ARTICLE IN PRESS
JPAA:106214

Journal of Pure and Applied Algebra ••• (••••) ••••••

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Vanishing of degree 3 cohomological invariants

Rebecca Black

Swarthmore College, 500 College Avenue, Swarthmore, PA 19081, United States of America

A R T I C L E   I N F O

Article history:
Received 21 December 2017
Received in revised form 5 August 2019
Available online xxxx
Communicated by C.A. Weibel

MSC:
14C15; 14F42; 20G10

Keywords:
Algebraic geometry
Cohomological invariant
Motivic cohomology
Chow ring

A B S T R A C T

For a complex algebraic variety X, we show that triviality of the degree three unramified cohomology $H^0(X, \mathcal{H}^3)$ (occurring on the second page of the Bloch-Ogus spectral sequence [1]) follows from a condition on the integral Chow group CH^2X and the integral cohomology group $H^3(X, \mathbb{Z})$. In the case that X is an appropriate approximation to the classifying stack BG of a finite p-group G, this result states that the group G has no degree three cohomological invariants. As a corollary we show that the nonabelian groups of order p^3 for odd prime p have no degree three cohomological invariants.

1. Introduction

Let G be a finite p-group of order p^n, considered as an algebraic group over $\mathbb{C}$. In this paper we employ the tools of the Bloch-Ogus spectral sequence and the motivic cohomology ring of the classifying space BG in order to examine in detail the relationship between the Chow ring of BG and the ring of cohomological invariants of G in low degree. In particular, our main result is that if the cycle class map

$$cl : CH^2BG \to H^4(BG, \mathbb{Z})$$

is an isomorphism, then there are no non-trivial degree three cohomological invariants of G. There has been a lot of progress recently in computing the Chow rings of various classes of p-groups, so we know that we have this isomorphism in certain cases. (See for example [9] for an excellent overview of recent progress.)

Ideally similar techniques could be employed to explicitly relate the Chow ring to vanishing of invariants in higher degree as well, but a more detailed computational understanding of the motivic cohomology of BG is necessary to extend this method to higher degrees.

E-mail address: rblack1@swarthmore.edu.

https://doi.org/10.1016/j.jpaa.2019.106214
0022-4049/© 2019 Published by Elsevier B.V.

Throughout the paper, varieties X are assumed to be smooth but not necessarily projective or proper.

2. Chow groups and cohomological invariants

For any linear algebraic group G over a field k, Totaro defines the Chow groups of BG in terms of finite-dimensional approximations to the classifying stack BG. Suppose that V is a representation of G, and let $S \subset V$ be the locus on which the stabilizers are non-trivial, with $\operatorname{codim} S = d$. Let $X = (V - S)/G$ be the quotient variety. Then

$$CH^i BG = CH^i X \text{ for } i < d.$$

Totaro proved the existence of a sequence of such representations V_n with the codimension of S_n going to infinity; see [8] for a good exposition. Throughout this paper we will freely assume that we have a variety X of this form where we have taken the representation to be of high enough dimension that X has the same invariants and cohomology as BG in low degree.

For a fixed base field k, let $H^1(-, G)$ denote the functor from fields over k to sets that takes K/k to the first nonabelian Galois cohomology set $H^1(K, G)$ (which can be thought of as isomorphism classes of G-torsors over K). Let $H^d(-, \mathbb{Z}/p)$ denote the functor that takes a field K/k to the degree d abelian Galois cohomology group $H^d(K, \mathbb{Z}/p)$. Then a degree d cohomological invariant of G is a natural transformation of functors

$$\eta : H^1(-, G) \to H^d(-, \mathbb{Z}/p).$$

(See [4] for a good introduction to the theory.) For our purposes, however, this is not the most convenient way to think of cohomological invariants. Given a quotient variety $X = (V - S)/G$ as above, with $\operatorname{codim} S \geq 2$, the generic fiber T of the map $V - S \to X$ is a versal G-torsor, meaning that any given cohomological invariant is actually completely defined by its value on that specific torsor (see discussion in [5]). Since T is defined over $\operatorname{Spec} k(X)$, its image under an invariant η will lie in the Galois cohomology group $H^d(k(X), \mathbb{Z}/p)$ for some degree d. Hence we can identify the group of degree d cohomological invariants of G with a certain subset of $H^d(k(X), \mathbb{Z}/p)$.

