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Determinants, permanents and some applications to statistical shape theory

Francisco J. Caro-Lopera [a], Graciela González-Farías [b,*], N. Balakrishnan [c]

[a] Departamento de Ciencias Básicas, Universidad de Medellín, Medellín, Colombia
[b] Centro de Investigación en Matemáticas, Monterrey, Mexico
[c] Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada

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ABSTRACT

A formula for the determinant of a matrix in terms of powers of traces is presented. Then, some expansions for powers of determinants of positive definite matrices in terms of zonal polynomials are derived. By making use of these, the associated elliptical families of matrix-variate distributions are obtained and applied in the framework of statistical shape theory, through the determination of the central non-isotropic configuration density. Finally, a relationship between the determinant and the permanent of a matrix is obtained.

1. Introduction

Determinants and their properties have been discussed quite extensively in the literature. Gottfried Wilhelm von Leibniz set the basis for their study around 1683, but some historians place the origin of the concept in ancient China and Japan, through its application in systems of linear equations. Since then, many works discussing its computation, properties and applications to many fields have been published.

Permanents are newer relative to determinants, and Augustin Louis Cauchy was the first to discuss the subject in 1812. This laid the foundation for an interesting field of study, with plenty of open problems still remaining with regard to its connection to determinants and its computation and properties. At present, no formula exists that expresses the permanent in terms of the determinant. The fact that the permanent does not share any classical algorithm with the determinant shows that their connection is more complex than it would appear superficially. Some modern aspects and applications of permanents are detailed in the elaborate review article by Balakrishnan [1] and the references contained therein.

The formal definition of the determinant of a $k \times k$ matrix $\mathbf{A} = (a_{i,j})$ is given by

$$|\mathbf{A}| = \sum_{\sigma \in S_k} sgn(\sigma) \prod_{i=1}^{k} a_{i,\sigma(i)}, \tag{1}$$

where the summation runs over all the permutations σ of $1, 2, \dots, k$. Here, $sgn(\sigma)$ denotes the sign of σ.

* Corresponding author.
E-mail address: gmfarias@gmail.com (G. González-Farías).

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On the other hand, the definition of the permanent of a $k \times k$ matrix $\mathbf{A} = (a_{i,j})$ is obtained by removing the signs in the definition of the determinant, and is simply given by

$$[\mathbf{A}] = \sum_{\sigma \in S_k} \prod_{i=1}^{k} a_{i,\sigma(i)}, \tag{2}$$

where the summation once again runs over all the permutations σ of $1, 2, \dots, k$.

Both definitions seem easy from the computational viewpoint, but the operations with subindices make their computation on higher dimensions a rather hard task. In fact, an explicit formula avoiding permutations, elementary operations, minors and factorizations, seems to be unavailable until now.

These days, modern computing technology can use existing definitions and algorithms to compute $|\mathbf{A}|$ and $[\mathbf{A}]$ quickly and efficiently. However, from a theoretical viewpoint, it is of interest to propose representations for $|\mathbf{A}|$ and $[\mathbf{A}]$ in terms of the matrix $\mathbf{A}$, instead of the usual representation in terms of the elements $a_{i,j}$ or submatrices. So, some functions f_d and f_p such that

$$|\mathbf{A}| = f_d(\mathbf{A})$$

and

$$[\mathbf{A}] = f_g(\mathbf{A}),$$

need to be developed, instead of the classical approach of finding functions g_d and g_p such that $|\mathbf{A}| = g_d(a_{ij})$ and $[\mathbf{A}] = g_p(a_{ij})$. Such representations would enable matrix transformations and facilitate developments that may not be possible if an element-wise representation was used instead. This is analogous to developments in matrix-variate distribution theory: the representation of the matrix-variate normal distribution in terms of the original random matrix, enabling the development of new results on Wishart, linear models, matrix-transformations, theory of integration, and Jacobians, would not be possible through work based on kernels expressed in terms of the elements of the random matrix.

From a different point of view, matrix polynomials such as determinants and traces play important roles in elliptical random matrix theory. However, most of the published work involves generator functions in terms of traces. A few distributions try to mix traces and determinants in the corresponding density, but a general theory of elliptical distributions indexed by kernels depending on determinants is unavailable until now. Such applications are clearly important, for example, in the statistical theory of shape.

In this paper, some solutions are provided to the problems described above. In Section 2, a new expansion for the determinant as a function of powers of traces, and a formula for integer powers of a determinant are presented. Both expressions are simplified for the case of positive definite matrices, and are expanded in terms of zonal polynomials. Section 3 uses these formulas to establish a connection between the determinant and the permanent. Section 4 applies the formulas for powers of determinants to random matrix theory based on elliptical distributions with kernels that depend on determinants. Some new integrals, similar to those given in the case of traces by Herz [12], are also derived. Finally, an application to statistical shape theory is discussed by deriving the central non-isotropic configuration density in terms of zonal polynomials.

2. The determinant

This section provides a function f_d for the determinant of a $k \times k$ matrix $\mathbf{A}$, such that

$$|\mathbf{A}| = f_d(\mathrm{tr}\,\mathbf{A}, k).$$

With regard to the problem described earlier in Section 1, some approaches involving order partitions and arrays have been attempted; see, for example [9]. However, this could also be achieved by the classical theory of symmetric functions; see, for example, [16].

For presenting the main result of this section, let $\kappa = (k^{\nu_k}, (k-1)^{\nu_{k-1}}, \dots, 2^{\nu_2}, 1^{\nu_1})$ be an ordered partition of the natural number k consisting of ν_1 ones, ν_2 twos, and so and on, which will be written as $\kappa = (p_k^{\nu_k}, p_{k-1}^{\nu_{k-1}}, \dots, p_2^{\nu_2}, p_1^{\nu_1})$, $p_j = j$, $j = 1, 2, \dots, k$. We should also write κ in the conventional way as usual, i.e., $\kappa = (k_1, k_2, \dots, k_\nu)$, where $k_1 \geq k_2 \geq \cdots \geq k_\nu \geq 1$ and $\nu = \sum_{i=1}^{k} \nu_i$ is the number of parts of κ. Note that $\sum_{i=1}^{k} p_i \nu_i = \sum_{i=1}^{\nu} k_i = k$. Recall that the partitions $\tau = (t_1, t_2, \dots)$ and $\lambda = (l_1, l_2, \dots)$ of k are lexicographically ordered (denoted by $\tau > \lambda$) if $t_j > l_j$ for the first index j for which the parts are unequal. Then, we have the following theorem.

Theorem 1. *The determinant of a $k \times k$ matrix $\mathbf{A}$ is given by*

$$|\mathbf{A}| = \sum_{\kappa} \frac{(-1)^{k+\nu} \prod_{j=1}^{k} (j-1)!^{\nu_j}}{\prod_{i=1}^{k} k_i! \nu_i!} \prod_{i=1}^{k} (\mathrm{tr}\,\mathbf{A}^{p_i})^{\nu_i}, \tag{3}$$

where the summation runs over all ordered partitions κ of the natural number k, denoted jointly by $\kappa = (p_k^{\nu_k}, p_{k-1}^{\nu_{k-1}}, \dots, p_2^{\nu_2}, p_1^{\nu_1})$ and $\kappa = (k_1, k_2, \dots, k_\nu)$, with $\nu = \sum_{i=1}^{k} \nu_i$ and $p_j = j, j = 1, 2, \dots, k$.

