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Strong Approximation of the Quantile Processes and Its Applications under Strong Mixing Properties

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Given some regularity conditions on the distribution $\mathbf{F}(\cdot)$ of a random $X_1, ..., X_n$ emanating from a strictly stationary sequence of random variables satisfying a strong mixing condition, it is shown that the sequence of quantile processes $\{n^{1/2}f(\mathbf{F}^{-1}(s))(\mathbf{F}_n^{-1}(s)-\mathbf{F}^{-1}(s)); 0<s<1\}$ behaves like a sequence of Brownian bridges $\{B_n(s); 0<s<1\}$. The latter is then utilized to construct (i) simultaneous bounds for the unknown quantile function $\mathbf{F}^{-1}(s)$, and (ii) a tolerance interval for predicting a future observation. Some numerical investigations of the results are also discussed. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let $\{X_n : n \in \mathbf{Z}\}$ be a real valued strictly stationary sequence of random variables taking values in a space $(\Omega, \mathscr{F}, P)$ with common distribution $F(x)$. The nth empirical and quantile measures are given by $F_n(x) = n^{-1} \sum_{i=1}^{n} I_{(-\infty, x]}(X_i)$ and $F_n^{-1}(s) = \inf\{X) \geqslant s\}$, respectively, and their processes are defined as

$$C_n(x) = n^{1/2}(F_n(x) - F(x)), \qquad -\infty < x \leqslant \infty,$$

and

$$Q_n(s) = n^{1/2}(F_n^{-1}(s) - F^{-1}(s)), \qquad 0 < s < 1.$$

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We will investigate how well $C_n(x)$ and $Q_n(x)$ can be approximated by the Gaussian Processes. The literature on the behavior of the strong approximation of $C_n(x)$ and $Q_n(x)$ for independently and identically sequences of random variables is extensive, with prominent contributions from Csáki (1977), Csörgő and Révész (1978), 1981), and Csörgő (1983), to name a few. For the dependence case, one needs to define a mixing coefficient as follows. For any collection X of random variables, let $\mathbf{B}(X)$ denote the Borel field generated by X. Thus, for $-\infty \leqslant m \leqslant n \leqslant \infty$, define $\mathscr{F}_m^n = \mathbf{B}(X_k : m \leqslant k \leqslant n)$. Hence, for each $n \geqslant 1$, define

$$\begin{aligned}\alpha_{r,s}(n) &= \alpha_{r,s}(\mathscr{F}^0_{-\infty}, \mathscr{F}_n^{\infty}) \\ &= \sup\left\{\frac{P(A \cap B) - P(A)\,P(B)}{P(A)^r\,P(B)^s} : A \in \mathscr{F}^0_{-\infty}, B \in \mathscr{F}_n^{\infty},\right.\end{aligned}$$

and

$$\left. P(A)\,P(B) \neq 0\right\} \downarrow 0, \tag{1.1}$$

where $0 \leqslant r, s \leqslant 1$. Since the process is stationary, it yields $\alpha_{r,s}(n) = \alpha_{r,r}(\mathscr{F}^j_{-\infty}, \mathscr{F}^{\infty}_{j+n})$ for any integer j. With $r = s = 0$, the process satisfying the (1.1) condition is called a strong mixing process, and $\alpha(n) = \alpha_{00}(n)$ is called a strong mixing coefficient; with $r = 1$ and $s = 0$, one obtains the uniform mixing process, and $\varphi(n) = \alpha_{10}(n)$ is the uniform mixing coefficient. For the sake of completeness, we also designate the absolute regular process to be when the absolute regular coefficient $\beta(n) = E\{\sup_{A \in \mathscr{F}_n^{\infty}} |P(A \mid \mathscr{F}^0_{-\infty}) - P(A)|\} \downarrow 0$. Extensive information about conditions of type (1.1) and absolute regular processes can be found in Bradley (1986). Philipp (1977), Berkes and Philipp (1977), and Yoshihara (1979) have studied the large sample strong approximation properties for the empirical process $C_n(x)$ for strictly stationary and strong mixing sequences of random variables with a strong mixing coefficient decreasing at polynomial order.

We have two main objectives in this work. The first, is to establish strong approximation results for the nth quantile process under a strictly stationary strong mixing coefficient decaying at polynomial rate. The second, and more important, aim will be to try to understand what these results represent; this is achieved by demonstrating how to construct simultaneous confidence intervals for the quantile function and to obtain one-step-ahead prediction intervals for future observations. In addition to the fact that these results are of considerable intrinsic interest, they are included here to show how much one can do from the understanding we shall develop. The results are attained by relating empirical processes with those of the quantile. The link between the empirical process and the quantile process is restricted to the strong mixing case. It is not hard to see that most of the argument also extends to other types of mixing cases, i.e., absolute regular and uniform mixing, although there are some non-trivial technical problems to overcome on the way. (One needs to establish similar lemmas to Lemma 1, 2 and 4 below, for absolute regular and uniform mixing.) The important fact to know, however, is that although some details change, the same intuition developed for the strong mixing case carries over qualitatively to uniform mixing and absolute regular processes.

2. Background and Results

In this section, our focus is on introducing some further notations, to define two types of Gaussian processes, which play an important role in our approximation, and to present the main results of the study.

We commence with notations. The symbol $\ll$ denotes that the left-hand side is bounded by an unspecified constant times the right-hand side; it is used instead of $0(\cdot)$ notation. In exactly the same way as in the i.i.d. case, we redefine the space for which the sequence $\{X_n : n \in \mathbf{Z}\}$ was generated to a space which is rich enough in the sense that a separable Gaussian process can be defined on it. Now, the separable Gaussian process will be called the Brownian bridge $\{B(s) : 0 \leqslant s \leqslant 1\}$, if $B(1) = B(0) = 0$, $E(B(s)) = 0$, and has covariance function $E(B(s)\, B(s')) = \Gamma(s, s')$, for $0 \leqslant s,\ s' \leqslant 1$. To define $\Gamma(s, s')$, we write

$$g_n(s) = I_{[0,s]}(U_n) - s, \qquad n \geqslant 1, \tag{2.1}$$

where $\{U_n : n \in \mathbf{Z}\}$ is a uniform on $[0, 1]$ strictly stationary strong mixing sequence of random variables.Then, for $0 \leqslant s,\ s' \leqslant 1$,

$$\Gamma(s, s') = E(g_1(s)\, g_1(s')) + \sum_{n=2}^{\infty} \{E(g_1(s)\, g_n(s')) + E(g_n(s)\, g_1(s'))\}, \tag{2.2}$$

such that the series on the right-hand side of $\Gamma(s, s')$ is absolutely convergent.

The second separable Gaussian process which will be utilized here is a Kiefer type process $\{K(s, t) : 0 \leqslant s \leqslant 1,\ t \geqslant 0\}$ with $K(s, 0) = K(1, t) = K(0, t) = 0$, $E(K(s, t)) = 0$, and covariance function $\Gamma^*(t, t', s, s') = \min(t, t')\, \Gamma(s, s')$, for $t, t' \geqslant 0$ and $0 \leqslant s,\ s' \leqslant 1$.

For convenient reference, the basic conditions on $F(\cdot)$, from which the various results are obtained, are gathered together here.

F_1: $F(x)$ is twice differentiable on (a, b), where $-\infty \leqslant a = \sup\{x : F(x) = 0)$ and $\infty \geqslant b = \inf\{x : F(x) = 1\}$,

F_2: $F' = f \neq 0$ on (a, b),

F_3: $\sup_{0<s<1} |f'(F^{-1}(s))| < \infty$,

F_4: $\sup_{0<s<1} |f''(F^{-1}(s))| < c$, for some constant $c > 0$,

F_5: $F(x)$ is strictly increasing on $[a, b]$,

F_6: $f(x)$ is unimodal,

F_7: $\sup_{s<x<b} F(x)(1-F(x))(|f'(x)|/f^2(x)) \leqslant \gamma$, for some $\gamma > 0$,

F_8: $A = \limsup_{x \downarrow -\infty} f(x) < \infty$, $B = \limsup_{x \uparrow \infty} f(x) < \infty$,

F_9: $\min(A, B) > 0$,

F_{10}: if $A = 0$ (resp. $B = 0$), then f is non-decreasing (resp. non-increasing) on an interval to the right of a (resp. to the left of b), and

F_{11}: for sufficiently large n, $f(F^{-1}(n^{-1/2}l(n))) > c$, where $c > 0$, and $l(n)$ is a slowly varying function of n with $l(n)^{1/\lambda} < \log n$.

THEOREM 1. *Let $\{X_i : i \in \mathbf{Z}\}$ be a strictly stationary real valued sequence of random variables satisfying the strong mixing property with strong mixing coefficient*

$$\alpha(n) \ll n^{-8}.$$

Suppose that F_1, F_2, and F_3 are satisfied. Then, there exists a Brownian bridge defined on the same probability space as the above sequence with covariance function $\Gamma(s, s')$, $0 \leqslant s, s' \leqslant 1$, and a positive constant λ such that with probability one

$$\sup_{0<s<1} |f(F^{-1}(s))\, Q_n(s) - B_n(s)| \ll (\log n)^{-\lambda}.$$

If we replace F_3 by milder conditions, then a Theorem 6 version of Csörgő and Révész (1978) can be formulated as follows.

THEOREM 2. *Let $\{X_i : i \in \mathbf{Z}\}$ be a strictly stationary real valued sequence of random variables satisfying the strong mixing property with strong mixing coefficient*

$$\alpha(n) \ll n^{-8}.$$

Suppose that F_1, F_2, and F_7 are satisfied. Then there exists a Kiefer process defined on the same probability space as the above sequence with covariance function $\min(n, n')\, \Gamma(s, s')$, *for* $0 < n, n' < \infty$, *and* $0 \leqslant s, s' \leqslant 1$, *such that*

$$\sup_{n^{-\mu} \leqslant s \leqslant 1-n^{-\mu}} |f(F^{-1}(s))\, Q_n(s) - K(s, n)/n^{1/2}| \ll (\log n)^{-\lambda},$$

for some $0 < \mu < d\alpha$, $d \in (0, 1/4)$, and $\alpha \in (0, 1/120)$ and $\lambda = 1/3840$.