In fact, we can say much more about that certain subset: Given a point $x \in X$ with $\operatorname{codim} \overline{\{x\}} = 1$, we get a residue map

$$\nu_x : H^d(k(X), \mathbb{Z}/p) \to H^{d-1}(k(x), \mathbb{Z}/p),$$

where $k(x)$ is the residue field of the local ring of x. If a class $\eta_T \in H^d(k(X), \mathbb{Z}/p)$ is the image of a versal torsor under an invariant, then $\nu_x(\eta_T) = 0$ for all such x; conversely, Totaro shows that if $\operatorname{codim} S \geq 2$, every class in the kernel of ν_x for all x does in fact define a cohomological invariant (letter to Serre, reprinted in [4]). Therefore we have the identification

$$\operatorname{Inv}^d G = \ker \left(H^d(k(X), \mathbb{Z}/p) \to \coprod_{x \in X^{(1)}} H^{d-1}(k(x), \mathbb{Z}/p) \right),$$

where $x \in X^{(1)}$ ranges over all codimension one points.

3. Bloch-Ogus spectral sequence and stable cohomology

In their 1974 paper [1], Bloch and Ogus showed that the product of residue maps considered above is part of a flasque resolution of the sheaf $\mathcal{H}^d$ on X, defined as the sheafification of the Zariski presheaf $U \mapsto H^d_{\text{ét}}(U, \mathbb{Z}/p)$. Therefore we can actually think of the kernel as a sheaf cohomology group, and we get

$$\operatorname{Inv}^d G = H^0(X, \mathcal{H}^d).$$

This sheaf cohomology group appears as the $E_2^{0,d}$ term of the Bloch-Ogus spectral sequence for X, which converges to the étale cohomology $H^*_{\text{ét}}(X, \mathbb{Z}/p)$. In the case that $X = (V - S)/G$ as above and we take the base field to be $\mathbb{C}$, we can in fact identify these étale cohomology groups with the group cohomology $H^*(G, \mathbb{Z}/p)$ in low degree (see e.g. [7] chapter 21).

The diagonal entries $E_2^{r,r}$ are isomorphic to the mod p Chow groups $CH^r X \otimes \mathbb{Z}/p \cong CH^r BG \otimes \mathbb{Z}/p$. Hence the differential $\delta : E_2^{0,3} \to E_2^{2,2}$ combined with the maps to and from the abutment give an exact sequence:

$$H^3(G, \mathbb{Z}/p) \to \operatorname{Inv}^3 G \xrightarrow{\delta} CH^2 BG \otimes \mathbb{Z}/p \to H^4(G, \mathbb{Z}/p).$$

Our basic plan of attack is to show that both the kernel and the image of δ are trivial, which forces $\operatorname{Inv}^3 G = 0$. Triviality of the image will follow immediately from the assumption on the integral Chow groups, since the map to $H^4(G, \mathbb{Z}/p)$ coincides with the mod p cycle class map. We will have to work a bit harder to show that the kernel is trivial. The kernel of δ is precisely the classes that survive to the group cohomology, also known as the stable cohomology (as discussed in [2]). Hence the vanishing of this kernel is equivalent to the vanishing of degree three stable cohomology, which is shown for several cases of p-groups in [2]. We will use a somewhat different argument that makes use of the relationship of the sheaf cohomology groups $H^r(X, \mathcal{H}^s)$ with the motivic cohomology ring $H^{*,*'}(X, \mathbb{Z}/p)$.

4. Motivic cohomology

This section summarizes a few important properties of the motivic cohomology ring $H^{*,*'}(X, \mathbb{Z}/p)$ associated to a variety X. We do not attempt a complete discussion of the definition of this ring here; see for example [10] or [6] for details. The beauty of the motivic cohomology ring for us is that it specializes for certain indices to both the mod p Chow groups and the étale cohomology groups. Specifically, Voevodsky and others have shown the following, for a smooth variety X:

$$H^{m,n}(X, \mathbb{Z}/p) \cong \begin{cases} 0 & \text{if } m > 2n; \\ CH^n X \otimes \mathbb{Z}/p & \text{if } m = 2n; \\ H^m_{\text{ét}}(X, \mathbb{Z}/p) & \text{if } m \le n. \end{cases}$$

Note that the $m \le n$ case here is known as the Beilinson-Lichtenbaum conjecture. Let τ denote a generator of $H^{0,1}(\operatorname{Spec}(k), \mathbb{Z}/p) \cong \mathbb{Z}/p$; then the cup product gives a map $\times\tau : H^{m,n}(X, \mathbb{Z}/p) \to H^{m,n+1}(X, \mathbb{Z}/p)$. For $n < m$, composing this map with the isomorphism from the Beilinson-Lichtenbaum conjecture gives a map

$$\times\tau^{m-n} : H^{m,n}(X, \mathbb{Z}/p) \to H^{m,m}(X, \mathbb{Z}/p) \cong H^m_{\text{ét}}(X, \mathbb{Z}/p).$$

