Journal of Multivariate Analysis **65**, 166–180 (1998)
Article No. MV971728

Approximation of the Power of Kurtosis Test for Multinormality

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Received June 20, 1996; revised September 18, 1997

In this paper we investigate performances of the test of multinormality introduced by Malkovich and Afifi. An approximation formula of the power of the test against elliptically symmetric distributions is derived. Examples which illustrate the present results are also discussed. © 1998 Academic Press

AMS subject classifications: primary 62H15; secondary 62H10.

Key words and phrases: elliptically symmetric distribution, Gaussian random field, multivariate kurtosis, power, tail probability, test for multinormality.

1. INTRODUCTION

Tests for multivariate normality (multinormality), which have received so much attention in statistical literature, are important since many multivariate statistical inferences are based on normality. Several tests for multinormality have been proposed and some of them have been summarized in the paper by Mardia (1980). Empirical studies of 10 representative tests were discussed by Romeu and Ozturk (1993).

This paper is devoted only to projection pursuit-type tests. Projection pursuit (PP) is well-surveyed in Huber (1985). The multivariate measures of skewness and kurtosis proposed in Malkovich and Afifi (1973) are known as two of the earliest practical applications of the PP technique (see Krzanowski and Marriott, 1994, Chap. 4). In this paper we investigate the asymptotic performances of the multivariate measure of kurtosis.

Malkovich and Afifi (1973) introduced

$$[\beta_2^M]^2 = \max_{\alpha \in \mathscr{S}^{p-1}} \left[\frac{E[(\alpha' X - \alpha' \mu)]^4}{(\alpha' \Sigma \alpha)^2} - 3 \right]^2$$

as a measure of multivariate kurtosis of a random p-dimensional column vector X having expectation μ and nonsingular covariance matrix Σ. Here

0047-259X/98 $25.00

$\mathscr{S}^{p-1}$ designates the unit sphere in R^p. Let $X_1, ..., X_N$ be N independent observations of X. The multivariate empirical kurtosis of $X_1, ..., X_N$ is defined as

$$[b_{2,N}^M]^2 = \max_{\alpha \in \mathscr{S}^{p-1}} [b_{2,N}(\alpha) - 3]^2, \tag{1.1}$$

where

$$b_{2,N}(\alpha) = \frac{N^{-1} \sum_{j=1}^N (\alpha' X_j - \alpha' \bar{X})^4}{(\alpha' S \alpha)^2}, \tag{1.2}$$

$\bar{X}$ is the sample mean vector and S is the sample covariance matrix. Suppose X has an elliptically symmetric distribution with mean vector μ and nonsingular covariance matrix Σ. Let

$$\theta = \frac{E[(\alpha' X - \alpha' \mu)]^4}{(\alpha' \Sigma \alpha)^2}$$

for $\alpha \in \mathscr{S}^{p-1}$ and

$$\mathscr{B}_{2,N}^M = \max_{\alpha \in \mathscr{S}^{p-1}} |b_{2,N}(\alpha) - \theta|.$$

We note that θ does not depend on α and that $\mathscr{B}_{2,N}^M = b_{2,N}^M$ if $\theta = 3$. Baringhaus and Henze (1991) discussed asymptotic theory of $\mathscr{B}_{2,N}^M$ when null hypothesis is "*ellipticity*." They showed that $\sqrt{N}(b_{2,N}(\alpha) - \theta)$ converges weakly to some Gaussian random field say, $W(\alpha)$ and that $\sqrt{N}\, \mathscr{B}_{2,N}^M$ converges weakly to $\max_{\alpha \in \mathscr{S}^{p-1}} |W(\alpha)|$ from the continuous mapping theorem. However, the result on the distribution of $\max_{\alpha \in \mathscr{S}^{p-1}} |W(\alpha)|$ was not developed since the distribution for the maxima of the modulus of Gaussian random field is generally hard to obtain.

Let z be a fixed positive real number. Then it holds that

$$P\{b_{2,N}^M \geqslant z\} = P\{[b_{2,N}^M]^2 \geqslant z^2\}. \tag{1.3}$$

In this paper, the power of the test for *multinormality* by $b_{2,N}^M$ or equivalently by $[b_{2,N}^M]^2$ against elliptically symmetric distributions is investigated. The power of the test against elliptically symmetric distributions is to evaluate (1.3) in the case that z is suitably chosen critical point under normality. Under elliptically symmetric distributed random vector X such that $\theta \neq 3$, it is not easy to evaluate the left-hand side of (1.3), but, the right-hand side of (1.3) is easily evaluated. That is, it will be shown that there exist positive constants a_X, b_X (both are independent of α) such that

$$\sqrt{N}\left[\frac{[b^M_{2,N}]^2-b_X}{a_X}\right]$$

converges weakly to the maxima of some nonsingular differentiable Gaussian random field. Further, if X satisfies $z>b_X$, the problem turns out to be that of tail probability of the maxima of Gaussian random field. Since its covariance function satisfies some regularity conditions, we can finally apply the result established by Sun (1993) to our problem. Therefore, as far as we evaluate the power against elliptically symmetric distributions, it suffices to advance asymptotic theory of $[b^M_{2,N}]^2$ rather than $b^M_{2,N}$. By this approach, we can obtain not only the weak convergence property of $b^M_{2,N}$ or $[b^M_{2,N}]^2$ as discussed in Baringhaus and Henze (1991), but also the information about the distribution of the random variable to which $[b^M_{2,N}]^2$ converges weakly after normalization by a_X and b_X.

