ j = 1, \ldots, k)$ are independent random variables with Bernoulli distribution $\mathcal{B}(p_j)$, $p_j \in]0,1]$, $j = 1, \ldots, k$. Also, Y_{ij} and I_{ij} are independent, which means that $\mathbb{E}(\Delta_{ij}) = \mathbb{E}(\Delta_{1j}) = p_j \mathbb{E}(I_{1j}) > 0$, $j = 1, \ldots, k$. Hereafter, we present an example illustrating the considered framework.

Example 3.1 (*Case* $k = 1$). Consider n independent copies of Y_1 with $\mathcal{B}(p_1)$ distribution, $p_1 \in]0,1]$ and $(W_1(t), W_2(t),\ t \in [0,1])$ where W_1 and W_2 are two independent C-valued processes. We set

$$Z_i(t) = W_{i1}(t)\mathbb{I}_{[0,T_{i1}[}(t) + W_{i2}(t)\mathbb{I}_{[T_{i1},1[}(t), \quad i = 1, \ldots, n,\ t \in [0,1[$$

with $T_{i1} = t_1 \in]0,1[$ if $Y_{i1} = 1$ and $T_{i1} = 1$ otherwise. In this case, intensities of jumps are given by $\left|Z_i(t_1) - Z_i(t_1^-)\right| = |W_{i1}(t_1) - W_{i2}(t_1)|\, Y_{i1}$ and each sample path has at most one jump located at t_1. Note that $p_1 = 1$ gives a systematic jump at t_1. Such modeling refers to short-term perturbations that can be interpreted as impulses: for example, we may think of treatments where impulses correspond to the periodic administration of some drugs.

Finally for convenience, we suppose that $\mathbb{E}\,\Delta_{\sigma(1)} > \cdots > \mathbb{E}\,\Delta_{\sigma(k)} > 0$ for some given permutation $\left(\sigma(1), \ldots, \sigma(k)\right)$ of $(1, \ldots, k)$. By this way, we denote by $t_{\sigma(j)}$ the jump time having the jth intensity $\Delta_{\sigma(j)}$, $j = 1, \ldots, k$. Our aim is to estimate the amplitudes $\mathbb{E}(I_{1j})$, $j = 1, \ldots, k$, on the basis of the discretely observed $X_1, \ldots, X_n$ from the model (2.1): $X_i\big(\frac{\ell}{q_n}\big)$, $\ell = 0, \ldots, q_n$, $i = 1, \ldots, n$, where ℓ and $q_n \ge 1$ are integers and $\lim_{n\to\infty} q_n = \infty$. First, we will estimate the times t_{jn}, $j = 1, \ldots, k$ defined as:

$$0 < \frac{\ell_{jn} - 1}{q_n} < t_j \le \frac{\ell_{jn}}{q_n} := t_{jn}, \quad j = 1, \ldots, k.$$

Here and throughout this part, we consider q_n sufficiently large to have $t_i \ne t_j \Rightarrow t_{i,n} \ne t_{j,n}$ and we use notation ℓ_j, $\ell_{\sigma(j)}$ for $\ell_{j,n}$, $\ell_{\sigma(j),n}$. Also, we set

$$\overline{\zeta}_{\ell,n} = \frac{1}{n}\sum_{i=1}^{n} \left|X_i\Big(\frac{\ell}{q_n}\Big) - X_i\Big(\frac{\ell-1}{q_n}\Big)\right|, \quad \ell = 1, \ldots, q_n$$

and make the assumption:

Assumption 3.1 (A3.1).

(i) The distribution of $\overline{\zeta}_{\ell,n}$, $\ell = 1, \ldots, q_n$ is continuous.
(ii) $I_{ij} \ge \delta_1 > 0$ (a.s.), $j = 1, \ldots, k$, $i = 1, \ldots, n$ where δ_1 is fixed.

Finally for $j = 1, \ldots, k$, or $\ell = 1, \ldots, q_n$, we set:

$$\overline{\Delta}_{j,n} = \frac{1}{n}\sum_{i=1}^{n} \left|X_i(t_j) - X_i(t_j^-)\right| \quad \text{and} \quad \overline{\zeta}^{(Z)}_{\ell,n} = \frac{1}{n}\sum_{i=1}^{n} \left|Z_i\Big(\frac{\ell}{q_n}\Big) - Z_i\Big(\frac{\ell-1}{q_n}\Big)\right|.$$

We begin with a result giving the proximity between $\overline{\zeta}^{(Z)}_{\ell_j,n}$ and $\overline{\Delta}_{j,n}$.

Lemma 3.1. *For all $j = 1, \dots, k$, A2.3-(ii) implies that* $\left|\overline{\zeta}^{(Z)}_{\ell_j,n} - \overline{\Delta}_{j,n}\right| \le 2\overline{M} q_n^{-\alpha}$ *with* $\overline{M} = \frac{1}{n}\sum_{i=1}^n M_i$.

Proof. First, note that we have the simple inequality

$$\big|\, |u-v| - |x-y| \,\big| \le |u-x| + |v-y|, \quad u,\ v,\ x,\ y \in \mathbb{R}. \tag{3.1}$$

Since $\overline{\Delta}_{j,n} = \frac{1}{n}\sum_{i=1}^n \left|X_i(t_j) - X_i(t_j^-)\right| = \frac{1}{n}\sum_{i=1}^n \left|Z_i(t_j) - Z_i(t_j^-)\right|$, this implies

$$\begin{aligned}\left|\overline{\zeta}^{(Z)}_{\ell_j,n} - \overline{\Delta}_{j,n}\right| &\le \frac{1}{n}\sum_{i=1}^n \left|\left|Z_i\Big(\frac{\ell_j}{q_n}\Big) - Z_i\Big(\frac{\ell_j - 1}{q_n}\Big)\right| - \left|Z_i(t_j) - Z_i(t_j^-)\right|\right| \\ &\le \frac{1}{n}\sum_{i=1}^n \left|Z_i\Big(\frac{\ell_j}{q_n}\Big) - Z_i(t_j)\right| + \left|Z_i(t_j^-) - Z_i\Big(\frac{\ell_j - 1}{q_n}\Big)\right| \le 2\overline{M} q_n^{-\alpha}\end{aligned}$$

from the condition A2.3-(ii) and the properties $\frac{\ell_j}{q_n} \in [t_j, \frac{\ell_j+1}{q_n}[$ and $t_j^- \in [\frac{\ell_j - 1}{q_n}, t_j[$. ■

3.2. Detection of jumps

Since k and δ_1 are unknown, we consider two sequences: $k_n \to \infty$ and $u_n \to 0$ such that $u_n q_n^\alpha \to \infty$, for $\alpha \in]0, 1]$ defined in Assumption 2.3. For example, if $q_n \simeq n^\beta$, $\beta > 0$, an omnibus choice for u_n is $u_n \simeq (\log n)^{-1}$. In order to detect the jumps, we need the following assumption.

Assumption 3.2 (A3.2)**.** Suppose that one of the following two conditions holds true:

(i) – $\mathbb{E}\,(a(X_1)) < \infty$, $\mathbb{E}\,(b(Z_1)) < \infty$, $\mathbb{E}\,(M_1) < \infty$,
 – $\mathbb{E}\,(\left|Z_1(t_j) - Z_1(t_j^-)\right|^4) < \infty$, $j = 1, \dots, k$,
 – $\sum_{n\ge 1} u_n^{-1} q_n^{-\alpha} < \infty$.
(ii) – $a(X_1) \le a_\infty < \infty$, $\mathbb{E}\,(\exp(c_1 b(Z_1))) < \infty$, $\mathbb{E}\,(\exp(c_2 M_1)) < \infty$, $(a_\infty > 0, c_1 > 0, c_2 > 0)$,
 – $\mathbb{E}\,(\exp(c_3 \left|Z_1(t_j) - Z_1(t_j^-)\right|)) < \infty$, $j = 1, \dots, k$, $(c_3 > 0)$.

Considering Example 2.2, A3.2(i) holds as soon as $\mathbb{E}\,(\int_0^1 |X_1(s)|\, ds) < \infty$ but the condition imposed on q_n implies that sample paths should be observed with high frequency, especially when α is small. Condition A3.2(ii) is more stringent since $a(X_1)$ is supposed to be bounded, but in this case, the only requirement $q_n \to \infty$ is sufficient to derive exponential rates of convergence.

Now, the jumps detection is carried as follows. We set $\widetilde{\ell}_{1,n} = \widetilde{\ell}_1 = \arg\max_{\ell=1,\dots,q_n} \overline{\zeta}_{\ell,n}$ and as $\overline{\zeta}_{\widetilde{\ell}_{j-1}} > u_n$:

$$\widetilde{\ell}_j = \arg \max_{\substack{\ell=1,\dots,q_n \\ \ell\neq\widetilde{\ell}_1,\dots,\ell\neq\widetilde{\ell}_{j-1}}} \overline{\zeta}_{\ell,n}, \quad j = 2, \dots, k_n.$$

The number of detected jumps is then given by

$$\widehat{k} := \widehat{k}_n = \min\left\{j = 1, \dots, k_n \ :\ \overline{\zeta}_{\widetilde{\ell}_j} \le u_n\right\} - 1.$$

Remark that the unique restriction on k_n is that k belongs to $\{1, \dots, k_n\}$ for n large enough: $k_n \to \infty$ is a sufficient condition. Hence if the above set is empty, it means that there exist at least k_n jumps: in this case, from a practical point of view, one has to replace k_n by k_n' with $k_n' > k_n$. Finally, detectors of jumps locations are given by $(\widehat{t}_{1,n}, \dots, \widehat{t}_{\widehat{k},n}) = (\widetilde{t}^*_{1,n}, \dots, \widetilde{t}^*_{\widehat{k},n})$ where $\widetilde{t}^*_{j,n}$ is the jth order statistic associated with $(\widetilde{t}_{1,n}, \dots, \widetilde{t}_{\widehat{k},n}) := (\frac{\widetilde{\ell}_1}{q_n}, \dots, \frac{\widetilde{\ell}_{\widehat{k}}}{q_n})$. Note that (a.s.) uniqueness of $\widehat{t}_{1,n}, \dots, \widehat{t}_{\widehat{k},n}$ is guaranteed by Assumption 3.1 and the next theorem shows that the times of jumps are detected with probability 1.

Theorem 3.1. *Suppose that Assumption 3.1 holds, then the condition A3.2-(i) implies:*

$$\mathbb{P}\Big(\bigcup_{j=1}^{\widehat{k}} \{\widehat{t}_{j,n} \neq t_{j,n}\}\Big) = \mathbb{P}\Big(\bigcup_{j=1}^{\widehat{k}} \{\widetilde{t}_{j,n} \neq t_{\sigma(j),n}\}\Big) = \mathcal{O}\big(n^{-2}\big) + \mathcal{O}\big(u_n^{-1} q_n^{-\alpha}\big); \tag{3.2}$$

while A3.2-(ii) gives:

$$\mathbb{P}\Big(\bigcup_{j=1}^{\widehat{k}} \{\widehat{t}_{j,n} \neq t_{j,n}\}\Big) = \mathbb{P}\Big(\bigcup_{j=1}^{\widehat{k}} \{\widetilde{t}_{j,n} \neq t_{\sigma(j),n}\}\Big) = \mathcal{O}\big(\exp(-c\, n)\big),\ c > 0. \tag{3.3}$$

The same bounds hold for $\mathbb{P}(\widehat{k} \neq k)$ so in both cases, we obtain that a.s. for n large enough, $\widehat{k} = k$ and for $j = 1, \dots, k$: $\widetilde{t}_{j,n} = t_{\sigma(j),n}$.

Proof. We may write $\mathbb{P}(\bigcup_{j=1}^{\widehat{k}}\{\widetilde{\ell}_j \neq \ell_{\sigma(j)}\}) \leq \mathbb{P}(\bigcup_{j=1}^{k}\{\widetilde{\ell}_j \neq \ell_{\sigma(j)}\}) + \mathbb{P}(\widehat{k} \neq k)$. First, we have

$$\mathbb{P}(\widehat{k} \neq k) \leq \mathbb{P}\left(\bigcup_{j=1}^{k}\{\overline{\zeta}_{\widetilde{\ell}_j,n} \leq u_n\}\right) + \mathbb{P}(\overline{\zeta}_{\widetilde{\ell}_{k+1},n} > u_n)$$

where $\mathbb{P}(\bigcup_{j=1}^{k}\{\overline{\zeta}_{\widetilde{\ell}_j,n} \leq u_n\}) \leq \mathbb{P}(\bigcup_{j=1}^{k}\{\overline{\zeta}_{\ell_{\sigma(j)},n} \leq u_n\}) + \mathbb{P}(\bigcup_{j=1}^{k}\{\widetilde{\ell}_j \neq \ell_{\sigma(j)}\})$ and

$$\mathbb{P}(\overline{\zeta}_{\widetilde{\ell}_{k+1,n}} > u_n) \leq \mathbb{P}(\overline{\zeta}_{\widetilde{l}_{k+1},n} > u_n, \cap_{j=1}^{k}\{\widetilde{\ell}_j = \ell_{\sigma(j)}\}) + \mathbb{P}\left(\bigcup_{j=1}^{k}\{\widetilde{\ell}_j \neq \ell_{\sigma(j)}\}\right).$$

Hence,

$$\mathbb{P}\left(\bigcup_{j=1}^{\widehat{k}}\{\widetilde{\ell}_j \neq \ell_{\sigma(j)}\}\right) \leq \sum_{j=1}^{k}\mathbb{P}(\overline{\zeta}_{\ell_{\sigma(j)},n} \leq u_n) + \mathbb{P}(\overline{\zeta}_{\widetilde{l}_{k+1},n} > u_n, \cap_{j=1}^{k}\{\widetilde{\ell}_j = \ell_{\sigma(j)}\}) + 3\,\mathbb{P}\left(\bigcup_{j=1}^{k}\{\widetilde{\ell}_j \neq \ell_{\sigma(j)}\}\right). \tag{3.4}$$

For the first term, we get from Lemmas 2.2 and 3.1 that for $j = 1, \ldots, k$

$$\overline{\zeta}_{\ell_{\sigma(j)},n} \geq \overline{\Delta}_{\sigma(j),n} - q_n^{-\alpha}(2\overline{M} + \bar{a}_X + \bar{b}_Z + c_m\,\|I-\rho\|_{\mathcal{L}}), \tag{3.5}$$

where $\bar{a}_X = \frac{1}{n}\sum_{i=1}^{n} a(X_i)$, $\bar{b}_Z = \frac{1}{n}\sum_{i=1}^{n} b(Z_i)$ and $\overline{M} = \frac{1}{n}\sum_{i=1}^{n} M_i$. Now, we set in all the following $\overline{\Lambda}_n = 2\overline{M} + \bar{a}_X + \bar{b}_Z + c_m\,\|I-\rho\|_{\mathcal{L}}$ for obtaining

$$\begin{aligned}\mathbb{P}(\overline{\zeta}_{\ell_{\sigma(j)},n} \leq u_n) &\leq \mathbb{P}(\overline{\Delta}_{\sigma(j),n} \leq u_n + q_n^{-\alpha}\overline{\Lambda}_n)\\ &\leq \mathbb{P}\left(\left|\overline{\Delta}_{\sigma(j),n} - \mathbb{E}\left(\Delta_{\sigma(j)}\right)\right| \geq \frac{\mathbb{E}\left(\Delta_{\sigma(j)}\right)}{2}\right) + \mathbb{P}\left(\overline{\Lambda}_n \geq \left(\frac{\mathbb{E}\left(\Delta_{\sigma(j)}\right)}{2} - u_n\right)q_n^{\alpha}\right).\end{aligned}$$

These terms are controlled by the following lemma whose proof is postponed to Appendix.