Our argument also makes use of the following long exact sequence, which relates this map to the sheaf cohomology groups that appear in the Bloch-Ogus spectral sequence (see [11]):

$$\cdots \to H^{m,n-1}(X,\mathbb{Z}/p) \xrightarrow{\times\tau} H^{m,n}(X,\mathbb{Z}/p) \to$$
$$H^{m-n}(X,\mathcal{H}^n) \to H^{m+1,n-1}(X,\mathbb{Z}/p) \xrightarrow{\times\tau} \cdots$$

Finally, we will also use the fact that there is motivic cohomology with integer coefficients as well, and in particular there is an isomorphism

$$H^{2n,n}(X,\mathbb{Z}) \cong CH^n X.$$

5. Main theorem

We are now ready to state and prove our main result on the vanishing of degree three cohomological invariants.

Theorem 5.1. *Let X be a smooth variety over $\operatorname{Spec}\mathbb{C}$ satisfying the following two properties:*

(i) $CH^2 X \cong H^4(X,\mathbb{Z})$;
(ii) *There is some power p^n with $p^n H^3(X,\mathbb{Z}) = 0$.*

Then $H^0(X,\mathcal{H}^3) = 0$. In particular, if X is an approximation of the classifying stack BG for an algebraic group G such that the above two conditions hold, then $\operatorname{Inv}^3 G = 0$.

Proof. The group $H^0(X,\mathcal{H}^3)$ fits into the following long exact sequence:

$$\cdots \to H^{3,2}(X,\mathbb{Z}/p) \xrightarrow{\times\tau} H^{3,3}(X,\mathbb{Z}/p) \to H^0(X,\mathcal{H}^2) \to$$
$$H^{4,2}(X,\mathbb{Z}/p) \xrightarrow{\times\tau} H^{4,3}(X,\mathbb{Z}/p) \to \cdots.$$

Therefore, we get our result if we can show that

(a) $\times\tau : H^{4,2}(X,\mathbb{Z}/p) \to H^{4,3}(X,\mathbb{Z}/p)$ is injective.
(b) $\times\tau : H^{3,2}(X,\mathbb{Z}/p) \to H^{3,3}(X,\mathbb{Z}/p)$ is surjective, and

The injectivity is easier to show, so we will do that first. We know that $H^{4,2}(X,\mathbb{Z}/p) \cong CH^2 X \otimes \mathbb{Z}/p$ is the mod p Chow group. The mod p cycle class map

$$c : CH^2 X \otimes \mathbb{Z}/p \to H^4(X,\mathbb{Z}/p)$$

agrees with the change of coefficients map $H^4(X,\mathbb{Z}) \to H^4(X,\mathbb{Z}/p)$ induced by the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p \to 0,$$

meaning its kernel is exactly pCH^2X. This shows that c is injective. Since we can identify c with the map $\times\tau^2$ on motivic cohomology (as discussed in e.g. [11]), we have shown (a).

For (b), denote by β the connecting homomorphism $\beta : H^3(X,\mathbb{Z}/p) \to H^4(X,\mathbb{Z})$; the plan of attack is first to show that

$$\ker(\beta) \subseteq \operatorname{im}(\times\tau) \subseteq H^{3,3}(X,\mathbb{Z}/p) \cong H^3(X,\mathbb{Z}/p),$$

and then to show that any class in $H^3(X,\mathbb{Z}/p)$ is equivalent to a class in $\ker(\beta)$ mod the image of $\times\tau$.

Please cite this article in press as: R. Black, Vanishing of degree 3 cohomological invariants, J. Pure Appl. Algebra (2019), https://doi.org/10.1016/j.jpaa.2019.106214

The key to the first step is that, for any exponent n the short exact sequence

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n \to 0$$

induces connecting maps on both étale cohomology and motivic cohomology:

$$\beta_{\mathrm{mot}} : H^{n,n}(X, \mathbb{Z}/p^n) \to H^{n+1,n}(X, \mathbb{Z}/p);$$
$$\beta_{\mathrm{et}} : H^n(X, \mathbb{Z}/p^n) \to H^{n+1}(X, \mathbb{Z}/p).$$

Under the isomorphisms from the Beilinson-Lichtenbaum conjecture, then, β_{et} maps from $H^{n,n}(X, \mathbb{Z}/p^n)$ to $H^{n+1,n+1}(X, \mathbb{Z}/p)$. By naturality of the isomorphisms arising from the Beilinson-Lichtenbaum conjecture we have that

$$\beta_{\mathrm{et}} = \times\tau \circ \beta_{\mathrm{mot}}.$$

Therefore, crucially for us, $\mathrm{im}(\beta_{\mathrm{et}}) \subseteq \mathrm{im}(\times\tau)$.

