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Exact distributions of MLEs of regression coefficients in GMANOVA–MANOVA model

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Abstract

This paper studies the exact distributions of the MLEs of the regression coefficient matrices in a GMANOVA–MANOVA model with normal error. The unique conditions for linear functions of the MLEs of regression coefficient matrices are presented, and the exact density functions or characteristic functions for these linear functions are derived.

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1. Introduction

Consider the following GMANOVA–MANOVA (generalized multivariate analysis of variance–multivariate analysis of variance) model with normal error:

$$\begin{cases} Y = XB_1Z_1^\tau + B_2Z_2^\tau + \mathcal{E}, \\ \mathcal{E} \sim N_{q\times n}(0, I_n \otimes \Sigma), \end{cases} \tag{1}$$

where Y is a $q \times n$ observable random response matrix, X is a $q \times p$ known constant matrix, Z_1 and Z_2 are the $n \times m$ and $n \times s$ known design matrices, respectively, B_1 and B_2 are the $p \times m$ and $q \times s$ unknown regression coefficient matrices, respectively, $\mathcal{E}$ is a $q \times n$ unobservable random error matrix, and A^τ denotes the transpose of matrix A. Model (1) was first proposed by Chinchilli and Elswick [2], where the matrix (Z_1, Z_2) was assumed to be of full column rank.

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They gave the MLEs of parameters and studied the goodness-of-fit test under their assumptions. MANOVA and GMANOVA models can be obtained as a special case of (1), therefore model (1) is a mixture of MANOVA and GMANOVA model. For the GMANOVA model, including extended GMANOVA model, the results about estimates of parameters and their distributions have been extensively studied in literatures, for example, see Potthoff and Roy [7], Rao [8], Grizzle and Allen [3], Reinsel [9], Kenward [4], von Rosen [10,11], Kollo and von Rosen [5] and among others. Kollo and von Rosen [5] gave a good summary for some results in model (1). They studied the MLEs of parameter matrices B_1, B_2 and Σ in this model and derived various moment formulae for these estimators. Bai [1] studied the MLEs of the parameters in model (1) and gave the exact distribution for MLE of covariance matrix. The MLEs given in [1] is listed as follows.

Lemma 1. *For model* (1), *if $n \geqslant rk(Z)+q$, the MLEs of B_1, B_2 and Σ are given* (*with probability* 1) *by*

$$\begin{cases} \hat{B}_1 = (X^\tau S^{-1} X)^- X^\tau S^{-1} Y Q_{Z_2} Z_1 (Z_1^\tau Q_{Z_2} Z_1)^-, \\ \hat{B}_2 = (Y - X\hat{B}_1 Z_1^\tau) Z_2 (Z_2^\tau Z_2)^-, \\ \hat{\Sigma} = \dfrac{1}{n}(Y - X\hat{B}_1 Z_1^\tau) Q_{Z_2} (Y - X\hat{B}_1 Z_1^\tau)^\tau, \end{cases} \tag{2}$$

respectively, where $S = Y Q_Z Y^\tau$, $Z \hat{=} (Z_1, Z_2)$, P_A denotes the orthogonal projection matrix onto the linear subspace $R(A)$ spanned by the columns of $p \times q$ matrix A, i.e., $P_A = A(A^\tau A)^- A^\tau = A(A^\tau A)^+ A^\tau$, $Q_A \hat{=} I_p - P_A$, A^- denotes the generalized inverse of matrix A such that $AA^-A = A$, A^+ denotes the Moore–Penrose inverse of matrix A and $rk(A)$ denotes the rank of A.

From the viewpoint of statistical inference, it is very important to study the distributions of $\hat{B}_1$ and $\hat{B}_2$ in Lemma 1, which are not discussed in current literatures. It is easy to see that $\hat{B}_1$ and $\hat{B}_2$ are not unique since they depend on the generalized inverses. Therefore we have to consider the linear functions of $\hat{B}_1$ and $\hat{B}_2$, say $K_1\hat{B}_1 L_1^\tau$, $K_2\hat{B}_2 L_2^\tau$ and $K_1\hat{B}_1 L_1^\tau + K_2\hat{B}_2 L_2^\tau$, respectively, where $K_i, L_i (\neq 0), i = 1, 2$ are some constant matrices such that these linear functions are unique. It is worth mentioning that, in general, all the above linear functions are nonlinear functions of normal matrix Y, hence it is difficult to study their distributions and some special techniques need to be developed. The remainder of this paper is arranged as follows. Section 2 gives the sufficient and necessary conditions for uniqueness of $K_1\hat{B}_1 L_1^\tau$ and $K_2\hat{B}_2 L_2^\tau$. Section 3 derives the density functions or characteristics functions for above linear functions of the MLEs.

2. Uniqueness

We first give a lemma which is useful to determine what kind of matrices $K_i, L_i (\neq 0), i = 1, 2$ ensure that $K_1\hat{B}_1 L_1^\tau$, $K_2\hat{B}_2 L_2^\tau$ or $K_1\hat{B}_1 L_1^\tau + K_2\hat{B}_2 L_2^\tau$ are unique. The obvious proofs are omitted for saving space.

Lemma 2. *Let A, B and C be given matrices, then*

(i) *$A(A^\tau A)^- B^\tau$ is unique $\Leftrightarrow R(B^\tau) \subset R(A^\tau)$.*
(ii) *$C(A^\tau A)^- A^\tau$ is unique $\Leftrightarrow R(C^\tau) \subset R(A^\tau)$.*

By applying Lemmas 1 and 2, we easily obtain:

Theorem 1. *Let* $K_i, L_i (\neq 0), i = 1, 2$ *be constant matrices, then*

(i) $K_1 \hat{B}_1 L_1^\tau$ *is unique* $\Leftrightarrow R(K_1^\tau) \subset R(X^\tau), R(L_1^\tau) \subset R(Z_1^\tau Q_{Z_2})$.
(ii) $K_2 \hat{B}_2 L_2^\tau$ *is unique* $\Leftrightarrow$ *one of the following conditions holds*:
 (i) $R(K_2^\tau) \subset R(Q_X), R(L_2^\tau) \subset R(Z_2^\tau)$. (ii) $R(L_2^\tau) \subset R(Z_2^\tau Q_{Z_1})$.

Remark 1. The proof of Lemma 2 can be found in Kollo and von Rosen [5]. The results in Theorem 1 when $K_2 = I$ was given in Kollo and von Rosen [5, p. 427].

Theorem 2. *For the constant matrices* $K_i, L_i (\neq 0), i = 1, 2$, *if there exists a nonnull d such that* $K_1 = dK_2X$ *and* $R((dL_1, L_2)^\tau) \subset R(Z^\tau)$, *then* $K_1 \hat{B}_1 L_1^\tau + K_2 \hat{B}_2 L_2^\tau$ *is unique.*

3. Distributions

In this section, we study the distributions of $K_1 \hat{B}_1 L_1^\tau$, $K_2 \hat{B}_2 L_2^\tau$ and $K_1 \hat{B}_1 L_1^\tau + K_2 \hat{B}_2 L_2^\tau$, respectively, where $K_i, L_i (\neq 0), i = 1, 2$ are assumed to satisfy the conditions in Theorem 1 or 2. We follow the symbols and notations in Muirhead [6] without specification. The obvious proofs are omitted for saving the space.