Lemma 3.2. (1) *Under the conditions A3.2-(i), we get* $\mathbb{P}\Big(\left|\overline{\Delta}_{\sigma(j),n} - \mathbb{E}\left(\Delta_{\sigma(j)}\right)\right| \geq \frac{\mathbb{E}(\Delta_{\sigma(j)})}{2}\Big) = \mathcal{O}\big(n^{-2}\big)$ *and* $\mathbb{P}\big(\overline{\Lambda}_n > \big(\frac{\mathbb{E}(\Delta_{\sigma(j)})}{2} - u_n\big)q_n^{\alpha}\big) = \mathcal{O}\big(q_n^{-\alpha}\big)$ *for* $j = 1, \ldots, k$.
(2) *If the conditions A3.2-(ii) hold,* $\mathbb{P}\Big(\left|\overline{\Delta}_{\sigma(j),n} - \mathbb{E}\left(\Delta_{\sigma(j)}\right)\right| \geq \frac{\mathbb{E}(\Delta_{\sigma(j)})}{2}\Big) = \mathcal{O}\big(e^{-c\,n}\big)$, *for some* $c > 0$ *and* $\mathbb{P}\big(\overline{\Lambda}_n > \big(\frac{\mathbb{E}(\Delta_{\sigma(j)})}{2} - u_n\big)q_n^{\alpha}\big) = \mathcal{O}\big(e^{-c\,nq_n^{\alpha}}\big)$ *for* $j = 1, \ldots, k$.

Concerning again (3.4), the term $\mathbb{P}(\overline{\zeta}_{\widetilde{l}_{k+1}} > u_n, \cap_{j=1}^{k}\{\widetilde{\ell}_j = \ell_{\sigma(j)}\})$ is controlled with

$$\left\{\overline{\zeta}_{\widetilde{l}_{k+1},n} > u_n, \cap_{j=1}^{k}\{\widetilde{\ell}_j = \ell_{\sigma(j)}\}\right\} \Rightarrow \left\{\bigcup_{\ell\notin\{\ell_{\sigma(1)},\ldots,\ell_{\sigma(k)}\}}\{\overline{\zeta}_{\ell,n} > u_n\}\right\}$$

and Corollary 2.1 implies that, for all $\ell \notin \{\ell_{\sigma(1)}, \ldots, \ell_{\sigma(k)}\}$:

$$\{\overline{\zeta}_{\ell,n} > u_n\} \Rightarrow \{q_n^{-\alpha}(\bar{a}_X + \bar{b}_Z + \overline{M} + c_m\,\|I-\rho\|_{\mathcal{L}}) > u_n\}.$$

This last event does not depend on ℓ, so

$$\mathbb{P}(\overline{\zeta}_{\widetilde{l}_{k+1},n}, \cap_{j=1}^{k}\{\widetilde{\ell}_j = \ell_{\sigma(j)}\}) \leq \mathbb{P}(\bar{a}_X + \bar{b}_Z + \overline{M} > u_n q_n^{\alpha} - c_m\,\|I-\rho\|_{\mathcal{L}}).$$

For this term, we obtain the bound $\mathcal{O}(u_n^{-1}q_n^{-\alpha})$ under the condition A2.3-(i) while A2.3-(ii) gives a $\mathcal{O}(e^{-cnu_nq_n^{\alpha}})$.

For the last term in (3.4), observe that the property $\mathbb{P}(A \cup B) = \mathbb{P}(A^c \cap B) + \mathbb{P}(A)$ implies for $k = 2$ the relation: $\mathbb{P}(\widetilde{\ell}_1 \neq \ell_{\sigma(1)} \cup \widetilde{\ell}_2 \neq \ell_{\sigma(2)}) = \mathbb{P}(\widetilde{\ell}_1 = \ell_{\sigma(1)}, \widetilde{\ell}_2 \neq \ell_{\sigma(2)}) + \mathbb{P}(\widetilde{\ell}_1 \neq \ell_{\sigma(1)})$. Next by induction and using the convention $\sum_1^0 \cdots = 0$, we obtain

$$\mathbb{P}\Big(\bigcup_{j=1}^{k}\{\widetilde{\ell}_j \neq \ell_{\sigma(j)}\}\Big) = \mathbb{P}(\widetilde{\ell}_1 \neq \ell_{\sigma(1)}) + \sum_{j=1}^{k-1}\mathbb{P}(\widetilde{\ell}_1 = \ell_{\sigma(1)}, \ldots, \widetilde{\ell}_j = \ell_{\sigma(j)}, \widetilde{\ell}_{j+1} \neq \ell_{\sigma(j+1)}).$$

First part: Study of $\mathbb{P}(\widetilde{\ell}_1 \neq \ell_{\sigma(1)})$.

Clearly, the relation $\overline{\zeta}_{\ell_{\sigma(1)},n} > \max_{\substack{\ell=1,\ldots,q_n\\ \ell\neq\ell_{\sigma(1)}}}\overline{\zeta}_{\ell,n} \Rightarrow \widetilde{\ell}_1 = \ell_{\sigma(1)}$ gives $\mathbb{P}(\widetilde{\ell}_1 \neq \ell_{\sigma(1)}) \leq \mathbb{P}\Big(\overline{\zeta}_{\ell_{\sigma(1)},n} \leq \max_{\substack{\ell=1,\ldots,q_n\\ \ell\neq\ell_{\sigma(1)}}}\overline{\zeta}_{\ell,n}\Big)$.

Setting $j=1$ in (3.5), we obtain $\mathbb{P}(\widetilde{\ell}_1 \neq \ell_{\sigma(1)}) \leq \mathbb{P}\big(\overline{\Delta}_{\sigma(1),n} \leq \max_{\substack{\ell=1,\dots,q_n\\ \ell\neq\ell_{\sigma(1)}}} \overline{\zeta}_{\ell,n} + q_n^{-\alpha}\Lambda_n\big)$. Next, we get

$$\max_{\substack{\ell=1,\dots,q_n\\ \ell\neq\ell_{\sigma(1)}}} \overline{\zeta}_{\ell,n} = \max\Big(\max_{\substack{\ell=1,\dots,q_n\\ \ell\neq\ell_{\sigma(1)},\dots,\ell\neq\ell_{\sigma(k)}}} \overline{\zeta}_{\ell,n}, \overline{\zeta}_{\ell_{\sigma(2)},n}, \dots, \overline{\zeta}_{\ell_{\sigma(k)},n}\Big)$$
$$\leq \max\big(q_n^{-\alpha}(\overline{a}_X + \overline{b}_Z + \overline{M} + c_m \|I-\rho\|_{\mathcal{L}}), \overline{\zeta}_{\ell_{\sigma(2)},n}, \dots, \overline{\zeta}_{\ell_{\sigma(k)},n}\big). \tag{3.6}$$

On the other hand, from Lemmas 2.2 and 3.1, we get for all $j \geq 2$,

$$\overline{\zeta}_{\ell_{\sigma(j)},n} \leq \overline{\Delta}_{\sigma(j),n} + q_n^{-\alpha}(2\overline{M} + \overline{a}_X + \overline{b}_Z + c_m\|I-\rho\|_{\mathcal{L}}). \tag{3.7}$$

We may deduce that $\max_{\substack{\ell=1,\dots,q_n\\ \ell\neq\ell_{\sigma(1)}}} \overline{\zeta}_{\ell,n} \leq \max_{j=2,\dots,k} \overline{\Delta}_{\sigma(j),n} + q_n^{-\alpha}\Lambda_n$ and, finally, we obtain that

$$\mathbb{P}(\widetilde{\ell}_1 \neq \ell_{\sigma(1)}) \leq \mathbb{P}(\overline{\Delta}_{\sigma(1),n} \leq \max_{j=2,\dots,k} \overline{\Delta}_{\sigma(j),n} + 2q_n^{-\alpha}\overline{\Lambda}_n) \tag{3.8}$$

and

$$\mathbb{P}(\widetilde{\ell}_1 \neq \ell_{\sigma(1)}) \leq \mathbb{P}(\overline{\Delta}_{\sigma(1),n} \leq \overline{\Delta}_{\sigma(2),n} + 2q_n^{-\alpha}\overline{\Lambda}_n) + \mathbb{P}(\overline{\Delta}_{\sigma(2),n} < \max_{j=3,\dots,k} \overline{\Delta}_{\sigma(j),n}).$$

Note that (3.8) reduces to $\mathbb{P}(\overline{\Delta}_{\sigma(1),n} \leq 2q_n^{-\alpha}\overline{\Lambda}_n)$ if $k=1$: this particular case will be handled in the second part of the proof. Here, since $\overline{\Delta}_{\sigma(2),n} < \max_{j=3,\dots,k} \overline{\Delta}_{\sigma(j),n} \Leftrightarrow \exists j = 3, \dots, k,\ \overline{\Delta}_{\sigma(j),n} > \overline{\Delta}_{\sigma(2),n}$, we get

$$\mathbb{P}(\widetilde{\ell}_1 \neq \ell_{\sigma(1)}) \leq \mathbb{P}\big(\overline{\Delta}_{\sigma(2),n} - \overline{\Delta}_{\sigma(1),n} - \mathbb{E}\,(\Delta_{\sigma(2)} - \Delta_{\sigma(1)}) \geq \mathbb{E}\,(\Delta_{\sigma(1)} - \Delta_{\sigma(2)}) - 2q_n^{-\alpha}\overline{\Lambda}_n\big)$$
$$+ \sum_{j=3}^{k} \mathbb{P}\big(\overline{\Delta}_{\sigma(j),n} - \overline{\Delta}_{\sigma(2),n} - \mathbb{E}\,(\Delta_{\sigma(j)} - \Delta_{\sigma(2)}) \geq \mathbb{E}\,(\Delta_{\sigma(2)} - \Delta_{\sigma(j)})\big). \tag{3.9}$$

By considering the event $\{\mathbb{E}\,(\Delta_{\sigma(1)} - \Delta_{\sigma(2)}) - 2q_n^{-\alpha}\overline{\Lambda}_n \geq \frac{1}{2}\mathbb{E}\,(\Delta_{\sigma(1)} - \Delta_{\sigma(2)})\}$, we may bound the first term of (3.9) by $\mathbb{P}\big(\overline{\Delta}_{\sigma(2),n} - \overline{\Delta}_{\sigma(1),n} - \mathbb{E}\,(\Delta_{\sigma(2)} - \Delta_{\sigma(1)}) \geq \frac{\mathbb{E}\,(\Delta_{\sigma(1)}-\Delta_{\sigma(2)})}{2}\big) + \mathbb{P}\big(\overline{\Lambda}_n > \frac{\mathbb{E}\,(\Delta_{\sigma(1)}-\Delta_{\sigma(2)})}{4} q_n^{\alpha}\big)$. These probabilities are controlled by the following lemma whose proof is postponed to Appendix.

Lemma 3.3. *For all $j = 2, \dots, k$, $j' = 1, \dots, j-1$ and $\eta > 0$, the following bounds hold.*

(1) *If the conditions given in A3.2-*(i) *are fulfilled,* $\mathbb{P}\big(\overline{\Lambda}_n > \eta q_n^{\alpha}\big) = \mathcal{O}\big(q_n^{-\alpha}\big)$ *and*

$$\mathbb{P}\big(\overline{\Delta}_{\sigma(j),n} - \overline{\Delta}_{\sigma(j'),n} - \mathbb{E}\,(\Delta_{\sigma(j)} - \Delta_{\sigma(j')}) \geq \eta\big) = \mathcal{O}\big(n^{-2}\big).$$

(2) *If the conditions A3.2-*(ii) *hold,* $\mathbb{P}\big(\overline{\Lambda}_n > \eta q_n^{\alpha}\big) = \mathcal{O}\big(e^{-c\,nq_n^{\alpha}}\big)$ *and for some $c > 0$:*

$$\mathbb{P}\big(\overline{\Delta}_{\sigma(j),n} - \overline{\Delta}_{\sigma(j'),n} - \mathbb{E}\,(\Delta_{\sigma(j)} - \Delta_{\sigma(j')}) \geq \eta\big) = \mathcal{O}\big(e^{-c\,n}\big).$$

Finally, the last term of (3.9) (which exists only for $k \geq 3$) is also derived from Lemma 3.3. Consequently, the control of $\mathbb{P}(\widetilde{\ell}_1 \neq \ell_{\sigma(1)})$ is achieved by collecting all the previous results and Borel–Cantelli's lemma implies that a.s. for n large enough, $\widetilde{\ell}_1 = \ell_{\sigma(1)}$.

Second part: Study of $\sum_{j=1}^{k-1} \mathbb{P}(\cap_{m=1}^{j}\{\widetilde{\ell}_m = \ell_{\sigma(m)}\} \cap \{\widetilde{\ell}_{j+1} \neq \ell_{\sigma(j+1)}\})$ *for* $k \geq 2$.