If, in addition to F_1, F_2, and F_7, we assume that F_8 and F_9 hold, then

$$\sup_{0<s<1} |f(F^{-1}(s))\, Q_n(s) - K(s, n)/n^{1/2}| \ll (\log n)^{-\lambda}.$$

Remarks. (i) The polynomial order of the mixing coefficient is the same as that used on Berkes and Phillip (1977), and consequently the results of the previous theorems are of the same order as in Berkes and Phillip (1977). (ii) If, in addition to F_1, F_2, and F_7, we assume that F_8, F_{10}, and F_{11} hold, then

$$\sup_{n^{-1/2}l(n)^{-1} < s < 1 - n^{1/2}l(n)^{-1}} |f(F^{-1}(s))\, Q_n(s) - K(s, n)/n^{1/2}| \ll (\log n)^{-\lambda},$$

where λ is as above.

The proof of this will be seen in the next section.

The next two theorems exploit the above results in showing how confidence contours for the quantile measures are obtained. As they stand, Theorems 1 and 2 do not provide an immediate confidence bound for the quantile function; this is because they depend on the unknown function $1/f(F^{-1}(s))$. For this reason, an estimator of $1/f(F^{-1}(s))$ is required. The estimator proposed in this study is shown to be strong consistent. But first, two sets of assumptions are needed.

The proposed estimator is a Kernel-type, which is similar to that suggested for the independence case by Csörgő and Révész (1984).

$$\phi_n(s) = \frac{1}{h_n} \int_0^1 K\left(\frac{s-u}{h_n}\right) dF_n^{-1}(s).$$

$K(\cdot)$ is a measurable Kernel-type function with the following assumptions

K_1: $K(\cdot)$ is a density function, which is absolutely continuous on $(-\infty, \infty)$ vanishing outside of the interval $(-1/2, 1/2)$,

K_2: $\int_{-1/2}^{1/2} xK(x)\, dx = 0$,

K_3: $\int_{-1/2}^{1/2} x^2K(x)\, dx < \infty$, and

K_4: $\sup_{-\infty<x<\infty} |K'(x)| < \infty$.

For the sequence of constants $\{h_n : n \in \mathbf{N}\}$, we assume the following:

H_1: $h_n \ll (\log\log n)^{1/6}/n^{1/6}$, and

H_2: $(\log n)^{\lambda} \ll h_n n^{1/2}/\log\log n$, where λ is defined in Theorem 1.

With this preparation in mind, the third result is now in order.

THEOREM 3. *Let $\{X_i : i \in \mathbf{Z}\}$ be a strictly stationary real valued sequence of random variables satisfying the strong mixing property with strong mixing coefficient*

$$\alpha(n) \ll n^{-8}.$$

Suppose that F_1–F_4, K_1–K_4, H_1 and H_2 are satisfied, then there exists a Brownian bridge defined on the same probability space with $E(B_n(s)) = 0$, $E(B_n(s), B_n(s')) = \Gamma(s, s')$, for $0 \leqslant s, s' \leqslant 1$, and a positive constant λ such that with probability one

$$\sup_{0<s<1} \left| \frac{1}{\phi_n(s)} Q_n(s) - B_n(s) \right| \ll (\log n)^{-\lambda},$$

and

$$\lim_{n \uparrow \infty} P(F_n^{-1}(s) - n^{1/2}\phi_n(s)c \leqslant F^{-1}(s)$$
$$\leqslant F_n^{-1}(s) + n^{1/2}\phi_n(s)c;\, 0 < s < 1) = K(c),$$

where $K(c) = P(\sup_{0 \leqslant s \leqslant 1} |B_n(s)| \leqslant c)$.

The above result is an analog of Kolmogorov's classical theorem on the Empirical distribution function. It should be pointed out that one-sided intervals can also be deduced from the above result. Theorems 1 and 2 may produce more direct and simpler confidence intervals for the quantile measure $F^{-1}(s)$ (the need of Kernel-type Estimator of $1/f(F^{-1}(s))$ is unnecessary), as can be seen below.

The following theorem is a Csörgő and Horváth (1990) analog for the stationary case.

THEOREM 4. *Let $\{X_i : i \in \mathbf{Z}\}$ be a strictly stationary real valued sequence of random variables satisfying the strong mixing property with mixing coefficient*

$$\alpha(n) \ll n^{-8}.$$

Then, for the sequence $\{\delta_n = n^{-\mu}$, $\mu \in (0, d\alpha)$, $d \in (0, 1/4)$, $\alpha \in (0, 1/120)$ and $n \in \mathbf{N}\}$ of positive constants, the following statements

$$\lim_{n \uparrow \infty} P(F_n^{-1}(s - n^{-1/2}c) \leqslant F^{-1}(s);\, \delta_n \leqslant s \leqslant 1 - \delta_n),$$
$$= \lim_{n \to \infty} P(F^{-1}(s) \leqslant F_n^{-1}(s + n^{1/2}c);\, \delta_n \leqslant s \leqslant 1 - \delta_n)$$
$$= P(\sup_{0 \leqslant s \leqslant 1} B(s) \leqslant c),$$

and

$$\lim_{n\to\infty} P(F_n^{-1}(s - n^{1/4}c) \leqslant F^{-1}(s) \leqslant F_n^{-1}(s + n^{1/2}c);\ \delta_n \leqslant s \leqslant 1-\delta_n)$$

$$= P(\sup_{0\leqslant s\leqslant 1} |B(s)| \leqslant c),$$

where $B(s)$ is a Gaussian process with covariance function $\Gamma(s, s')$, $0 \leqslant s$, $s' \leqslant 1$, hold true without assuming any further conditions on F.

The next result discussed here, which ties in with the proposed method, is the randomly located tolerance interval. The general theory of constructing prediction intervals for future observations was developed and expanded in detail by Butler (1981). To give a fully detailed, self-contained version of showing some weak convergence results similar to lemmas found in Butler's work (2.1, 2.3, etc.) under now dependence, we would have to copy a few pages of detailed and essentially uninteresting calculations. Also, Cho and Miller (1987) have produced some of these results under strictly stationary uniform mixing sequences. Since this seems to be a somewhat unjustified addition, we assume that the reader is familiar with the Butler (1981) and Cho and Miller (1987) studies, and merely point out some results and ideas developed there and omit the proofs.

The class of $100\alpha\%$ tolerance intervals for X_{n+1} is given by

$$\{I(\delta) = [F^{-1}(\delta), F^{-1}(\delta+\alpha)] : 0 \leqslant \delta \leqslant 1-\alpha\}. \tag{2.4}$$

The one which supports the smallest trimmed variance, $\sigma^2(\delta)$, say, of (2.4), i.e., $I(\delta^*)$, is the chosen interval for predicting the one-step-ahead, X_{n+1}, future observation. Since $F(\cdot)$ is unknown, an estimator of δ^*, $\hat{\delta}^*$ say, is obtained by substituting $F^{-1}(\cdot)$ with $F_n^{-1}(\cdot)$. The $100\alpha\%$ prediction interval for X_{n+1} is then given by

$$\hat{I}(\hat{\delta}^*) = [F_n^{-1}(\hat{\delta}^*), F_n^{-1}(\hat{\delta}^*+\alpha)]. \tag{2.5}$$

The property, which can be verified by using Theorem 1 as in Cho and Miller (1987), is expressed in the following corollary.

COROLLARY 1. *Let $\{X_i : i \in \mathbf{Z}\}$ be a strictly stationary strong mixing sequence of real-valued random variables with strong mixing coefficient*

$$\alpha(n) \ll n^{-8}.$$

Suppose that F_1–F_6 are satisfied, then

(i) $\hat{\delta}^* \to \delta^*$ *in probability, and*

(ii) $P_n(\alpha) = P(F_n^{-1}(\hat{\delta}^*) \leqslant X_{n+1} \leqslant F_n^{-1}(\hat{\delta}^*+\alpha)) \to \alpha$ *in probability.*

Some numerical investigation of the coverage probability $P_n(\alpha)$ is discussed in the next section.

Discussion. The method provided in this study has the advantage of being used directly without assuming any prior structure of the process $\{X_t; t \in \mathbf{Z}\}$. It is flexible and accurate and can be applied to a wide variety of realistic situations. This is indicated by the numerical investigation given below. Specifically, the corollary enables us to obtain nonparametric prediction intervals without fitting parametrix models. Therefore we do not have to worry about specification of the process X_{n+1}, when we have non-Gaussian marginals such as exponential and Laplacian. This is because the maximum likelihood estimator under exponential or Laplacian has a boundary problem (see, e.g., Smith 1986). It might be argued that for an autocorrelated process $\{X_t; t \in \mathbf{Z}\}$ the last observation X_n may carry more weight than $X_{n-1}, X_{n-2}, ...$ as in the case of exponential smoothing. However, exponential smoothing lacks the theoretical background, especially in assessing the forecast error (see, e.g., Chatfield, 1993). A Monte Carlo simulation study in Fotopoulos *et al.* (1993) shows that the prediction interval suggested in the above corollary performs well for an ARMA (1, 1) process when the innovation process is drawn from a standard Cauchy distribution, where the second moment does not exist. The use of prediction intervals instead of point prediction is advocated in Keyfitz (1972), Butler (1981), and Cho and Miller (1987), among others.