Definition 1 (*Wang [12]*). If the density function of a $p \times q$ random matrix *X* is given by

$$p_X(x) = \pi^{-pq/2} \frac{\Gamma_p[(\gamma + p + q - 1)/2]}{\Gamma_p[(\gamma + p - 1)/2]} |\Sigma|^{-q/2} |V|^{-p/2} \times |I_q + (x - \mu)^\tau \Sigma^{-1} (x - \mu) V^{-1}|^{-(\gamma + p + q - 1)/2}, \quad x \in R^{p \times q},$$

where $\mu \in R^{p\times q}$, $\Sigma > 0$, $V > 0$, $\gamma > 0$, then we say that *X* has a matrix-variate *t*-distribution with parameters (μ, Σ, V, γ) and denote by $X \sim t_{p\times q}(\mu, \Sigma, V, \gamma)$.

Lemma 3. *Let*

$$\begin{cases} X|T \sim N_{p\times q}(\mu, V \otimes T^{-1}), \\ T \sim W_p(\gamma + p - 1, \Sigma), \end{cases}$$

then $X \sim t_{p\times q}(\mu, \Sigma^{-1}, V, \gamma)$.

Lemma 4. *Let* $X \sim W_p(n, \Sigma)$,$A \geqslant 0$, $a \geqslant 0$ *and*

$$h_a(A) = E(|I_p + X^{-1}A|^{-a}), \tag{3}$$

then

$$h_a(A) = \frac{\Gamma_{rk(A)}(n/2 + a)}{\Gamma_{rk(A)}((n - p + rk(A))/2)} \left| \frac{1}{2}\Lambda \right|^{(n-p+rk(A))/2} \times \Psi\left(\frac{n}{2} + a, \frac{1}{2}(n + rk(A) + 1); \frac{1}{2}\Lambda \right), \tag{4}$$

where Λ *is a* $rk(A) \times rk(A)$ *diagonal matrix with the nonnull eigenvalues of* $\Sigma^{-1}A$ *as its diagonal elements, and* $\Psi(a, c; \cdot)$ *is a confluent function defined by* (12) *in* [6, p. 472].

Proof. Let the spectral decomposition of $\Sigma^{-1/2}A\Sigma^{-1/2}$ be

$$\Sigma^{-1/2}A\Sigma^{-1/2} = P\begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}P^\tau,$$

where P is a $p \times p$ orthogonal matrix, Λ is a $rk(A) \times rk(A)$ diagonal matrix with the nonnull eigenvalues of $\Sigma^{-1/2}A\Sigma^{-1/2}$ as its diagonal elements. Let

$$\tilde{X} = P^\tau\Sigma^{-1/2}X\Sigma^{-1/2}P = \begin{pmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} \end{pmatrix},$$

where $\tilde{X}_{11}$ is $rk(A) \times rk(A)$, then $\tilde{X} \sim W_p(n, I_p)$ and

$$|I_p + X^{-1}A| = \begin{vmatrix} I_{rk(A)} + \tilde{X}_{11\cdot 2}^{-1}\Lambda & 0 \\ -\tilde{X}_{22}^{-1}\tilde{X}_{21}\tilde{X}_{11\cdot 2}^{-1}\Lambda & I_{p-rk(A)} \end{vmatrix} = |I_{rk(A)} + \tilde{X}_{11\cdot 2}^{-1}\Lambda|,$$

where $\tilde{X}_{11\cdot 2} = \tilde{X}_{11} - \tilde{X}_{12}\tilde{X}_{22}^{-1}\tilde{X}_{21}$. It follows from $\tilde{X} \sim W_p(n, I_p)$ and Theorem 3.2.10 in [6] that $\tilde{X}_{11\cdot 2} \sim W_{rk(A)}(n - p + rk(A), I_{rk(A)})$. Let $Y = \Lambda^{-1/2}\tilde{X}_{11\cdot 2}\Lambda^{-1/2}$, then $Y \sim W_{rk(A)}(n - p + rk(A), \Lambda^{-1})$ and

$$|I_p + X^{-1}A| = |Y|^{-1}|I_{rk(A)} + Y|,$$

thus we have

$$h_a(A) = E(|Y|^a|I_{rk(A)} + Y|^{-a}) = \frac{|\Lambda/2|^{(n-p+rk(A))/2}}{\Gamma_{rk(A)}((n-p+rk(A))/2)} \times \int_{Y>0} |I_{rk(A)} + Y|^{-a}|Y|^{(n-p-1)/2+a} etr\left(-\frac{1}{2}\Lambda Y\right)(dY);$$

this and the definition of function $\Psi(a, c; \cdot)$ (see (12) in [6, p. 472]) mean that (4) holds. Finally, the nonnull eigenvalues of $\Sigma^{-1/2}A\Sigma^{-1/2}$ are the same as those of $\Sigma^{-1}A$. $\square$

Remark 2. Based on Lemma 10.6.4 in [11], the confluent function $\Psi(a, c; \cdot)$ can be obtained by the Gaussian hypergeometric function ${}_2F_1(a, b; c; \cdot)$.

Lemma 5. *Let G be an $m \times n$ random matrix with the density function $p_G(g)$, $g \in R^{m\times n}$. Let*

$$F = PGQ^\tau, \tag{5}$$

where P and Q are $p \times m$ and $q \times n$ nonrandom column orthogonal matrix, respectively, then the density function of the $p \times q$ random matrix F is given by

$$p_G(P^\tau fQ), \quad R(f) \subset R(P), \quad R(f^\tau) \subset R(Q). \tag{6}$$

Proof. Let M and N be $p \times (p - m)$ and $q \times (q - n)$ nonrandom column orthogonal matrices such that (P, M) and (Q, N) are nonrandom orthogonal matrices, respectively. Again let G_1, G_2 and G_3 be, respectively, $m \times (q - n)$, $(p - m) \times n$ and $(p - m) \times (q - n)$ random matrices such that G, G_1, G_2 and G_3 are independent and $P(G_i = 0) = 1$, $i = 1, 2, 3$. Let

$$\tilde{F} = (P, M)\begin{pmatrix} G & G_1 \\ G_2 & G_3 \end{pmatrix}(Q, N)^\tau, \tag{7}$$

then from (5), we know that $\tilde{F} = F$, a.s., thus $\tilde{F}$ and F have the identical distribution. Note that the Jacobian determinant of the transformation from (G, G_1, G_2, G_3) to $\tilde{F}$ is equal to 1 and

$$G = P^\tau \tilde{F} Q, \quad G_1 = P^\tau \tilde{F} N, \quad G_2 = M^\tau \tilde{F} Q, \quad G_3 = M^\tau \tilde{F} N,$$

hence the density function of $\tilde{F}$ is given by

$$p_G(P^\tau f Q), \quad P^\tau f N = 0, \quad M^\tau f Q = 0, \quad M^\tau f N = 0. \tag{8}$$