In Section 2, the weak convergence theorem of a random field on $\mathscr{S}^{p-1}$ related to $[b^M_{2,N}]^2$ is given. The result is straightforwardly obtained from weak convergence property of $\sqrt{N}(b_{2,N}(\alpha)-\theta)$ given in Baringhaus and Henze (1991). Further an approximation formula of the power of the test by $[b^M_{2,N}]^2$ against elliptically symmetric distributions is derived under some conditions by using the result of Sun (1993). Similar approximation formula of the test by $b^M_{2,N}$ is obtained by using (1.3). Illustration of examples are provided in Section 3.

2. MAIN RESULTS

Suppose that X is distributed as spherically symmetric with $E[XX']=I_p$. This implies that $E[|X|^2]=p$, where $|\cdot|$ stands for the Euclidean norm. We define $m_k=E[|X|^k]$, for $k\geqslant 1$.

Let $C(\mathscr{S}^{p-1})$ be the separable Banach space of real-valued continuous functions defined on $\mathscr{S}^{p-1}$, endowed with the supremum norm. We first introduce a random field

$$W_N(\alpha)=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\left[(\alpha'X_j)^4-\theta\{2(\alpha'X_j)^2-1\}\right],$$

which is a random elements of $C(\mathscr{S}^{p-1})$. In what follows, $W_N(\cdot)\Rightarrow W(\cdot)$ means that the distribution of a random element $W_N(\cdot)$ of $C(\mathscr{S}^{p-1})$ converges weakly to the distribution of a random element $W(\cdot)$ of $C(\mathscr{S}^{p-1})$. Note that $E[W_N(\alpha)]=0$ for $\alpha\in\mathscr{S}^{p-1}$, and the covariance function of $W_N(\alpha)$ is

$$
\begin{aligned}
c(\alpha, \bar{\alpha}) &\equiv E[W_N(\alpha)\, W_N(\bar{\alpha})] \\
&= \frac{24m_8}{p(p+2)(p+4)(p+6)} (\alpha'\bar{\alpha})^4 \\
&\quad + 72\left[\frac{m_8}{p(p+2)(p+4)(p+6)} - \frac{2m_4m_6}{p^2(p+2)^2\,(p+4)}\right. \\
&\quad \left. + \frac{m_4^3}{p^3(p+2)^3}\right] (\alpha'\bar{\alpha})^2 \\
&\quad + 9\left[\frac{m_8}{p(p+2)(p+4)(p+6)} - \frac{4m_4m_6}{p^2(p+2)^2\,(p+4)}\right. \\
&\quad \left. + \frac{4m_4^3}{p^3(p+2)^3} - \frac{m_4^2}{p^2(p+2)^2}\right] \qquad (2.1)
\end{aligned}
$$

for $\alpha, \bar{\alpha} \in \mathscr{S}^{p-1}$. At first, we note the following result under spherical terms which was obtained in Baringhaus and Henze (1991).

THEOREM 2.1 (Baringhaus and Henze, 1991, Theorem 3.1, Lemma 3.2). *Let $X = (X_1, ..., X_p)'$ have a spherically symmetric distribution with unit covariance matrix such that $m_8 < \infty$. Let $\theta = E[X_1^4]$. Then there exists a zero-mean Gaussian random field $W(\alpha)$, $\alpha \in \mathscr{S}^{p-1}$, with continuous sample paths and covariance kernel $c(\alpha, \bar{\alpha})$ such that*

$$W_N(\cdot) \Rightarrow W(\cdot).$$

Further,

$$\sup_{\alpha \in \mathscr{S}^{p-1}} |W_N(\alpha) - \sqrt{N}\,\{b_{2,N}(\alpha) - \theta\}|$$

converges to zero in probability, which leads to

$$\sqrt{N}\{b_{2,N}(\cdot) - \theta\} \Rightarrow W(\cdot).$$

Next, let X be an elliptically symmetric distributed random p-vector with mean μ and covariance matrix Σ and let σ^2 be $c(\alpha, \alpha)$ given by (2.1) with

$$m_{2k} = E[\{(X-\mu)'\, \Sigma^{-1}(X-\mu)\}^k]$$

for $k = 2, 3, 4$. Note that σ^2 does not depend on α. Define that $Z(\alpha) = W(\alpha)/\sigma$. Obviously $Z(\alpha)$ is also Gaussian with mean zero and the covariance function $\rho(\alpha, \bar{\alpha}) \equiv E[Z(\alpha)\, Z(\bar{\alpha})] = c(\alpha, \bar{\alpha})/\sigma^2$, for $\alpha, \bar{\alpha} \in \mathscr{S}^{p-1}$. From the continuous mapping theorem and the affine invariance of $b_{2,N}^M$, we obtain the following result.