Proof. Here, we present a proof based on symmetric function theory, which is a shorter approach than the one used by González-Farías and Caro-Lopera [9] through C-arrays.

The determinant $|\mathbf{A}|$ can be considered as the skew symmetrization of the lower monomial symmetric function $M_{1^k}(\mathbf{A})$; see, for example, [16, p. 40]. But, $k!M_{1^k}$ is the augmented symmetric function $[1^k]$ defined by David and Kendall [3] and it can be expressed in terms of products of power sums (see [4]), which, after some simplification, is the expansion presented in (3). □

Next, we present the expressions of $|\mathbf{A}|$ for a few cases, and for simplicity, we denote $\prod_{i=1}^{k}(\text{tr } \mathbf{A}^{p_i})^{\nu_i}$ by $(k^{\nu_k}, (k-1)^{\nu_{k-1}}, \dots, 2^{\nu_2}, 1^{\nu_1})$, and avoid the fractions by listing $k!|\mathbf{A}|$:

$$1!|\mathbf{A}| = (1),$$
$$2!|\mathbf{A}| = -(2) + (1^2),$$
$$3!|\mathbf{A}| = 2(3) - 3(21) + (1^3),$$
$$4!|\mathbf{A}| = -6(4) + 8(31) + 3(2^2) - 6(21^2) + (1^4).$$

Now, the expression (3) is valid for any matrix, and it gives a direct expansion in terms of the elements of the matrix. However, for some special matrices, the expression becomes much simpler. For example, consider a positive definite $k \times k$ matrix $\mathbf{A}$, then, as we shall see, $|\mathbf{A}|$ can be expressed in terms of the well-known zonal polynomials $C_\kappa(\mathbf{A})$; see [13]. In this case, the relation is quite simple through the use of a normalization constant to express $M_{1^k}(\mathbf{A})$ in terms of $C_{1^k}(\mathbf{A})$.

Explicitly (see, for example Muirhead (1982)), let $\mathbf{Y}$ be an $m \times m$ symmetric matrix with latent roots $y_1, \dots, y_m$, and let $\kappa = (k_1, \dots, k_m)$ be a partition of k into no more than m parts. So, the zonal polynomial $C_\kappa(Y)$ is a symmetric and homogeneous polynomial of degree k in the latent roots of Y, such that the term of the highest weight in $C_\kappa(Y)$ is $y_1^{k_1} \cdots y_m^{k_m}$, which means that for some constant d_κ (depending on κ)

$$C_\kappa(Y) = d_\kappa(y_1^{k_1} \cdots y_m^{k_m} + \text{symmetric terms}) + \text{terms of lower weight.} \tag{4}$$

In terms of monomial symmetric functions, we have

$$C_\kappa(\mathbf{Y}) = d_\kappa M_\kappa(\mathbf{Y}) + \sum_{\lambda<\kappa} d_\lambda M_\lambda(\mathbf{Y}),$$

where the summation runs over all the lexicographical ordered partitions $\lambda < \kappa$. The coefficient of the highest weight d_κ can be computed for non-degenerate zonal polynomials ($\mathbf{Y} > 0$) in a closed-form given by James [14, Eqs. (5.11), (5.14)]. Then, the remaining coefficients $d'_\lambda s$ of lower partitions can be computed in a recursive way [14].

For the lower ordered partition 1^k of k, the above expression reduces to

$$C_{1^k}(\mathbf{Y}) = d_{1^k}(y_1^1 \cdots y_m^1 + \text{symmetric terms}),$$

where $y_1, \dots, y_m$ are the latent roots of $\mathbf{Y}$. In the special case of a $k \times k$ positive definite matrix $\mathbf{Y}$, we just take $m = k$ and $\mathbf{Y} = \mathbf{A}$, with $y_1, y_2, \dots, y_k$ being the latent roots. Then, the monomial symmetric function $M_{1^k}(\mathbf{A})$ in the parenthesis consists of only one summand, since there does not remain any more possible products of the k latent roots given that the full product is already included, that is,

$$C_{1^k}(\mathbf{A}) = d_{1^k}(y_1^1 \cdots y_k^1),$$

where d_{1^k} is obtained by taking $\kappa = 1^k$ in [14, Eqs. (5.11), (5.14)]. Then, noting that $|\mathbf{A}| = y_1 y_2 \cdots y_k$, we have established the following result.

Corollary 2. *If $\mathbf{A}$ is a $k \times k$ positive definite matrix, then*

$$|\mathbf{A}| = \frac{\prod_{i=1}^{k}(2+k-i)!}{2^k k!^2 \prod_{i=1}^{k-1}\prod_{j=i+1}^{k}(-i+j)} C_{1^k}(\mathbf{A}), \tag{5}$$

where $C_{1^k}(\mathbf{A})$ is the zonal polynomial indexed by the partition 1^k in the k latent roots of $\mathbf{A}$.

For example, for $k = 1, \ldots, 4$, we have the following:

$$k = 1: \ |\mathbf{A}| = C_1(\mathbf{A}),$$
$$k = 2: \ |\mathbf{A}| = \frac{3}{4}C_{1^2}(\mathbf{A}),$$
$$k = 3: \ |\mathbf{A}| = \frac{1}{2}C_{1^3}(\mathbf{A}),$$
$$k = 4: \ |\mathbf{A}| = \frac{5}{16}C_{1^4}(\mathbf{A}).$$

Finally, the expansion of $|\mathbf{A}|^t$, for a positive integer t, is obtained through the multinomial theorem. In this regard, first note that (3) runs over the N_k lexicographically ordered partitions κ of k, say $\kappa_1 > \kappa_2 > \cdots > \kappa_{N_k}$. So, let us denote $x_l = \frac{(-1)^{k+\nu}\prod_{j=1}^{k}(j-1)!^{\nu_j}}{\prod_{i=1}^{k}k_i!\nu_i!}\prod_{i=1}^{k}(\mathrm{tr}\,\mathbf{A}^{p_i})^{\nu_i}$ for the l-th summand of (3), which is associated with the corresponding partition κ_l implicit in $\prod_{i=1}^{k}(\mathrm{tr}\,\mathbf{A}^{p_i})^{\nu_i}$. We then have

$$|\mathbf{A}| = \sum_{\kappa}\frac{(-1)^{k+\nu}\prod_{j=1}^{k}(j-1)!^{\nu_j}}{\prod_{i=1}^{k}k_i!\nu_i!}\prod_{i=1}^{k}(\mathrm{tr}\,\mathbf{A}^{p_i})^{\nu_i} = \sum_{i=1}^{N_k}x_{\kappa_i}.$$

Thus, we obtain the following theorem.