3. Numerical Investigation in Prediction

The class of strictly stationary processes is very broad. Obviously, the one-sided linear processes expressed by the form

$$X_t = \sum_{i=0}^{\infty} g_i Z_{t-i} \tag{3.1}$$

is included in this class, if $\sum |g_i| < \infty$ and the sequence $\{\mathbf{Z}_i, i \in \mathbf{Z}\}$ is an i.i.d. with $E\mathbf{Z}_i = 0$ and $E\mathbf{Z}_1^2 < \infty$. Further, by Corollary 4 in Withers (1981), it can be shown that if the common density $p(x)$, say, of an independently and identically distributed sequence $\{Z_i : i \in \mathbf{Z}\}$ satisfies

(a) $\int |p(x) - p(x+y)| \, dx \leqslant c \, |y|$, where $c > 0$,

(b) $EZ_i = 0$, $E\mathbf{Z}_1^2 = 1$,

(c) $\sum_{k=0}^{\infty} g_k z^k \neq 0$, for $|z| \leqslant 1$ and $g_k \ll k^{-v}$, $v > 3/2$,

then $\{X_t : t \in \mathbf{Z}\}$ is also a strong mixing sequence with $\alpha(n) \ll n^{-\varepsilon}$, where ε depends upon the moment condition of $\mathbf{Z}_i$'s and v. Similarly, if, in addition to the above conditions, $g_k \ll e^{-\nu k}$ (instead of polynomial order), then the conclusion remains the same, but now $\alpha(n) \ll e^{-\nu\lambda k}$ and λ depends upon the moment conditions. However, if some of the conditions stated above are violated, then there are counter examples that even if the process $\{X_t : t \in \mathbf{Z}\}$ satisfies the one-sided linear form, it is not a strong mixing sequence, i.e., $X_t = 1/2X_{t-1} + Z_t = \sum_{i=0}^{\infty} 2^{-i} Z_{t-i}$, where $P(Z_t = 1) = P(Z_t = -1) = 1/2$ (see, e.g., Bradley, 1986), is not a strong mixing process. It is apparent that when the innovative process $\{Z_t : t \in \mathbf{Z}\}$ is an i.i.d. Gaussian process satisfying the above conditions, then one can show that the sequence $\{X_t : t \in \mathbf{Z}\}$ is strictly stationary strong mixing. Obviously, the ARMA (p, q), $p \geqslant 0$, $q \geqslant 0$ is a particular case of (3.1).

To see how this confidence interval for the one-step-ahead new observations works, we performed the simulation by generating observations from AR(2)

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$$

and ARMA(2, 1)

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t - \theta Z_{t-1},$$

respectively, where ϕ_1, ϕ_2, and θ are the autoregressive and moving average parameters and Z_t is a Gaussian innovation with mean 0 and constant variance. The parameter values are selected within the triangular region so that they satisfy the stationary condition given in Box and Jenkings (1976).

Following the same procedure as in Cho and Miller (1987), we generated the 151 observations using the RNNOR routine of IMSL, and discarded the first 50 observations to minimize the effect of starting values. Using the remaining 100 observations, we constructed the 90% P.I. leaving the last observation to check the coverage. The coverage percentages were

TABLE I

Coverage Percentage of One-Step-Ahead P.I., AR(2)

ϕ_2 \ ϕ_1	−1.8	−1.5	−1.2	−0.9	−0.6	−0.3	0.0	0.3	0.6	0.9	1.2	1.5	1.8
0.9							0.837						
0.6						0.875	0.905	0.832					
0.3					0.845	0.905	0.912	0.867	0.855				
0.0				0.890	0.897	0.885	0.887	0.902	0.917	0.857			
−0.3			0.887	0.897	0.872	0.895	0.892	0.880	0.885	0.892	0.867		
−0.6		0.897	0.855	0.892	0.902	0.915	0.885	0.875	0.887	0.890	0.892	0.870	
−0.9	0.855	0.865	0.887	0.862	0.880	0.875	0.895	0.880	0.882	0.887	0.900	0.855	0.895

TABLE II

Coverage Percentage of One-Step-Ahead P.I., ARMA(2, 1), $\theta = 0.4$

ϕ_2 \ ϕ_1	-1.8	-1.5	-1.2	-0.9	-0.6	-0.3	0.0	0.3	0.6	0.9	1.2	1.5	1.8
0.9							0.837						
0.6						0.875	0.900	0.867					
0.3					0.872	0.892	0.897	0.870	0.860				
0.0				0.872	0.920	0.902	0.897	0.882	0.915	0.862			
0.3			0.880	0.887	0.882	0.907	0.897	0.857	0.887	0.905	0.875		
0,6		0.880	0.870	0.890	0.882	0.907	0.890	0.885	0.880	0.890	0.897	0.875	
0.9	0.860	0.877	0.892	0.852	0.882	0.890	0.915	0.875	0.885	0.892	0.885	0.872	0.892

obtained from 400 replications and are reported in Tables I and II. It is observed that most of the coverage percentages ranged from 87 to 92%, which is similar to the result reported in Cho and Miller. Although we did the simulation only for the AR(2) and ARMA(2, 1) processes, considering that most of the time series data can be approximated by the low order ARMA models, e.g., ARMA(p, q), $p \leqslant 2$ and $q \leqslant 2$, we arrived at the same conclusions as did Cho and Miller for ARMA$(1, q)$, $q \geqslant 0$.

4. Proofs

Since $\{U_i = F(X_i) : i \in \mathbf{Z}\}$ forms a uniform on $[0, 1]$ sequence of strictly stationary random variables, and since the problem under uniformity can be handled much more easily, the use of the following sequences is now adopted.

$\{E_n(s) : 0 \leqslant s \leqslant 1, n \in N\}$ is the uniform empirical distribution function,

$\{E_n^{-1}(s) : 0 < s < 1, n \in N\}$ is the uniform quantile function,

$\{U_n(s) = n^{1/2}(E_n(s) - s) : 0 \leqslant s \leqslant 1, n \in N\}$ is the uniform empirical process,

$\{V_n(s) = n^{1/2}(E_n^{-1}(s) - s) : 0 \leqslant s \leqslant 1, n \in N\}$ is the uniform quantile process, and

$\{R_n^*(s) = U_n(s) + V_n(s) : 0 < s < 1, n \in N\}$ is the uniform Bahadur–Kiefer process.

Proof of Theorem 1. In establishing Thoerems 1–4, we were aided by some ideas found in Berkes and Philipp (1977), Csörgő (1983), Csörgő and Révész (1978), Philipp (1977), and Philipp and Pinzur (1980), but first we start with some standard results. From the mean value theorem, it is obvious that under F_1 the following result is in order:

$$\begin{aligned}
Q_n(s) &= n^{1/2}(F_n^{-1}(s) - F^{-1}(s)), \qquad s \in (0,1) \\
&= n^{1/2}(F^{-1}(E_n^{-1}(s) - F^{-1}(s))) \\
&= n^{1/2}(E_n^{-1}(s) - s)/f(F^{-1}(\theta_{s,n})) \\
&= V_n(s)/f(F^{-1}(s)) + V_n(s)\left\{\frac{f(F^{-1}(s))}{f(F^{-1}(\theta_{s,n}))} - 1\right\}\Big/ f(F^{-1}(s)) \\
&= V_n(s)/f(F^{-1}(s)) + \frac{V_n(s)(s-\theta_{s,n})\, f'(F^{-1}(\delta_{s,n}))}{f(F^{-1}(\delta_{s,n}))\, f(F^{-1}(\theta_{s,n}))\, f(F^{-1}(s))} \\
&= (V_n(s) + \varepsilon_n(s))/f(F^{-1}(s)), \qquad (4.1)
\end{aligned}$$

where $\theta_{s,n} \in (E_n^{-1}(s) \wedge s, E_n^{-1} \vee s)$, $\delta_{s,n} \in (\theta_{s,n} \vee s,\ \theta_{s,n} \vee s)$ and $\varepsilon_n(s) = V_n(s)(s-\theta_{s,n})\, f'(F^{-1}(\delta_{s,n}))/f(F^{-1}(\delta_{s,n}))\, f(F^{-1}(\theta_{s,n}))$. Here, the conventions $\wedge$ and $\vee$ used above mean $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$, respectively.

From the definition of $R_n^*(s)$, it then follows that

$$\begin{aligned}
f(F^{-1}(s))\, Q_n(s) &= V_n(s) + \varepsilon_n(s) \\
&= -U_n(s) + R_n^*(s) + \varepsilon_n(s), \qquad (4.2)
\end{aligned}$$

which, in turn, yields

$$\begin{aligned}
\sup_{0<s<1} |f(F^{-1}(s))\, Q_n(s) - B_n'(s)| &\leqslant \sup_{0 \leqslant s \leqslant 1} |U_n(s) - B_n(s)| \\
&\quad + \sup_{0<s<1} |\varepsilon_n(s)| + \sup_{0<s<1} |R_n^*(s)|, \qquad (4.3)
\end{aligned}$$

where $B_n'(s) = -B_n(s)$ is a Brownian Bridge.

It remains to show that under the condition stated in Theorem 1, each of the terms in the right-hand side of (4.3) is, at most, of order $(\log n)^{-\lambda}$, for some $\lambda > 0$.

For the first term, the result by Philipp and Pinzur (1980) ($d = 1$) is needed. It is stated as follows.

LEMMA 1. *Let $\{U_i : i \in \mathbf{Z}\}$ be a uniform on $[0, 1]$ strictly stationary sequence of random variables satisfying the strong mixing condition with mixing coefficient*

$$\alpha(n) \ll n^{-5-\varepsilon},$$

for some $\varepsilon \in (0, 1/4]$. Let $\Gamma(s, s')$ be the covariance function. Then, without changing its distribution, we redefine the empirical process $\{U_k(s) : s \in [0, 1],\ k \geqslant 0\}$ of $\{U_n : n \in \mathbf{Z}\}$ on a richer probability space on which there *exists a Kiefer Process* $\{K(s, k) : s \in [0, 1], k \geqslant 0\}$ *with a covariance function* $\Gamma^*(s, s', t, t') = \min(t, t')\, \Gamma(s, s')$ *and a constant* $\lambda > 0$ *depending only on* ε, *such that with probability one*

$$\sup_{k \leqslant n} \sup_{s \in [0,1]} |k^{1/2} U_k(s) - K(s, k)| \ll n^{1/2} (\log n)^{-\lambda}.$$

For the latter result, it follows that

$$\sup_{0 \leqslant s \leqslant 1} \left| U_n(s) - \frac{K(s, n)}{n^{1/2}} \right| = \sup_{0 \leqslant s \leqslant 1} |U_n(s) - B_n(s)| \ll (\log n)^{-\lambda} \qquad \text{a.s.,} \qquad (4.4)$$

where $B_n(s) = n^{-1/2} K(s, n)$ is a Brownian Bridge.