Note that when $P^\tau f N = 0, M^\tau f N = 0$, we have $R(f^\tau P) \subset R(Q), R(f^\tau M) \subset R(Q)$. Now for any $c \in R(f^\tau)$, we can find d such that $c = f^\tau d$, thus $c = f^\tau(PP^\tau + MM^\tau)d = f^\tau P(P^\tau d) + f^\tau M(M^\tau d) \in R(Q)$. This implies that $R(f^\tau) \subset R(Q)$. Conversely, when $R(f^\tau) \subset R(Q)$, there exits a D such that $f^\tau = QD^\tau$, which gives that $P^\tau f N = 0$ and $M^\tau f N = 0$. Therefore $P^\tau f N = 0, M^\tau f N = 0 \Leftrightarrow R(f^\tau) \subset R(Q)$. Similarly, we have $M^\tau f Q = 0, M^\tau f N = 0 \Leftrightarrow R(f) \subset R(P)$. Thus $P^\tau f N = 0, M^\tau f Q = 0, M^\tau f N = 0 \Leftrightarrow R(f) \subset R(P), R(f^\tau) \subset R(Q)$. Since $\tilde{F}$ and F have the same distribution, it follows from (8) that (6) holds. □

3.1. The distribution of $K_1 \hat{B}_1 L_1^\tau$

Theorem 3. *For model* (1), *if $\hat{F}_1 \hat{=} K_1 \hat{B}_1 L_1^\tau$ is unique, then its density function is given by*

$$\begin{aligned}
p_{\hat{F}_1}(f_1) = {} & (2\pi)^{-rk(K_1)rk(L_1)/2} \frac{\Gamma_{rk(K_1)}[(n - rk(Z) + rk(K_1))/2]}{\Gamma_{rk(K_1)}[(n - rk(Z) + rk(K_1) + rk(L_1))/2]} \\
& \times \frac{\Gamma_{rk(K_1)}[(n - rk(Z) + rk(K_1) + rk(L_1) - q + rk(X))/2]}{\Gamma_{rk(K_1)}[(n - rk(Z) + rk(K_1) - q + rk(X))/2]} \\
& \times |\Lambda_1^2|^{-rk(L_1)/2} |\Delta_1^2|^{-rk(K_1)/2} \\
& \times etr\{-\tfrac{1}{2}(f_1 - F_1)^\tau [K_1(X^\tau \Sigma^{-1} X)^+ K_1^\tau]^+ (f_1 - F_1)[L_1(Z_1^\tau Q_{Z_2} Z_1)^+ L_1^\tau]^+\} \\
& \times {}_1F_1(\tfrac{1}{2}(q - rk(X)); \tfrac{1}{2}(n - rk(Z) + rk(K_1) + rk(L_1)); \\
& \tfrac{1}{2}(f_1 - F_1)^\tau [K_1(X^\tau \Sigma^{-1} X)^+ K_1^\tau]^+ (f_1 - F_1)[L_1(Z_1^\tau Q_{Z_2} Z_1)^+ L_1^\tau]^+), \\
& R(f_1 - F_1) \subset R(K_1), \quad R((f_1 - F_1)^\tau) \subset R(L_1),
\end{aligned} \tag{9}$$

where $F_1 = K_1 B_1 L_1^\tau$, Λ_1^2 is a *$rk(K_1) \times rk(K_1)$ diagonal matrix with the nonnull eigenvalues of $K_1(X^\tau \Sigma^{-1} X)^+ K_1^\tau$ as its diagonal elements, and Δ_1^2 is a $rk(L_1) \times rk(L_1)$ diagonal matrix with the nonnull eigenvalues of $L_1(Z_1^\tau Q_{Z_2} Z_1)^+ L_1^\tau$ as its diagonal elements.*

Proof. When $\hat{F}_1$ is unique, it follows from Theorem 1 that there exist C_1 and D_1 such that

$$K_1^\tau = X^\tau C_1^\tau, \quad L_1^\tau = Z_1^\tau Q_{Z_2} D_1^\tau. \tag{10}$$

From Lemma 1 and (1), we have

$$\hat{F}_1 = F_1 + C_1 X (X^\tau S^{-1} X)^- X^\tau S^{-1} \mathcal{E} P_{Q_{Z_2} Z_1} D_1^\tau, \tag{11}$$

where $S = \mathcal{E} Q_Z \mathcal{E}^\tau$. Note that $rk(\Sigma^{-1/2} X) = rk(X)$, hence $\Sigma^{-1/2} X$ can be written as

$$\Sigma^{-1/2} X = \tilde{P} \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} (M, N)^\tau, \tag{12}$$

where $\tilde{P} \hat{=} (\tilde{P}_1, \tilde{P}_2)$ and (M, N) are, respectively, $q \times q$ and $p \times p$ orthogonal matrices, $\tilde{P}_1$ and M are, respectively, $q \times rk(X)$ and $p \times rk(X)$ matrices, and Λ is a $rk(X) \times rk(X)$ diagonal matrix with positive diagonal elements. Let

$$T \hat{=} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \tilde{P}^\tau \Sigma^{-1/2} S \Sigma^{-1/2} \tilde{P}, \tag{13}$$

where T_{11} is a $rk(X) \times rk(X)$ matrix, then

$$T^{-1} = \begin{pmatrix} T_{11\cdot 2}^{-1} & -T_{11\cdot 2}^{-1} V \\ -V^\tau T_{11\cdot 2}^{-1} & T_{22\cdot 1}^{-1} \end{pmatrix},$$

where $T_{11\cdot 2} = T_{11} - T_{12} T_{22}^{-1} T_{21}$, $T_{22\cdot 1} = T_{22} - T_{21} T_{11}^{-1} T_{12}$, $V = T_{12} T_{22}^{-1}$. From (12) and (13), we have

$$X^\tau S^{-1} X = (M, N) \begin{pmatrix} \Lambda T_{11\cdot 2}^{-1} \Lambda & 0 \\ 0 & 0 \end{pmatrix} (M, N)^\tau,$$

which means that

$$(X^\tau S^{-1} X)^- = (M, N) \begin{pmatrix} \Lambda^{-1} T_{11\cdot 2} \Lambda^{-1} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} (M, N)^\tau.$$

Therefore

$$X (X^\tau S^{-1} X)^- X^\tau S^{-1} = \Sigma^{1/2} \tilde{P}_1 (I_{rk(X)}, -V) \tilde{P}^\tau \Sigma^{-1/2}. \tag{14}$$