Theorem 3. *For any $k \times k$ matrix $\mathbf{A}$, and a positive integer n, we have*

$$|\mathbf{A}|^n = \sum_{\sum_{i=1}^{N_k} r_i = n}\frac{n!}{\prod_{i=1}^{N_k} r_i!}\prod_{1\le t\le N_k}x_t^{r_t}. \tag{6}$$

A useful special case is when $\mathbf{A} > 0$, in which case we obtain the following simple consequence of Corollary 2.

Corollary 4. *If $\mathbf{A}$ is a $k \times k$ positive definite matrix, then*

$$|\mathbf{A}|^n = \left[\frac{\prod_{i=1}^{k}(2+k-i)!}{2^k k!^2\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}(-i+j)}\right]^n C_{n^k}(\mathbf{A}), \tag{7}$$

where $C_{n^k}(\mathbf{A})$ is the zonal polynomial indexed by the partition n^k of nk in the k latent roots of $\mathbf{A}$.

Proof. If $\mathbf{A}$ is a $k\times k$ positive definite matrix, the n-th power of $C_{1^k}(\mathbf{A})$ is trivial. It is expanded in terms of only one monomial symmetric function ($M_{n^k}(\mathbf{A})$) with the maximum possible number of parts k (i.e., there are no monomial symmetric functions associated with partitions of lower weight, since they exceed the k parts and there are only k latent roots). Moreover, $M_{n^k}(\mathbf{A})$ consists of only one summand $y_1^n\cdots y_k^n$, since there are no other symmetric terms given that the monomial already consists of the full product of the k latent roots. Thus,

$$\begin{aligned}(C_{1^k}(\mathbf{A}))^n &= d_{1^k}^n(y_1^n\cdots y_k^n)\\ &= d_{1^k}^n(y_1^n\cdots y_k^n + 0 \text{ symmetric terms}) + 0 \text{ terms of lower weight}\\ &= C_{n^k}(\mathbf{A}),\end{aligned}$$

or

$$C_{n^k}(\mathbf{A}) = d_{1^k}^n(y_1^n\cdots y_k^n) = d_{1^k}^n(y_1^1\cdots y_k^1)^n = d_{1^k}^n|\mathbf{A}|^n.$$

Strictly, taking $m = nk$ in the definition (4) of the zonal polynomials, we have $\mathbf{Y}$ to be a $nk\times nk$ symmetric matrix of rank k, with the non-null latent roots being $y_1, \ldots, y_k$ and $y_{k+1} = y_{k+2} = \cdots = y_{nk} = 0$. This means that, in the particular case of n^k, any partition $\lambda < n^k$ has more than k parts, since each part of n^k is n and necessarily the next close lower weight partition

$(n^{k-1}, (n-1), 1)$ must increase the number of parts by one and so the remaining lower partitions will have at least $k+1$ parts. Thus, the corresponding monomial symmetric functions $M_\lambda(\mathbf{Y}) = 0$ for all $\lambda < n^k$, and consequently we have

$$\begin{aligned}
C_{n^k}(\mathbf{Y}) &= \sum_{\lambda \leq n^k} d_\lambda M_\lambda(\mathbf{Y}) \\
&= d_{n^k} M_{n^k}(\mathbf{Y}) + \sum_{\lambda < n^k} d_\lambda M_\lambda(\mathbf{Y}) \\
&= d_{n^k} M_{n^k}(\mathbf{Y}) + 0 \\
&= d_{n^k} M_{n^k}(\mathbf{Y}) \\
&= d_{n^k}(y_1^n \cdots y_k^n + 0 \text{ symmetric terms}) \\
&= d_{n^k}(y_1^n \cdots y_k^n) \\
&= d_{n^k}(y_1^1 \cdots y_k^1)^n \\
&= d_{n^k} |\mathbf{A}|^n.
\end{aligned}$$

In this case, d_{n^k} corresponds to a degenerate case given that $\mathbf{Y}$ is not positive definite; then, it cannot be computed using the results of [14] and must be calculated by the results of [5], resulting in

$$d_{n^k} = d_{1^k}^n,$$

as it must be. $\square$

3. The permanent

First, recall that the SM tables in [3] transform the product of power-sums (S) of weight w in terms of the augmented symmetric matrix (M). They can be seen as an inferior triangular matrix, which we shall denote by $(\mathbf{PS}_k) := \mathbf{S}_k$.

In this section, we are interested in the inverse MS tables [3], which can be seen as an upper triangular matrix, denoted by $[\mathbf{PS}_k] := [\mathcal{M}]$ (the MS tables by David and Kendall [3]), and which simply is the transpose of $(\mathbf{PS}_k)^{-1} = (\mathbf{S}_k)'^{-1}$.

Thus, if $\mathbf{S}_k$ is the lower triangular SM matrix, then the corresponding upper triangular MS matrix is given by

$$[\mathbf{M}_k] = \mathbf{S}_k'^{-1}, \tag{8}$$

which means that

$$([k], [k-1\,1], \ldots, [1^k])' = \mathbf{S}_k^{-1}((k), (k-1\,1), \ldots, (1^k))', \tag{9}$$

with both vectors being given in a lexicographical order.

Recall that the last column of $[\mathbf{M}_k]$ gives the expansion of the determinant (see the preceding section).

Now, we establish by induction the following lemma.

Lemma 5. *Let N_k be the number of lexicographical ordered partitions of k. Then, given the $N_k \times N_k$ lower triangular matrix $\mathbf{S}_k$, its inverse, the lower triangular matrix $[\mathbf{M}_k]' = (x_{ij})$ satisfies the property that, for each $i = 2, 3, \ldots, N_k$,*

$$\sum_{j=1}^{N_k} x_{ij} = 0,$$

and trivially,

$$\sum_{j=1}^{N_k} x_{ij} = 1$$

for $i = 1$.

Proof. Recall that $\mathbf{S}_k$ is a lower triangular matrix with diagonal and first column constituted by ones (the remaining elements are non-negative integers). So, $|\mathbf{S}_k| = 1 \neq 0$, and its inverse is a lower triangular matrix with ones in the diagonal and the remaining entries as integers.

The induction is over the N_k rows of $[\mathbf{M}_k]'$.

Then, given a $\mathbf{S}_k$, by the definitions of M's, S's and symmetric functions, and by the Gauss–Jordan process, it is easy to see that the first two rows satisfy the above mentioned property; specifically, if $\mathbf{S}_k = \begin{pmatrix} 1 & 0 & \cdots \\ 1 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$, then $\begin{pmatrix} [k] \\ [k-1,1] \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots \\ -1 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} (k) \\ (k-1)(1) \\ \vdots \end{pmatrix}$. Now by elementary row operations, the augmented matrix $[\mathbf{S}_k\ \mathbf{I}_{N_k}]$ becomes $\begin{bmatrix} \mathbf{A} & \mathbf{a} & \mathbf{X} & \mathbf{y} \\ \mathbf{b} & 1 & \mathbf{z} & 1 \end{bmatrix}$, where $\mathbf{A}$ is the $N_k - 1$ identity matrix, $\mathbf{a}$ and $\mathbf{y}$ are $(N_k - 1) \times 1$ vectors of zeros, $\mathbf{z}$ is a $1 \times (N_k - 1)$ vector of zeros, $\mathbf{b} = (b_1, \ldots, b_{N_k-1})$, where $b_j = s_{N_k,j}$ for $j = 1, 2, \ldots, N_k - 1$, and $\mathbf{X}$ is a $(N_k - 1) \times (N_k - 1)$ lower triangular matrix with ones in the diagonal and the elements below the diagonal are denoted by $x_{i,j}$. Note that the ones in the diagonal of $\mathbf{X}$ follow from the definition of the S and M functions.