For this term, we have

$$\cap_{m=1}^{j}\{\widetilde{\ell}_m = \ell_{\sigma(m)}\} \cap \{\widetilde{\ell}_{j+1} \neq \ell_{\sigma(j+1)}\} = \cap_{m=1}^{j}\{\widetilde{\ell}_m = \ell_{\sigma(m)}\} \cap \Big\{\arg \max_{\substack{\ell=1,\dots,q_n\\ \ell\neq\ell_{\sigma(1)},\dots,\ell\neq\ell_{\sigma(j)}}} \overline{\zeta}_{\ell,n} \neq \ell_{\sigma(j+1)}\Big\}.$$

As $\{\overline{\zeta}_{\ell_{\sigma(j+1)},n} > \max_{\substack{\ell=1,\dots,q_n\\ \ell\neq\ell_{\sigma(1)},\dots,\ell\neq\ell_{\sigma(j+1)}}} \overline{\zeta}_{\ell,n}\} \Rightarrow \{\arg\max_{\substack{\ell=1,\dots,q_n\\ \ell\neq\ell_{\sigma(1)},\dots,\ell\neq\ell_{\sigma(j)}}} \overline{\zeta}_{\ell,n} = \ell_{\sigma(j+1)}\}$, we deduce that the probability of interest is bounded by $\mathbb{P}\big(\overline{\zeta}_{\ell_{\sigma(j+1)},n} \leq \max_{\substack{\ell=1,\dots,q_n\\ \ell\neq\ell_{\sigma(1)},\dots,\ell\neq\ell_{\sigma(j+1)}}} \overline{\zeta}_{\ell,n}\big)$. Then, using the convention $\sum_1^0 \cdots = 0$, it is sufficient to control the terms

$$\sum_{j=1}^{k-2} \mathbb{P}\Big(\overline{\zeta}_{\ell_{\sigma(j+1)},n} \leq \max\Big(\max_{\substack{\ell=1,\dots,q_n\\ \ell\neq\ell_{\sigma(1)},\dots,\ell\neq\ell_{\sigma(k)}}} \overline{\zeta}_{\ell,n}, \overline{\zeta}_{\ell_{\sigma(j+2)},n}, \dots, \overline{\zeta}_{\ell_{\sigma(k)},n}\Big)\Big) + \mathbb{P}\Big(\overline{\zeta}_{\ell_{\sigma(k)},n} \leq \max_{\substack{\ell=1,\dots,q_n\\ \ell\neq\ell_{\sigma(1)},\dots,\ell\neq\ell_{\sigma(k)}}} \overline{\zeta}_{\ell,n}\Big).$$

Using the bounds established in (3.5)–(3.7), we arrive at

$$\sum_{j=1}^{k-2} \mathbb{P}\Big(\overline{\Delta}_{\sigma(j+1),n} \leq \max_{m=j+2,\dots,k} \overline{\Delta}_{\sigma(m),n} + 2q_n^{-\alpha}\overline{\Lambda}_n\Big) + \mathbb{P}\Big(\mathbb{E}\,\Delta_{\sigma(k)} - \overline{\Delta}_{\sigma(k),n} \geq \mathbb{E}\,\Delta_{\sigma(k)} - 2q_n^{-\alpha}\overline{\Lambda}_n\Big)$$

where again $\overline{\Lambda}_n = 2\overline{M} + \bar{a}_X + \bar{b}_Z + c_m \|I - \rho\|_{\mathcal{L}}$. The study of the first term is analogous to that performed for the term given in (3.8). Details are left to the reader. The second one is handled similarly to (3.9) for obtaining: $\mathbb{P}\big(\big|\overline{\Delta}_{\sigma(k),n} - \mathbb{E}\,\Delta_{\sigma(k)}\big| \geq \frac{\mathbb{E}\,\Delta_{\sigma(k)}}{2}\big) + \mathbb{P}\big(\overline{\Lambda}_n \geq \frac{\mathbb{E}\,\Delta_{\sigma(k)}}{4} q_n^{\alpha}\big)$ and the upper bounds are the same as those established in Lemma 3.3. Collecting all the results, Borel–Cantelli's lemma applied to (3.4) implies that $\sum_n \mathbb{P}(\widehat{k} \neq k) < \infty$ and $\sum_n \mathbb{P}\Big(\bigcup_{j=1}^{\widehat{k}}\{\widetilde{\ell}_j \neq \ell_{\sigma(j)}\}\Big) < \infty$ leading to the final result. ■

3.3. Estimation of intensity

Since a.s. for n large enough, $\widehat{k} = k$ and consecutive times of jumps are detected with $(\widehat{\ell}_1, \ldots, \widehat{\ell}_{\widehat{k}}) = (\widetilde{\ell}_1^*, \ldots, \widetilde{\ell}_{\widehat{k}}^*)$ the associated order statistic, we may evaluate their corresponding intensities $\mathbb{E}\,(I_j), j = 1, \ldots, k$. We start by estimating $\mathbb{E}\,(\Delta_j)$ with

$$\widehat{\Delta}_j = \frac{1}{n}\sum_{i=1}^{n}\Big|X_i\Big(\frac{\widehat{\ell}_j}{q_n}\Big) - X_i\Big(\frac{\widehat{\ell}_j - 1}{q_n}\Big)\Big|, \quad j = 1, \ldots, \widehat{k}.$$

Since $\mathbb{E}\,(\Delta_j) = p_j \mathbb{E}\,(I_j)$, estimators of $\mathbb{E}\,(I_j)$ are given by

$$\widehat{I}_j = \frac{\widehat{\Delta}_j}{\widehat{p}_j} \quad \text{where } \widehat{p}_j = \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_{\left\{|X_i(\frac{\widehat{\ell}_j}{q_n}) - X_i(\frac{\widehat{\ell}_j - 1}{q_n})| > u_n\right\}}, \quad j = 1, \ldots, \widehat{k}$$

with the same u_n as in Section 3.2, satisfying again the condition: $u_n \to 0$ such that $u_n q_n^{\alpha} \to \infty$. Note that $\widehat{p}_j \xrightarrow[n\to\infty]{\text{a.s.}} p_j$ so, a.s. for n large enough, the denominator is not zero.

For the almost sure behavior, we study the quantity

$$\widehat{I}_j - \mathbb{E}\,(I_j) = \frac{(\widehat{\Delta}_j - \mathbb{E}\,(\Delta_j)) - (\widehat{p}_j - p_j)\mathbb{E}\,(I_j)}{\widehat{p}_j}, \quad j = 1, \ldots, \widehat{k},$$

and for $\varepsilon > 0$, we get

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{j=1}^{\widehat{k}} \big|\widehat{I}_j - \mathbb{E}\,(I_j)\big| \geq \varepsilon\right) &\leq \mathbb{P}(\widehat{k} \neq k) + \sum_{j=1}^{k}\mathbb{P}\left(\big|\widehat{I}_j - \mathbb{E}\,(I_j)\big| \geq \varepsilon\right)\\
&\leq \mathbb{P}(\widehat{k} \neq k) + \sum_{j=1}^{k}\mathbb{P}\left(\big|\widehat{\Delta}_j - \mathbb{E}\,(\Delta_j)\big| \geq \frac{\varepsilon\widehat{p}_j}{2}\right) + \mathbb{P}\left(\big|\widehat{p}_j - p_j\big| \geq \frac{\varepsilon\widehat{p}_j}{2\mathbb{E}\,(I_j)}\right)
\end{aligned}$$

and for all $\eta \in]0, p_j[$, we have

$$\leq \mathbb{P}(\widehat{k} \neq k) + \sum_{j=1}^{k}\mathbb{P}\left(\big|\widehat{\Delta}_j - \mathbb{E}\,(\Delta_j)\big| \geq \frac{\varepsilon(p_j - \eta)}{2}\right) + \mathbb{P}\left(\big|\widehat{p}_j - p_j\big| \geq \frac{\varepsilon(p_j - \eta)}{2\mathbb{E}\,(I_j)}\right) + 2\,\mathbb{P}\left(\big|\widehat{p}_j - p_j\big| \geq \eta\right) \tag{3.10}$$

where the latter term doest not depend on ε. Then we may derive the following result whose proof is postponed to Appendix.

Theorem 3.2. *Under Assumption 3.1, we obtain*

(1) *if the condition A3.2-*(i) *holds, and $u_n = (\log n)^{-1}$, $q_n = n^{\beta}$ with $\beta > \frac{5}{4\alpha}$, then almost surely for n large enough*

$$\big|\widehat{I}_j - \mathbb{E}\,(I_j)\big| = \mathcal{O}\Big(\frac{(\log n)^c}{n^{\frac{1}{4}}}\Big), \quad c > \frac{1}{4}, \ j = 1, \ldots, k;$$

(2) *if the condition A3.2-*(ii) *holds, and $u_n q_n^{\alpha} \to \infty$, then almost surely for n large enough*

$$\big|\widehat{I}_j - \mathbb{E}\,(I_j)\big| = \mathcal{O}\Big(\sqrt{\frac{\log n}{n}}\Big), \quad j = 1, \ldots, k.$$

We conclude that, under the mild conditions A3.2-(i), one needs to observe each sample path with high frequency to estimate the intensities of jumps with some given accuracy. Recall that α is linked with regularity of the process between two jumps. Looking at the condition $\beta > \frac{5}{4\alpha}$, it appears, as expected, that more α is small, more the estimation will be difficult without a high sampling rate. Under A3.2-(ii) with the boundedness of $a(X_1)$, we are close to the classical root-n rate of convergence. Finally by examining the proof of Theorem 3.2, it appears that the strong consistency of $\widehat{I}_j$ holds as soon as $q_n \to \infty$.

4. Random jumps

4.1. Detection of jumps

Now, we suppose that $Z_i, i = 1, \dots, n$, has K_i jumps at random instants, with K_i a nonnegative integer-valued random variable and $0 < T_{i,1} < \cdots < T_{i,K_i} < 1$ almost surely if $K_i \geq 1$. Here the sequence $\left((\left|Z_i(T_{ij}) - Z_i(T_{ij}^-)\right|, K_i)\ i = 1, \dots, n\right)$, with $Z(T_{i,j}^-) = \lim_{\eta \searrow 0} Z(T_{i,j} - \eta)$, is i.i.d. We set $p_k = \mathbb{P}(K_i = k)$ for $k \geq 0$, $p_0 \neq 1$, and also, we suppose that K_i is independent from $\left|Z_i(T_{ij}) - Z_i(T_{ij}^-)\right|, j = 1, \dots, K_i, i = 1, \dots, n$. Assumptions 2.1–2.2 lead to $\left|Z_i(T_{ij}) - Z_i(T_{ij}^-)\right| = \left|X_i(T_{ij}) - X_i(T_{ij}^-)\right|$. Recall that the trajectories satisfy a Hölder condition between two consecutive jumps. The main difference with the previous section is that *times of jumps differ from one sample path to the other*. By this way, we have to consider separately the X_i's for their detection. Finally, we associate to each $T_{i,j}$ an integrable intensity of jump:

$$\Delta_{ij} = \left|X_i(T_{i,j}) - X_i(T_{i,j}^-)\right| = \left|Z_i(T_{i,j}) - Z_i(T_{i,j}^-)\right|, \quad j = 1, \dots, K_i;\ i = 1, \dots, n$$

with $\mathbb{P}(\Delta_{1j} > \delta_1) = 1$ for some $\delta_1 > 0$.

Example 4.1. Let $0 = T_{i0} < T_{i1} < \cdots$ be a strictly increasing sequence of random variables (almost surely). Let us set $K_i = \sum_{j=1}^{\infty} \mathbb{I}_{T_{ij} \leq 1}$ and $Z_i(t) = \sum_{j=1}^{k+1} Y_{i,j-1} \mathbb{I}_{[T_{i,j-1}, T_{ij}[}(t)$ if $K_i = k$ with $T_{i,k+1} = 1$, $0 \leq t \leq 1$, where Y_{ij} is $\mathcal{A} - \mathcal{B}_{\mathbb{R}}$ measurable and for each $j = 1, \dots, k$, $Y_{1j}, \dots, Y_{nj}$ are i.i.d. Note that an example of such a model is the *compound Poisson process*.

Now, for $\ell = 1, \dots, q_n, j = 1, \dots, K_i$ and $i = 1, \dots, n$, we set:

$$\zeta_{i\ell n} = \left|X_i\Big(\frac{\ell}{q_n}\Big) - X_i\Big(\frac{\ell - 1}{q_n}\Big)\right|, \qquad \zeta_{i\ell n}^{(Z)} = \left|Z_i\Big(\frac{\ell}{q_n}\Big) - Z_i\Big(\frac{\ell - 1}{q_n}\Big)\right|$$

and we consider the integer-valued variables L_{ijn} defined as:

$$\frac{L_{ijn} - 1}{q_n} < T_{i,j} \leq \frac{L_{ijn}}{q_n}, \quad j = 1, \dots, K_i, \ i = 1, \dots, n. \tag{4.1}$$

We associate them with the increments $\zeta_{iL_{jn}} = \left|X_i\big(\frac{L_{ijn}}{q_n}\big) - X_i\big(\frac{L_{ijn}-1}{q_n}\big)\right|$ and $\zeta_{iL_{jn}}^{(Z)} = \left|Z_i\big(\frac{L_{ijn}}{q_n}\big) - Z_i\big(\frac{L_{ijn}-1}{q_n}\big)\right|$. Thus, these variables correspond to the increments including a jump. To detect these jumps, the following conditions will be useful instead of Assumptions 3.1 and 3.2.

Assumption 4.1 (A4.1).

(i) $W_{ij} = T_{i,j} - T_{i,j-1} \geq \delta_0, j = 1, \dots, K_i + 1$, where $T_{i,0} = 0$, $T_{i,K_i+1} = 1$ and δ_0 is a positive constant.
(ii) $\Delta_{ij} \geq \delta_1 > 0$ (a.s.), $j = 1, \dots, K_i, i = 1, \dots, n$ where δ_1 is fixed.

Assumption 4.1 means that W_{ij} and Δ_{ij} are not too small. Here and throughout this section, we take n large enough (namely such that $\frac{1}{q_n} < \delta_1$) to make sure that all intervals $[\frac{\ell-1}{q_n}, \frac{\ell}{q_n}]$, $\ell = 1, \dots, q_n$ include at most one jump. The first condition can be relaxed as shown by the following remark.