THEOREM 2.2. *Let X be an elliptically symmetric distributed random p-vector with mean μ and covariance matrix Σ. Let $Y=\Sigma^{-1/2}(X-\mu)=(Y_1, ..., Y_p)'$, $\theta=E[Y_1^4]$. Suppose that $m_8=E[|Y|^8]=E[\{(X-\mu)'\Sigma^{-1}(X-\mu)\}^4]<\infty$ and $\theta\neq 3$. Then we have*

$$\sqrt{N}\left[\frac{[b_{2,N}^M]^2-\{\theta-3\}^2}{2\,|\theta-3|\,\sigma}\right]\Rightarrow \max_{\alpha\in\mathscr{S}^{p-1}} Z(\alpha).$$

Proof. Let $Y_j=\Sigma^{-1/2}(X_j-\mu)$ for $j=1, ..., N$, where $\Sigma^{-1/2}$ is a positive definite square root of Σ^{-1}. Let $\tilde{b}_{2,N}^M$ and $\tilde{b}_{2,N}(\alpha)$ be the ones as in (1.1) and (1.2), respectively, with Y_j instead of X_j. Since $b_{2,N}^M$ is affine invariant and both of θ and σ^2 do not depend on α, it follows that

$$\begin{aligned}\sqrt{N}\left[\frac{[b_{2,N}^M]^2-\{\theta-3\}^2}{2\,|\theta-3|\,\sigma}\right]&=\sqrt{N}\left[\frac{[\tilde{b}_{2,N}^M]^2-\{\theta-3\}^2}{2\,|\theta-3|\,\sigma}\right]\\&=\max_{\alpha\in\mathscr{S}^{p-1}}\sqrt{N}\left[\frac{\{\tilde{b}_{2,N}(\alpha)-3\}^2-\{\theta-3\}^2}{2\,|\theta-3|\,\sigma}\right]\\&=\max_{\alpha\in\mathscr{S}^{p-1}}\frac{\{\tilde{b}_{2,N}(\alpha)+\theta-6\}\sqrt{N}\{\tilde{b}_{2,N}(\alpha)-\theta\}}{2\,|\theta-3|\,\sigma}.\end{aligned}$$

Here $\tilde{b}_{2,N}(\alpha)$ converges, as a member of $C(\mathscr{S}^{p-1})$, to θ in probability. Similarly, let

$$\tilde{W}_N(\alpha)=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}[(\alpha' Y_j)^4-\theta\{2(\alpha' Y_j)^2-1\}],\qquad \alpha\in\mathscr{S}^{p-1}.$$

Since Y_j's are independently distributed as spherical, it holds from Theorem 2.1 that

$$\sup_{\alpha\in\mathscr{S}^{p-1}}|\tilde{W}_N(\alpha)-\sqrt{N}\{\tilde{b}_{2,N}(\alpha)-\theta\}|$$

converges to zero in probability and $\tilde{W}_N(\cdot)\Rightarrow W(\cdot)$, where $W(\cdot)$ is a zero mean Gaussian random field with the covariance function $c(\alpha,\bar{\alpha})$ given by (2.1) with

$$m_{2k}=E[\{(X-\mu)'\,\Sigma^{-1}(X-\mu)\}^k]=E[|Y|^{2k}]$$

for $k=2, 3, 4$. These imply

$$\frac{\{\tilde{b}_{2,N}(\cdot)+\theta-6\}\sqrt{N}\{\tilde{b}_{2,N}(\cdot)-\theta\}}{2\,|\theta-3|\,\sigma}\Rightarrow Z(\cdot)\qquad(=W(\cdot)/\sigma).$$

Therefore, by the continuous mapping theorem,

$$\sqrt{N}\left[\frac{[b_{2,N}^{M}]^2-\{\theta-3\}^2}{2\,|\theta-3|\,\sigma}\right]\Rightarrow \max_{\alpha\in\mathscr{S}^{p-1}} Z(\alpha),$$

which completes the proof. ∎

Some modifications of Proposition 3.3 in Baringhaus and Henze (1991) gives the following representation of $Z(\cdot)$. This result will be used to derive an approximation formula of tail probability.