Now, in order to reduce $\mathbf{b}$ as a vector of zeros, just multiply the i-th row of the augmented array by $-b_i$, $i = 1, 2, \ldots, N_k - 1$, respectively, and add to the last row in order to get the $N_k \times N_k$ identity matrix on the left hand side of the augmented array and the matrix on the right side is the $N_k \times N_k$ lower triangular $\mathbf{M}_k = (x_{i,j})$, where $x_{i,j} = 0, i < j$; $x_{i,i} = 1, i = 1, \ldots, N_k$; $x_{1,1} = -s_{N_k,1}$; the i-th row $(i = 2, \ldots, N_k - 1)$ is $-s_{N_k,i}(x_{i,1}, x_{i,2}, \ldots, x_{i,i-1}, 1)$; and the last row is $\left(-s_{N_k,1} - \sum_{i=2}^{N_k-1} s_{N_k,i}x_{i,1},\ -s_{N_k,2} - \sum_{i=3}^{N_k-1} s_{N_k,i}x_{i,2}, \ldots, -s_{N_k,N_k-1},\ 1\right)$.

Assume that the first $N_k - 1$ rows of $\mathbf{M}_k$ satisfy the proposition, namely,

$$\sum_{j=1}^{i-1} x_{i,j} + 1 = 0, \quad i = 2, \ldots, N_k - 1 \quad \text{and} \quad \sum_{j=1}^{1} x_{1,j} = 1. \tag{10}$$

Then, by applying this induction hypothesis and recalling that $s_{N_k,1} = 1$, we have

$$\begin{aligned}\sum_{j=1}^{N_k} x_{N_k,j} &= -s_{N_k,1} - \sum_{i=2}^{N_k-1} s_{N_k,i}x_{i,1} - s_{N_k,2} - \sum_{i=3}^{N_k-1} s_{N_k,i}x_{i,2} - \cdots - s_{N_k,N_k-1} + 1 \\ &= -s_{N_k,1} - s_{N_k,2}(x_{2,1} + 1) - s_{N_k,3}(x_{3,1} + x_{3,2} + 1) - \cdots - s_{N_k,N_k-1}\left(\sum_{j=1}^{N_k-2} x_{N_k-1,j} + 1\right) + 1 \\ &= 0,\end{aligned}$$

as required. Thus, the lemma follows for every N_k. □

With these results and properties, the permanent of a matrix can now be studied.

The definition of the permanent of $\mathbf{A}$ in (2) differs from that of the determinant of $\mathbf{A}$ in (1) in that the signs of the permutations are not taken into account. Unlike the determinant, the permanent has no easy geometrical interpretation; it is mainly used in combinatorics and in treating boson Green's functions in quantum field theory. The permanent seems to be more difficult to compute than the determinant. While the determinant can be computed by Gaussian elimination, whose execution time is bound by polynomial expressions, this and other known polynomial time methods cannot be used to calculate the permanent. Note that the definition in (2) can be expressed in terms of the Hadamard product as follows.

Now, starting with the formula (3) for the determinant, as mentioned earlier, the last column of $\mathbf{M}_k$, $((x_{N_k,1} \cdots x_{N_k,N_k})')$, is given by $\frac{(-1)^{k+\nu}\prod_{j=1}^{k}(j-1)!^{\nu_j}}{\prod_{i=1}^{k} k_i!\nu_i!}$, where the row runs over the lexicographical ordered partitions $(k), (k-1, 1), \ldots, (1^k)$; see [4].

Thus, in terms of Lemma 5, we have

$$\sum_{\kappa} \frac{(-1)^{k+\nu}\prod_{j=1}^{k}(j-1)!^{\nu_j}}{\prod_{i=1}^{k} k_i!\nu_i!} = 0.$$

So, if the factor $(-1)^{k+\nu}$ is suppressed in the expansion of the determinant, clearly the permanent plus polynomials $(P(\kappa, \mathbf{A}))$ indexed by other partitions of k are obtained. Only the coefficient of each monomial in the permanent needs to be obtained now. By a suitable modification to the argument of [4], it can be proved that

$$\sum_{\kappa} \frac{\prod_{j=1}^{k}(j-1)!^{\nu_j}}{\prod_{i=1}^{k} k_i!\nu_i!} = 1;$$

recall that the number of terms in the permanent is $k!$, the number of permutations of k. Then, we obtain

$$\begin{aligned}\sum_{\kappa} \frac{\prod_{j=1}^{k}(j-1)!^{\nu_j}}{\prod_{i=1}^{k} k_i!\nu_i!}\prod_{i=1}^{k}(\operatorname{tr}\mathbf{A}^{p_i})^{\nu_i} &= \left(\sum_{\kappa} \frac{\prod_{j=1}^{k}(j-1)!^{\nu_j}}{\prod_{i=1}^{k} k_i!\nu_i!}\right)[\mathbf{A}] + P(\kappa, \mathbf{A}) \\ &= (1)[\mathbf{A}] + P(\kappa, \mathbf{A}),\end{aligned}$$

arriving finally with the following theorem.

Theorem 6. *The permanent* $[\mathbf{A}]$ *of the* $k \times k$ *matrix* $\mathbf{A}$ *is the polynomial constituted by monomials indexed by the lower partition* 1^k *with coefficient* 1 *in the expansion of*

$$\sum_{\kappa} \frac{\prod_{j=1}^{k} (j-1)!^{\nu_j}}{\prod_{i=1}^{k} k_i! \nu_i!} \prod_{i=1}^{k} (\mathrm{tr}\, \mathbf{A}^{p_i})^{\nu_i}, \tag{11}$$

where the summation runs over all ordered partitions κ *of the natural number* k*, denoted jointly by* $\kappa = (p_k^{\nu_k}, p_{k-1}^{\nu_{k-1}}, \ldots, p_2^{\nu_2}, p_1^{\nu_1})$ *and* $\kappa = (k_1, k_2, \ldots, k_\nu)$*, with* $p_j = j, j = 1, 2, \ldots, k$.