This completes the proof of the first term being bounded above by $(\log n)^{-\lambda}$.

For the following result, we first define $f \colon \mathbf{R}^\infty \to \mathbf{R}$ to be measurable, with $Y_n = f(U_n, U_{n+1}, \dots)$, $n \geqslant 1$. Then, by defining $U_n^{(1)}(s)$ to be the uniform Empirical Process formed by the sequence $\{Y_n : n \in \mathbf{N}\}$, similar to $U_n(s)$, Berkes and Philipp (1977) have shown the following result.

LEMMA 2. *Under* $\alpha(j) \ll j^{-8}$, *the sequence* $\{(2 \log \log n)^{-1/2}\, U_n^{(1)}(s), n \geqslant 1\}$ *of functions on* $[0, 1] \times [0, 1]$ *is with probability one, relative compact, in the supremum norm, and has the unit ball B in the reproducing Kernel Hilbert space* $H(\Gamma^*)$ *as its set of limits, where* $\Gamma^*(s, s', t, t') = \min(t, t')\, \Gamma(s, s')$, *for* $0 \leqslant s, s' \leqslant 1$, *and* $t, t' \geqslant 0$.

An implication of Lemma 2 is the fact that

$$\limsup_{n \uparrow \infty} \sup_{0 \leqslant s \leqslant 1} |U_n^{(1)}(s)| / (2 \log \log n)^{1/2} \leqslant c \qquad \text{a.s.} \qquad (4.5)$$

for some constant $c > 0$.

For $f(U_1, U_2, \dots) = U_1$, (4.5) is deduced to

$$\limsup_{n \uparrow \infty} \sup_{0 \leqslant s \leqslant 1} |U_n(s)| / (2 \log \log n)^{1/2} \leqslant c, \qquad \text{a.s.} \qquad (4.6)$$

The following lemma is due to Babu and Singh (1978).

LEMMA 3. *If* (4.6) *holds, then*

$$\limsup_{n \uparrow \infty} \sup_{0 \leqslant s \leqslant 1} |V_n(s)| / (2 \log \log n)^{1/2} \leqslant c, \qquad \text{a.s.,}$$

for the same constant c defined in (4.6).

Now, under F_1, F_2, and F_3, there exists $c_2>0$ such that

$$\begin{aligned}&\sup_{0<s<1}|\varepsilon_n(s)|\\&\quad=\sup_{0<s<1}|V_n(s)|\,|s-\theta_{s,n}|\,|f'(F^{-1}(s))|/f(F^{-1}(\delta_{s,n}))\,f(F^{-1}(\theta_{s,n}))\\&\quad\leqslant c_2 n^{-1/2}\sup_{0<s<1}|V_n(s)|^2.\end{aligned} \tag{4.7}$$

Thus, using Lemma 3, it yields

$$\sup_{0<s<1}|\varepsilon_n(s)|\ll n^{-1/2}\log\log n \qquad \text{a.s.} \tag{4.8}$$

It remains to evaluate how $R_n^*(s)$ behaves. We observe that

$$\begin{aligned}R_n^*(s)&=U_n(s)+V_n(s)\\&=U_n(s)-U_n(E_n^{-1}(s))+n^{1/2}(E_n(E_n^{-1}(s))-s).\end{aligned}$$

Hence, from the definition of $E_n^{-1}(s)$ and Lemma 3, it follows that

$$\begin{aligned}&\sup_{0<s<1}|R_n^*(s)|\\&\quad\leqslant\sup_{0<s<1}|U_n(s)-U_n(E_n^{-1}(s))|+\sup_{0<s<1}n^{1/2}\,|E_n(E_n^{-1}(s))-s|\\&\quad\leqslant\sup_{0<s<1}\ \sup_{|t-s|\leqslant c\lambda_s}|U_n(s)-U_n(t)|+n^{-1/2}\qquad\text{a.s.,}\end{aligned} \tag{4.9}$$

where $\lambda_n=n^{-1/2}(\log\log n)^{1/2}$.

For the coming lemmas, the following definitions, notations, and results are required.

Set $n_k=[2^{k^{1-\varepsilon}}\log k]$, for some $0<\varepsilon\leqslant 1/4$ and let $\mathbf{N}_k=\{n:n=n_k, n_k+1, ..., n_{k+1}-1;\ k=1,2,...\}$. It is clear that $\lambda_{n_k}^{-1}=n_k^{1/2}/(\log\log n_k)^{1/2}\ll 2^{k^{1-\varepsilon}/2}$. Set $S_{n,k}^{(c)}=\{j/c2^n: j=0,1,2,...,c2^n-1;\ n\geqslant[k^{1-\varepsilon}/2]\}$, for any $k\in N$. Now, for $k\in\mathbf{N}$ sufficiently large, there exists $n\in\mathbf{N}_k$, such that $n_k\leqslant n<n_{k+1}$. For $|t-s|<c\lambda_n$, it follows that $|t-s|<c2^{-[k^{1-\varepsilon}/2]}$; there exists $s_j\in S_{n,k}^{(c)}$, such that $|t-s_j|<c2^{-[k^{1-\varepsilon}/2]}$ and $|s-s_j|<c2^{-[k^{1-\varepsilon}/2]}$.

By setting $x_i(s,t)=g_i(t)-g_i(s)=I_{(s,t]}(U_i)-(t-s)$, it is clear that

$$n^{1/2}\,|U_n(s)-U_n(t)|\leqslant n^{1/2}\,|U_n(s)-U_n(s_j)|+n^{1/2}\,|U_n(t)-U_n(s_j)| \tag{4.10}$$

and

$$
\begin{aligned}
&n^{1/2}\,|U_n(s)-U_n(s_j)| \\
&\quad=\Big|\sum_{i\leqslant n} x_i(s_j,s)\Big| \\
&\quad\leqslant\Big|\sum_{i\leqslant n_k} x_i(s_j,s)\Big|+\Big|\sum_{n_k<i\leqslant n} x_i(0,s)\Big|+\Big|\sum_{n_k<i\leqslant n} x_i(0,s_j)\Big|. \qquad (4.11)
\end{aligned}
$$

In conjunction with (4.10) and (4.11), it can be shown that

$$
\begin{aligned}
&n^{1/2}\sup_{0\leqslant s\leqslant 1}\ \sup_{|t-s|<c\lambda_s}|U_n(s)-U_n(t)| \\
&\quad\leqslant 2\max_{1\leqslant j\leqslant c2^{[k^{1-\varepsilon/2}]}}\ \sup_{s_j\leqslant s<s_{j+1}}\Big|\sum_{i\leqslant n_k} x_i(s_j,s)\Big| \\
&\qquad+4\max_{n_k\leqslant n<n_{k+1}}\ \sup_{0\leqslant s\leqslant 1}\Big|\sum_{n_k<i\leqslant n} x_i(0,s)\Big|. \qquad (4.12)
\end{aligned}
$$

The following lemma is the key result in determining the order of $R_n^*(s)$. This is due to Philipp (1977). In this work, the function f will be used in the restrictive form $f(U_1, U_2, \dots)=U_1$.

Lemma 4. *Let $H\geqslant 0$, $N\geqslant 1$ be integers and let $R\geqslant 1$. Suppose that $l=t-s\geqslant N^2$, and that $\alpha(n)\ll n^{-8}$. Then, as $N\uparrow\infty$, for $0\leqslant s<t\leqslant 1$,*

$$
\begin{aligned}
&P\Big(\Big|\sum_{j=H+1}^{H+N} x_j(s,t)\Big|\geqslant \mathrm{AR}\, l^{\alpha}(N\log\log N)^{1/2}\Big) \\
&\qquad\ll \exp(-6Rl^{-\alpha}\log\log N)+R^{-2}N^{-(1+\beta)},
\end{aligned}
$$

where $A(\geqslant 1)$, and α and β are positive constants ($\alpha=1/120$, $\beta=0.03$).

The next two lemmas are similar to Lemmas 5.1 and 5.2 in Berkes and Philipp (1977).

Lemma 5. *For $\alpha(j)\ll j^{-8}$ and for any k sufficiently large, there exists $n\in \mathbf{N}_k$ and $s_j\in S_{n,k}^{(c)}$ such that*

$$
P\Big(\max_{1\leqslant j\leqslant c2^{[k^{1-\varepsilon/2}]}}\ \sup_{s_j\leqslant s\leqslant s_{j+1}}\Big|\sum_{i\leqslant n_k} x_i(s_j,s)\Big|>n^{1/2-\lambda}\Big)\ll\exp(-k^{\lambda}),
$$

where λ is a strictly positive constant depending only on ε.

Proof. For the most part, the analysis of the proof is based on arguments similar to those presented in Lemma 5.1 of Berkes and Philipp (1977). We thus prefer to proceed with the same notation.

For $0 \leqslant s < s' \leqslant 1$ and integers $P \geqslant 0$, $Q \geqslant 1$, call $F(P, Q, s, s') = |\sum_{n=P+1}^{P+Q} x_n(s, s')|$. Clearly,

$$F(P, Q, s, s'') \leqslant F(P, Q, s, s') + F(P, Q, s', s''), \qquad \text{for} \quad 0 \leqslant s < s' < s'' \leqslant 1. \tag{4.13}$$

Put $m = [k^{1-\varepsilon}(1/2+\gamma)]$ for some $\gamma \in (0, 1/4)]$. Then, for any $s \in [s_j, s_{j+1})$, we have that

$$s = s_j + \frac{1}{c}\left\{\sum_{v=r+1}^{m} t_v 2^{-v} + \theta 2^{-m}\right\}, \qquad \text{for} \quad r = [k^{1-\varepsilon}/2],$$

where $t_v = 0$, or 1 and $\theta \in [0, 1)$.