Remark 4.1. The condition A4.1-(i) excludes in particular gamma-distributed interarrival times. By adding the condition $\sum_{n\geq 1} nq_n^{-1} < \infty$, observe that all subsequent results of this part hold true as soon as $\mathbb{P}(T_{i,j+1} - T_{i,j} < q_n^{-1} \mid K_i = k) \leq \psi(k) q_n^{-1}$ with ψ such that $\mathbb{E}\left(K_1 \Psi(K_1)\right) < \infty$. A compound Poisson process satisfies this condition since we have $\mathbb{P}(T_{i,j+1} - T_{i,j} < q_n^{-1} \mid K_i = k) = 1 - (1 - q_n^{-1})^k \leq k q_n^{-1}$ and $\mathbb{E}\left(K_1^2\right) < \infty$.

Assumption 4.2 (A4.2). Suppose that for some $p \geq 1$:

(i) $\mathbb{E}\left(a(X_1)\right)^p < \infty$, $\mathbb{E}\left(b(Z_1)\right)^p < \infty$, $\mathbb{E}\left(M_1\right)^p < \infty$,
(ii) $\sum_{n\geq 1} nq_n^{-\alpha p} u_n^{-p} < \infty$.

The condition A4.2-(ii) implies that more α is small (more the sample paths are irregular), more p should be chosen large enough.

Now, to detect the jumps we consider the random set $\mathcal{L}_{in}$ defined by $\mathcal{L}_{in} = \left\{L_{ijn},\ j = 1, \dots, K_i,\ K_i \geq 1\right\},\ i = 1, \dots, n$ and we predict this set with

$$\widehat{\mathcal{L}}_{in} = \left\{\ell \in \{1, \dots, q_n\}\ :\ \left|X_i\Big(\frac{\ell}{q_n}\Big) - X_i\Big(\frac{\ell - 1}{q_n}\Big)\right| > u_n\right\}$$

still with $u_n \to 0$ such that $u_n q_n^{\alpha} \to \infty$ for $\alpha \in]0, 1]$ defined in Assumption 2.3. Again an omnibus choice is $u_n = (\log n)^{-1}$ for $q_n = n^{\beta}$, $\beta > 0$. Moreover we denote by $\widehat{K}_i$ the cardinal of the set $\widehat{\mathcal{L}}_{in}$ and $\{\widehat{L}_{i1}, \dots, \widehat{L}_{i\widehat{K}_i}\}$ its elements. We begin with a result enlightening the fact that for each sample path and n large enough, one may identify the K_i jumps with probability 1.

Theorem 4.1. *If Assumptions 4.1 and 4.2 hold,* $\mathbb{P}\left(\bigcup_{i=1}^{n} \widehat{\mathcal{L}}_{in} \not\equiv \mathcal{L}_{in}\right) = \mathcal{O}(nq_n^{-\alpha p}u_n^{-p})$.

Proof. We have

$$\widehat{\mathcal{L}}_{in} \equiv \mathcal{L}_{in} \Leftrightarrow \begin{array}{l}(\text{a})\, \forall j = 1, \dots, K_i,\ L_{ijn} \in \widehat{\mathcal{L}}_{in} \quad (K_i \geq 1)\\ (\text{b})\, \forall\, \ell \notin \mathcal{L}_{in}, \quad \ell \notin \widehat{\mathcal{L}}_{in}.\end{array}$$

We may deduce that $\bigcup_{i=1}^{n}\left\{\widehat{\mathcal{L}}_{in} \not\equiv \mathcal{L}_{in}\right\} = \bigcup_{i=1}^{n}\left\{\bigcup_{j=1}^{K_i}\{L_{ijn} \notin \widehat{\mathcal{L}}_{in}\} \cup \left\{\bigcup_{\ell=1, \ell\notin\mathcal{L}_{in}}^{q_n}\{\ell \in \widehat{\mathcal{L}}_{in}\}\right\}\right\}$. Moreover $\bigcup_{j=1}^{K_i}\{L_{ijn} \notin \widehat{\mathcal{L}}_{in}\} \Leftrightarrow \left\{\bigcup_{j=1}^{K_i}\left|X_i(\frac{L_{ijn}}{q_n}) - X_i(\frac{L_{ijn}-1}{q_n})\right| \leq u_n\right\}$ and $\bigcup_{\substack{\ell=1\\ \ell\notin\mathcal{L}_{in}}}^{q_n}\{\ell \in \widehat{\mathcal{L}}_{in}\} \Leftrightarrow \left\{\bigcup_{\substack{\ell=1\\ \ell\notin\mathcal{L}_{in}}}^{q_n}\{\left|X_i(\frac{\ell}{q_n}) - X_i(\frac{\ell-1}{q_n})\right| > u_n\}\right\}$. Hence $\mathbb{P}(\bigcup_{i=1}^{n}\{\widehat{\mathcal{L}}_{in} \not\equiv \mathcal{L}_{in}\}) \leq \sum_{i=1}^{n} p_{i1n} + p_{i2n}$ with $p_{i1n} := \mathbb{P}(\bigcup_{j=1}^{K_i}\{\left|X_i(\frac{L_{ijn}}{q_n}) - X_i(\frac{L_{ijn}-1}{q_n})\right| \leq u_n\})$ and $p_{i2n} := \mathbb{P}(\bigcup_{\ell=1,\ell\notin\mathcal{L}_{in}}^{q_n}\{\left|X_i(\frac{\ell}{q_n}) - X_i(\frac{\ell-1}{q_n})\right| > u_n\})$.

Let us begin by p_{i1n}: from Lemma 2.2, we have:

$$-(a(X_{i-1}) + b(Z_{i-1}) + c_m \|I-\rho\|_{\mathcal{L}})q_n^{-\alpha} + \left|Z_i\left(\frac{L_{ijn}}{q_n}\right) - Z_i\left(\frac{L_{ijn}-1}{q_n}\right)\right| \leq \left|X_i\left(\frac{L_{ijn}}{q_n}\right) - X_i\left(\frac{L_{ijn}-1}{q_n}\right)\right|$$
$$\leq (a(X_{i-1}) + b(Z_{i-1}) + c_m \|I-\rho\|_{\mathcal{L}})q_n^{-\alpha} + \left|Z_i\left(\frac{L_{ijn}}{q_n}\right) - Z_i\left(\frac{L_{ijn}-1}{q_n}\right)\right|.$$

Moreover similarly to Lemma 3.1 we may deduce from (3.1) that

$$\left|\left|Z_i\left(\frac{L_{ijn}}{q_n}\right) - Z_i\left(\frac{L_{ijn}-1}{q_n}\right)\right| - \left|Z_i(T_{ij}) - Z_i(T_{ij}^-)\right|\right| \leq 2M_i q_n^{-\alpha}. \tag{4.2}$$

Setting $\Lambda_i = a(X_{i-1}) + b(Z_{i-1}) + 2M_i + c_m \|I-\rho\|_{\mathcal{L}}$, we get

$$-\Lambda_i q_n^{-\alpha} + \left|Z_i(T_{ij}) - Z_i(T_{ij}^-)\right| \leq \left|X_i\left(\frac{L_{ijn}}{q_n}\right) - X_i\left(\frac{L_{ijn}-1}{q_n}\right)\right| \leq \Lambda_i q_n^{-\alpha} + \left|Z_i(T_{ij}) - Z_i(T_{ij}^-)\right| \tag{4.3}$$

and p_{i1n} is bounded as follows:

$$\begin{aligned} p_{i1n} &\leq \mathbb{P}\left(\bigcup_{j=1}^{K_i}\{\left|Z_i(T_{ij}) - Z_i(T_{ij}^-)\right| \leq u_n + \Lambda_i q_n^{-\alpha}\}\right) \leq \mathbb{P}\left(\bigcup_{j=1}^{K_i}\{\left|Z_i(T_{ij}) - Z_i(T_{ij}^-)\right| \leq 2u_n\}\right) + \mathbb{P}(\Lambda_i > q_n^{\alpha}u_n)\\ &\leq \sum_{k=0}^{\infty}\mathbb{P}\left(\bigcup_{j=1}^{k}\{\left|Z_i(T_{ij}) - Z_i(T_{ij}^-)\right| \leq 2u_n\} \mid K_i = k\right)\mathbb{P}(K_i = k) + \mathbb{P}(\Lambda_i > q_n^{\alpha}u_n)\\ &\leq \sum_{k=1}^{\infty}\sum_{j=1}^{k}\mathbb{P}(\Delta_{ij} \leq 2u_n)\mathbb{P}(K_i = k) + \mathbb{P}(\Lambda_i > q_n^{\alpha}u_n)\end{aligned}$$

because K_i is independent from Δ_{ij}. Next Assumption 4.1-(ii) implies the nullity of the first term for n large enough (namely such that $2u_n \leq \delta_1$) and the second term is controlled by Markov's inequality and Assumption 4.2-(i). Hence, we arrive at $p_{i1n} = \mathcal{O}(q_n^{-\alpha p}u_n^{-p})$ uniformly in i.

Now, we turn to p_{i2n}. From Corollary 2.1, we know that

$$\left|X_i\left(\frac{\ell}{q_n}\right) - X_i\left(\frac{\ell-1}{q_n}\right)\right| \leq (a(X_{i-1}) + b(Z_{i-1}) + M_i + c_m \|I-\rho\|_{\mathcal{L}})q_n^{-\alpha}$$

so $\bigcup_{\ell=1,\ell\notin\mathcal{L}_{in}}^{q_n}\left\{\left|X_i(\frac{\ell}{q_n}) - X_i(\frac{\ell-1}{q_n})\right| > u_n\right\} \Rightarrow \left\{a(X_{i-1}) + b(Z_{i-1}) + M_i + c_m\|I-\rho\|_{\mathcal{L}} > q_n^{\alpha}u_n\right\}$ and,

$$p_{i2n} \leq \mathbb{P}(a(X_{i-1}) > v_n) + \mathbb{P}(b(Z_{i-1}) > v_n) + \mathbb{P}(M_i > v_n)$$

with $v_n = \frac{u_n q_n^{\alpha} - c_m\|I-\rho\|_{\mathcal{L}}}{3}$ and Markov's inequality gives that p_{i2n} has a similar order as p_{i1n}. ■

4.2. Estimation of intensity

As $\left((\Delta_{ij}, K_i),\ i = 1, \dots, n\right)$ is supposed to be i.i.d., we have

$$\mathbb{E}(\Delta_{ij}) = \mathbb{E}\left|X_i(T_{ij}) - X_i(T_{ij}^-)\right| \equiv \mathbb{E}(\Delta_{1j}), \quad j = 1, \dots, K_i,\ i = 1, \dots, n.$$

So, the ordering of jumps' intensities is the same for each sample path; but contrary to the deterministic case, two distinct jumps may have the same intensity. Again Assumptions 2.1 and 2.2 guarantee that for each $j = 1, \dots, K_i$, $\left|X_i(T_{ij}) - X_i(T_{ij}^-)\right| = \left|Z_i(T_{ij}) - Z_i(T_{ij}^-)\right|$ are independent variables. For some fixed $k \geq 1$, it is possible to construct an estimator of the k-first jumps $\mathbb{E}(\Delta_1), \dots, \mathbb{E}(\Delta_k)$ by selecting the X_i's having at least k jumps. To this end, we set for $j = 1, \dots, k$:

$$\widehat{\Delta}_j := \widehat{\Delta}_{jn} = \begin{cases} \dfrac{\sum_{i=1}^{n} \left| X_i\left(\frac{\widehat{L}_{ij}}{q_n}\right) - X_i\left(\frac{\widehat{L}_{ij-1}}{q_n}\right) \right| \mathbb{I}_{\{\widehat{K}_i \geq j\}}}{\sum_{i=1}^{n} \mathbb{I}_{\{\widehat{K}_i \geq j\}}}, & \text{if } \sum_{i=1}^{n} \mathbb{I}_{\{\widehat{K}_i \geq j\}} > 0, \\ 0, & \text{if } \sum_{i=1}^{n} \mathbb{I}_{\{\widehat{K}_i \geq j\}} = 0, \end{cases}$$

still with $\widehat{K}_i = \left|\widehat{\mathcal{L}}_{in}\right|$ and $\widehat{\mathcal{L}}_{in} = \left\{\widehat{L}_{i1}, \dots, \widehat{L}_{i\widehat{K}_i}\right\}$, $i = 1, \dots, n$. The strong consistency and rates of convergence are given in the following theorem.