PROPOSITION 2.3. *The limiting Gaussian random field* $Z(\alpha)$, $\alpha\in\mathscr{S}^{p-1}$, *can be represented in the form*

$$Z(\alpha)=\eta_1^{1/2}\sum_{l=1}^{\nu(4)}\varphi_{4,l}(\alpha)\,N_{4,l}+\eta_2^{1/2}\sum_{l=1}^{\nu(2)}\varphi_{2,l}(\alpha)\,N_{2,l}+\eta_3^{1/2}N_0, \tag{2.2}$$

where N_0; $N_{2,l}$, $l=1, ..., \nu(2)$; $N_{4,l}$, $l=1, ..., \nu(4)$ *are independent standard normal random variables and* $\{\varphi_{2,l}\colon l=1, ..., \nu(2)\}$, $\{\varphi_{4,l}\colon l=1, ..., \nu(4)\}$ *are linearly independent surface harmonics of degree* 2 *and* 4, *respectively, being orthonormal with respect to the uniform distribution on* $\mathscr{S}^{p-1}$, *where*

$$\nu(4)=\frac{p(p-1)(p+1)(p+6)}{24},\qquad \nu(2)=\frac{(p-1)(p+2)}{2}.$$

Further, $\eta_k=\delta_k/\sigma^2$ *for* $k=1, 2, 3$ *with* δ_k*'s are all given in Proposition* 3.3 *of Baringhaus and Henze* (1991) *and*

$$\sigma^2=\frac{105m_8}{p(p+1)(p+4)(p+6)}-\frac{180m_4m_6}{p^2(p+2)^2\,(p+4)}+\frac{108m_4^3}{\{p(p+2)\}^3}-\frac{9m_4^2}{\{p(p+2)\}^2}.$$

The fact that the limiting random field derived from $[b_{2,N}^{M}]^2$ is Gaussian with mean zero and unit variance as shown in Theorem 2.2 motivates us to use the following theorem established by Sun (1993).

THEOREM 2.4 (Sun, 1993). *Suppose* $Z(t)$ *is a* d*-dimensional nonsingular differentiable Gaussian random field on a bounded* d*-dimensional set* I, *with mean zero, unit variance and covariance function* $v(s,t)$. *Under some regularity conditions for* $v(s,t)$, *as* $z\to\infty$,

$$P\{\max_{t\in I} Z(t)\geqslant z\}=\kappa_0\psi\left(\frac{z^2}{2},\frac{d+1}{2}\right)+\kappa_2\psi\left(\frac{z^2}{3},\frac{d-1}{2}\right)\{1+o(1)\},$$

where

$$\psi(x, d) = \int_x^\infty y^{d-1} \exp(-y)\, dy$$

is an incomplete gamma function and κ_0, κ_2 *are two geometric constants represented as*

$$\kappa_0 = \frac{1}{2\pi^{(d+1)/2}} \int_I \|V(t)\|^{1/2}\, dt_1 \cdots dt_d, \tag{2.3}$$

$$\kappa_2 = \frac{1}{4\pi^{(d+1)/2}} \int_I \frac{1}{2} \{-S(t) - d(d-1)\} \, \|V(t)\|^{1/2}\, dt_1 \cdots dt_d. \tag{2.4}$$

Here $\|V(t)\|$ *is the determinant of the* $d \times d$ *matrix*

$$V(t) = \left\{ \frac{\partial^2 v(s, t)}{\partial s_i\, \partial t_j} \bigg|_{s=t} \right\}$$

and $S(t)$ *is the scalar curvature of the manifold which has* $V(t)$ *as its metric tensor.*

Let us define a hemisphere

$$\mathscr{S}^{p-1}/2 = \{\alpha = (\alpha_1, ..., \alpha_p)' \in \mathscr{S}^{p-1} : \alpha_p \geqslant 0\}.$$

Since $b_{2,N}(\alpha) = b_{2,N}(-\alpha)$, its maxima is unchanged if α is restricted to a hemisphere $\mathscr{S}^{p-1}/2$. Note again that both of θ and σ do not depend on α.

Let z be a fixed positive real number. Assume that X satisfies the conditions of Theorem 2.2 and $z > (\theta - 3)^2$. For using Theorem 2.4, we evaluate the probability as

$$\begin{aligned} P\{[b_{2,N}^M]^2 \geqslant z\} &= P\{\max_{\alpha \in \mathscr{S}^{p-1}/2} [\{b_{2,N}(\alpha) - 3\}^2 - (\theta - 3)^2] \geqslant z - (\theta - 3)^2\} \\ &= P\left\{ \max_{\alpha \in \mathscr{S}^{p-1}/2} \sqrt{N} \left[\frac{\{b_{2,N}(\alpha) - 3\}^2 - (\theta - 3)^2}{\{2\,|\theta - 3|\,\sigma\}} \right] \geqslant x \right\} \\ &\approx P\{\max_{\alpha \in \mathscr{S}^{p-1}/2} Z(\alpha) \geqslant x\} \\ &= P\{\max_{\phi \in I_\phi} Z(\phi) \geqslant x\}, \end{aligned} \tag{2.5}$$

where $Z(\alpha)$ is that in Theorem 2.2 and $x = \sqrt{N}\,\{z - (\theta - 3)^2\}/(2\,|\theta - 3|\,\sigma)$. Since $z > (\theta - 3)^2$, x can be regarded as a point in tail of the distribution of $\max Z(\alpha)$ for large N. The parameter space of the Gaussian random field