Since $x_n(s, s') < x_n(s, s'') + (s'' - s)$, for any $h = 0, 1, ..., 2^m - 1$,

$$F\left(P, Q, \frac{h2^{-m}}{c}, \frac{h + \theta 2^{-m}}{c}\right) \leqslant F\left(P, Q, \frac{h2^{-m}}{c}, \frac{(h+1)2^{-m}}{c}\right) + Q2^{-m}c. \tag{4.14}$$

By repeating (4.13), it follows that

$$F(P, Q, s_j, s) \leqslant \sum_{v=r+1}^{m+1} F(P, Q, s_{\alpha,v}, s_{\alpha+1,v}) + Q2^{-m}/c \tag{4.15}$$

for any $s_{\alpha,v}, s_{\alpha+1,v} \in S_{n,k}^{(c)}$ and $v = v+1, ..., m$.

In exactly the same way as Berkes and Philipp (1977), define $\varphi(n) = 2A(n \log\log n)^{1/2}$, and define the events

$$E_k(v, \alpha) \equiv \{F(0, n_k, s_{\alpha,v}, s_{\alpha+1,v}) \geqslant 2^{-\alpha v}\varphi(n_k)\}$$

and

$$E_k = \bigcup_{r < v \leqslant m+1} \; \bigcup_{0 < \alpha < 2^v} E_k(v, \alpha).$$

Hence, with $H = 0$, $N = n_k$, $R = 2c$, and $l = 2^{-v}$, according to Lemma 4,

$$\begin{aligned} P(E_k(v, \alpha)) &\ll \exp(-6 \cdot 2^{v\alpha} \log\log n_k) + n_k^{-(1+\beta)} \\ &\ll \exp(-3 \cdot 2^{v\alpha} \log k) + n_k^{-1}. \end{aligned} \tag{4.16}$$

This implies that

$$\begin{aligned}P(E_k) &\ll \sum_{r<\nu\leqslant m+1} 2^{\nu}\exp(-3\cdot 2^{\alpha\nu}\log k)+2^{m}n_k^{-1}\\ &\ll \sum_{\nu>r}\exp(-2^{\alpha\nu}\log k)+n_k^{-(1/2-\gamma)}\\ &\ll 2^{-k^{1-\varepsilon}(1/2-\gamma)}.\end{aligned} \tag{4.17}$$

Thus, on the complement of E_k, it follows that with probability one

$$\begin{aligned}F(0, n_k, s_k, s) &\leqslant 2A\varphi(n_k)\sum_{\nu=r+1}^{m+1} 2^{\alpha\nu}+n_k 2^{-m}\\ &\ll \varphi(n_k)\, 2^{-\alpha r}+n_k^{1/2-\gamma}\\ &\ll n^{1/2}-\gamma.\end{aligned}$$

This completes the proof of the lemma. ∎

For the preceeding result, define

$$S_\nu=\left\{\frac{k}{2^\nu}: k=0, ..., 2^\nu-1, \nu\geqslant 1\right\}$$

Next, we have the following result.

LEMMA 6. *For $\alpha(j)\ll j^{-8}$ and for k sufficiently large, there exists $n\in \mathbf{N}_k$, such that*

$$P\Big(\max_{n_k\leqslant n<n_{k+1}}\ \sup_{0\leqslant s\leqslant 1}\Big|\sum_{n_k<i\leqslant m} x_i(0, s)\Big|\geqslant n_k^{1/2}(\log n_k)^{-\lambda}\Big)\ll k^{-2},$$

where λ is a strictly positive constant depending only on ε.

Proof. The proof of this lemma is also highly dependent upon similar arguments to those in (5.2) in Berkes and Philipp (1977). So, for these reasons, we proceed with the same notation.

As before, we can express any $n: n_k\leqslant n<n_{k+1}$, in dyadic form as

$$n=n_k+\sum_{0\leqslant j\leqslant q} p_\nu 2^j=n_k+\sum_{p<j\leqslant q} p_\nu 2^j+\theta 2^p,$$

where $\theta\in[0, 1)$, $p_\nu=0$ or 1, $p=[(1/2-\gamma)\,k^{1-\varepsilon}]$, and $q=[\log(n_{k+1}-n_k)/\log 2]$.

For any $P\geqslant 0$, $0\leqslant Q<R$ (integers) and $s, s': 0\leqslant s<s'\leqslant 1$, the following subadditive property is satisfied.

$$F(P, R, s, s')\leqslant F(P, Q, s, s')+F(P+Q, R-Q, s, s'). \tag{4.18}$$

Hence, for $0 \leqslant s,\ s' \leqslant 1$ and n such that $n_k \leqslant n < n_{k+1}$. Applying (4.18), it follows that

$$F(n_k, n-n_k, s, s') \leqslant \sum_{p<j\leqslant q} F(n_k + h_j 2^{j+1}, 2^j, s, s') + 2^p$$

where $h_j = 0, 1, ..., 2^{q-j}-1$, for $j = p+1, ..., q$. Since

$$s = \sum_{\nu=1}^{\infty} t_\nu 2^{-\nu} = \sum_{\nu\leqslant m} t_\nu 2^{-\nu} + \theta 2^{-m}, \qquad \theta \in [0, 1),$$

where m is similar to that in Lemma 5, but now is related to j (for an exact definition, see below). As in Lemma 5,

$$F(P, Q, o, s) \leqslant \sum_{\nu=1}^{m+1} F(P, Q, s_{\alpha,\nu}, s_{\alpha+1,\nu})\, Q 2^{-m}, \tag{4.19}$$

where $s_{\alpha,\nu}, s_{\alpha+1,\nu} \in S_\nu$, $\nu = 1, 2, ..., m+1$, and $0 \leqslant \alpha < 2^\nu$.

Define the following events

$$H_h(\nu, \alpha, j, h) = \{F(n_k + h_j 2^{q-j}, 2^j, s_{\alpha,\nu}, s_{\alpha+1,\nu}) \geqslant 2^{-\alpha\nu} 2^{1/8(j-q)\beta} \theta(2^q)\}$$

and

$$H_k = \bigcup_{p<j\leqslant q} \bigcup_{\nu \leqslant (1/2+\gamma)j} \bigcup_{0\leqslant \alpha < 2^\nu} \bigcup_{0 \leqslant h < 2^{q-j}} H_k(\nu, \alpha, j, h).$$

By Lemma 4, with $R = 2 \cdot 2^{(1/2)(q-j)(1-(1/4)\beta)}$, $H = n_k + h 2^{j+1}$, $N = 2^j$, and $l = 2^{-\nu}$

$$\begin{aligned} &P(H_k(\nu, \alpha, j, h)) \\ &\qquad \leqslant \exp(-6 \cdot 2^{\alpha\nu} 2^{(1/2)(q-j)(1-(1/4)\beta)} \log j) + 2^{-(q-j)(1-(1/4)\beta) - j(1+\beta)}. \end{aligned}$$

Moreover,

$$\begin{aligned} P(H_k) &\leqslant \sum_{p<j\leqslant q} \sum_{\nu\leqslant(1/2+\gamma)j} 2^\nu 2^{q-j} \exp(-6 \cdot 2^{\alpha\nu + (1/2)(q-j)(1-(1/4)\beta)} \log j) \\ &\quad + \sum_{p<j\leqslant q} \sum_{\nu \leqslant (1/2+\gamma)j} 2^\nu 2^{q-j} 2^{-(q-j)(1-(1/4)\beta) - j(1+\beta)} \\ &\leqslant \exp(-5 \log p) + 2^{-p(1/2 + (5/4)\beta - \gamma) + \alpha\beta/4} \\ &\leqslant k^{-2}. \end{aligned} \tag{4.20}$$

Hence, on H_i^c, we have that with probability one

$$\begin{aligned} & F(n_k + h_j 2^{j+1}, 2^j, 0, s) \\ & \quad \leqslant \sum_{v \leqslant (1/2+\gamma)j} F(n_k + h_j 2^{j+1}, 2^j, s_{\alpha,v}, s_{\alpha+1,v}) + 2^{j - j(1/2+\gamma)} \\ & \quad \ll \varphi(2^q) \sum_{v \leqslant (1/2+\gamma)j} 2^{-\alpha v - (1/8)(q-j)\beta} + 2^{j(1/2-\gamma)} \\ & \quad \ll \varphi(2^q)\, 2^{-(1/8)(q-j)\beta}, \end{aligned}$$

which yields

$$\begin{aligned} F(n_k, n - n_k, 0, s) & \ll \varphi(2^q) \sum_{p < j \leqslant q} 2^{-(1/8)(q-j)\beta} + \sum_{j \leqslant q} 2^{j(1/2-\gamma)} + n_k^{(1/2)-\gamma} \\ & \ll \varphi(2^q) + 2^{q((1/2)-\gamma)} + n_k^{(1/2)-\gamma} \\ & \ll n_k^{1/2} (\log n_k)^{-\varepsilon}, \end{aligned} \tag{4.21}$$

because $\varphi(2^q)^2 = 2^q \log q \ll n_{k+1} - n_k \ll n_k k^{-\varepsilon} \ll n_k (\log n_k)^{-\varepsilon}$.
This completes the proof of Lemma 6.

In connection with (4.9), (4.12), and Lemmas 5 and 6, it is shown that

$$\sup_{0 < s < 1} |R_n^*(s)| \ll (\log n)^{-\lambda} \qquad \text{for} \quad \lambda > 0. \tag{4.22}$$

Combining (4.4), (4.8), and (4.22), the proof of Theorem 1 is now completed.