Theorem 4.2. *Suppose that Assumptions 4.1 and 4.2 (with $p = 1$) are fulfilled, and that for $j = 1, \dots, k$ $\mathbb{E}(\exp(c_0\, \Delta_{1j})) < \infty$ with $c_0 > 0$. We have for all $\varepsilon > 0$;*

$$\mathbb{P}\left(\left|\widehat{\Delta}_j - \mathbb{E}(\Delta_{1j})\right| \geq \varepsilon\right) = \mathcal{O}(n q_n^{-\alpha} u_n^{-1}) + \mathcal{O}(\exp(-c_1\, n\varepsilon^2)) + \mathcal{O}\left(\frac{n \log n}{q_n^{\alpha}\varepsilon}\right), \quad c_1 > 0.$$

Proof. We have to study $\mathbb{P}\left(\left|\widehat{\Delta}_j - \mathbb{E}(\Delta_{1j})\right| \geq \varepsilon\right)$, $j = 1, \dots, k$, $k \geq 1$, $\varepsilon > 0$. First, this term is equal to

$$\mathbb{P}\left(\left|\widehat{\Delta}_j - \mathbb{E}(\Delta_{1j})\right| \geq \varepsilon, \bigcup_{i=1}^{n}\left\{\widehat{\mathcal{L}}_i \not\equiv \mathcal{L}_{in}\right\}\right) + \mathbb{P}\left(\left|\widehat{\Delta}_j - \mathbb{E}(\Delta_{1j})\right| \geq \varepsilon, \bigcap_{i=1}^{n}\left\{\widehat{\mathcal{L}}_i \equiv \mathcal{L}_{in}\right\}\right)$$

so it may be bounded with $\mathbb{P}(\bigcup_{i=1}^{n}\{\widehat{\mathcal{L}}_i \not\equiv \mathcal{L}_{in}\}) + \mathbb{P}(\left|\widetilde{\Delta}_j - \mathbb{E}(\Delta_{1j}) \geq \varepsilon\right|)$ where we have set

$$\widetilde{\Delta}_j = \frac{\sum_{i=1}^{n} \left| X_i\left(\frac{L_{ij}}{q_n}\right) - X_i\left(\frac{L_{ij-1}}{q_n}\right)\right| \mathbb{I}_{\{K_i \geq j\}}}{\sum_{i=1}^{n} \mathbb{I}_{\{K_i \geq j\}}} \mathbb{I}_{\left\{\sum_{i=1}^{n} \mathbb{I}_{\{K_i \geq j\} > 0}\right\}}$$

using the convention $\frac{0}{0} = 0$. The first term is controlled with Theorem 4.1 and gives a $\mathcal{O}(n q_n^{-\alpha} u_n^{-1})$. Next from (4.3) and after some derivations, we may write $\mathbb{P}(\left|\widetilde{\Delta}_j - \mathbb{E}(\Delta_{1j})\right| \geq \varepsilon) \leq p_{1n} + p_{2n}$ with

$$p_{1n} := \mathbb{P}\left(\left|\frac{\sum_{i=1}^{n} \Delta_{ij}\mathbb{I}_{\{K_i \geq j\}}}{\sum_{i=1}^{n} \mathbb{I}_{\{K_i \geq j\}}} \mathbb{I}_{\left\{\sum_{i=1}^{n} \mathbb{I}_{\{K_i \geq j\} > 0}\right\}} - \mathbb{E}(\Delta_{1j})\right| \geq \frac{\varepsilon}{2}\right)$$

$$p_{2n} := \mathbb{P}\left(\frac{\sum_{i=1}^{n} \Lambda_i \mathbb{I}_{\{K_i \geq j\}}}{\sum_{i=1}^{n} \mathbb{I}_{\{K_i \geq j\}}} \mathbb{I}_{\left\{\sum_{i=1}^{n} \mathbb{I}_{\{K_i \geq j\} > 0}\right\}} \geq q_n^{\alpha}\frac{\varepsilon}{2}\right).$$

Concerning the first term p_{1n}, we have

$$p_{1n} = \sum_{m=0}^{n} \mathbb{P}\left(\left|\frac{\sum_{i=1}^{n} \Delta_{i,j}\mathbb{I}_{\{K_i \geq j\}}}{m}\mathbb{I}_{m>0} - \mathbb{E}(\Delta_{1j})\right| \geq \frac{\varepsilon}{2} \mid \sum_{i=1}^{n} \mathbb{I}_{\{K_i \geq j\}} = m\right) \times \mathbb{P}\left(\sum_{i=1}^{n} \mathbb{I}_{\{K_i \geq j\}} = m\right).$$

As $\sum_{i=1}^{n} \mathbb{I}_{\{K_i \geq j\}} \sim \mathcal{B}(n, \sum_{i \geq j} p_i)$ and, since $\left\{\sum_{i=1}^{n} \mathbb{I}_{\{K_i \geq j\}} = m\right\}$ is equivalent to have exactly m indicators equal to 1, the i.i.d assumption on the $\Delta_{i,j}$'s and independence from K_i give

$$p_{1n} = \mathbb{I}_{\{\varepsilon \leq 2\mathbb{E}(\Delta_{1j})\}}\mathbb{P}\left(\sum_{i=1}^{n} \mathbb{I}_{\{K_i \geq j\}} = 0\right) + \sum_{m=1}^{n} \mathbb{P}\left(\left|\frac{\sum_{i=1}^{m} \Delta_{i,j}}{m} - \mathbb{E}(\Delta_{1j})\right| \geq \frac{\varepsilon}{2}\right) \times \mathbb{P}\left(\sum_{i=1}^{n} \mathbb{I}_{\{K_i \geq j\}} = m\right).$$

Now, one may use Bernstein's inequality, stated as in e.g. [13, p. 297], to obtain:

$$p_{1n} \leq \left(1 - \sum_{i\geq j} p_i\right)^n + 2\sum_{m=1}^{n}\binom{n}{m}\left(1-\sum_{i\geq j} p_i\right)^{n-m}\left(\sum_{i\geq j} p_i\right)^m \exp\Bigl(-\frac{m\varepsilon^2}{8\sigma_j^2+4H_j\varepsilon}\Bigr)$$

with $\sigma_j^2 = \operatorname{Var}(\Delta_{1j})$ and H_j a constant linked to the central moments of Δ_{1j}. The last expression is bounded by $2\Bigl(1 - \sum_{i\geq j} p_i + \sum_{i\geq j} p_i \exp\bigl(-\frac{\varepsilon^2}{8\sigma_j^2+4H_j\varepsilon}\bigr)\Bigr)^n$. Since $\ln(1-a) \leq -a$ for $0 < a < 1$ and $1-e^{-a} \geq a - \frac{a^2}{2}$ for all $a \geq 0$, we successively obtain for all j such that $\sum_{i\geq j} p_i > 0$:

$$p_{1n} \leq 2\exp\left(-n\sum_{i\geq j} p_i\left(1-\exp\left(-\frac{\varepsilon^2}{8\sigma_j^2+4H_j\varepsilon}\right)\right)\right) \leq 2\exp\left(-\frac{n\sum_{i\geq j} p_i\varepsilon^2}{8\sigma_j^2+4H_j\varepsilon}\left(1-\frac{\varepsilon^2}{16\sigma_j^2+8H_j\varepsilon}\right)\right).$$

Next, there exists $0 < c_2 < 1$ such that $p_{1n} \leq 2\exp\Bigl(-c_2 n(\sum_{i\geq j} p_i)\frac{\varepsilon^2}{8\sigma_j^2+4H_j\varepsilon}\Bigr)$.

Finally, for the term p_{2n} we may write:

$$p_{2n} = \sum_{m=1}^{n}\mathbb{P}\left(\frac{\sum_{i=1}^{n}\Lambda_i\mathbb{I}_{\{K_i\geq j\}}}{m} \geq q_n^{\alpha}\frac{\varepsilon}{2}, \sum_{i=1}^{n}\mathbb{I}_{\{K_i\geq j\}} = m\right) \leq \sum_{m=1}^{n}\mathbb{P}\left(\sum_{i=1}^{n}\Lambda_i\mathbb{I}_{\{K_i\geq j\}} \geq \frac{m}{2}q_n^{\alpha}\varepsilon\right).$$

We conclude with Markov's inequality and the condition A4.2-(i), $p = 1$, to get the bound $\mathcal{O}(\frac{n\log n}{q_n^{\alpha}\varepsilon})$. ■

Remark 4.2. We may observe that the choices $u_n = (\log n)^{-1}$, $q_n = n^{\beta}$, $\varepsilon = \varepsilon_0 n^{-\frac{1}{2}}(\log n)^{\gamma}$ ($\varepsilon_0 > 0$), with $\gamma > 2$, $\beta \geq \frac{5}{2\alpha}$ entail $\sum_n \mathbb{P}\bigl(\bigl|\widehat{\Delta}_j - \mathbb{E}(\Delta_{1j})\bigr| \geq \varepsilon_0 n^{-\frac{1}{2}}(\log n)^{\gamma}\bigr) < \infty$. So in Theorem 4.2, an expected rate of convergence to estimate the jumps' intensities is $\mathcal{O}\bigl((\log n)^{\gamma}\, n^{-\frac{1}{2}}\bigr)$.

4.3. Estimation of the maximal jump

Suppose that there exists a unique integer $k_{\max}$ such that $\mathbb{E}(\Delta_{1k_{\max}}) > \max_{\substack{j=1,\ldots,k\\ j\neq k_{\max}}} \mathbb{E}(\Delta_{1j})$. Then, an estimator of the maximal intensity of jump is $\widehat{\Delta}_{\max} = \max_{j=1,\ldots,k_n}\widehat{\Delta}_j$ with $k_n \to \infty$ as $n \to \infty$. From $\max_{j=1,\ldots,k_n}\bigl|\widehat{\Delta}_j - \mathbb{E}(\Delta_{1j})\bigr| \geq \bigl|\max_{j=1,\ldots,k_n}|\widehat{\Delta}_j| - \max_{j=1,\ldots,k_n}|\mathbb{E}(\Delta_{1j})|\bigr|$, we get that for all $\varepsilon > 0$:

$$\mathbb{P}\bigl(\bigl|\widehat{\Delta}_{\max} - \mathbb{E}(\Delta_{1k_{\max}})\bigr| \geq \varepsilon\bigr) \leq \sum_{j=1}^{k_n}\mathbb{P}\bigl(\bigl|\widehat{\Delta}_j - \mathbb{E}(\Delta_{1j})\bigr| \geq \varepsilon\bigr).$$

Now, for K_i with a finite support $\{0, \ldots, k_0\}$ and unknown $k_0 \geq 1$, we clearly have $\widehat{\Delta}_{\max} = \max_{j=1,\ldots,k_0}\widehat{\Delta}_j$ almost surely for n large enough (as a consequence of $\widehat{K}_i = K_i$ giving in turn that $\widehat{\Delta}_j = 0$ for n large enough and $j \geq k_0 + 1$). Also, remark that $\max_{j=1,\ldots,k_n}\mathbb{E}(\Delta_{1j}) = \max_{j=1,\ldots,k_0}\mathbb{E}(\Delta_{1j})$ and $\mathbb{E}(\Delta_{1j}) = 0$ for $j \geq k_0 + 1$. Hence the summation ranges over $[[0, k_0]]$ and one may obtain a similar rate of convergence as in Remark 4.2 for the estimation of the maximal jump. If K_i is a $\mathbb{N}$-valued random variable, we can also derive a rate of convergence with the same methodology as in [7] and with sequences k_n increasing slowly to infinity. Finally, it can also be shown that $\widehat{k}_{\max} = \arg\max_{j=1,\ldots,k_n}\widehat{\Delta}_j$ is a consistent estimator of $k_{\max}$.

5. The completely random case

5.1. The considered framework

In this part, for a fixed $k \geq 1$, we denote by $(\Delta_{\sigma(1)}, \ldots, \Delta_{\sigma(k)})$, k independent intensities of jumps which are ordered in decreasing average: $\mathbb{E}\,\Delta_{\sigma(1)} > \cdots > \mathbb{E}\,\Delta_{\sigma(k)}$. We associate them to k independent continuous variables $(T_{\sigma(1)}, \ldots, T_{\sigma(k)})$: by this way, $T_{\sigma(j)}$ corresponds to the jump with highest jth average intensity. Next, with the ordered statistics $(T_1, \ldots, T_k) = (T^*_{\sigma(1)}, \ldots, T^*_{\sigma(k)})$, $T^*_{\sigma(1)} < \cdots < T^*_{\sigma(k)}$, we consider a sample path Z with jumps at times $(T_1, \ldots, T_k)$. Then, we work with n i.i.d copies of Z, say $Z_1, \ldots, Z_n$. Here, the key difference with the random case is that intensities of jumps have not the same order from one sample path to the other and the difficulty is to estimate them. The latter construction is resumed with the following hypothesis.

Assumption 5.1 (A5.1)**.** For each $i = 1, \ldots, n$, there exists a permutation $\big(\sigma_i(1), \ldots, \sigma_i(k)\big)$ of $(1, \ldots, k)$ such that $\mathbb{E}\,\Delta_{i\sigma_i(j)} = \mathbb{E}\,\Delta_{\sigma(j)}$ with $\mathbb{E}\,\Delta_{\sigma(1)} > \cdots > \mathbb{E}\,\Delta_{\sigma(k)}$. Moreover $(\Delta_{i\sigma_i(j)},\ j = 1, \ldots, k,\ i = 1, \ldots, n)$ is a collection of independent random variables and, for each $j = 1, \ldots, k$, the $(\Delta_{i\sigma_i(j)},\ i = 1, \ldots, n)$ are identically distributed.

We make use of the L_{ijn}'s defined in Eq. (4.1), linked with the arrival times of jumps (in chronological order) and, we consider their independent counterparts $L_{i\sigma_i(j)n}$ with $\frac{L_{i\sigma_i(j)n}-1}{q_n} < T_{i\sigma_i(j)} \le \frac{L_{i\sigma_i(j)n}}{q_n}$, $i = 1, \ldots, n, j = 1, \ldots, k$ (associated with jumps ordered by intensities). Now, we suppose that

Assumption 5.2 (A5.2)**.**

(i) $(T_{i\sigma_i(j)},\ i = 1, \ldots, n,\ j = 1, \ldots, k)$ are globally independent with respective bounded densities $f_1, \ldots, f_k$ on $[0, 1]$.
(ii) $\sum_{n\ge 1} nq_n^{-1} < \infty$.
(iii) $\Delta_{ij} \ge \delta_1, j = 1, \ldots, k, i = 1, \ldots, n$ where δ_1 is a positive constant.

The next lemma establishes that with probability one, two consecutive instants are not in the same interval.

Lemma 5.1. *If the conditions* (i)–(ii) *of Assumption 5.2 hold, for all $i = 1, \ldots, n$, the $(T_{ij},\ j = 1, \ldots, k)$ do not belong to the same interval a.s. for n large enough:* $\mathbb{P}\big(\bigcup_{i=1}^n \bigcup_{j=1}^k \{T_{i,j+1} - T_{ij} \le \frac{1}{q_n}\}\big) = \mathcal{O}(nq_n^{-1})$.