$Z(\alpha)$ is $d = (p-1)$-dimension. It is possible to reparametrize α in terms of ϕ by the spherical polar coordinate transformation,

$$\alpha = \alpha(\phi) = (\alpha_1(\phi), ..., \alpha_p(\phi))',$$

$$\phi = (\phi_1, ..., \phi_{p-1}) \in [0, \pi] \times \cdots \times [0, \pi] \equiv I_\phi \subset R^{p-1},$$

where $\phi_{p-1} = 0$ and $\phi_{p-1} = \pi$ represent the same point. Thus by putting $Z(\alpha) = Z(\alpha(\phi)) \equiv Z(\phi)$, for $\phi \in I_\phi$, $Z(\phi)$ is a $(p-1)$-dimensional non-singular differentiable Gaussian random field with mean zero, unit variance and covariance function $\rho(\phi, \bar{\phi}) = \rho(\alpha(\phi), \alpha(\bar{\phi}))$, for $\phi, \bar{\phi} \in I_\phi$, where the expression of $\rho(\cdot, \cdot)$ is given in the paragraph before Theorem 2.2. The regularity conditions for $\rho(\cdot, \cdot)$ given in Sun (1993) can be easily checked. Especially the fact that $Z(\cdot)$ has finite Karhunen–Loève expansion as given in (2.2) guarantees that the critical radius of the tube of the manifold derived from $\rho(\cdot, \cdot)$ is positive, so that Theorem 2.4 is applicable to $Z(\phi)$ in (2.5).

Now the covariance function of $Z(\phi)$ is, by (2.1),

$$\rho(\phi, \bar{\phi}) = \{\tau_1(\alpha(\phi)' \alpha(\bar{\phi}))^4 + \tau_2(\alpha(\phi)' \alpha(\bar{\phi}))^2 + \tau_3\}/\sigma^2,$$

where

$$\tau_1 = \frac{24 m_8}{p(p+2)(p+4)(p+6)},$$

$$\tau_2 = 72\left[\frac{m_8}{p(p+2)(p+4)(p+6)} - \frac{2 m_4 m_6}{p^2(p+2)^2\,(p+4)^2} + \frac{m_4^3}{p^3(p+2)^3}\right],$$

$$\tau_3 = 9\left[\frac{m_8}{p(p+2)(p+4)(p+6)} - \frac{4 m_4 m_6}{p^2(p+2)^2\,(p+4)^2}\right.$$
$$\left. + \frac{4 m_4^3}{p^3(p+2)^3} - \frac{m_4^2}{p^2(p+2)^2}\right].$$

The metric tensor matrix is obtained as the following diagonal matrix:

$$R(\phi) = \left\{ \left.\frac{\partial^2 \rho(\phi, \bar{\phi})}{\partial \phi\, \partial \bar{\phi}}\right|_{\phi = \bar{\phi}} \right\}$$
$$= \left(\frac{4\tau_1 + 2\tau_2}{\sigma^2}\right) \operatorname{diag}\left\{ \sum_{u=1}^{p} \left(\frac{\partial \alpha_u}{\partial \phi_1}\right)^2, ..., \sum_{u=1}^{p} \left(\frac{\partial \alpha_u}{\partial \phi_{p-1}}\right)^2 \right\}.$$

Thus, straightforward computations give the scalar curvature

$$S(\phi)=-(p-1)(p-2)\left(\frac{4\tau_1+2\tau_2}{\sigma^2}\right)^{-1}.$$

Detailed calculations are found in Naito (1996). By using these, two geometric constants corresponding to (2.3) and (2.4) can be obtained respectively as

$$\kappa_0=\frac{1}{2\pi^{p/2}}\int_{I_\phi}\|R(\phi)\|^{1/2}\,d\phi_1\cdots d\phi_{p-1}=\frac{\omega_{p-1}}{4\pi^{p/2}}\left(\frac{4\tau_1+2\tau_2}{\sigma^2}\right)^{(p-1)/2}, \tag{2.6}$$

$$\begin{aligned}\kappa_2&=\frac{1}{4\pi^{p/2}}\int_{I_\phi}\left\{-\frac{S(\phi)}{2}-\frac{(p-1)(p-2)}{2}\right\}\|R(\phi)\|^{1/2}\,d\phi_1\cdots d\phi_{p-1}\\&=\frac{(p-1)(p-2)\,\omega_{p-1}}{16\pi^{p/2}}\left(\frac{4\tau_1+2\tau_2}{\sigma^2}\right)^{(p-1)/2}\left[\left(\frac{4\tau_1+2\tau_2}{\sigma^2}\right)^{-1}-1\right],\end{aligned} \tag{2.7}$$

where ω_{d-1} is the surface area of $\mathscr{S}^{d-1}$ given as $\omega_{d-1}=2\pi^{d/2}/\Gamma(d/2)$.