Proof of Theorem 2. The following preliminary discussion is necessary for achieving the establishment of the theorem. First, some strong law behavior of the weighted uniform-[0, 1] empirical process under stationary and strong mixing is required. Let ψ denote a preassigned non-negative function on [0, 1]. The functional form considered here is $\psi(s) = (s(1-s))^{-1/2}$. The weighted empirical process is then defined by

$$\tilde{U}_n(s) = n^{1/2} \psi(s)(E_n(s) - s), \qquad s \in [0, 1].$$

In Lemma 2 of Berkes and Philipp (1977) it is established that for $\psi(s) = 1$, on a set of probability one, the sequence $\{(2 \log \log n)^{-1/2} \tilde{U}_n(s)\}$ is relatively compact in the topology of uniform convergence on [0, 1] with limit set the unit ball B in the reproducing kernel Hilbert space $H(\Gamma^*)$, where $\Gamma^*(s, s', n, n') = \min(n, n')\, \Gamma(s, s')$. If $\psi(\cdot)$ is bounded on [0, 1], then the result remains unchanged. On the other hand, if ψ is not bounded, we should expect that relative compactness may occur only for those ψ's which are bounded on every interior interval of [0, 1]. We, therefore, have to focus our attention on the behavior of ψ near 0 and 1. We consider the weight ψ_δ defined by

$$\psi_\delta(s)=\begin{cases}(s(1-s))^{-1/2} & \text{if } s\in[\delta, 1-\delta]\\ 0 & \text{otherwise.}\end{cases}$$

Since ψ_δ is bounded, the sequence $\{(2\log\log n)^{-1/2}\,\tilde{U}_n(s), n\geqslant 3\}$ is relative compact in the topology of uniform convergence on $[\delta, 1-\delta]$. We then ask to know what happens if δ is replaced by $\{\delta_n, n\geqslant 1\}$ with $\delta_n\downarrow 0$ as $n\uparrow\infty$. The following proposition is then in order.

PROPOSITION. *Under* $\alpha(j)\ll j^{-8}$, *the sequence* $\{(2\log\log n)^{1/2}\,\psi_{n^{-\mu}}(s)\,U_n(s), n\geqslant 3\}$, *for some* $0<\mu<d\alpha$, $d\in(\alpha, 1/4)$, *and* $\alpha\in(0, 1/120)$, *of random functions on* $[0,1]^2$ *is with probability one relative compact, in the supremum norm.*

Remark. The proposition implies

$$\begin{aligned}&\limsup_{n\uparrow\infty}\ \sup_{s\in[0,1]}\ \psi_{n^{-\mu}}(s)(\log\log n)^{-1/2}\,|U_n(s)|\\ &\quad=\limsup_{n\uparrow\infty}\ \sup_{n^{-\mu}<s<1-n^{-\mu}}\ [s(1-s)\log\log n]^{-1/2}\,|U_n(s)|\leqslant c \qquad \text{a.s.}\end{aligned}\tag{4.23}$$

for some $\mu\in(0, 1/480)$.

Discussion. Choose an increasing sequence $\{n_k; k\geqslant 1\}$ of positive integers such that $n_k\uparrow\infty$ almost exponentially. Next, for fixed k, we partition the unit interval $s\in[0,1]$ into d_k intervals $[s_j, s_{j+1})$ $(1\leqslant j\leqslant d_k)$, each of length $1/d_k$, where $d_k\approx n_k^\tau$ for some positive small τ. Then,

$$Z(j,k)=n_{k+1}^{1/2}\,\psi_{n_{k+1}^{-\mu}}(s_j)\;U_{n_{k+1}}(s_j)-n_k^{1/2}\psi_{n_k^{-\mu}}(s_j)\;U_{n_k}(s_j)$$

may be considered as the component $Z(j,k)$ of random vector $Z_k\in\mathbf{R}^{d_k}$. The sequence $\{Z_k, k\geqslant 1\}$ is the skeleton process of $\psi_{\delta_n}(s)\;U_n(s)$, where $\delta_n=n^{-\mu}$.

We show as in Berkes and Philipp (1977), that oscillation of $\psi_{\delta_n}(s)\;U_n(s)$ over the rectangles $\{(s,n): s_j\leqslant s\leqslant s_{j+1}, n_k\leqslant n\leqslant n_{k+1}\}$ is with probability one $\ll(\log n)^{-\lambda}$ $(\lambda=1/3840)$, uniformly for all j, with $1\leqslant j\leqslant d_k$. The latter statement shows that in order to prove (4.23) we should investigate the behavior of the skeleton process of $\psi_{\delta_n}(s)\;U_n(s)$, since it contains all the needed information about the process.

Set $D(s,n)=n^{1/2}\psi_{\delta_n}(s)\;U_n(s)=\psi_{\delta_n}(s)\sum_{i\leqslant n}x_i(0,s)$. We choose n_k as

$$n_k=[2^{k-\varepsilon}]\qquad\text{for some small}\quad\varepsilon>0.$$

Put $r_k=[dk^{1-\varepsilon}]$, $d\in(0, 1/4)$, and define $s_j=(j-1)\,2^{-r_k}$ for $j=1,2,\ldots,2^{r_k}$.

It can be seen that for $s \in [s_j, s_{j+1})$ and $n_k \leqslant n \leqslant n_{k+1}$,

$$|D(s,n) - D(s_j, n_k)| \leqslant \psi_{\delta_{n_k}}(s) \left| \sum_{i \leqslant n_k} x_i(s_j, s) \right| + \psi_{\delta_{n_k}}(s) \left| \sum_{n_k < i \leqslant n} x_i(0, s) \right|$$
$$+ |\psi_{\delta_n}(s) - \psi_{\delta_n}(s_j)| \left| \sum_{i \leqslant n_k} x_i(0, s_j) \right|.$$

We need to show that

$$G(k) = \max_{n_k \leqslant n \leqslant n_{s+1}} \max_{1 \leqslant j \leqslant 2^{rk}} \sup_{s_j \leqslant s \leqslant s_{j+1}} |D(s,n) - D(s_j, n_k)|$$

is bounded above a.s. by $n_k^{1/2} (\log n_k)^{-\lambda}$ for some $\lambda > 0$.

We choose δ in $\psi_\delta(s)$ such that

$$\delta = 2^{-\beta r} \qquad \text{for some} \quad \beta < \alpha.$$

It is required that $s_j \in [\delta, 1-\delta]$; this implies that $j \in [l_k, u_k] \cap \mathbf{Z}$, where $l_k = 2^{(1-\beta)rk}$ and $u_k = 2^{rk}(1 - 2^{-\beta rk})$.

LEMMA 7. *Under $\alpha(n) \ll n^{-8}$, and for k sufficiently large,*

$$P\left(\max_{j \in [l_k, u_k] \cap Z} \sup_{s \in [s_j, s_{j+1})} (s(1-s)^{1/2}) \left| \sum_{i \leqslant n_k} x_i(s_j, s) \right| \geqslant n_k^{1/2} (\log n_k)^{-\lambda} \right) \ll k^{-2},$$

for some $\lambda > 0$, independent of k.

Proof. For sufficiently large k, set $m = [1/2 + \gamma) k^{1-\varepsilon}]$, $\gamma \in (0, 1/4]$; then, in conjunction with (4.13) and (4.14), we observe that for $\alpha_v \in [l_v, u_v] \cap \mathbf{Z}$, $v = r+1, ..., m+1$, $P \geqslant 0$, and $Q \geqslant 1$

$$F(P, Q, s_j, s) \leqslant \sum_{v=r+1}^{m+1} F(P, Q, \alpha_v 2^{-v}, (\alpha_v + 1) 2^{-v}) + Q 2^{-m}.$$

Define as before,

$$\begin{aligned} \varphi(n) &= 2A(n \log\log n)^{1/2} \\ E_k(v, \alpha) &= \{F(0, n_k, \alpha 2^{-v}, (\alpha+1) 2^{-v}) > 2^{-\beta_1 v} 2^{-\beta r} \varphi(n_k)\} \\ &\subseteq \{F(0, n_k, \alpha 2^{-v}, (\alpha+1) 2^{-v}) > 2^{-(\beta_1 + \beta) v} \varphi(n_k)\} \\ &= E'_k(v, \alpha), \qquad \text{where} \quad \beta_1 + \beta = \alpha, \text{ the same as in Lemma 4,} \end{aligned}$$

and

$$E_k = \bigcup_{r < v \leqslant m+1} \bigcup_{l_v < \alpha < u_v} E'_k(v, \alpha) \subseteq E'_k.$$

Now, as in Berkes and Philipp (1977), for $P \geqslant 0$, $N = n_k$, $R = 2$, and $\beta = 2^{-\nu}$,

$$P(E_k(\nu, \alpha)) \ll \exp(-3 \cdot 2^{\alpha\nu} \log k) + n_k^{-1} 2^m$$

and

$$P(E_k) \ll \sum_{r < \nu \leqslant m+1} 2^\nu \exp(-3 \cdot 2^{\alpha\nu} \log k) + n_k^{-1/2}$$

for $m = [(1/2 + \gamma) k^{1-\varepsilon}]$.

Hence, by the Borel–Cantelli Lemma

$$\begin{aligned} \psi_{\delta_{n_k}}(s) \left| \sum_{i \leqslant n_k} x_i(s_j, s) \right| &\leqslant 2A\varphi(n_k) \left| \sum_{\nu = r+1}^{m} 2^{-\beta,\nu} + n_k 2^{-m} \right| \qquad \text{a.s.} \\ &\ll n_k^{1/2} (\log n_k)^{\beta_1}, \qquad \text{a.s.} \quad ∎ \end{aligned}$$

LEMMA 8. *Under $\alpha(n) \ll n^{-8}$, and for sufficiently large k,*

$$P\left(\max_{n_k \leqslant n \leqslant n_{k+1}} \sup_{s \in [0,1]} \psi_{\delta_{n_k}}(s) \left| \sum_{n_k < n} x_i(0, s) \right| > n_k^{1/2} (\log n_k)^{-\lambda} \right) \ll k^{-2}$$

for some λ independent of k.