Proof. Note that $\Big\{\bigcup_{i=1}^n \bigcup_{j=1}^k T_{i,j+1} - T_{ij} \le \frac{1}{q_n}\Big\} \Rightarrow \Big\{\bigcup_{i=1}^n \bigcup_{\substack{j,j'=1\\ j'\neq j}}^k \bigcup_{\ell=1}^{q_n} \{T_{i\sigma_i(j)} \in [\frac{\ell-1}{q_n}, \frac{\ell}{q_n}] \cap T_{i\sigma_i(j')} \in [\frac{\ell-1}{q_n}, \frac{\ell}{q_n}]\}\Big\}$. Using independence and boundedness of the densities of $T_{i\sigma(j)}$'s, we get that $\sum_{i=1}^n \sum_{j\neq j'} \sum_{\ell=1}^{q_n} \mathbb{P}(T_{i\sigma_i(j)} \in [\frac{\ell-1}{q_n}, \frac{\ell}{q_n}] \cap T_{i\sigma_i(j')} \in [\frac{\ell-1}{q_n}, \frac{\ell}{q_n}]) = \mathcal{O}(nq_n^{-1})$. ■

5.2. *Detection of jumps*

We begin with a result enlightening the fact that for each sample path, one may identify the k jumps with probability 1 for n large enough. Again in this part, the set $\mathcal{L}_{in}$ is defined by $\mathcal{L}_{in} = \big\{L_{ijn},\ j = 1, \ldots, k\big\},\ i = 1, \ldots, n,\ \zeta_{i\ell n} = \big|X_i(\frac{\ell}{q_n}) - X_i(\frac{\ell-1}{q_n})\big|$ and we note $\zeta_{iL_jn} := \zeta_{i,L_{ijn},n}$.

Theorem 5.1. *Suppose that Assumptions 4.2, 5.1 and 5.2 are fulfilled, then a.s. for n large enough, we get that $\zeta_{i\ell n} < \zeta_{iL_jn}, j = 1, \ldots, k, i = 1, \ldots, n, \ell = 1, \ldots, q_n$ with $\ell \notin \mathcal{L}_{in}$. More precisely,*

$$\mathbb{P}\Big(\bigcup_{i=1}^n \bigcup_{\substack{\ell=1\\ \ell\notin\mathcal{L}_{in}}}^{q_n} \bigcup_{j=1}^k \big\{\zeta_{i\ell n} \ge \zeta_{iL_jn}\big\}\Big) = \mathcal{O}(nq_n^{-\alpha p}) + \mathcal{O}(nq_n^{-1}).$$

Proof. The desired probability is clearly bounded by

$$\sum_{i=1}^n \sum_{j=1}^k \mathbb{P}\left(\max_{\substack{\ell=1,\ldots,q_n\\ \ell\notin\mathcal{L}_{in}}} \zeta_{i\ell n} \ge \zeta_{iL_jn}, \bigcap_{i=1}^n \bigcap_{j=1}^k \Big\{T_{i,j+1} - T_{ij} > \frac{1}{q_n}\Big\}\right) + \mathbb{P}\left(\bigcup_{i=1}^n \bigcup_{j=1}^k \Big\{T_{i,j+1} - T_{ij} \le \frac{1}{q_n}\Big\}\right).$$

Next, from Lemma 2.2, we may write for all $\ell = 1, \ldots, q_n$, and $i = 1, \ldots, n$:

$$\Big\{\max_{\substack{\ell=1,\ldots,q_n\\ \ell\notin\mathcal{L}_{in}}} \zeta_{i\ell n} \ge \zeta_{iL_jn}\Big\} \Rightarrow \Big\{\max_{\substack{\ell=1,\ldots,q_n\\ \ell\notin\mathcal{L}_{in}}} \zeta_{i\ell n}^{(Z)} + q_n^{-\alpha}(a(X_{i-1}) + b(Z_{i-1}) + c_m \|I - \rho\|_{\mathcal{L}}) \ge \zeta_{iL_jn}\Big\}.$$

As for $\ell \notin \mathcal{L}_{in}$, there is no jump in $[\frac{\ell-1}{q_n}; \frac{\ell}{q_n}]$, the condition A2.3-(ii) gives $\max_{\substack{\ell=1,\ldots,q_n\\ \ell\notin\mathcal{L}_{in}}} \zeta_{i\ell n}^{(Z)} \le M_i q_n^{-\alpha}$, so

$$\Big\{\max_{\substack{\ell=1,\ldots,q_n\\ \ell\notin\mathcal{L}_{in}}} \zeta_{i\ell n} \ge \zeta_{iL_jn}\Big\} \Rightarrow \Big\{q_n^{-\alpha}(M_i + a(X_{i-1}) + b(Z_{i-1}) + c_m \|I - \rho\|_{\mathcal{L}}) \ge \zeta_{iL_jn}\Big\}. \tag{5.1}$$

Next we may use (4.2) (since $\bigcap_{i=1}^n \bigcap_{j=1}^k \{T_{i,j+1} - T_{ij} > \frac{1}{q_n}\}$ implies that two consecutive jumps cannot belong to the same interval) and deduce with Lemma 2.2 that $\zeta_{iL_jn} \ge \Delta_{ij} - q_n^{-\alpha}(2M_i + a(X_{i-1}) + b(Z_{i-1}) + c_m \|I - \rho\|_{\mathcal{L}})$. Hence, (5.1) may be rewritten as

$$\Big\{\max_{\substack{\ell=1,\ldots,q_n\\ \ell\notin\mathcal{L}_{in}}} \zeta_{i\ell n} \ge \zeta_{iL_jn}\Big\} \Rightarrow \Big\{q_n^{-\alpha}(3M_i + 2a(X_{i-1}) + 2b(Z_{i-1}) + 2c_m \|I - \rho\|_{\mathcal{L}}) \ge \Delta_{ij}\Big\}.$$

Finally the condition A5.2-(iii) gives that $\mathbb{P}\big(\max_{\substack{\ell=1,\dots,q_n\\ \ell\notin\mathcal{L}_{in}}} \zeta_{i\ell n} \geq \zeta_{iL_jn}\big)$ is bounded with

$$\mathbb{P}\left(M_i \geq \frac{\delta_1 q_n^\alpha - 6c_m \|I-\rho\|_{\mathcal{L}}}{9}\right) + \mathbb{P}\left(a(X_{i-1}) \geq \frac{\delta_1 q_n^\alpha - 2c_m \|I-\rho\|_{\mathcal{L}}}{6}\right) + \mathbb{P}\left(b(Z_{i-1}) \geq \frac{\delta_1 q_n^\alpha - 2c_m \|I-\rho\|_{\mathcal{L}}}{6}\right).$$

The result follows with Markov's inequality, the condition A4.2-(i) and the conclusion is a straightforward consequence of Borel–Cantelli's lemma with A4.2-(ii). ■

Remark 5.1. Theorem 5.1 implies that almost surely for n large enough, $\zeta_{i\ell n} < \zeta_{iL_jn}$ for all $\ell \notin \mathcal{L}_{in}, i = 1,\dots,n, j = 1,\dots,k$. Hence for each sample path, the jumps are almost surely identified and we have at our disposal n sets of k values: $\{\widetilde{L}_{i1},\dots,\widetilde{L}_{ik}\}$. Here, note that the $\widetilde{L}_{ij}$ are not ordered either with respect to jumps intensities or arrival times. By considering the associated order statistics: $(\widehat{L}_{i1},\dots,\widehat{L}_{ik}) := (\widetilde{L}^*_{i1},\dots,\widetilde{L}^*_{ik})$, the set $\widehat{\mathcal{L}}_{in} := \{\widehat{L}_{i1},\dots,\widehat{L}_{ik}\}$ represents the arrival times of jumps and one gets $\widehat{\mathcal{L}}_{in} \equiv \mathcal{L}_{in}, i = 1,\dots,n$ a.s. for n large enough.

5.3. *Estimation of the jumps' intensities*

Since we may identify a.s. for n large enough the k jumps of each X_i, we are in position to estimate the intensities $\mathbb{E}(\Delta_{\sigma(j)}), j = 1,\dots,k$. We begin with the estimation of coefficients $a_0,\dots,a_k, a_k = 1$, of the polynomial of degree k with the distinct roots $\mathbb{E}(\Delta_{\sigma(j)})$:

$$\prod_{j=1}^{k}(x - \mathbb{E}\,\Delta_{\sigma(j)}) = \sum_{j=0}^{k} a_{k-j}x^{k-j} = 0.$$

Using Viète's formula and independence of the jumps, we get for $j = 1,\dots,k$:

$$a_{k-j} = (-1)^j \sum_{1\leq\ell_1<\cdots<\ell_j\leq k} \mathbb{E}(\Delta_{\sigma(\ell_1)})\cdots\mathbb{E}(\Delta_{\sigma(\ell_j)}) = (-1)^j \sum_{1\leq\ell_1<\cdots<\ell_j\leq k} \mathbb{E}(\Delta_{\sigma(\ell_1)}\cdots\Delta_{\sigma(\ell_j)}).$$

Here, the key point is that we have to consider the sum of k jumps, the sum of their product in pairs, ..., and finally their products. All these sums are exhaustive, hence we observe that we may use the jumps estimated by chronological order to estimate each term. The next example illustrates this fact for $k = 2$ and $k = 3$.

Example 5.1.
- For $k = 2$, we get $a_0 = \mathbb{E}(\Delta_{\sigma(1)}\Delta_{\sigma(2)}) = \mathbb{E}(\Delta_1\Delta_2)$, $a_1 = \mathbb{E}(\Delta_{\sigma(1)} + \Delta_{\sigma(2)}) = \mathbb{E}(\Delta_1 + \Delta_2)$, $a_2 = 1$;
- for $k = 3$, $a_0 = \mathbb{E}(\Delta_{\sigma(1)}\Delta_{\sigma(2)}\Delta_{\sigma(3)}) = \mathbb{E}(\Delta_1\Delta_2\Delta_3)$, $a_1 = \mathbb{E}(\Delta_{\sigma(1)}\Delta_{\sigma(2)} + \Delta_{\sigma(1)}\Delta_{\sigma(3)} + \Delta_{\sigma(2)}\Delta_{\sigma(3)}) = \mathbb{E}(\Delta_1\Delta_2 + \Delta_1\Delta_3 + \Delta_2\Delta_3)$, $a_2 = \mathbb{E}(\Delta_{\sigma(1)} + \Delta_{\sigma(2)} + \Delta_{\sigma(3)}) = \mathbb{E}(\Delta_1 + \Delta_2 + \Delta_3)$, $a_3 = 1$.

Hence, we compute the k estimators of $a_{k-j}, j = 1,\dots,k$ by setting

$$\widehat{a}_{k-j} = (-1)^j \sum_{1\leq\ell_1<\cdots<\ell_j\leq k} \frac{1}{n}\sum_{i=1}^{n} \left|X_i\Big(\frac{\widehat{L}_{i\ell_1 n}}{q_n}\Big) - X_i\Big(\frac{\widehat{L}_{i\ell_1 n}-1}{q_n}\Big)\right| \cdots \left|X_i\Big(\frac{\widehat{L}_{i\ell_j n}}{q_n}\Big) - X_i\Big(\frac{\widehat{L}_{i\ell_j n}-1}{q_n}\Big)\right|.$$

To study their behavior, we use Remark 5.1 and the property that summations are exhaustive to obtain below the strong consistency of these estimators as well as their rates of convergence.

5.3.1. *Convergence of the* $\widehat{a}_{k-j}, j = 1,\dots,k$

For $\widehat{a}_{k-1} = -\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{k}\left|X_i(\frac{\widehat{L}_{ijn}}{q_n}) - X_i(\frac{\widehat{L}_{ijn}-1}{q_n})\right|$ and $a_{k-1} = -\mathbb{E}(\sum_{j=1}^{k}\Delta_{\sigma(j)}) = -\mathbb{E}(\sum_{j=1}^{k}\Delta_j)$, we obtain the following result proved in the Appendix.

Proposition 5.1. *Suppose that Assumptions 4.2, 5.1, and 5.2 are fulfilled, then*

(1) $\widehat{a}_{k-1} \xrightarrow[n\to\infty]{a.s.} a_{k-1}$ *if either* $a(\cdot)$ *is bounded or* $\sum_n q_n^{-\alpha} < \infty$*;*

(2) $\left|\widehat{a}_{k-1} - a_{k-1}\right| = \mathcal{O}\big((\log n)^c\, n^{-\frac{1}{4}}\big), c > \frac{1}{4}$ *a.s. if* $\mathbb{E}\left|Z_1(T_{1\sigma_1(j)}) - Z_1(T^-_{1\sigma_1(j)})\right|^4 < \infty, j = 1,\dots,k$ *and* $q_n = n^\beta$ *with* $\beta > \max(\frac{5}{4\alpha}, \frac{2}{\min(1,\alpha p)})$.

Note that if Assumption 4.2-(i) is fulfilled with $p \geq 2$, the condition $\beta > \max(\frac{5}{4\alpha}, \frac{2}{\min(1,\alpha p)})$ may be reduced to $\beta > \max(\frac{5}{4\alpha}, \frac{5}{2})$. Finally to simplify the study of our estimators, we add an assumption of boundedness and derive the following result for the coefficients $a_{k-j}, j = 2,\dots,k$.

Proposition 5.2. *Under the hypotheses of Proposition 5.1, we suppose in addition that* $\|X\| \leq C$ *and that functions* a, b *and* M_i *are bounded. Then, for* $j = 2,\dots,k$ *and* $\varepsilon_n > 0$ *such that* $q_n^\alpha\varepsilon_n \to \infty$

$$\mathbb{P}\big(\left|\widehat{a}_{k-j} - a_{k-j}\right| > \varepsilon_n\big) = \mathcal{O}(nq_n^{-\min(1,\alpha p)}) + \mathcal{O}(\exp(-nc\varepsilon_n^2)), \quad c > 0.$$

5.3.2. The special case of $k = 2$ and conclusion

Collecting the previous results, we obtain that $\widehat{a}_{k-j} \xrightarrow[n\to\infty]{\text{a.s.}} a_{k-j}$ for each $j = 1, \dots, k$. Now the problem consists in solving the equation $\sum_{j=0}^{k} \widehat{a}_{k-j,n} x^j = 0$ (with $\widehat{a}_{k,n} \equiv 1$) to recover the roots $\mathbb{E}\,(\Delta_{\sigma(j)}), j = 1, \dots, k$. For $k = 2$, the resolution is straightforward and gives the solutions: $\widetilde{\Delta}_{\sigma(1)n} = \frac{1}{2}(\widehat{S} + \sqrt{\widehat{S}^2 - 4\widehat{P}})$ and $\widetilde{\Delta}_{\sigma(2)n} = \frac{1}{2}(\widehat{S} - \sqrt{\widehat{S}^2 - 4\widehat{P}})$ with $\widehat{S} := \widehat{a}_1 = \frac{1}{n}\sum_{i=1}^{n} \left|X_i\big(\frac{\widehat{L}_{i1n}}{q_n}\big) - X_i\big(\frac{\widehat{L}_{i1n}-1}{q_n}\big)\right| + \left|X_i\big(\frac{\widehat{L}_{i2n}}{q_n}\big) - X_i\big(\frac{\widehat{L}_{i2n}-1}{q_n}\big)\right|$ and $\widehat{P} := \widehat{a}_0 = \frac{1}{n}\sum_{i=1}^{n} \left|X_i\big(\frac{\widehat{L}_{i1n}}{q_n}\big) - X_i\big(\frac{\widehat{L}_{i1n}-1}{q_n}\big)\right|\left|X_i\big(\frac{\widehat{L}_{i2n}}{q_n}\big) - X_i\big(\frac{\widehat{L}_{i2n}-1}{q_n}\big)\right|$. We easily derive the strong consistency of these estimators with the help of Propositions 5.1 and 5.2. The cases $k = 3$ and $k = 4$ are again rather easy to handle but for $k > 4$, the use of numerically approximated solutions should be considered. Simulations should be carried out to see how estimation is involved in the accuracy of this approximation.