For any fixed $q\in(0,1)$, let z_q be the upper $100q$ percent point of $[b^M_{2,N}]^2$ under normality. Then, from above arguments, we obtain the following result.

Proposition 2.5. *The power of the* $100q$ *level test for multinormality by* $[b^M_{2,N}]^2$ *against elliptically symmetric distributions satisfying the conditions of Theorem* 2.2 *and* $z_q>(\theta-3)^2$ *is approximately given as*

$$P\{[b^M_{2,N}]^2\geqslant z_q\}\approx\kappa_0\psi\left(\frac{x^2}{2},\frac{p}{2}\right)+\kappa_2\psi\left(\frac{x^2}{2},\frac{p-2}{2}\right), \tag{2.8}$$

where $x=N^{1/2}\{z_q-(\theta-3)^2\}/(2\,|\theta-3|\,\sigma)$; κ_0 *and* κ_2 *are given in* (2.6) *and* (2.7), *respectively*.

Let $\tilde{z}_q$ be the upper $100q$ percentile of $b^M_{2,N}$ (not $[b^M_{2,N}]^2$) under normality. We see from (1.3) that Proposition 2.5 simultaneously gives an approximation formula of the power of the test by $b^M_{2,N}$. It is summarized as follows.

Proposition 2.6. *The power of the* $100q$ *level test for multinormality by* $b^M_{2,N}$ *against elliptically symmetric distributions satisfying the conditions of Theorem* 2.2 *and* $\tilde{z}_q^2>(\theta-3)^2$ *is approximately given as*

$$P\{b^M_{2,N}\geqslant\tilde{z}_q\}\approx\kappa_0\psi\left(\frac{y^2}{2},\frac{p}{2}\right)+\kappa_2\psi\left(\frac{y^2}{2},\frac{p-2}{2}\right),$$

where $y = N^{1/2}\{\tilde{z}_q^2 - (\theta-3)^2\}/(2\,|\theta-3|\,\sigma)$; κ_0 *and* κ_2 *are given in* (2.6) *and* (2.7), *respectively*.

3. EXAMPLES

In this section, we provide some examples to illustrate the results in Section 2. Some elliptical distributions to which Proposition 2.5 is applicable are considered. The quantities which are necessary to obtain geometric constants κ_0 and κ_2 are given. As in Proposition 2.5, z_q stands for the upper $100q$ percentile of $[b_{2,N}^M]^2$ under normality.

EXAMPLE 3.1 (Contaminated normal distribution). If the random vector X has the density

$$f(x) = \frac{(1-\varepsilon)}{(2\pi)^{p/2}\,|\Delta|^{1/2}} \exp\left[-\frac{1}{2}(x-\mu)'\,\Delta^{-1}(x-\mu)\right] + \frac{\varepsilon}{(2\pi)^{p/2}\,|c\Delta|^{1/2}} \exp\left[-\frac{1}{2c}(x-\mu)'\,\Delta^{-1}(x-\mu)\right]$$

for some constant vector $\mu \in R^p$, some symmetric positive definite matrix Δ and some positive $c(\neq 1)$, we say that X has a contaminated normal distribution and denote it by $X \sim CN_p(c, \mu, \Delta)$ for $0 < \varepsilon < 1$ (see Anderson, 1993, p. 9). We have

$$E[X] = \mu, \qquad E[(X-\mu)(X-\mu)'] = \{1 + \varepsilon(c-1)\}\,\Delta = \Sigma.$$

For $X \sim CN_p(c, \mu, \Delta)$, since $(X-\mu)'\,\Delta^{-1}(X-\mu)$ has the density

$$\frac{(1-\varepsilon)}{2^{p/2}\Gamma(p/2)}\,t^{p/2-1}\exp(-t/2) + \frac{\varepsilon}{(2c)^{p/2}\,\Gamma(p/2)}\,t^{p/2-1}\exp(-t/(2c)),$$

we obtain

$$m_4 = p(p+2)\,\frac{\{1+\varepsilon(c^2-1)\}}{\{1+\varepsilon(c-1)\}^2},$$

$$m_6 = p(p+2)(p+4)\,\frac{\{1+\varepsilon(c^3-1)\}}{\{1+\varepsilon(c-1)\}^3},$$

$$m_8 = p(p+2)(p+4)(p+6)\,\frac{\{1+\varepsilon(c^4-1)\}}{\{1+\varepsilon(c-1)\}^4}.$$