Proof. For $p = [(1/2 - \gamma) k^{1-\varepsilon}]$ and $q = [\log_2(n_{k+1} - n_k)]$, and in exactly the same way as 5.15–5.17 in Berkes and Philipp (1977), we may write that for fixed k

$$s = \sum_{\nu=1}^{\infty} \sigma_\nu 2^{-\nu} = \sum_{i \leqslant \nu \leqslant m} \sigma_\nu 2^{-\nu} + g 2^{-m},$$

for $m = [(1/2 + \gamma) k^{1-\varepsilon}]$, and for $P \geqslant 0$, $Q \geqslant 1$, and $0 < \beta < 1/4$

$$\psi_{\delta_n}(s) F(P, Q, 0, s) \leqslant \psi_{2^{-\beta r}}(s) \left(\sum_{\nu=1}^{m} F(P, Q, \alpha_\nu 2^{-\nu}, (\alpha_\nu + 1) 2^{-\nu}) + Q 2^{-m} \right),$$

where $\alpha_\nu \in [l_\nu, u_n] \cap \mathbf{Z}$.

As before, we define the events

$$\begin{aligned} &H'_k(\nu, \alpha, j, h) \\ &\quad = \{F(n_k + h 2^{j+1}, 2^j, \alpha 2^{-\nu}, (\alpha+1) 2^{-\nu}) > 2^{-\beta,\nu} 2^{-\beta r} 2^{1/8(j-q)\lambda} \varphi(2^q)\} \\ &\quad \subseteq \{F(n_k + h 2^{n+1}, 2^j, \alpha 2^{-\nu}, (\alpha+1) 2^{-\nu}) > 2^{-\alpha\nu} 2^{1/8(j-q)\lambda} \varphi(2^q)\} \\ &\quad = H_k(\nu, \alpha, j, h) \end{aligned}$$

and

$$H'_k = \bigcup_{j\in[p,q]\cap Z} \bigcup_{v\in[1,(1/2+\gamma)j]\cap Z} \bigcup_{\alpha\in[l_v,u_v]\cap Z} \bigcup_{h\in[0,2^{q-j}]\cap Z} H'_k(v,\alpha,j,h),$$
$$\subseteq H_k$$

which is exactly as in Lemma 6. This completes the proof of the lemma. ∎

LEMMA 9. *Under $\alpha(n) \ll n^{-8}$, and for sufficiently large n,*

$$P\Bigg(\max_{j\in[l_k,u_k]\cap Z} \sup_{s\in[s_j,s_{j+1}]} |\psi_{\delta_{n_k}}(s) - \psi_{\delta_{n_k}}(s_j)| \times \Bigg|\sum_{i\leqslant n_k} x_i(0,s_i)\Bigg| \geqslant n_k^{1/2}(\log n_k)^{-\lambda}\Bigg) \ll k^{-2},$$

for some $\lambda > 0$, independent of k.

Proof. It is clear that $m = [(1/2+\gamma)\,k^{1-\varepsilon}]$

$$\begin{aligned} |\psi_{\delta_{n_k}}(s) - \psi_{\delta_{n_k}}(s_j)| \Bigg|\sum_{i\leqslant n_k} x_i(0,s_j)\Bigg| &\leqslant 2\psi_{\delta_{n_k}}(s_j)\, F(0,n_k,0,s_j) \\ &\leqslant 2\psi_{\delta_{n_k}}(l_k) \Bigg\{\sum_{v=[\beta r]+1}^{m+1} F(0,n_k,\alpha_v 2^{-v},(\alpha_v+1)\,2^{-v})\Bigg\} + n_k 2^{-m}, \end{aligned}$$

where $r = [dk^{1-\varepsilon}]$.

The rest of the proof follows from Lemma 7. ∎

Combining Lemmas 7–9, the proof of (4.23), or the Proposition, is now in order. An implication of (4.23) is the following result. The proof of this can be done using similar arguments to Lemma 2.3 in Babu and Singh (1978).

COROLLARY 2. *If* (4.23) *holds, then*

$$\limsup_{n\uparrow\infty} \sup_{n^{-\mu}<s<1-n^{-\mu}} (s(1-s)\log\log n)^{-1/2}\,|V_n(s)| \leqslant c \qquad a.s.,$$

for some $0<\mu<d\alpha$, $d\in(0,1/4)$, and $\alpha\in(0,1/120)$.

With this preparation in mind, the proof of Theorem 2 follows. As in Csörgő and Révész (1978),

$$|f(F^{-1}(s))\,Q_n(s) - V(s)| \leqslant (1/2)\,n^{-1/2}V_n^2(s)\,f(F^{-1}(s))\,\frac{|f'(F^{-1}(\xi)|}{f^3(F^{-1}(\xi))} \qquad \text{a.s.,} \tag{4.24}$$

for $\xi \in (s \wedge (s+n^{-1/2}V_n(s)), s \vee (s+n^{-1/2}V_n(s)))$ and $s \in [\delta_n, 1-\delta_n]$ where $\delta_n = n^{-\mu}$.

In view of Corollary 2, (4.24) can be majorized by

$$|f(F^{-1}(s))\, Q_n(s) - V_n(s)| \leqslant (1/2)\, n^{-1/2} \log\log n \left[\frac{s(1-s)}{\xi(1-\xi)}\right] \times \left[\xi(1-\xi)\frac{|f'(F^{-1}(s))|}{f^2(F^{-1}(\xi))}\right]\left[\frac{f(F^{-1}(s))}{f(F^{-1}(\xi))}\right] \quad \text{a.s.} \tag{4.25}$$

Arguing as in Csörgő and Révész (1978), all the bracketed terms are bounded above by a constant.

Since

$$\sup_{n^{-\mu}<s<1-n^{-\mu}} |f(F^{-1}(s))\, Q_n(s) + K(s,n)/n^{1/2}| \leqslant \sup_{n^{-\mu}<s<1-n^{-\mu}} |f(F^{-1}(s))\, Q_n(s) - V_n(s)| + \sup_{n^{-\mu}<s<1-n^{-\mu}} |R_n^*(s)| + \sup_{n^{-\mu}<s<1-n^{-\mu}} |U_n(s) - K(s,n)/n^{1/2}| \quad \text{a.s.,} \tag{4.26}$$

and since the second and the third term in (4.26) are bounded above by $(\log n)^{-\lambda}$ (see, e.g., (4.22) and (4.4), respectively), the result follows immediately.

For the second part of the theorem, it is sufficient to show that

$$\sup_{s\in[0,\delta_n]\cup[1-\delta_n,1]} |f(F^{-1}(s))\, Q_n(s) - V_n(s)| \ll (\log n)^{-\lambda} \quad \text{a.s.} \tag{4.27}$$

We show the result only for $s \in [0, \delta_n]$; similar arguments are applied for the other tail.

As in 3.2.14 in Csörgő (1983),

$$|f(F^{-1}(s))\, Q_n(s) - V_n(s)| \leqslant |V_n(s)| \left|1 - \frac{f(F^{-1}(s))}{f(F^{-1}(\xi))}\right|, \tag{4.28}$$

where $s \in [0, \delta_n]$ and $|s-\xi| \leqslant n^{-1/2}\, |V_n(s)|$.

Obviously, $|1-f(F^{-1}(s))/f(F^{-1}(\xi))|$ is bounded because of the additional conditions. As far as $V_n(s)$ is concerned, we have that

$$\sup_{0<s<\delta_n} |V_n(s)| \leqslant \sup_{0<s<\delta_n} |R_n^*(s)| + \sup_{0<s<1} \sup_{|s-t|<\delta_n} |U_n(s)-U_n(t)|, \tag{4.29}$$

for $\delta_n = n^{-\mu}$.

Applying Lemmas 5 and 6 by replacing λ_n with δ_n, the result shows that

$$\sup_{0<s<\delta_n} |V_n(s)| \ll (\log n)^{-\lambda} \qquad \text{a.s.} \tag{4.30}$$

This completes the proof of Theorem 2.

Proof of the Remark. As in the second part of the proof of Theorem 2, we need to show that

$$\sup_{0<s<\delta_n} |f(F^{-1}(s))\, Q_n(s) - V_n(s)| \ll (\log \mathrm{n})^{-\lambda} \qquad \text{a.s.}$$

It is clear that

$$f(F^{-1}(s))\, Q_n(s) - V_n(s) = n^{1/2} \int_{U_{k:n} \wedge s}^{U_{k:n} \vee s} \left(\frac{f(F^{-1}(s))}{f(F^{-1}(u))} - 1 \right) du, \tag{4.31}$$

where $U_{k:n}$ is the kth order statistic of the uniform variates $U_1, ..., U_n$, $s \in ((k-1)/n, k/n] \cap [\varepsilon_n, \delta_n]$, and $\varepsilon_n = n^{-1/2} l(n)^{-1}$.

If $s \leqslant U_{k:n}$, then the right-hand side of (4.30) is bounded above, for $s \in [0, \delta_n]$, by

$$n^{1/2} \int_s^{U_{k:n}} \left(1 + \frac{f(F^{-1}(s))}{f(F^{-1}(u))} \right) du \leqslant 2\, |V_n(s)| \ll (\log n)^{-\lambda},$$

since f is non-decreasing to the right of α and (4.30).

On the other hand, if $s \geqslant U_{k:n}$, then the right-hand side of (4.31) is bounded by

$$\begin{aligned} &n^{-1/2} \int_0^{s-U_{k:n}} \left(\frac{f(F^{-1}(s))}{f(F^{-1}(u))} + 1 \right) du \\ &\quad \leqslant n^{-1/2} \int_0^{n^{-1/2}(\log n)^{-\lambda}} \left(\frac{f(F^{-1}(s))}{f(F^{-1}(s-u))} + 1 \right) du \qquad \text{a.s.} \end{aligned}$$

Substituting $u = n^{1/2}(\log n)^{-\lambda}$, and since f is now decreasing to the right of a, the result follows at once.

Proof of Theorem 3. The sketch of the proof of this theorem is based on three simple lemmas. Lemma 1 shows that the law of iterated logarithms is satisfied for the quantile process; this, in turn, will play an auxiliary role in showing a strong consistency of the Kernel-type estimator of $1/f(F^{-1}(s))$, which is shown in Lemma 3. Lemma 2 indicates how fast the bias of this estimator approaches zero. We proceed as follows.