Note that $P_{Z_2} P_{Q_{Z_2} Z_1} = P_{Q_{Z_2} Z_1} P_{Z_2} = 0$, hence P_{Z_2} and $P_{Q_{Z_2} Z_1}$ can be simultaneously diagonalized by an orthogonal matrix. Since P_{Z_2} and $P_{Q_{Z_2} Z_1}$ are $n \times n$ idempotent matrices with ranks $rk(Z_2)$ and $rk(Z) - rk(Z_2)$, respectively, there exits an $n \times n$ orthogonal matrix $\tilde{Q} \hat{=} (\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3)$ such that

$$P_{Z_2} = \tilde{Q}_1 \tilde{Q}_1^\tau, \quad P_{Q_{Z_2} Z_1} = \tilde{Q}_2 \tilde{Q}_2^\tau, \tag{15}$$

where $\tilde{Q}_1$, $\tilde{Q}_2$ and $\tilde{Q}_3$ are, respectively, $n \times rk(Z_2)$, $n \times (rk(Z) - rk(Z_2))$ and $n \times (n - rk(Z))$ matrices. Let

$$\mathcal{E}_1 = \tilde{P}^\tau \Sigma^{-1/2} \mathcal{E} \tilde{Q}_1, \quad \mathcal{E}_2 = \tilde{P}^\tau \Sigma^{-1/2} \mathcal{E} \tilde{Q}_2, \quad \mathcal{E}_3 = \tilde{P}^\tau \Sigma^{-1/2} \mathcal{E} \tilde{Q}_3, \tag{16}$$

then $S = \Sigma^{1/2} \tilde{P} \mathcal{E}_3 \mathcal{E}_3^\tau \tilde{P}^\tau \Sigma^{1/2}$ (since $Q_Z = \tilde{Q}_3 \tilde{Q}_3^\tau$). From (11), (14–16), we have

$$\hat{F}_1 = F_1 + C_1 \Sigma^{1/2} \tilde{P}_1 (I_{rk(X)}, -V) \mathcal{E}_2 \tilde{Q}_2^\tau D_1^\tau. \tag{17}$$

Let the singular value decompositions of $C_1 \Sigma^{1/2} \tilde{P}_1$ and $\tilde{Q}_2^\tau D_1^\tau$ be

$$C_1 \Sigma^{1/2} \tilde{P}_1 = P_1 \Lambda_1 Q_1^\tau, \quad \tilde{Q}_2^\tau D_1^\tau = P_2 \Delta_1 Q_2^\tau, \tag{18}$$

where P_i, Q_i, $i = 1, 2$ are column orthogonal matrices, and Λ_1 and Δ_1 are, respectively, $r_1 \times r_1$ and $s_1 \times s_1$ diagonal matrices with positive diagonal elements, $r_1 \hat{=} rk(C_1 \Sigma^{1/2} \tilde{P}_1)$ and $s_1 \hat{=} rk(\tilde{Q}_2^\tau D_1^\tau)$. By substituting (18) into (17) we have

$$\hat{F}_1 = F_1 + P_1 \Lambda_1 Q_1^\tau (I_{rk(X)}, -V) \mathcal{E}_2 P_2 \Delta_1 Q_2^\tau \hat{=} F_1 + P_1 \hat{G}_1 Q_2^\tau. \tag{19}$$

From (1) we know

$$(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3) = \tilde{P}^\tau \Sigma^{-1/2} \mathcal{E} \tilde{Q} \sim N_{q\times n}(0, I_n \otimes I_q),$$

which indicates that

$$\begin{cases} \mathcal{E}_1 \sim N_{q\times rk(Z_2)}(0, I_{rk(Z_2)} \otimes I_q), \\ \mathcal{E}_2 \sim N_{q\times(rk(Z)-rk(Z_2))}(0, I_{rk(Z)-rk(Z_2)} \otimes I_q), \\ \mathcal{E}_3 \sim N_{q\times(n-rk(Z))}(0, I_{n-rk(Z)} \otimes I_q), \end{cases} \tag{20}$$

and $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ are independent. Note that $S = \Sigma^{1/2}\tilde{P}\mathcal{E}_3\mathcal{E}_3^\tau \tilde{P}^\tau \Sigma^{1/2}$, hence S and $\mathcal{E}_2$ are independent and from (20) we have $S \sim W_q(n - rk(Z), \Sigma)$. Furthermore, V is independent of $\mathcal{E}_2$ (since V is a function of S) and from (13) we have $T \sim W_q(n - rk(Z), I_q)$. Therefore, it follows from Lemma 3 and Theorem 3.2.10 in [6] that

$$V \sim t_{rk(X)\times(q-rk(X))}(0, I_{rk(X)}, I_{q-rk(X)}, n - rk(Z) - q + rk(X) + 1). \tag{21}$$

Let $\tilde{V} = Q_1^\tau V$, then

$$\tilde{V} \sim t_{r_1\times(q-rk(X))}(0, I_{r_1}, I_{q-rk(X)}, n - rk(Z) - q + rk(X) + 1). \tag{22}$$

From the definition of $\hat{G}_1$ (see (19)), we have

$$\hat{G}_1 = \Lambda_1(Q_1^\tau, -\tilde{V})\mathcal{E}_2 P_2 \Delta_1. \tag{23}$$

It follows from (20) and the independence between $\tilde{V}$ and $\mathcal{E}_2$ (since V and $\mathcal{E}_2$ are independent) that

$$\hat{G}_1|\tilde{V} \sim N_{r_1\times s_1}(0, \Delta_1^2 \otimes [\Lambda_1(I_{r_1} + \tilde{V}\tilde{V}^\tau)\Lambda_1]), \tag{24}$$

which means that the conditional density function of $\hat{G}_1$ given $\tilde{V} = \tilde{v}$ is

$$\begin{aligned} p_{\hat{G}_1|\tilde{V}}(g_1|\tilde{v}) = {} & (2\pi)^{-r_1 s_1/2}|\Lambda_1|^{-s_1}|\Delta_1|^{-r_1}|I_{r_1} + \tilde{v}\tilde{v}^\tau|^{-s_1/2} \\ & \times etr[-\tfrac{1}{2}g_1^\tau \Lambda_1^{-1}(I_{r_1} + \tilde{v}\tilde{v}^\tau)^{-1}\Lambda_1^{-1} g_1 \Delta_1^{-2}], \quad g_1 \in R^{r_1\times s_1}. \end{aligned}$$

Thus the density function of $\hat{G}_1$ is

$$\begin{aligned} p_{\hat{G}_1}(g_1) = {} & (2\pi)^{-r_1 s_1/2}|\Lambda_1|^{-s_1}|\Delta_1|^{-r_1} E\{|I_{r_1} + \tilde{V}\tilde{V}^\tau|^{-s_1/2} \\ & \times etr[-\tfrac{1}{2}g_1^\tau \Lambda_1^{-1}(I_{r_1} + \tilde{V}\tilde{V}^\tau)^{-1}\Lambda_1^{-1} g_1 \Delta_1^{-2}]\}, \quad g_1 \in R^{r_1\times s_1}. \end{aligned} \tag{25}$$