Some examples of this result are as follows:

(1) $k = 1$: trivially, $|\mathbf{A}| = |a| = \frac{1}{1!}(1(\mathrm{tr}\,(\mathbf{A}^1))^1) = a$ and $\frac{1}{1!}(1(\mathrm{tr}\,(\mathbf{A}^1))^1) = a$, and therefore $[\mathbf{A}] = a$;

(2) $k = 2$:

$$|\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \frac{1}{2!}(-(\mathrm{tr}\,(\mathbf{A}^2))^1 + (\mathrm{tr}\,(\mathbf{A}))^2) = ad - bc$$

and without negative factors,

$$\frac{1}{2!}((\mathrm{tr}\,(\mathbf{A}^2))^1 + (\mathrm{tr}\,(\mathbf{A}))^2) = (a^2 + d^2) + 1(bc + ad),$$

and therefore

$$[\mathbf{A}] = ad + bc;$$

(3) $k = 3$:

$$\begin{aligned} |\mathbf{A}| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= \frac{1}{3!}\{2(\mathrm{tr}\,(\mathbf{A}^3))^1 - 3(\mathrm{tr}\,(\mathbf{A}^2))^1(\mathrm{tr}\,(\mathbf{A}))^1 + (\mathrm{tr}\,(\mathbf{A}))^3\} \\ &= -ceg + bfg + cdh - afh - bdi + aei, \end{aligned}$$

and removing the negative signs, we get

$$\begin{aligned} &\frac{1}{3!}\{2(\mathrm{tr}\,(\mathbf{A}^3))^1 + 3(\mathrm{tr}\,(\mathbf{A}^2))^1(\mathrm{tr}\,(\mathbf{A}))^1 + (\mathrm{tr}\,(\mathbf{A}))^3\} \\ &\quad = (a^3 + e^3 + i^3) + (a^2e + ae^2 + a^2i + e^2i + ai^2 + ei^2) + 2(abd + bde + acg + efh + cgi + fhi) \\ &\qquad + 1(ceg + bfg + cdh + afh + bdi + aei), \end{aligned}$$

and therefore

$$[\mathbf{A}] = ceg + bfg + cdh + afh + bdi + aei;$$

(4) $k = 4$:

$$\begin{aligned} |\mathbf{A}| = \begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} &= \frac{1}{4!}\{-6\mathrm{tr}\,(\mathbf{A}^4) + 8\mathrm{tr}\,(\mathbf{A}^3)\mathrm{tr}\,(\mathbf{A}) + 3(\mathrm{tr}\,\mathbf{A}^2)^2 - 6\mathrm{tr}\,(\mathbf{A}^2)\mathrm{tr}\,(\mathbf{A})^2 + \mathrm{tr}\,(\mathbf{A})^4\} \\ &= dgjm - chjm - dfkm + bhkm + cflm - bglm - dgin + chin + dekn - ahkn - celn \\ &\quad + agln + dfio - bhio - dejo + ahjo + belo - aflo - cfip + bgip + cejp - agjp - bekp + afkp. \end{aligned}$$

Without negative signs, we have

$$\begin{aligned} &\frac{1}{4!}\{6\mathrm{tr}\,(\mathbf{A}^4) + 8\mathrm{tr}\,(\mathbf{A}^3)\mathrm{tr}\,(\mathbf{A}) + 3(\mathrm{tr}\,\mathbf{A}^2)^2 + 6\mathrm{tr}\,(\mathbf{A}^2)\mathrm{tr}\,(\mathbf{A})^2 + \mathrm{tr}\,(\mathbf{A})^4\} \\ &\quad = (a^4 + f^4 + k^4 + p^4) + (a^3f + af^3 + a^3k + f^3k + ak^3 + fk^3 + a^3p + f^3p + k^3p + ap^3 + fp^3 + kp^3) \\ &\qquad + (b^2e^2 + a^2f^2 + c^2i^2 + g^2j^2 + a^2k^2 + f^2k^2 + d^2m^2 + h^2n^2 + l^2o^2 + a^2p^2 + f^2p^2 + k^2p^2) \\ &\qquad + 3(a^2be + bef^2 + a^2ci + f^2gj + cik^2 + gjk^2 + a^2dm + f^2hn + k^2lo + dmp^2 + hnp^2 + lop^2) \\ &\qquad + (cf^2i + a^2gj + a^2fk + af^2k + +bek^2 + afk^2 + df^2m + dk^2m + a^2hn + hk^2n + a^2lo + f^2lo \\ &\qquad + a^2fp + af^2p + a^2kp + f^2kp + ak^2p + fk^2p + bep^2 + afp^2 + cip^2 + gjp^2 + akp^2 + fkp^2) \\ &\qquad + 4(abef + acik + acik + fgjk + admp + fhnp + klop) + 2(bcei + acfi + abgi + bfgi \\ &\qquad + acej + cefj + acip + aclm + begj + afgj + cgij + abek + befkcfik + cikp + bgik \end{aligned}$$

$$
\begin{aligned}
&+cejk+agjk+bdem+adfm+abhm+bfhm+cdim+adkm+cklm+aden+defn\\
&+behn+afhn+ghjn+fhkn+fgln+gkln+dhmn+adio+fhjo+diko+hjko+cilo\\
&+gjlo+aklo+fklo+dlmo+hlno+abep+befp+fgjp+gjkp+dfmp+bhmp\\
&+dkmp+clmp+denp+ahnp+hknp+glnp+diop+hjop+alop+flop)\\
&+1(dgjm+chjm+dfkm+bhkm+cflm+bglm+dgin+chin+dekn+ahkn+celn\\
&+agln+dfio+bhio+dejo+ahjo+belo+aflo+cfip+bgip+cejp+agjp+bekp+afkp),
\end{aligned}
$$

and therefore

$$
\begin{aligned}
[\mathbf{A}] = {}& dgjm+chjm+dfkm+bhkm+cflm+bglm+dgin+chin+dekn+ahkn+celn\\
&+agln+dfio+bhio+dejo+ahjo+belo+aflo+cfip+bgip+cejp+agjp+bekp+afkp.
\end{aligned}
$$

4. Determinantal elliptical distributions

Noncentral elliptical multivariate distributions involve a number of general integrals that need to be determined depending on the transformations under consideration. Yet, they all stem from the important fact that the elliptically contoured distribution is characterized by a symmetric function, say, $h(\mathbf{U})$, i.e., $h(\mathbf{AB}) = h(\mathbf{BA})$, for any square matrices $\mathbf{A}$ and $\mathbf{B}$. The simplest function is $h(\operatorname{tr}\mathbf{AB})$; the zonal polynomials arise naturally and many distributional results have been developed using them during the last five decades. However, if the function takes the form $h(|\mathbf{AB}|)$, no general distributional results are available in this case in the setting of zonal polynomials.

Here, we develop an application of the expansion of determinants in the context of elliptical distributions. This would facilitate the development of new kernels which depend on determinants that can be expanded in terms of zonal polynomials, and the classical integrals over positive definite spaces and orthogonal groups can then be used through this expansion.