Therefore,

$$\theta = 3\left[\frac{\{1+\varepsilon(c^2-1)\}}{\{1+\varepsilon(c-1)\}^2}\right],$$

$$\sigma^2 = \frac{105\{1+\varepsilon(c^4-1)\}}{\{1+\varepsilon(c-1)\}^4} - \frac{180\{1+\varepsilon(c^2-1)\}\{1+\varepsilon(c^3-1)\}}{\{1+\varepsilon(c-1)\}^5}$$
$$+ \frac{108\{1+\varepsilon(c^2-1)\}^3}{\{1+\varepsilon(c-1)\}^6} - \frac{9\{1+\varepsilon(c^2-1)\}^2}{\{1+\varepsilon(c-1)\}^4},$$

$$4\tau_1 + 2\tau_2 = \frac{240\{1+\varepsilon(c^4-1)\}}{\{1+\varepsilon(c-1)\}^4} - \frac{144\{1+\varepsilon(c^2-1)\}\{1+\varepsilon(c^3-1)\}}{\{1+\varepsilon(c-1)\}^5}$$
$$+ \frac{72\{1+\varepsilon(c^2-1)\}^3}{\{1+\varepsilon(c-1)\}^6}.$$

By using these, we can obtain κ_0 and κ_2 in (2.8). Further, for any given ε, we can apply Proposition 2.5 to $X \sim CN_p(c, \mu, \Delta)$ with c satisfying $(\theta-3)^2 < z_q$ which is equivalent to

$$1 < \frac{\{1+\varepsilon(c^2-1)\}}{\{1+\varepsilon(c-1)\}^2} < 1 + \frac{\sqrt{z_q}}{3}.$$

EXAMPLE 3.2 (Symmetric Kotz Type distribution). The random vector X is said to have a symmetric Kotz Type distribution if X has the density

$$f(x) = \frac{s r^{(2a+p-1)/(2s)} \Gamma(p/2)}{\pi^{p/2} \Gamma((2a+p-2)/(2s))} |\Delta|^{-1/2} [(x-\mu)' \Delta^{-1}(x-\mu)]^{a-1}$$
$$\times \exp[-r\{(x-\mu)' \Delta^{-1}(x-\mu)\}^s],$$

$r, s > 0$, $2a + p > 2$, for some vector $\mu \in R^p$ and some symmetric positive definite matrix Δ (see Fang *et al.* (1989, p. 76)). We shall denote this by $X \sim MK_p(a, r, s, \mu, \Delta)$.

We have

$$E[X] = \mu, \qquad E[(X-\mu)(X-\mu)'] = \frac{\Gamma((2a+p)/(2s))}{p r^{1/2} \Gamma((2a+p-2)/(2s))} \Delta = \Sigma.$$

In the special case $s = 1$ and $a = 1$, these family of distributions include the multivariate normal distribution. For $X \sim MK_p(a, r, s, \mu, \Delta)$, since $(X-\mu)'\Delta^{-1}(X-\mu)$ has the density

$$\frac{s r^{(2a+p-2)/(2s)} \Gamma(p/2)}{\Gamma((2a+p-2)/(2s))} t^{p/2+a-2} \exp(-rt^s), \qquad t > 0,$$

direct computations give, for $X \sim MK_p(a, r, 1, \mu, \Delta)$ such that $a \neq 1$,

$$m_4 = p^2 \frac{2a+p}{2a+p-2},$$

$$m_6 = p^3 \frac{(2a+p+2)(2a+p)}{(2a+p-2)^2},$$

$$m_8 = p^4 \frac{(2a+p+4)(2a+p+2)(2a+p)}{(2a+p-2)^3}.$$

Thus

$$\theta = 3\left[\frac{p(2a+p)}{(p+2)(2a+p-2)}\right],$$

$$\sigma^2 = \frac{(2a+p)\,p^3}{(p+2)(2a+p-2)^3}\left[\frac{105(2a+p+4)(2a+p+2)}{(p+4)(p+6)} - \frac{180(2a+p+2)(2a+p)}{(p+2)(p+4)} + \frac{108(2a+p)^2}{(p+2)^2}\right] - \left\{\frac{3p(2a+p)}{(p+2)(2a+p-2)}\right\}^2,$$

$$4\tau_1 + 2\tau_2 = \left\{\frac{p}{(2a+p-2)}\right\}^3 \left[\frac{240(2a+p+4)(2a+p+2)(2a+p)}{(p+2)(p+4)(p+6)} - \frac{288(2a+p+2)(2a+p)^2}{(p+2)^2\,(p+4)^2} + \frac{144(2a+p)^3}{(p+2)^3}\right].$$