Calling upon Lemma 3, (4.8), and (4.2), the following lemma is easily formed.

LEMMA 10. *Under the same provisos of Lemma* 2, *there exists* $c > 0$, *such that*

$$\limsup_{n\uparrow\infty} \sup_{0<s<1} \frac{f(F^{-1}(s))\,|Q_n(s)|}{(\log\log n)^{1/2}} \leqslant c \qquad a.s.$$

The next result is a modified version of Csörgő and Révész (1984). Since the proof of the following lemma does not differ much from Lemma 3 in Csörgő and Révész (1984), it is omitted.

LEMMA 11. *Under* K_1–K_4, H_1, *and* F_1–F_4, *the following result holds*,

$$\limsup_{n\uparrow\infty} h_n^{-2} \sup_{0<s<1} \left|\frac{1}{h_n}\int_0^1 K\left(\frac{s-u}{h_n}\right) dF_n^{-1}(u) - \frac{1}{f(F^{-1}(s))}\right| \leqslant c,$$

for some constant $c > 0$.

Next, we have the following.

LEMMA 12. *Under* K_1–K_4, H_1, H_2, *and* F_1–F_4, *it follows that*

$$\limsup_{n\uparrow\infty} \frac{h_n n^{1/2}}{(\log\log n)^{1/2}} \sup_{0<s<1} \left|\phi_n(s) - \frac{1}{f(F^{-1}(s))}\right| \leqslant c \qquad a.s.,$$

where $\phi_n(s) = (1/h_n)\int_0^1 K((s-u)/h_n)\,dF_n^{-1}(s) > 0$, *and* c *is a positive constant.*

Proof. Via the triangle inequality, it follows that

$$\begin{aligned}
&\sup_{0<s<1} \left|\frac{1}{h_n}\int_0^1 K\left(\frac{s-u}{h_n}\right) dF_n^{-1}(u) - \frac{1}{f(F^{-1}(s))}\right| \\
&\quad\leqslant \sup_{0<s<1} \left|\frac{1}{h_n}\int_0^1 K\left(\frac{s-u}{h_n}\right) d(F_n^{-1}(s) - F^{-1}(s))\right| \\
&\qquad + \sup_{0<s<1} \left|\frac{1}{h_n}\int_0^1 K\left(\frac{s-u}{h_n}\right) dF^{-1}(s) - \frac{1}{f(F^{-1}(s))}\right|.
\end{aligned}$$

Invoking the fact that $h_n \ll (\log\log n/n)^{1/6}$ and Lemma 11, the second term on the R.H.S. of the latter inequality tends to zero for sufficiently large n. It remains to show that the other term is approaching zero with probability one.

In connection with Lemma 10, it follows that for sufficiently large n,

$$\begin{aligned}
&\sup_{0<s<1}\left|\frac{1}{h_n}\int_0^1 K\left(\frac{s-n}{h_n}\right) d(F_n^{-1}(s)-F^{-1}(s))\right| \\
&\quad=\frac{(\log\log n)^{1/2}}{h_n^2 n^{1/2}} \sup_{0<s<1}\left|\int_0^1 \frac{f(F^{-1}(u))\,Q_n(u)}{f(F^{-1}(u))} K'\left(\frac{s-u}{h_n}\right) du\right| \\
&\quad\leqslant \frac{(\log\log n)^{1/2}}{h_n n^{1/2}} \sup_{0<s<1} \frac{f(F^{-1}(s))\,Q_n(s)}{(\log\log n)^{1/2}} \sup_{0<s<1} |K'(s)| \inf_{0<s<1} f(F^{-1}(s)) \\
&\quad\leqslant \frac{(\log\log n)^{1/2}}{h_n n^{1/2}} \qquad \text{a.s.}
\end{aligned}$$

This completes the proof of Lemma 12. ∎

The rest of Theorem 2 follows from the triangle inequality.

$$\begin{aligned}
|f(F^{-1}(s))\,Q_n(s)-B_n(s)| &\leqslant \left|\frac{1}{\phi_n(s)} Q_n(s)-B_n(s)\right| \\
&\quad+\frac{f(F^{-1}(s))}{\phi_n(s)}\left|\phi_n(s)-\frac{1}{f(F^{-1}(s))}\right|. \qquad (4.32)
\end{aligned}$$

This completes the proof of the theorem.

Proof of Theorem 4. The steps required to show this theorem are exactly the same as in Csörgő and Horváth (1990). The additional information needed here is to show that

$$\sup_{[0,\delta_n]\cup[1-\delta_n,1]} |B(s)| \downarrow 0 \qquad \text{as} \quad n\uparrow\infty,$$

where $B(s)$ is a Gaussian Process with covariance function $\Gamma(s,s)$. This will be accomplished below, but first some discussion.

For any $0<n_1<n_2<\cdots$ define

$$B_k(s)=(K(s,n_{k+1})-K(s,n_k))(n_{k+1}-n_k)^{-1/2}, \qquad k\in N.$$

The sequence $\{B_k(s),\ k\in \mathbf{N}$ and $s\in[0,1]\}$ is a Gaussian process with independent increments and covariance function $\Gamma(s,s')$ (see, e.g., Lemma 6.1 in Berkes and Philipp, 1977).

We need to show that, for $s \in [0, \delta_n] \cup [1-\delta_n, 1]$, and for $\delta_n = n^{-\mu}$ (μ introduced above), the process $K(s, n)/n^{1/2}$ is bounded above almost surely by a sequence of constants and tending to zero uniformly in s as $n \uparrow \infty$.

We may write that for $n_k \leqslant n \leqslant n_{k+1}$ and $s \in [0, \delta_n] \cup [1-\delta_n, 1]$,

$$K(s, n) = \sum_{i=0}^{k-1} (n_{i+1} - n_i)^{1/2} B_i(s) + (n - n_k)^{1/2} \bar{B}_k(s), \tag{4.33}$$

where $\bar{B}_k(s) = (K(s, n) - K(s, n_k))(n - n_k)^{-1/2}$.

LEMMA 13. *For sufficiently large n,*

$$\sup_{s \in [0, \delta_n] \cup [1-\delta_n, 1]} |K(s, n)|/n^{1/2} \ll (\log n)^{-\lambda} \qquad a.s.$$

Proof. We show the validity of this result only for $s \in [0, \delta_n]$; the same is applied for $s \in [1-\delta_n, 1]$. It is easy to check that for $0 \leqslant s < s' \leqslant 1$,

$$\begin{aligned} \Gamma(s, s') &= Eg_1^2(s) + 2 \sum_{n=2}^{\infty} Eg_1(s)\, g_n(s) \\ &\quad + Eg_1^2(s') + 2 \sum_{n=2}^{\infty} Eg_1(s')\, g_n(s') \\ &\quad - Eg_1^2(s, s') - 2 \sum_{n=1}^{\infty} Eg_1(s, s')\, g_n(s, s') \\ &= \sigma^2(0, s) + \sigma^2(0, s') - \sigma^2(s, s'), \end{aligned}$$

where $g_n(s, s') = I_{[s, s']}(U_n) - (s' - s)$.

In conjuction with $\alpha \ll n^{-8}$ and Lemma 6 in Davydov (1970) for $0 \leqslant s < s' \leqslant 1$,

$$\begin{aligned} \sigma^2(s, s') &\leqslant (s'-s)(1-(s'-s)) + 10 \sum_{n=2}^{\infty} \alpha(n)^{1/r_3} ((s'-s)(1-(s'-s)))^{1/r_1 + 1/r_2} \\ &\ll (s'-s)^{3/4}. \end{aligned}$$

The last statement follows by assuming that $r_3 = 4$ and $1/r_1 + 1/r_2 = 3/4$.

Let $\{n_k = [2^{k^{1-\varepsilon}}],\ k \in \mathbf{N},\ 0 < \varepsilon < 1/8\}$ be a sequence of integers and $\{s_{jk} = s_j : s_j = (j-1)/2^{r_k} \text{ for } r_k = d\alpha k^{1-\varepsilon}\}$. It is clear that

$$\{B(s) : 0 < s < \delta\} \overset{D}{=} \{X(u) = B^*(u) - B^*(0), \text{for } u \in [0, 1]\},$$

where $B^*(u) = B(s + u(s' - s))$ and for $s' - s = \delta$.

By setting in the process, $X(u)$, s_j, and s_{j+1} instead of s and s', it follows that for $0 \leqslant u,\ u' \leqslant 1$,

$$E[(X_k(u) - X_k(u'))^2] \ll |u-u'|^{3/4}\, 2^{-3r_k/4}.$$

Now, applying the same arguments as in Lemma 6.2 in Berkes and Philipp, it follows that

$$\begin{aligned} P(&\sup_{0<s<2^{-r_k}} |B(s)| > (\log n_k)^{-\lambda}) \\ &= P(\sup_{s_j \leqslant s \leqslant s_{j+1}} |B(s) - B(s_j)| > (\log n_k)^{-\lambda}) \\ &= P(\sup_{0\leqslant u \leqslant 1} |X(u)| > \log n_k)^{-\lambda}) \ll k^{-3}. \end{aligned} \tag{4.34}$$

Thus

$$P\left(\sum_{i=1}^{k-1} (n_{i+1} - n_i) \sup_{s_{ji} \leqslant s \leqslant s_{j+1,i}} |B(s) - B(s_{ji})| > (\log n_k)^{-\lambda}\right) \ll k^{-2}. \tag{4.35}$$

Next, applying Lemma 6.3 in Berkes and Philipp, for $s \in [0, \delta_n]$, we have that

$$P(\sup_{n_k \leqslant n \leqslant n_{k+1}} \sup_{s\in[0,\delta_n]} |K(s,n) - K(s,n_k)| > n_k(\log n_k)^{-\lambda}) \ll k^{-2}. \tag{4.36}$$

Hence, using the Borel–Cantelli theorem for both (4.35) and (4.36), the proof of the lemma follows immediately. This proves Theorem 4. ∎

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