For example, Herz [12] studied the following integral representations:

$$
{}_1F_1(a;c;\mathbf{X}) = \frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)}\int_{\mathbf{0}<\mathbf{Y}<\mathbf{I}_m}\operatorname{etr}(\mathbf{XY})|\mathbf{Y}|^{a-(m+1)/2}\times|\mathbf{I}-\mathbf{Y}|^{c-a-(m+1)/2}(d\mathbf{Y}) \tag{12}
$$

which is valid for all symmetric $\mathbf{X}$, and $\operatorname{Re}(a), \operatorname{Re}(c), \operatorname{Re}(c-a) > \frac{1}{2}(m-1)$;

$$
{}_2F_1(a,b;c;\mathbf{X}) = \frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)}\int_{\mathbf{0}<\mathbf{Y}<\mathbf{I}_m}|\mathbf{I}-\mathbf{XY}|^{-b}|\mathbf{Y}|^{a-(m+1)/2}\times|\mathbf{I}-\mathbf{Y}|^{c-a-(m+1)/2}(d\mathbf{Y}) \tag{13}
$$

which is valid for $\operatorname{Re}(\mathbf{X}) < \mathbf{I}$, and $\operatorname{Re}(a), \operatorname{Re}(c-a) > \frac{1}{2}(m-1)$, wherein the multivariate gamma function is defined as $\Gamma_m(a) = \pi^{m(m-1)/4}\prod_{i=1}^{m}\Gamma\left[a-\frac{1}{2}(i-1)\right]$, with $\operatorname{Re}(a) > \frac{1}{2}(m-1)$.

These lead to the well-known Kummer and Euler relations, respectively [12]:

$$
{}_1F_1(a;c;\mathbf{X}) = \operatorname{etr}(\mathbf{X})\,{}_1F_1(c-a;c;-\mathbf{X}), \tag{14}
$$

$$
\begin{aligned}
{}_2F_1(a,b;c;\mathbf{X}) &= |\mathbf{I}-\mathbf{X}|^{-b}\,{}_2F_1(c-a,b;c;-\mathbf{X}(\mathbf{I}-\mathbf{X})^{-1})\\
&= |\mathbf{I}-\mathbf{X}|^{c-a-b}\,{}_2F_1(c-a,c-b;c;\mathbf{X}).
\end{aligned} \tag{15}
$$

The extension of (12) via an elliptical model $h(\operatorname{tr}\mathbf{XY})$ is possible when a Taylor expansion of the general function in terms of zonal polynomials is assumed, in which case the matrix generalized Kummer relation is obtained [6].

However, the extension of the Euler relation that substitutes $|\mathbf{I}-\mathbf{XY}|$ in (13) for any elliptical model $h(|\mathbf{I}-\mathbf{XY}|)$, or in fact for any other particular model, is unavailable in the literature. This generalization will be handled here through the use of zonal determinantal expansion in (5).

First, let P_r be the set of all ordered partitions ρ's of r; $(a)_\kappa$, the generalized hypergeometric coefficient associated to ordered partition $\kappa = (k_1,\ldots,k_m)$ and a complex, be defined as $(a)_\kappa = \prod_{i=1}^{m}\left(a-\frac{1}{2}(i-1)\right)_{k_i}$, where $(a)_k = a(a+1)\cdots(a+k-1)$, $(a)_0 = 1$; $\binom{\kappa}{\sigma}$, the generalized binomial coefficient, be defined by $\frac{C_\kappa(\mathbf{I}_m+\mathbf{Y})}{C_\kappa(\mathbf{I}_m)} = \sum_{s=0}^{k}\sum_\sigma\binom{\kappa}{\sigma}\frac{C_\sigma(\mathbf{Y})}{C_\sigma(\mathbf{I}_m)}$, where the last summation runs over all ordered partitions σ of the integer s (see [17]).

Theorem 7. *Let* $\operatorname{Re}(\mathbf{X}) < \mathbf{I}$ *and* $\operatorname{Re}(a), \operatorname{Re}(c-a) > \frac{1}{2}(m-1)$. *Then, we have*

$$
\begin{aligned}
&\int_{\mathbf{0}<\mathbf{Y}<\mathbf{I}_m} h(|\mathbf{I}-\mathbf{XY}|)|\mathbf{Y}|^{a-(m+1)/2}|\mathbf{I}-\mathbf{Y}|^{c-a-(m+1)/2}(d\mathbf{Y})\\
&\quad= \frac{\Gamma_m(a)\Gamma_m(c-a)}{\Gamma_m(c)}\sum_{k=0}^{\infty}\frac{h^{(k)}(0)}{k!}\left[\frac{\prod_{i=1}^{m}(2+m-i)!}{2^m m!^2\prod_{i=1}^{m-1}\prod_{j=i+1}^{m}(-i+j)}\right]^k C_{k^m}(\mathbf{I})\sum_{r=0}^{km}\sum_{\rho\in P_r}\binom{(k^m)}{\rho}\frac{(-1)^{km}}{C_\rho(\mathbf{I})}\frac{(a)_\rho}{(c)_\rho}C_\rho(\mathbf{X}).
\end{aligned} \tag{16}
$$

Proof. Expanding $h(\cdot)$ in a Taylor series, applying (7), and then using the generalized binomial expansion of [17, p. 267], we obtain

$$\int_{\mathbf{0}<\mathbf{Y}<\mathbf{I}_m} h(|\mathbf{I}-\mathbf{XY}|)|\mathbf{Y}|^{a-(m+1)/2}|\mathbf{I}-\mathbf{Y}|^{c-a-(m+1)/2}(d\mathbf{Y})$$

$$= \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} \int_{\mathbf{0}<\mathbf{Y}<\mathbf{I}_m} |\mathbf{I}-\mathbf{XY}|^k |\mathbf{Y}|^{a-(m+1)/2}|\mathbf{I}-\mathbf{Y}|^{c-a-(m+1)/2}(d\mathbf{Y})$$

$$= \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} \left[\frac{\prod_{i=1}^{m}(2+m-i)!}{2^m m!^2 \prod_{i=1}^{m-1}\prod_{j=i+1}^{m}(-i+j)} \right]^k$$

$$\int_{\mathbf{0}<\mathbf{Y}<\mathbf{I}_m} C_{k^m}(\mathbf{I}-\mathbf{XY})|\mathbf{Y}|^{a-(m+1)/2}|\mathbf{I}-\mathbf{Y}|^{c-a-(m+1)/2}(d\mathbf{Y})$$

$$= \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} \left[\frac{\prod_{i=1}^{m}(2+m-i)!}{2^m m!^2 \prod_{i=1}^{m-1}\prod_{j=i+1}^{m}(-i+j)} \right]^k C_{k^m}(\mathbf{I}) \sum_{r=0}^{km}\sum_{\rho\in P_r} \binom{(k^m)}{\rho}$$

$$\times \frac{(-1)^{km}}{C_\rho(\mathbf{I})} \int_{\mathbf{0}<\mathbf{Y}<\mathbf{I}_m} C_\rho(\mathbf{XY})|\mathbf{Y}|^{a-(m+1)/2}|\mathbf{I}-\mathbf{Y}|^{c-a-(m+1)/2}(d\mathbf{Y}).$$

Now, by using the result of [17, p. 254], the required result is obtained. □

Note that the above expansion can be computed efficiently through the algorithms of [15] and the general expressions for the k-th derivative of the generator function. Formulas for these derivatives under the classical elliptical models such as Kotz-type, Person VII, Bessel, and Jensen-logistic are all given in [2].