It follows from (22) and Definition 1 that the density function of $\tilde{V}$ is given by

$$\begin{aligned} p_{\tilde{V}}(\tilde{v}) = {} & \pi^{-r_1(q-rk(X))/2} \frac{\Gamma_{r_1}[(n - rk(Z) + r_1)/2]}{\Gamma_{r_1}[(n - rk(Z) + r_1 - q + rk(X))/2]} \\ & \times |I_{r_1} + \tilde{v}\tilde{v}^\tau|^{-(n-rk(Z)+r_1)/2}, \quad \tilde{v} \in R^{r_1\times(q-rk(X))}; \end{aligned}$$

this and (25) imply that the density function of $\hat{G}_1$ is

$$\begin{aligned} p_{\hat{G}_1}(g_1) = {} & (2\pi)^{-r_1 s_1/2} \frac{\Gamma_{r_1}[(n - rk(Z) + r_1)/2]}{\Gamma_{r_1}[(n - rk(Z) + r_1 + s_1)/2]} \\ & \times \frac{\Gamma_{r_1}[(n - rk(Z) + r_1 + s_1 - q + rk(X))/2]}{\Gamma_{r_1}[(n - rk(Z) + r_1 - q + rk(X))/2]} \\ & \times |\Lambda_1|^{-s_1}|\Delta_1|^{-r_1} E\{etr[-\tfrac{1}{2}g_1^\tau \Lambda_1^{-1}(I_{r_1} + \bar{V}\bar{V}^\tau)^{-1}\Lambda_1^{-1} g_1 \Delta_1^{-2}]\}, \\ & g_1 \in R^{r_1\times s_1}, \end{aligned} \tag{26}$$

where

$$\bar{V} \sim t_{r_1 \times (q-rk(X))}(0, I_{r_1}, I_{q-rk(X)}, n - rk(Z) + s_1 - q + rk(X) + 1). \tag{27}$$

From (26) and (27) there is

$$\begin{aligned} p_{\hat{G}_1}(g_1) = {} & (2\pi)^{-r_1 s_1/2} \frac{\Gamma_{r_1}[(n - rk(Z) + r_1)/2]}{\Gamma_{r_1}[(n - rk(Z) + r_1 + s_1)/2]} \\ & \times \frac{\Gamma_{r_1}[(n - rk(Z) + r_1 + s_1 - q + rk(X))/2]}{\Gamma_{r_1}[(n - rk(Z) + r_1 - q + rk(X))/2]} \\ & \times |\Lambda_1|^{-s_1} |\Delta_1|^{-r_1} etr(-\tfrac{1}{2} g_1^\tau \Lambda_1^{-2} g_1 \Delta_1^{-2}) \\ & \times {}_1F_1(\tfrac{1}{2}(q - rk(X)); \tfrac{1}{2}(n - rk(Z) + r_1 + s_1); \tfrac{1}{2} g_1^\tau \Lambda_1^{-2} g_1 \Delta_1^{-2}), \\ & g_1 \in R^{r_1 \times s_1}; \end{aligned}$$

this, (19) and Lemma 5 indicate that the density function of $\hat{F}_1$ is

$$\begin{aligned} p_{\hat{F}_1}(f_1) = {} & (2\pi)^{-r_1 s_1/2} \frac{\Gamma_{r_1}[(n - rk(Z) + r_1)/2]}{\Gamma_{r_1}[(n - rk(Z) + r_1 + s_1)/2]} \\ & \times \frac{\Gamma_{r_1}[(n - rk(Z) + r_1 + s_1 - q + rk(X))/2]}{\Gamma_{r_1}[(n - rk(Z) + r_1 - q + rk(X))/2]} \\ & \times |\Lambda_1|^{-s_1} |\Delta_1|^{-r_1} etr[-\tfrac{1}{2}(f_1 - F_1)^\tau P_1 \Lambda_1^{-2} P_1^\tau (f_1 - F_1) Q_2 \Delta_1^{-2} Q_2^\tau] \\ & \times {}_1F_1(\tfrac{1}{2}(q - rk(X)); \tfrac{1}{2}(n - rk(Z) + r_1 + s_1); \\ & \tfrac{1}{2}(f_1 - F_1)^\tau P_1 \Lambda_1^{-2} P_1^\tau (f_1 - F_1) Q_2 \Delta_1^{-2} Q_2^\tau), \\ & R(f_1 - F_1) \subset R(P_1), \quad R((f_1 - F_1)^\tau) \subset R(Q_2). \end{aligned} \tag{28}$$

It follows from (10), (12) and (18) that

$$K_1(X^\tau \Sigma^{-1} X)^+ K_1^\tau = P_1 \Lambda_1^2 P_1^\tau, \quad L_1(Z_1^\tau Q_{Z_2} Z_1)^+ L_1^\tau = Q_2 \Delta_1^2 Q_2^\tau,$$

hence the diagonal elements of Λ_1^2 and Δ_1^2 are, respectively, the nonnull eigenvalues of $K_1(X^\tau \Sigma^{-1} X)^+ K_1^\tau$ and $L_1(Z_1^\tau Q_{Z_2} Z_1)^+ L_1^\tau$, and

$$[K_1(X^\tau \Sigma^{-1} X)^+ K_1^\tau]^+ = P_1 \Lambda_1^{-2} P_1^\tau, \quad [L_1(Z_1^\tau Q_{Z_2} Z_1)^+ L_1^\tau]^+ = Q_2 \Delta_1^{-2} Q_2^\tau. \tag{29}$$

From (10), (12) and (18) we have

$$\begin{aligned} R(P_1) &= R(C_1 X M \Lambda^{-1}) = R(C_1 X M) = R(C_1 X M M^\tau X^\tau C_1^\tau) \\ &= R(C_1 X X^\tau C_1^\tau) = R(C_1 X) = R(K_1). \end{aligned}$$

Therefore, from definitions of r_1 and P_1 (see (18)), we obtain

$$r_1 = rk(P_1) = dim(R(P_1)) = dim(R(K_1)) = rk(K_1), \tag{30}$$

where $dim(R)$ denotes the dimension of R. It follows from $P_{Q_{Z_2} Z_1} Q_{Z_2} Z_1 = Q_{Z_2} Z_1$ that $R(D_1 Q_{Z_2} Z_1) = R(D_1 P_{Q_{Z_2} Z_1})$, thus from (10) and (18), we have

$$\begin{aligned} R(Q_2) &= R(D_1 \tilde{Q}_2) = R(D_1 P_{Q_{Z_2} Z_1} D_1^\tau) = R(D_1 P_{Q_{Z_2} Z_1}) \\ &= R(D_1 Q_{Z_2} Z_1) = R(L_1); \end{aligned}$$

this and the definition of s_1 and Q_2 (see (18)) indicate that

$$s_1 = rk(Q_2) = dim(R(Q_2)) = dim(R(L_1)) = rk(L_1). \tag{31}$$

Substituting (29)–(31) into (28) leads to (9). □

Remark 3. For the GMANOVA model (corresponding to the case of $Z_2 = 0$ in model (1)), when X and Z_1 are full rank in column, K_1 and L_1 are full rank in row, it follows from Theorem 3 that