We can apply Proposition 2.5 to $X \sim MK_p(a, r, 1, \mu, \Delta)$ with the parameter a satisfying

$$a \neq 1, \qquad a > -\frac{p}{2} + 1, \qquad -\frac{\sqrt{z_q}}{3} < \frac{4(1-a)}{(p+2)(2a+p-2)} < \frac{\sqrt{z_q}}{2}.$$

EXAMPLE 3.3 (Symmetric multivariate Pearson Type VII distribution). If the random vector X has the density

$$f(x) = \frac{\Gamma(a)}{\Gamma(a-p/2)(\pi m)^{p/2}\,|\Delta|^{1/2}} \left\{1 + \frac{1}{m}(x-\mu)'\,\Delta^{-1}(x-\mu)\right\}^{-a}$$

for some vector $\mu \in R^p$ and some symmetric positive definite matrix Δ, we say that X has a symmetric multivariate Pearson Type VII distribution and we shall denote it by $X \sim MPVII_p(m, a, \mu, \Delta)$ $(a > p/2,\ m > 0$; see Fang *et al.*, (1989, p. 81)). Note that in the case m is a positive integer and $a = (p+m)/2$, then it is multivariate t-distribution. In Examples 2.9 and 3.8 of Baringhaus and Henze (1991), m_{2k} for $k = 2, 3, 4$ are presented. From these we have

$$\theta = 3\left[\frac{2a-p-2}{2a-p-4}\right],$$

$$\sigma^2 = \frac{105(2a-p-2)^3}{(2a-p-4)(2a-p-6)(2a-p-8)} - \frac{180(2a-p-2)^3}{(2a-p-4)^2\,(2a-p-6)} + 108\left\{\frac{2a-p-2}{2a-p-4}\right\}^3 - 9\left\{\frac{2a-p-2}{2a-p-4}\right\}^2,$$

$$4\tau_1 + 2\tau_2 = \frac{240(2a-p-2)^3}{(2a-p-4)(2a-p-6)(2a-p-8)} - \frac{144(2a-p-2)^3}{(2a-p-4)^2\,(2a-p-6)} + 72\left\{\frac{2a-p-2}{2a-p-4}\right\}^3.$$

If we choose $X \sim MPVII_p(m, a, \mu, \Delta)$ with the parameter a satisfying

$$a > \max\left\{\frac{p}{2} + 4,\ \frac{p}{2} + 2 + \frac{3}{\sqrt{z_q}}\right\},$$

we can apply Proposition 2.5 to such X.

EXAMPLE 3.4 (Symmetric multivariate Pearson Type II distribution). The random vector X is said to have a symmetric multivariate Pearson Type II distribution if X has the density

$$f(x) = = \frac{\Gamma(p/2+m+1)}{\Gamma(m+1)\,\pi^{p/2}\,|\Delta|^{1/2}} \times \{1-(x-\mu)'\,\Delta^{-1}(x-\mu)\}^m \times I\{(x-\mu)'\,\Delta^{-1}(x-\mu) \leqslant 1\}$$

for some vector $\mu \in R^p$ and some symmetric positive definite matrix Δ $(m \in R,\ m > -1$ see Fang *et al.* (1989, p. 89)). We shall denote this by $X \sim MPII_p(m, \mu, \Delta)$.

By using Examples 2.8 and 3.7 of Baringhaus and Henze (1991), we have for $X \sim MPII_p(m, \mu, \Delta)$,

$$\theta = 3\left[\frac{p+2m+2}{p+2m+4}\right],$$

$$\sigma^2 = \frac{105(p+2m+2)^3}{(p+2m+4)(p+2m+6)(p+2m+8)}$$
$$-\frac{180(p+2m+2)^3}{(p+2m+4)^2\,(p+2m+6)}$$
$$+108\left\{\frac{p+2m+2}{p+2m+4}\right\}^3 - 9\left\{\frac{p+2m+2}{p+2m+4}\right\}^2,$$

$$4\tau_1 + 2\tau_2 = \frac{240(p+2m+2)^3}{(p+2m+4)(p+2m+6)(p+2m+8)}$$
$$-\frac{144(p+2m+2)^3}{(p+2m+4)^2\,(p+2m+6)} + 72\left\{\frac{p+2m+2}{p+2m+4}\right\}^3.$$

We can check the conditions of Proposition 2.5 through the parameter m. If we choose $X \sim MPII_p(m, \mu, \Delta)$ with the parameter m satisfying

$$m > -\frac{(p+4)}{2} + \frac{3}{\sqrt{z_q}},$$

we can apply Proposition 2.5 to X, since $m_8 < \infty$ and $(\theta-3)^2 < z_q$.

ACKNOWLEDGMENT

The author is grateful to Professor Y. Fujikoshi of Hiroshima University for his comments and he also wants to thank an associate editor and referees for their useful comments which led to substantial improvements in the presentation.

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