Another application of the established results arises in the context of shape theory; for example, see [10] (under Gaussian models) and [2,7] (under elliptical models). Incidentally, all these works are in terms of traces.

First, let $\mathbf{X} : N \times K$ have a determinantal matrix-variate elliptically contoured distribution if its density with respect to the Lebesgue measure is given by

$$f_{\mathbf{X}}(\mathbf{X}) = \frac{1}{|\boldsymbol{\Sigma}|^{K/2}|\boldsymbol{\Theta}|^{N/2}} h\left\{\left|(\mathbf{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})\boldsymbol{\Theta}^{-1}\right|\right\},$$

where $\boldsymbol{\mu} : N\times K$, $\boldsymbol{\Sigma} : N\times N$, $\boldsymbol{\Theta} : K\times K$, and $\boldsymbol{\Sigma}$ positive definite ($\boldsymbol{\Sigma} > \mathbf{0}$), $\boldsymbol{\Theta} > \mathbf{0}$. The function $h : \Re \to [0, \infty)$ is termed the generator function, and is such that $\int_0^\infty u^{NK-1}h(u^2)du < \infty$. In this case, we shall use the notation $\mathbf{X} \sim \mathcal{E}_{N\times K}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\Theta}, h)$. This class of matrix-variate distributions possessing determinantal kernels defines a number of new densities, including the Gaussian, Contaminated Gaussian, Pearson type II and type VI, Kotz-type, Jensen-logistic, Power exponential, and Bessel distributions, among others; they have tails that are heavy or light, and/or distributions with greater or smaller degree of kurtosis than the Gaussian model. A list of classical versions, in terms of traces which can be replaced by the new distributions in terms of determinants, can be seen in [8,11].

There are several approaches to statistical shape theory. In this section, we focus on the configuration or affine shape. Let us consider the following definition from [10]: two figures $\mathbf{X} : N \times K$ and $\mathbf{X}_1 : N \times K$ have the *same configuration*, or *affine shape*, if $\mathbf{X}_1 = \mathbf{XE} + \mathbf{1}_N\mathbf{e}'$, for some translation $\mathbf{e} : K \times 1$ and non-singular matrix $\mathbf{E} : K \times K$.

Statistical shape theory compares shapes of objects in the presence of randomness, and so under the assumption that a figure $\mathbf{X}$, comprising N landmarks in K dimensions, follows a determinantal elliptical distribution $\mathbf{X} \sim \mathcal{E}_{N\times K}(\boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\Sigma}_{\mathbf{X}}, \boldsymbol{\Theta}, h)$, it is of natural interest to remove translation, scaling, and uniform shear from $\mathbf{X}$, resulting in the determinantal configuration density of this random figure.

To obtain non-isotropic densities, the matrix transformations proposed by Díaz-García and Caro-Lopera [7] are followed, i.e., we assume that

$$\mathbf{X} \sim \mathcal{E}_{N\times K}(\boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\Sigma}_{\mathbf{X}}, \boldsymbol{\Theta}, h). \tag{17}$$

If $\boldsymbol{\Theta}^{1/2}$ is the positive definite square root of the matrix $\boldsymbol{\Theta}$, i.e., $\boldsymbol{\Theta} = (\boldsymbol{\Theta}^{1/2})^2$, with $\boldsymbol{\Theta}^{1/2} : K \times K$, and noting that

$$\mathbf{X}\boldsymbol{\Theta}^{-1}\mathbf{X}' = \mathbf{X}(\boldsymbol{\Theta}^{1/2}\boldsymbol{\Theta}^{1/2})^{-1}\mathbf{X}' = \mathbf{X}\boldsymbol{\Theta}^{-1/2}(\mathbf{X}\boldsymbol{\Theta}^{-1/2})' = \mathbf{ZZ}',$$

where

$$\mathbf{Z} = \mathbf{X}\boldsymbol{\Theta}^{-1/2},$$

we then have

$$\mathbf{Z} \sim \mathcal{E}_{N\times K}(\boldsymbol{\mu}_{\mathbf{Z}}, \boldsymbol{\Sigma}_{\mathbf{X}}, \mathbf{I}_K, h)$$

with $\boldsymbol{\mu}_{\mathbf{Z}} = \boldsymbol{\mu}_{\mathbf{X}}\boldsymbol{\Theta}^{-1/2}$; see [11, p. 20].

The standard starting point in shape theory is thus achieved, where the original landmark matrix is replaced by $\mathbf{Z} = \mathbf{X}\boldsymbol{\Theta}^{-1/2}$. Thus, the usual procedure is subsequently pursued by removing from $\mathbf{Z}$, translation, scale, and uniform shear in order to obtain the configuration of $\mathbf{Z}$ (or $\mathbf{X}$) via the affine decomposition (see [2] in the case of kernels in terms of traces).

Thus, the determinantal configuration coordinates are constructed in the three steps indicated in the expression

$$\boldsymbol{LX\Theta}^{-1/2} = \mathbf{LZ} = \mathbf{Y} = \mathbf{UE}. \tag{18}$$

The matrix $\mathbf{U} : (N-1)\times K$ contains configuration coordinates of $\mathbf{X} : N \times K$. Let $\mathbf{Y}_1 : K \times K$ be non-singular and $\mathbf{Y}_2 : q \times K$, with $q = N - K - 1 \geq 1$, such that $\mathbf{Y} = (\mathbf{Y}_1' \mid \mathbf{Y}_2')'$. Define also $\mathbf{U} = (\mathbf{I} \mid \mathbf{V}')'$, then $\mathbf{V} = \mathbf{Y}_2\mathbf{Y}_1^{-1}$ and $\mathbf{E} = \mathbf{Y}_1$, where $\mathbf{L} : (N-1)\times N$ is a Helmert sub-matrix.

The following theorem presents the result for the central case.

Theorem 8. *Let $\mathbf{Y} \sim \mathcal{E}_{N-1\times K}(\mathbf{0}, \boldsymbol{\Sigma}, \mathbf{I}, h)$, where $\boldsymbol{\Sigma} : N-1 \times N-1$ is a positive definite matrix, $\mathbf{0}$ is a $N-1\times K$ matrix of zeros, $\mathbf{I}$ is a $K\times K$ identity matrix, and $\mathbf{G} : K \times K$ is an arbitrary positive definite matrix. Then, the central determinantal configuration density is given by*

$$\frac{|\mathbf{G}|^{(q+K)/2}\Gamma_K\left(\frac{N-1}{2}\right)\Gamma_K\left(\frac{K+1}{2}\right)}{\pi^{-K^2/2}|\boldsymbol{\Sigma}|^{\frac{K}{2}}\Gamma_K\left(\frac{K}{2}\right)\Gamma_K\left(\frac{N+K}{2}\right)}\sum_{t=0}^{\infty}\frac{h^{(t)}(0)}{t!}\left[\frac{\prod_{i=1}^{K}(2+K-i)!}{2^K K!^2\prod_{i=1}^{K-1}\prod_{j=i+1}^{K}(-i+j)}\right]^t\frac{\left(\frac{N-1}{2}\right)_{t^K}}{\left(\frac{N+K}{2}\right)_{t^K}}C_{t^K}(\mathbf{GU}'\boldsymbol{\Sigma}^{-1}\mathbf{U}).$$