$$\begin{aligned} p_{\hat{F}_1}(f_1) = {} & (2\pi)^{-r(K_1)r(L_1)/2} \frac{\Gamma_{r(K_1)}[(n-m+r(K_1))/2]}{\Gamma_{r(K_1)}[(n-m+r(K_1)+r(L_1))/2]} \\ & \times \frac{\Gamma_{r(K_1)}[(n-m+r(K_1)+r(L_1)-q+p)/2]}{\Gamma_{r(K_1)}[(n-m+r(K_1)-q+p)/2]} |K_1(X^\tau\Sigma^{-1}X)^{-1}K_1^\tau|^{-r(L_1)/2} \\ & \times |L_1(Z_1^\tau Z_1)^{-1}L_1^\tau|^{-r(K_1)/2} etr\{-\tfrac{1}{2}(f_1-F_1)^\tau[K_1(X^\tau\Sigma^{-1}X)^{-1}K_1^\tau]^{-1}(f_1-F_1) \\ & \times [L_1(Z_1^\tau Z_1)^{-1}L_1^\tau]^{-1}\}\,{}_1F_1(\tfrac{1}{2}(q-p); \tfrac{1}{2}(n-m+r(K_1)+r(L_1)); \\ & \tfrac{1}{2}(f_1-F_1)^\tau[K_1(X^\tau\Sigma^{-1}X)^{-1}K_1^\tau]^{-1}(f_1-F_1)[L_1(Z_1^\tau Z_1)^{-1}L_1^\tau]^{-1}), \\ & f_1 \in R^{r(K_1)\times r(L_1)}, \end{aligned}$$

where $r(A)$ denotes the number of rows of matrix A. This is the result obtained by Kenward [4] but with different notations.

Theorem 4. *Follow the notations in Theorem 3, the characteristic function of $\hat{F}_1$ is given by*

$$\begin{aligned} \varphi_{\hat{F}_1}(t) = {} & etr[it^\tau F_1 - \tfrac{1}{2}t^\tau K_1(X^\tau\Sigma^{-1}X)^+K_1^\tau t L_1(Z_1^\tau Q_{Z_2}Z_1)^+L_1^\tau] \\ & \times |\tfrac{1}{2}\Lambda_1(t,\Sigma)|^{(n-rk(Z)-q+rk(X)+r_1(t,\Sigma))/2} \\ & \times \frac{\Gamma_{rk(K_1)}((n-rk(Z)+rk(K_1))/2)}{\Gamma_{rk(K_1)}((n-rk(Z)-q+rk(X)+r_1(t,\Sigma))/2)} \Psi(\tfrac{1}{2}(n-rk(Z)+rk(K_1)), \\ & \tfrac{1}{2}(n-rk(Z)-q+rk(X)+rk(K_1)+r_1(t,\Sigma)+1); \tfrac{1}{2}\Lambda_1(t,\Sigma)), \\ & t \in R^{r(K_1)\times r(L_1)}, \end{aligned} \tag{32}$$

where $r_1(t,\Sigma) = rk(t^\tau K_1(X^\tau\Sigma^{-1}X)^+K_1^\tau t L_1(Z_1^\tau Q_{Z_2}Z_1)^+L_1^\tau)$, $\Lambda_1(t,\Sigma)$ is a $r_1(t,\Sigma)\times r_1(t,\Sigma)$ diagonal matrix with the nonnull eigenvalues of $t^\tau K_1(X^\tau\Sigma^{-1}X)^+K_1^\tau t L_1(Z_1^\tau \cdot Q_{Z_2}Z_1)^+L_1^\tau$ as its diagonal elements.

Proof. It follows from (19) and (24) that

$$\hat{F}_1|\tilde{V} \sim N_{r(K_1)\times r(L_1)}(F_1, (Q_2\Delta_1^2Q_2^\tau)\otimes[P_1\Lambda_1(I_{r_1}+\tilde{V}\tilde{V}^\tau)\Lambda_1P_1^\tau]), \tag{33}$$

which indicates that the characteristics function of $\hat{F}_1$ is

$$\begin{aligned} \varphi_{\hat{F}_1}(t) &= E[etr(it^\tau\hat{F}_1)] = E\{E[etr(it^\tau\hat{F}_1)|\tilde{V}]\} \\ &= etr(it^\tau F_1 - \tfrac{1}{2}t^\tau P_1\Lambda_1^2P_1^\tau t Q_2\Delta_1^2Q_2^\tau) \\ &\quad \times E[etr(-\tfrac{1}{2}\tilde{V}^\tau\Lambda_1P_1^\tau tQ_2\Delta_1^2Q_2^\tau t^\tau P_1\Lambda_1\tilde{V})], \quad t \in R^{r(K_1)\times r(L_1)}. \end{aligned} \tag{34}$$

From (22) and Lemma 3, there exits a random matrix $\tilde{T}$ such that

$$\begin{cases} \tilde{V}|\tilde{T} \sim N_{r_1\times(q-rk(X))}(0, I_{q-rk(X)}\otimes\tilde{T}^{-1}), \\ \tilde{T} \sim W_{r_1}(n-rk(Z)-q+rk(X)+r_1, I_{r_1}). \end{cases} \tag{35}$$

Therefore, we have

$$\begin{aligned}&E[etr(-\tfrac{1}{2}\tilde{V}^\tau\Lambda_1 P_1^\tau t Q_2\Delta_1^2 Q_2^\tau t^\tau P_1\Lambda_1\tilde{V})]\\&\quad = E\{E[etr(-\tfrac{1}{2}\tilde{V}^\tau\Lambda_1 P_1^\tau t Q_2\Delta_1^2 Q_2^\tau t^\tau P_1\Lambda_1\tilde{V})|\tilde{T}]\}\\&\quad = E[|I_{r_1}+\tilde{T}^{-1}\Lambda_1 P_1^\tau t Q_2\Delta_1^2 Q_2^\tau t^\tau P_1\Lambda_1|^{-(q-rk(X))/2}],\\&\qquad t\in R^{r(K_1)\times r(L_1)}.\end{aligned}\tag{36}$$

It follows from (34)–(36) and Lemma 4 that (32) holds. □

3.2. The distribution of $K_2\hat{B}_2L_2^\tau$

Theorem 5. *For model* (1), *if* $\hat{F}_2\hat{=}K_2\hat{B}_2L_2^\tau$ *is unique*, *then*:
(i) *When* $R(K_2^\tau)\subset R(Q_X)$ *and* $R(L_2^\tau)\subset R(Z_2^\tau)$, *we have*

$$\hat{F}_2\sim N_{r(K_2)\times r(L_2)}(F_2,[L_2(Z_2^\tau Z_2)^+L_2^\tau]\otimes(K_2\Sigma K_2^\tau)),\tag{37}$$

where $F_2=K_2B_2L_2^\tau$.
(ii) *When* $R(L_2^\tau)\subset R(Z_2^\tau Q_{Z_1})$, *the characteristic function of* $\hat{F}_2$ *is*