Acknowledgments

We want to thank the Reviewers and the Managing Guest Editor for improving the first version of this paper.

Appendix. Auxiliary proofs

The proofs of Lemmas 3.2 and 3.3 being similar, we only give the derivation of the latter one.

Proof of Lemma 3.3. (1) Suppose that the conditions given in A3.2-(i) are fulfilled. Similarly to the proof of Theorem 1 p. 388–389 in [53], we get that

$$P\left(\Big|\frac{1}{n}\sum_{i=1}^{n} Y_{i,n}\Big| \geq \eta\right) \leq \frac{3c}{n^2\eta^4}, \quad \eta > 0,\ n \geq 1, \tag{A.1}$$

for independent and centered random variables $Y_{i,n}, i = 1, \dots, n$ such that $\mathbb{E}\,(Y_{in}^4) \leq c$ with some finite constant c not depending on n. Next, as $\left|X_1(t_j) - X_1(t_j^-)\right| = \left|Z_1(t_j) - Z_1(t_j^-)\right|, j = 1, \dots, k$, the variables $|X_i(t_{\sigma(j)}) - X_i(t^-_{\sigma(j)})| - |X_i(t_{\sigma(j')}) - X_i(t^-_{\sigma(j')})|$ are independent with finite fourth moment thanks to the condition A3.2-(i). For the term $\mathbb{P}\big(\overline{\Lambda}_n > \eta q_n^\alpha\big)$, we apply the Markov's inequality and get for n large enough that this term is a $\mathcal{O}(q_n^{-\alpha})$.

(2) If the conditions A3.2-(ii) are fulfilled, exponential moments do exist and we have $a(X_1) < a_\infty$, so we may use Bernstein inequality to get the claimed exponential bound. Concerning the term involving $\overline{\Lambda}_n$: we first bound it with

$$\mathbb{P}\Big(\sum_{i=1}^{n} b(Z_i) > \frac{\eta}{2} n q_n^\alpha - \frac{n}{2}(c_m \|I-\rho\|_{\mathcal{L}} + a_\infty)\Big) + \mathbb{P}\Big(\sum_{i=1}^{n} M_i > \frac{\eta}{4} n q_n^\alpha - \frac{n}{4}(c_m \|I-\rho\|_{\mathcal{L}} + a_\infty)\Big).$$

Next, since the $b(Z_i)$ and M_i are independent random variables with exponential moment, we obtain by Markov's inequality that these two terms are of order $\mathcal{O}\big(e^{-c\, n q_n^\alpha}\big)$ for some $c > 0$.

Proof of Theorem 3.2. We start from the relation (3.10) with three terms to study. The first one, $\mathbb{P}(\widehat{k} \neq k)$, is controlled in Theorem 3.1. For the second term, we set $\eta = \frac{p_j}{2}$ and $\varepsilon_1 = \frac{\varepsilon p_j}{4}$, so for $j = 1, \dots, k$

$$\mathbb{P}\left(\big|\widehat{\Delta}_j - \mathbb{E}\,(\Delta_j)\big| \geq \varepsilon_1\right) \leq \mathbb{P}\left(\Big|\frac{1}{n}\sum_{i=1}^{n} \Big|X_i\left(\frac{\ell_j}{q_n}\right) - X_i\left(\frac{\ell_j - 1}{q_n}\right)\Big| - \mathbb{E}\,(\Delta_j)\Big| \geq \varepsilon_1\right) + \mathbb{P}(\widehat{\ell}_j \neq \ell_j).$$

The term $\mathbb{P}(\widehat{\ell}_j \neq \ell_j)$ is also controlled with Theorem 3.1. Next, from a similar bound as in (3.7), the first probability is bounded by

$$\mathbb{P}\big(\big|\overline{\Delta}_{j,n} - \mathbb{E}\,\Delta_j\big| > \frac{\varepsilon_1}{2}\big) + P\left(\overline{\Lambda}_n \geq \frac{q_n^\alpha \varepsilon_1}{2}\right) \tag{A.2}$$

with again $\overline{\Lambda} = \overline{a}_X + \overline{b}_Z + 2\overline{M} + c_m \|I-\rho\|_{\mathcal{L}}$.

Following the beginning of the proof of Lemma 3.3, the condition A3.2-(i) gives the bounds $\mathcal{O}\big(n^{-2}\varepsilon_1^{-4}\big) + \mathcal{O}\big(\varepsilon_1^{-1} q_n^{-\alpha}\big)$ prevailing those obtained in (3.2) for $\mathbb{P}(\widehat{\ell}_j \neq \ell_j)$ and $\mathbb{P}(\widehat{k} \neq k)$ as soon as $\varepsilon_1 u_n^{-1} \to 0$. On the other hand, under A3.2-(ii) and following the second part of Lemma 3.3, the obtained bounds are $\mathcal{O}\big(e^{-c\, n\varepsilon_1^2}\big) + \mathcal{O}\big(e^{-c\, n q_n^\alpha \varepsilon_1}\big)$ for some $c > 0$. Again, the bound obtained in the relation (3.3) is negligible when $\varepsilon_1 \to 0$.

Finally, for the two last terms of (3.10), the choice $\eta = \frac{p_1}{2}$ gives that $\mathbb{P}(\big|\widehat{p}_j - p_j\big| \geq \frac{p_j}{2})$ is negligible with respect to $\mathbb{P}(\big|\widehat{p}_j - p_j\big| \geq \frac{\varepsilon_1}{\mathbb{E}\,(I_j)})$ as soon as $\varepsilon_1 \to 0$. This latter term is bounded with

$$\mathbb{P}\left(\Big|\frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{\left\{\left|X_i\left(\frac{\ell_j}{q_n}\right) - X_i\left(\frac{\ell_j-1}{q_n}\right)\right| > u_n\right\}} - p_j\Big| \geq \frac{\varepsilon_1}{\mathbb{E}\,(I_j)}\right) + \mathbb{P}(\widehat{\ell}_j \neq \ell_j). \tag{A.3}$$

From the relation $\mathbb{I}_A = \mathbb{I}_B + \mathbb{I}_{A\cap B^c} - \mathbb{I}_{A^c\cap B}$, we may write that

$$\mathbb{I}_{\left\{\left|X_i\left(\frac{\ell_j}{q_n}\right)-X_i\left(\frac{\ell_j-1}{q_n}\right)\right|>u_n\right\}} = \mathbb{I}_{\{Y_{ij}=1\}} + \mathbb{I}_{\left\{\left|X_i\left(\frac{\ell_j}{q_n}\right)-X_i\left(\frac{\ell_j-1}{q_n}\right)\right|>u_n, Y_{ij}=0\right\}} - \mathbb{I}_{\left\{\left|X_i\left(\frac{\ell_j}{q_n}\right)-X_i\left(\frac{\ell_j-1}{q_n}\right)\right|\le u_n, Y_{ij}=1\right\}}.$$

Then, the left probability of (A.3) is bounded with

$$\mathbb{P}\Big(\Big|\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_{\{Y_{ij}=1\}} - p_j\Big| > \frac{\varepsilon_1}{3\mathbb{E}\,(I_{1j})}\Big) + \mathbb{P}\Big(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_{\left\{\left|X_i\left(\frac{\ell_j}{q_n}\right)-X_i\left(\frac{\ell_j-1}{q_n}\right)\right|>u_n, \left|Z_i(t_j)-Z_i(t_j^-)\right|=0\right\}} > \frac{\varepsilon_1}{3\mathbb{E}\,(I_{1j})}\Big)$$
$$+\,\mathbb{P}\Big(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_{\left\{\left|X_i\left(\frac{\ell_j}{q_n}\right)-X_i\left(\frac{\ell_j-1}{q_n}\right)\right|\le u_n, \left|Z_i(t_j)-Z_i(t_j^-)\right|=I_{ij}\right\}} > \frac{\varepsilon_1}{3\mathbb{E}\,(I_{1j})}\Big).$$

The first term is a $\mathcal{O}(\exp(-2n\frac{\varepsilon_1^2}{9\mathbb{E}\,(I_{1j})^2}))$ by Hoeffding's inequality. For the others, we have for $\Lambda_i = a(X_{i-1}) + b(Z_{i-1}) + 2M_i + c_m\,\|I-\rho\|_{\mathcal{L}}$:

$$\left|Z_i(t_j) - Z_i(t_j^-)\right| - \Lambda_i q_n^{-\alpha} \le \left|X_i\Big(\frac{\ell_j}{q_n}\Big) - X_i\Big(\frac{\ell_j-1}{q_n}\Big)\right| \le \left|Z_i(t_j) - Z_i(t_j^-)\right| + \Lambda_i q_n^{-\alpha}$$

so we get the two implications:

$$\Big\{\Big\{|X_i\Big(\frac{\ell_j}{q_n}\Big) - X_i\Big(\frac{\ell_j-1}{q_n}\Big)| > u_n\Big\}, \Big\{\left|Z_i(t_j) - Z_i(t_j^-)\right| = 0\Big\}\Big\} \Rightarrow \{\Lambda_i > u_n q_n^{\alpha}\}$$
$$\Big\{\Big\{\Big|X_i\Big(\frac{\ell_j}{q_n}\Big) - X_i\Big(\frac{\ell_j-1}{q_n}\Big)\Big| \le u_n\Big\}, \Big\{\left|Z_i(t_j) - Z_i(t_j^-)\right| = I_{ij}\Big\}\Big\} \Rightarrow \{\Lambda_i \ge q_n^{\alpha}(I_{ij} - u_n)\} \Rightarrow \{\Lambda_i \ge q_n^{\alpha}(\delta_1 - u_n)\}$$

since $\mathbb{P}(I_{ij} > \delta_1) = 1$. Under the condition A3.2-(i), we arrive at a bound of order $\mathcal{O}(u_n^{-1}q_n^{-\alpha}\varepsilon_1^{-1}) + \mathcal{O}(\exp(-cn\varepsilon_1^2)) + \mathcal{O}(u_n^{-1}q_n^{-\alpha}) + \mathcal{O}(n^{-2})$ for the term given in (A.3). Finally, collecting all the results, the predominant bounds are of order $\mathcal{O}(u_n^{-1}q_n^{-\alpha}\varepsilon_1^{-1}) + \mathcal{O}(n^{-2}\varepsilon_1^{-4})$. Next setting $\varepsilon_1 = (\log n)^c\, n^{-\frac{1}{4}}$, $c > \frac{1}{4}$, and $q_n = n^{\beta}$ with $\beta > \frac{5}{4\alpha}$, $u_n = (\log n)^{-1}$, we may apply Borel–Cantelli's lemma to derive the claimed result. If the condition A3.2-(ii) is fulfilled, the predominant bound is now transformed in $\mathcal{O}(\exp(-cn\varepsilon_1^2))$, so we may derive the rate of convergence with the choice $\varepsilon_1 = \varepsilon_0\sqrt{\frac{\log n}{n}}$ for a sufficiently large enough ε_0 and all $q_n \to \infty$ (since $n\varepsilon_1^2 = o(nq_n^{\alpha}\varepsilon_1)$).

Proof of Proposition 5.1. (1) To get the strong consistency, we notice that a.s. for n large enough

$$\widehat{a}_{k-1} = -\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{k}\zeta_{iL_jn} := -\sum_{j=1}^{k}\overline{\zeta}_{L_jn}$$

as all possible summations in j are considered. From Lemmas 2.2 and 3.1 (whose proof is exactly the same for random instants of jumps), we obtain the same bound for each $j = 1, \ldots, k$:

$$\Big|\left|\overline{\zeta}_{L_jn} - \mathbb{E}\,\Delta_j\right| - \left|\overline{\Delta}_j - \mathbb{E}\,\Delta_j\right|\Big| \le 2\overline{\Lambda}_n q_n^{-\alpha} \tag{A.4}$$

with again $\overline{\Lambda}_n = 2\overline{M} + \overline{a}_X + \overline{b}_Z + c_m\,\|I-\rho\|_{\mathcal{L}}$. We conclude with the law of large numbers (applying Markov's inequality to control $\overline{a}_X$ in the case where $a(\cdot)$ is not bounded).