Proof. The density of $\mathbf{Y}$ is given by

$$\frac{1}{|\boldsymbol{\Sigma}|^{\frac{K}{2}}}h\left(|\mathbf{Y}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}|\right). \tag{19}$$

Let $(\mathbf{F}^{1/2})^2 = \mathbf{F} > 0$ and $\mathbf{0} < \mathbf{F} < \mathbf{G}$ for an arbitrary positive definite $K\times K$ matrix $\mathbf{G}$. If $\mathbf{H} \in O(K)$ and $\mathbf{E} = \mathbf{F}^{1/2}\mathbf{H}$, then for $\mathbf{Y} = \mathbf{UF}^{1/2}\mathbf{H}$, we have

$$(d\mathbf{Y}) = 2^{-K}|\mathbf{F}|^{(q-1)/2}(d\mathbf{V})(d\mathbf{F})(\mathbf{H}'d\mathbf{H}), \tag{20}$$

where $q = N - K - 1$ (see [2]). If $\mathbf{Y} = \mathbf{UF}^{1/2}\mathbf{H}$ is factorized, then the joint density of $\mathbf{U}$, $\mathbf{F}$ and $\mathbf{H}$ is

$$\frac{2^{-K}|\mathbf{F}|^{(q-1)/2}}{|\boldsymbol{\Sigma}|^{\frac{K}{2}}}h\left(|\mathbf{FU}'\boldsymbol{\Sigma}^{-1}\mathbf{U}|\right)(\mathbf{H}'d\mathbf{H})(d\mathbf{F})(d\mathbf{V}). \tag{21}$$

Assuming now that h admits a Taylor expansion, the joint density of $\mathbf{U}$, $\mathbf{F}$ and $\mathbf{H}$ can be expressed as

$$\frac{2^{-K}|\mathbf{F}|^{(q-1)/2}}{|\boldsymbol{\Sigma}|^{\frac{K}{2}}}\sum_{t=0}^{\infty}\frac{h^{(t)}(0)}{t!}|\mathbf{FU}'\boldsymbol{\Sigma}^{-1}\mathbf{U}|^t(\mathbf{H}'d\mathbf{H})(d\mathbf{F})(d\mathbf{V}).$$

Now, recall that

$$(d\mathbf{H}) = \frac{1}{Vol[O(K)]}(\mathbf{H}'d\mathbf{H}) = \frac{1}{\frac{2^K\pi^{K^2/2}}{\Gamma_K\left(\frac{K}{2}\right)}}(\mathbf{H}'d\mathbf{H}), \tag{22}$$

and so integration with respect to $\mathbf{H}$ gives the joint density of $\mathbf{F}$ and $\mathbf{U}$ as

$$\frac{\pi^{K^2/2}|\mathbf{F}|^{(q-1)/2}}{|\boldsymbol{\Sigma}|^{\frac{K}{2}}\Gamma_K\left(\frac{K}{2}\right)}\sum_{t=0}^{\infty}\frac{h^{(t)}(0)}{t!}|\mathbf{FU}'\boldsymbol{\Sigma}^{-1}\mathbf{U}|^t(d\mathbf{F})(d\mathbf{V}). \tag{23}$$

Upon using (7), the joint density of $\mathbf{F}$ and $\mathbf{U}$ finally takes on the form

$$\frac{\pi^{K^2/2}|\mathbf{F}|^{(q-1)/2}}{|\boldsymbol{\Sigma}|^{\frac{K}{2}}\Gamma_K\left(\frac{K}{2}\right)}\sum_{t=0}^{\infty}\frac{h^{(t)}(0)}{t!}\left[\frac{\prod_{i=1}^{K}(2+K-i)!}{2^K K!^2\prod_{i=1}^{K-1}\prod_{j=i+1}^{K}(-i+j)}\right]^t C_{t^K}(\mathbf{FU}'\boldsymbol{\Sigma}^{-1}\mathbf{U})(d\mathbf{F})(d\mathbf{V}).$$

Let $\mathbf{0} < \mathbf{F} < \mathbf{G}$ for an arbitrary positive definite $k \times k$ matrix $\mathbf{G}$ and define $\mathbf{W} = \mathbf{G}^{-1/2}\mathbf{F}\mathbf{G}^{-1/2}$, so that $(d\mathbf{F}) = |\mathbf{G}|^{(K+1)/2}(d\mathbf{W})$ and $\mathbf{0} < \mathbf{W} < \mathbf{I}_K$.

Then, the joint density function of $\mathbf{W}$ and $\mathbf{U}$ (given $\mathbf{G}$) has the form

$$\frac{\pi^{K^2/2}|\mathbf{G}|^{(q+K)/2}|\mathbf{W}|^{(q-1)/2}}{|\boldsymbol{\Sigma}|^{\frac{K}{2}}\Gamma_K\left(\frac{K}{2}\right)} \sum_{t=0}^{\infty} \frac{h^{(t)}(0)}{t!} \left[\frac{\prod_{i=1}^{K}(2+K-i)!}{2^K K!^2 \prod_{i=1}^{K-1}\prod_{j=i+1}^{K}(-i+j)}\right]^t C_{t^K}(\mathbf{W}\mathbf{G}^{1/2}\mathbf{U}'\boldsymbol{\Sigma}^{-1}\mathbf{U}\mathbf{G}^{1/2})(d\mathbf{W})(d\mathbf{V}).$$

Thus, integration over $\mathbf{0} < \mathbf{W} < \mathbf{I}_K$ gives the central determinantal configuration density as

$$\frac{\pi^{K^2/2}|\mathbf{G}|^{(q+K)/2}}{|\boldsymbol{\Sigma}|^{\frac{K}{2}}\Gamma_K\left(\frac{K}{2}\right)} \sum_{t=0}^{\infty} \frac{h^{(t)}(0)}{t!} \left[\frac{\prod_{i=1}^{K}(2+K-i)!}{2^K K!^2 \prod_{i=1}^{K-1}\prod_{j=i+1}^{K}(-i+j)}\right]^t \int_{\mathbf{0}<\mathbf{W}<\mathbf{I}_K} |\mathbf{W}|^{(q-1)/2} C_{t^K}(\mathbf{W}\mathbf{G}^{1/2}\mathbf{U}'\boldsymbol{\Sigma}^{-1}\mathbf{U}\mathbf{G}^{1/2})(d\mathbf{W})(d\mathbf{V})$$

from which the required result follows by using the result of Muirhead (1982, p. 254). □

Recall that the classical central configuration density (in terms of traces) is invariant under the elliptical family [2], but the preceding determinantal central configuration density is not invariant, and this can shed some additional insight and information, which is not apparent in the case of traces.

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