$$\begin{aligned}\varphi_{\hat{F}_2}(t)=\;&etr\{it^\tau F_2-\tfrac{1}{2}t^\tau K_2\Sigma K_2^\tau tL_2(Z_2^\tau Z_2)^+L_2^\tau\\&-\tfrac{1}{2}t^\tau K_2X(X^\tau\Sigma^{-1}X)^+X^\tau K_2^\tau tL_2[(Z_2^\tau Q_{Z_1}Z_2)^+-(Z_2^\tau Z_2)^+]L_2^\tau\}\\&\times|\tfrac{1}{2}\Lambda_2(t,\Sigma)|^{(n-rk(Z)-q+rk(X)+r_2(t,\Sigma))/2}\\&\times\frac{\Gamma_{rk(K_2X)}((n-rk(Z)+rk(K_2X))/2)}{\Gamma_{rk(K_2X)}((n-rk(Z)-q+rk(X)+r_2(t,\Sigma))/2)}\\&\times\Psi(\tfrac{1}{2}(n-rk(Z)+rk(K_2X)),\\&\tfrac{1}{2}(n-rk(Z)-q+rk(X)+rk(K_2X)+r_2(t,\Sigma)+1);\tfrac{1}{2}\Lambda_2(t,\Sigma)),\\&t\in R^{r(K_2)\times r(L_2)},\end{aligned}\tag{38}$$

where $r_2(t,\Sigma)=rk(t^\tau K_2X(X^\tau\Sigma^{-1}X)^+X^\tau K_2^\tau tL_2[(Z_2^\tau Q_{Z_1}Z_2)^+-(Z_2^\tau Z_2)^+]L_2^\tau)$, $\Lambda_2\,(t,\Sigma)$ *is a* $r_2(t,\Sigma)\times r_2(t,\Sigma)$ *diagonal matrix with the nonnull eigenvalues of* $t^\tau K_2X(X^\tau\cdot\Sigma^{-1}X)^+X^\tau K_2^\tau tL_2$ $[(Z_2^\tau Q_{Z_1}Z_2)^+-(Z_2^\tau Z_2)^+]L_2^\tau$ *as its diagonal elements.*

Proof. (i) When $R(K_2^\tau)\subset R(Q_X)$ and $R(L_2^\tau)\subset R(Z_2^\tau)$, there exist C_2 and D_2 such that

$$K_2^\tau=Q_XC_2^\tau,\quad L_2^\tau=Z_2^\tau D_2^\tau,$$

hence from Lemma 1 and (1), we have

$$\hat{F}_2=F_2+K_2\mathcal{E}P_{Z_2}D_2^\tau,$$

this and (1) imply that

$$\hat{F}_2\sim N_{r(K_2)\times r(L_2)}(F_2,(D_2P_{Z_2}D_2^\tau)\otimes(K_2\Sigma K_2^\tau)).$$

Thus (37) holds.
(ii) It follows from Lemma 1 that

$$\hat{F}_2=K_2YZ_2(Z_2^\tau Z_2)^-L_2^\tau-K_2X\hat{B}_1Z_1^\tau Z_2(Z_2^\tau Z_2)^-L_2^\tau\hat{=}\tilde{F}_1-\tilde{F}_2.\tag{39}$$

When $R(L_2^\tau) \subset R(Z_2^\tau Q_{Z_1})$, there exits a D_3 such that

$$L_2^\tau = Z_2^\tau Q_{Z_1} D_3^\tau. \tag{40}$$

From (1), (14)–(16), we have

$$\begin{cases} \tilde{F}_1 = K_2 X B_1 Z_1^\tau Z_2 (Z_2^\tau Z_2)^- L_2^\tau + F_2 + K_2 \Sigma^{1/2} \tilde{P} \mathcal{E}_1 \tilde{Q}_1^\tau Q_{Z_1} D_3^\tau, \\ \tilde{F}_2 = K_2 X B_1 Z_1^\tau Z_2 (Z_2^\tau Z_2)^- L_2^\tau - K_2 \Sigma^{1/2} \tilde{P}_1 (I_{rk(X)}, -V) \mathcal{E}_2 \tilde{Q}_2^\tau Q_{Z_1} D_3^\tau, \end{cases} \tag{41}$$

which shows that $\tilde{F}_1$ and $\tilde{F}_2$ are independent (since $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_3$ are independent, and V is a function of $\mathcal{E}_3$), hence from (39) we have

$$\varphi_{\hat{F}_2}(t) = \varphi_{\tilde{F}_1}(t) \varphi_{\tilde{F}_2}(-t), \quad t \in R^{r(K_2) \times r(L_2)}. \tag{42}$$

It follows from (15), (20) and (41) that

$$\tilde{F}_1 \sim N_{r(K_2) \times r(L_2)}(K_2 X B_1 Z_1^\tau Z_2 (Z_2^\tau Z_2)^+ L_2^\tau + F_2, [L_2 (Z_2^\tau Z_2)^+ L_2^\tau] \otimes (K_2 \Sigma K_2^\tau)), \tag{43}$$

which means that

$$\begin{aligned} \varphi_{\tilde{F}_1}(t) = etr\{& it^\tau [K_2 X B_1 Z_1^\tau Z_2 (Z_2^\tau Z_2)^+ L_2^\tau + F_2] \\ & - \tfrac{1}{2} t^\tau K_2 \Sigma K_2^\tau t L_2 (Z_2^\tau Z_2)^+ L_2^\tau\}, \quad t \in R^{r(K_2) \times r(L_2)}. \end{aligned} \tag{44}$$

Further, from (39) and (40) we have

$$\tilde{F}_2 = -K_2 X \hat{B}_1 Z_1^\tau Q_{Z_2} Q_{Z_1} D_3^\tau.$$

Thus from Theorem 4 we have (by taking $K_1 = -K_2 X$ and $L_1 = D_3 Q_{Z_1} Q_{Z_2} Z_1$ in Theorem 4)

$$\begin{aligned} \varphi_{\tilde{F}_2}(t) = etr\{& it^\tau K_2 X B_1 Z_1^\tau Z_2 (Z_2^\tau Z_2)^+ L_2^\tau - \tfrac{1}{2} t^\tau K_2 X (X^\tau \Sigma^{-1} X)^+ X^\tau K_2^\tau t \\ & \times L_2 [(Z_2^\tau Q_{Z_1} Z_2)^+ - (Z_2^\tau Z_2)^+] L_2^\tau\} |\tfrac{1}{2} \Lambda_2(t, \Sigma)|^{(n - rk(Z) - q + rk(X) + r_2(t, \Sigma))/2} \\ & \times \frac{\Gamma_{rk(K_2 X)}((n - rk(Z) + rk(K_2 X))/2)}{\Gamma_{rk(K_2 X)}((n - rk(Z) - q + rk(X) + r_2(t, \Sigma))/2)} \\ & \times \Psi(\tfrac{1}{2}(n - rk(Z) + rk(K_2 X)), \\ & \tfrac{1}{2}(n - rk(Z) - q + rk(X) + rk(K_2 X) + r_2(t, \Sigma) + 1); \tfrac{1}{2} \Lambda_2(t, \Sigma)), \\ & t \in R^{r(K_2) \times r(L_2)}; \end{aligned}$$