(2) We have

$$\mathbb{P}\big(|\widehat{a}_{k-1} - a_{k-1}| \ge \varepsilon\big) = \mathbb{P}\Big(|\widehat{a}_{k-1} - a_{k-1}| \ge \varepsilon, \bigcap_{i=1}^{n}\{\widehat{\mathcal{L}}_{in} \equiv \mathcal{L}_{in}\}\Big) + \mathbb{P}\Big(|\widehat{a}_{k-1} - a_{k-1}| \ge \varepsilon, \bigcup_{i=1}^{n}\{\widehat{\mathcal{L}}_{in} \not\equiv \mathcal{L}_{in}\}\Big)$$

that can be bounded by $\mathbb{P}\Big(\Big|\sum_{j=1}^{k}\overline{\zeta}_{L_jn} - \sum_{j=1}^{k}\mathbb{E}\,\Delta_j\Big| \ge \varepsilon\Big) + \mathbb{P}\Big(\bigcup_{i=1}^{n}\bigcup_{\substack{\ell=1\\ \ell\notin\mathcal{L}_{in}}}^{q_n}\bigcup_{j=1}^{k}\{\zeta_{i\ell n} \ge \zeta_{iL_jn}\}\Big)$. First, the second term does not depend on ε and it is a $\mathcal{O}(nq_n^{-\alpha p}) + \mathcal{O}(nq_n^{-1})$ by Theorem 5.1. Next from (A.4), we may derive analogously to Eq. (A.2):

$$\mathbb{P}\Big(\Big|\sum_{j=1}^{k}\overline{\zeta}_{L_jn} - \sum_{j=1}^{k}\mathbb{E}\,\Delta_j\Big| \ge \varepsilon\Big) \le \sum_{j=1}^{k}\mathbb{P}\Big(\left|\overline{\Delta}_{jn} - \mathbb{E}\,\Delta_j\right| \ge \frac{\varepsilon}{2k}\Big) + \mathbb{P}\Big(\overline{\Lambda}_n \ge \frac{q_n^{\alpha}}{2k}\varepsilon\Big).$$

The first term is handled with the help of the relation (A.1), it yields to a $\mathcal{O}(n^{-2}\varepsilon^{-4})$. For the second term, Markov's inequality and the condition A4.1-(i) applied with $p = 1$ give the bound $\mathcal{O}(q_n^{-\alpha}\varepsilon^{-1})$. Finally, we obtain a bound of order $\mathcal{O}(n^{-2}\varepsilon^{-4}) + \mathcal{O}(\varepsilon^{-1}q_n^{-\alpha}) + \mathcal{O}(nq_n^{-\min(1,\alpha p)})$. The rate is obtained for $\varepsilon_n = (\log n)^c n^{-\frac{1}{4}}$, $c > \frac{1}{4}$, $q_n = n^{-\beta}$ with $\beta > \max(\frac{5}{4\alpha}, \frac{2}{\min(1,\alpha p)})$ and Borel–Cantelli's lemma.

Proof of Proposition 5.2. First, we state the following lemma.

Property 1. *Let* $(u_j,\ j=1,\dots,p)$ *and* $(v_j,\ j=1,\dots,p)$ *be positive numbers such that* $\max_{j=1,\dots,p}(u_j \vee v_j) \le d$, *then*

$$\left|\prod_{j=1}^{p} u_j - \prod_{j=1}^{p} v_j\right| \le d^{p-1}\sum_{j=1}^{p}\left|u_j - v_j\right|, \quad p \ge 2.$$

Proof of Property 1. If $p = 2$,

$$|u_1u_2 - v_1v_2| = |u_1(u_2 - v_2) + v_2(u_1 - v_1)| \le d\big[|u_1 - v_1| + |u_2 - v_2|\big]. \tag{A.5}$$

Now, set $\alpha_{p-1} = u_1 \cdots u_{p-1}$ and $\beta_{p-1} = v_1 \cdots v_{p-1}$, then from (A.5) and by induction

$$\begin{aligned}\left|\alpha_{p-1}u_p - \beta_{p-1}v_p\right| &\le \alpha_{p-1}\left|u_p - v_p\right| + v_p\left|\alpha_{p-1} - \beta_{p-1}\right| \\ &\le d^{p-1}\left|u_p - v_p\right| + d\left(d^{p-2}\sum_{j=1}^{p-1}\left|u_j - v_j\right|\right) \le d^{p-1}\sum_{j=1}^{p}\left|u_j - v_j\right|.\end{aligned}$$

Hence the result. ■

Next for proving Proposition 5.2, we begin as in the proof of Proposition 5.1-(2). Setting $\zeta_{ijn} = \left|X_i\left(\frac{L_{ijn}}{q_n}\right) - X_i\left(\frac{L_{ijn}-1}{q_n}\right)\right|$ and since $\|X\|$ is bounded, we obtain by Property 1:

$$\left|\prod_{p=1}^{j}\zeta_{i\sigma_i(\ell_p)n} - \prod_{p=1}^{j}\mathbb{E}\,\Delta_{\sigma(\ell_p)}\right| \le (2C)^{j-1}\sum_{p=1}^{j}\left|\zeta_{i\sigma_i(\ell_p)n} - \mathbb{E}\,\Delta_{\sigma(\ell_p)}\right|.$$

The proof is concluded with the classical approximations of X_i by Z_i and $L_{i\sigma_i(j)n}/q_n$ by $T_{i\sigma_i(j)}$. Here, all the quantities are a.s. bounded so we may make use of the Hoeffding's inequality to derive the claimed exponential bounds. Details are left to the reader.

References

[1] Y. Aït-Sahalia, J. Jacod, Fisher's information for discretely sampled Lévy processes, Econometrica 76 (4) (2008) 727–761.
[2] Y. Aït-Sahalia, J. Jacod, Estimating the degree of activity of jumps in high frequency data, Ann. Statist. 37 (5A) (2009) 2202–2244.
[3] A. Antoniadis, X. Brossat, J. Cugliari, J.-M. Poggi, Prévision d'un processus à valeurs fonctionnelles en présence de non stationnarités. Application à la consommation d'électricité, J. SFdS 153 (2) (2012) 52–78.
[4] O.E. Barndorff-Nielsen, F.E. Benth, A.E. Veraart, Modelling electricity futures by ambit fields, Adv. Appl. Probab. 46 (3) (2014) 719–745.
[5] P. Besse, H. Cardot, D. Stephenson, Autoregressive forecasting of some climatic variation, Scand. J. Stat. 27 (4) (2000) 673–687.
[6] P. Billingsley, Convergence of Probability Measures, second ed., in: Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons Inc., New York, 1999.
[7] D. Blanke, D. Bosq, Exponential bounds for intensity of jumps, Math. Methods Statist. 23 (3) (2014) 1–17.
[8] T. Bollerslev, V. Todorov, Estimation of jump tails, Econometrica 79 (6) (2011) 1727–1783.
[9] E.G. Bongiorno, E. Salinelli, A. Goia, P. Vieu, Contributions in Infinite-Dimensional Statistics and Related Topics, Società Editrice Esculapio, 2014.
[10] A.V. Borisov, Analysis and estimation of the states of special Markov jump processes. II. Optimal filtering in the presence of Wiener noise, Avtomat. i Telemekh. 5 (2004) 61–76.
[11] D. Bosq, Linear Processes in Function Spaces, in: Lecture Notes in Statistics, vol. 149, Springer-Verlag, New-York, 2000.
[12] D. Bosq, Estimating and detecting jumps. Applications to $D[0,1]$-valued linear processes, in: M. Hallin, D.M. Mason, D. Pfeifer, J.G. Steinebach (Eds.), Festschrift in Honour of Paul Deheuvels, Springer, 2015, pp. 41–66 (Chapter 4).
[13] D. Bosq, D. Blanke, Prediction and Inference in Large Dimensions, in: Wiley Series in Probability and Statistics, John Wiley & Sons, Ltd., Chichester, 2007, Dunod, Paris.
[14] P.J. Brockwell, R.A. Davis, Y. Yang, Estimation for nonnegative Lévy-driven Ornstein–Uhlenbeck processes, J. Appl. Probab. 44 (4) (2007) 977–989.
[15] H. Cardot, C. Crambes, P. Sarda, Ozone pollution forecasting using conditional mean and conditional quantiles with functional covariates, in: Statistical Methods for Biostatistics and Related Fields, Springer, Berlin, 2007, pp. 221–243.
[16] J. Chiquet, N. Limnios, Dynamical systems with semi-Markovian perturbations and their use in structural reliability, in: Stochastic Reliability and Maintenance Modeling, in: Springer Ser. Reliab. Eng., vol. 9, Springer, London, 2013, pp. 191–218.
[17] E. Clément, S. Delattre, A. Gloter, Asymptotic lower bounds in estimating jumps, Bernoulli 20 (3) (2014) 1059–1096.
[18] F. Comte, C. Duval, V. Genon-Catalot, Nonparametric density estimation in compound Poisson processes using convolution power estimators, Metrika 77 (1) (2014) 163–183.
[19] F. Comte, V. Genon-Catalot, Nonparametric adaptive estimation for pure jump Lévy processes, Ann. Inst. Henri Poincaré Probab. Stat. 46 (3) (2010) 595–617.
[20] R. Cont, P. Tankov, Financial Modelling with Jump Processes, in: Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton, FL, 2004.
[21] C. Crambes, A. Mas, Asymptotics of prediction in functional linear regression with functional outputs, Bernoulli 19 (5B) (2013) 2627–2651.
[22] S. Dabo-Niang, A. Laksaci, Nonparametric quantile regression estimation for functional dependent data, Comm. Statist. Theory Methods 41 (7) (2012) 1254–1268.
[23] J. Damon, S. Guillas, Estimation and simulation of autoregressive hilbertian processes with exogenous variables, Stat. Inference Stoch. Process. 8 (2) (2005) 185–204.
[24] B. de Saporta, J.-F. Yao, Tail of a linear diffusion with Markov switching, Ann. Appl. Probab. 15 (1B) (2005) 992–1018.
[25] S. Djebali, L. Górniewicz, A. Ouahab, Topological structure of solution sets for impulsive differential inclusions in Fréchet spaces, Nonlinear Anal. 74 (6) (2011) 2141–2169.
[26] C. Duval, Density estimation for compound Poisson processes from discrete data, Stochastic Process. Appl. 123 (11) (2013) 3963–3986.
[27] L. El Hajj, Théorèmes limites pour les processus autorégressifs à valeurs dans $D[0,1]$ (Ph.D. thesis), Université Paris, 6, Paris, 2013, décembre.

[28] L. El Hajj, Estimation et prévision des processus à valeurs dans $D[0, 1]$, Ann. I.S.U.P. 58 (3) (2014) 61–76.
[29] N. El Karoui, E. Gobet, Les outils Stochastiques des Marchés Financiers, Éditions de l'École Polytechnique, Palaiseau, 2012.
[30] F. Ferraty, Y. Romain (Eds.), The Oxford Handbook of Functional Data Analysis, Oxford University Press, Oxford, 2011.
[31] F. Ferraty, P. Vieu, Nonparametric Functional Data Analysis, in: Springer Series in Statistics, Springer, New York, 2006, Theory and practice.
[32] A. Goia, A functional linear model for time series prediction with exogenous variables, Statist. Probab. Lett. 82 (5) (2012) 1005–1011.
[33] R. Guy, C. Larédo, E. Vergu, Approximation of epidemic models by diffusion processes and their statistical inference, J. Math. Biol. 70 (3) (2015) 621–646.
[34] X. Guyon, S. Iovleff, J.-F. Yao, Linear diffusion with stationary switching regime, ESAIM Probab. Stat. 8 (2004) 25–35.
[35] L. Horváth, P. Kokoszka, Inference for Functional Data With Applications, in: Springer Series in Statistics, Springer, New York, 2012.
[36] R. Ignaccolo, S. Ghigo, S. Bande, Functional zoning for air quality, Environ. Ecol. Stat. 20 (1) (2013) 109–127.
[37] V. Jacq, P. Albert, R. Delorme, Le mistral. Quelques aspects des connaissances actuelles, La météorologie 50 (2005) 30–38.
[38] S. Janson, S. Kaijser, 2012. Higher moments of banach space valued random variables. ArXiv e-prints. URL http://arxiv.org/abs/1208.4272v2.
[39] M. Jeanblanc, M. Yor, M. Chesney, Mathematical Methods for Financial Markets, in: Springer Finance, Springer-Verlag London, Ltd., London, 2009.
[40] D. Kannan, V. Lakshmikantham (Eds.), Handbook of Stochastic Analysis and Applications, in: Statistics: Textbooks and Monographs, vol. 163, Marcel Dekker, Inc., New York, 2002.
[41] V. Kargin, A. Onatski, Curve forecasting by functional autoregression, J. Multivariate Anal. 99 (10) (2008) 2508–2526.
[42] V.S. Koroliuk, N. Limnios, Poisson approximation of processes with locally independent increments with Markov switching, Theory Stoch. Process. 15 (1) (2009) 40–48.
[43] O.O. Kurchenko, Convergence of a sequence of random fields in the space $D([0, 1]^d)$, Teor. Ĭmovīr. Mat. Stat. (64) (2001) 82–91.
[44] P. Lévy, Fonctions aléatoires à corrélation linéaire, C. R. Acad. Sci. Paris 242 (1956) 2095–2097.
[45] J.M. Marion, B. Pumo, Comparaison des modèles ARH(1) et ARHD(1) sur des données physiologiques, Ann. I.S.U.P. 48 (3) (2004) 29–38.
[46] G. Nason, A test for second-order stationarity and approximate confidence intervals for localized autocovariances for locally stationary time series, J. R. Stat. Soc. Ser. B. Stat. Methodol. 75 (5) (2013) 879–904.
[47] W.R. Pestman, Measurability of linear operators in the Skorokhod topology, Bull. Belg. Math. Soc. Simon Stevin 2 (4) (1995) 381–388.
[48] D. Preston, P. Protopapas, C. Brodley, Discovering arbitrary event types in time series, Stat. Anal. Data Min. 2 (5–6) (2009) 396–411.
[49] N. Privault, Stochastic Finance, in: Chapman & Hall/CRC Financial Mathematics Series, CRC Press, Boca Raton, FL, 2014, An introduction with market examples.
[50] J.O. Ramsay, B. Silverman, Functional Data Analysis, Springer Verlag, 2005.
[51] Y. Shimizu, Threshold selection in jump-discriminant filter for discretely observed jump processes, Stat. Methods Appl. 19 (3) (2010) 355–378.
[52] Y. Shimizu, N. Yoshida, Estimation of parameters for diffusion processes with jumps from discrete observations, Stat. Inference Stoch. Process. 9 (3) (2006) 227–277.
[53] A.N. Shiryaev, Probability, second ed., in: Graduate Texts in Mathematics, vol. 95, Springer-Verlag, New York, 1996, Translated from the first (1980) Russian edition by R. P. Boas.
[54] P. Tankov, E. Voltchkova, Jump-diffusion models: a practitioner's guide, Banque et Marchés 99 (2009) 1–24.
[55] N.M. Tanushev, Superpositions and higher order Gaussian beams, Commun. Math. Sci. 6 (2) (2008) 449–475.
[56] L. Torgovitski, A Darling–Erdős-type CUSUM-procedure for functional data, Metrika 78 (1) (2015) 1–27.