this, (42) and (44) show that (38) holds. $\square$

3.3. *The distribution of* $K_1 \hat{B}_1 L_1^\tau + K_2 \hat{B}_2 L_2^\tau$

Theorem 6. *For model* (1), *let* $K_i, L_i, i = 1, 2$ *satisfy the conditions in Theorem* 2, *then the characteristic function of* $\hat{F}_3 \hat{=} K_1 \hat{B}_1 L_1^\tau + K_2 \hat{B}_2 L_2^\tau$ *is*

$$\begin{aligned} \varphi_{\hat{F}_3}(t) = etr\{& it^\tau F_3 - \tfrac{1}{2} t^\tau K_2 \Sigma K_2^\tau t L_2 (Z_2^\tau Z_2)^+ L_2^\tau - \tfrac{1}{2} t^\tau K_2 X (X^\tau \Sigma^{-1} X)^+ X^\tau K_2^\tau t \\ & \times [(dL_1, L_2)(Z^\tau Z)^+ (dL_1, L_2)^\tau - L_2 (Z_2^\tau Z_2)^+ L_2^\tau]\} \\ & \times |\tfrac{1}{2} \Lambda_3(t, \Sigma)|^{(n - rk(Z) - q + rk(X) + r_3(t, \Sigma))/2} \\ & \times \frac{\Gamma_{rk(K_2 X)}((n - rk(Z) + rk(K_2 X))/2)}{\Gamma_{rk(K_2 X)}((n - rk(Z) - q + rk(X) + r_3(t, \Sigma))/2)} \end{aligned}$$

$$\times\Psi(\tfrac{1}{2}(n-rk(Z)+rk(K_2X)),$$
$$\tfrac{1}{2}(n-rk(Z)-q+rk(X)+rk(K_2X)+r_3(t,\Sigma)+1);\ \tfrac{1}{2}\Lambda_3(t,\Sigma)),$$
$$t\in R^{r(K_1)\times r(L_1)}, \tag{45}$$

where $F_3=K_1B_1L_1^\tau+K_2B_2L_2^\tau$, *d is a nonnull value determined by* $K_1=dK_2X$, $r_3(t,\Sigma)=rk(t^\tau K_2X(X^\tau\Sigma^{-1}X)^+X^\tau K_2^\tau t[(dL_1,L_2)(Z^\tau Z)^+(dL_1,L_2)^\tau-L_2(Z_2^\tau Z_2)^+\cdot L_2^\tau])$, $\Lambda_3(t,\Sigma)$ *is a* $r_3(t,\Sigma)\times r_3(t,\Sigma)$ *diagonal matrix with the nonnull eigenvalues of* $t^\tau K_2X(X^\tau\Sigma^{-1}X)^+X^\tau K_2^\tau t$ $[(dL_1,L_2)(Z^\tau Z)^+(dL_1,L_2)^\tau-L_2(Z_2^\tau Z_2)^+L_2^\tau]$ *as its diagonal elements.*

Proof. When $K_i, L_i, i=1,2$ satisfy the conditions in Theorem 2, there exists a nonnull value d and D_4 such that

$$K_1=dK_2X,\quad dL_1^\tau=Z_1^\tau D_4^\tau,\quad L_2^\tau=Z_2^\tau D_4^\tau, \tag{46}$$

hence from Lemma 1, we have

$$\hat{F}_3=K_2YP_{Z_2}D_4^\tau+K_2X\hat{B}_1Z_1^\tau Q_{Z_2}D_4^\tau\hat{=}\tilde{F}_3+\tilde{F}_4. \tag{47}$$

Substituting (1), (14)–(16) into the above equality leads to

$$\begin{cases}\tilde{F}_3=K_2XB_1Z_1^\tau P_{Z_2}D_4^\tau+K_2B_2L_2^\tau+K_2\Sigma^{1/2}\tilde{P}\mathcal{E}_1\tilde{Q}_1^\tau D_4^\tau,\\ \tilde{F}_4=K_2XB_1Z_1^\tau Q_{Z_2}D_4^\tau+K_2\Sigma^{1/2}\tilde{P}_1(I_{rk(X)},-V)\mathcal{E}_2\tilde{Q}_2^\tau D_4^\tau,\end{cases} \tag{48}$$

which shows that $\tilde{F}_3$ and $\tilde{F}_4$ are independent (since $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ are independent and V is a function of $\mathcal{E}_3$). Thus from (47) we have

$$\varphi_{\hat{F}_3}(t)=\varphi_{\tilde{F}_3}(t)\varphi_{\tilde{F}_4}(t),\quad t\in R^{r(K_1)\times r(L_1)}. \tag{49}$$

It follows from (1) and (47) that

$$\tilde{F}_3\sim N_{r(K_1)\times r(L_1)}(K_2XB_1Z_1^\tau P_{Z_2}D_4^\tau+K_2B_2L_2^\tau,[L_2(Z_2^\tau Z_2)^+L_2^\tau]\otimes(K_2\Sigma K_2^\tau)), \tag{50}$$

which indicates that

$$\begin{aligned}\varphi_{\tilde{F}_3}(t)=&\,etr\{it^\tau(K_2XB_1Z_1^\tau P_{Z_2}D_4^\tau+K_2B_2L_2^\tau)\\&-\tfrac{1}{2}t^\tau K_2\Sigma K_2^\tau tL_2(Z_2^\tau Z_2)^+L_2^\tau\},\quad t\in R^{r(K_1)\times r(L_1)}.\end{aligned} \tag{51}$$

From (47) and Theorem 4, the characteristic function of $\tilde{F}_4$ is (by taking $K_1=K_2X$ and $L_1=D_4Q_{Z_2}Z_1$ in Theorem 4)

$$\begin{aligned}\varphi_{\tilde{F}_4}(t)=&\,etr\{it^\tau K_2XB_1Z_1^\tau Q_{Z_2}D_4^\tau-\tfrac{1}{2}t^\tau K_2X(X^\tau\Sigma^{-1}X)^+X^\tau K_2^\tau t\\&\times[(dL_1,L_2)(Z^\tau Z)^+(dL_1,L_2)^\tau-L_2(Z_2^\tau Z_2)^+L_2^\tau]\}\\&\times|\tfrac{1}{2}\Lambda_3(t,\Sigma)|^{(n-rk(Z)-q+rk(X)+r_3(t,\Sigma))/2}\\&\times\frac{\Gamma_{rk(K_2X)}((n-rk(Z)+rk(K_2X))/2)}{\Gamma_{rk(K_2X)}((n-rk(Z)-q+rk(X)+r_3(t,\Sigma))/2)}\\&\times\Psi(\tfrac{1}{2}(n-rk(Z)+rk(K_2X)),\\&\tfrac{1}{2}(n-rk(Z)-q+rk(X)+rk(K_2X)+r_3(t,\Sigma)+1);\ \tfrac{1}{2}\Lambda_3(t,\Sigma)),\\&t\in R^{r(K_1)\times r(L_1)}.\end{aligned} \tag{52}$$

Therefore, from (49), (51) and (52) we know that (45) holds. □

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