Now let $x \in \ker(\beta) \subseteq H^3(X, \mathbb{Z}/p)$. Then we can pull x back to a class $\tilde{x} \in H^3(X, \mathbb{Z})$. By assumption, $H^3(X, \mathbb{Z})$ is p^n-torsion for some n, meaning that $\tilde{x}$ in turn comes from a class $\overline{x} \in H^2(X, \mathbb{Z}/p^n)$. Then we have $x = \beta_{\mathrm{et}}(\overline{x}) \in \mathrm{im}(\times\tau)$ as desired.

For the general case, we now assume that $\beta(x) \neq 0 \in H^4(X, \mathbb{Z})$. Recall that by assumption $H^4(X, \mathbb{Z}) \cong CH^2X \cong H^{4,2}(X, \mathbb{Z})$; we write $y \in H^{4,2}(X, \mathbb{Z})$ for the image of $\beta(x)$ under this isomorphism. Since $py = 0$, we have $y = \beta_{\mathrm{mot}}(x')$ for some $x' \in H^{3,2}(X, \mathbb{Z}/p)$. Then $\beta(\tau x') = \beta(x) \in H^4(X, \mathbb{Z})$ (where we abuse notation a bit by conflating $\tau x'$ and its image under the isomorphism $H^{3,3}(X, \mathbb{Z}/p) \cong H^3(X, \mathbb{Z}/p)$). By the previous case, then, $\beta(x - \tau x') = 0$, so $x - \tau x' \in \mathrm{im}(\times\tau)$; therefore we also have $x \in \mathrm{im}(\times\tau)$ as desired. □

In the case that $X = (V - S)/G$ is an approximation to BG as described above, with $|G| = p^n$, we do automatically have that $H^3(X, \mathbb{Z}) \cong H^3(G, \mathbb{Z})$ is p^n-torsion, so the second condition of the theorem is automatically satisfied. Therefore we have shown that for finite p-groups G, if the degree two cycle class map is an isomorphism then G has no nontrivial degree three cohomological invariants. For example, Yagita proved that the cycle class map is an isomorphism in all degrees for the two nonabelian groups of order p^3 for odd primes p [11], meaning by our result these groups have no cohomological invariants of degree three. As noted above, for these specific examples the vanishing of degree three cohomological invariants can be proved by a variety of methods (for example by using the results of [2]). In fact it is known that for any p-group G of order at most p^4, the approximations $X = (V - S)/G$ to BG are rational, therefore if $\tilde{X}$ is a smooth compactification of such an approximation its unramified cohomology will be trivial. This provides another possible method of attack: If we can show that the unramified cohomology of such a compactification matches that of BG in low degrees, then the vanishing of the unramified cohomology (equivalently, cohomological invariants) would follow.

Another interesting angle is the connection with a result of Colliot-Thélène and Voisin ([3], theorem 3.7), which implies that for a smooth, connected, projective variety X, condition (i) from Theorem 5.1 implies that

$$H^0(X, \mathcal{H}^3(\mathbb{Z})) \cong H^0(X, \mathcal{H}^3),$$

where $\mathcal{H}^3(\mathbb{Z})$ is constructed the same way as the sheaf $\mathcal{H}^3$ but with integer coefficients. (Specifically, condition (i) implies the vanishing of the bottom row of the diagram of exact sequences at the beginning of section 5 of [3].) Hence in this case vanishing of the degree three unramified cohomology with finite coefficients is equivalent to the vanishing of $H^0(X, \mathcal{H}^3(\mathbb{Z}))$. This group is trivial in the case that X is rationally

Please cite this article in press as: R. Black, Vanishing of degree 3 cohomological invariants, J. Pure Appl. Algebra (2019), https://doi.org/10.1016/j.jpaa.2019.106214

connected, which again is a property satisfied by compactifications of the varieties $X = (V - S)/G$ that we have been considering. Again, therefore, if we can construct such a compactification whose unramified cohomology matches that of BG, then the vanishing result follows directly. An advantage of leveraging the machinery of motivic cohomology is that it allowed us to work directly with the quotient variety $(V - S)/G$ rather than a projective variety, but exploring the connection between these results seems like an interesting avenue for further investigation.

Acknowledgements

Infinite thanks to Patrick Brosnan for his help and patience as I struggled to work through the last details of this proof; and thank you also to Nobuaki Yagita and Masaki Kameko for extremely helpful suggestions and clarifications. Thank you also to the reviewers for many helpful corrections, and for pointing out the connection to the results of [3] and to the rationality of the quotient varieties.

References

[1] S. Bloch, A. Ogus, Gersten's conjecture and the homology of schemes, Ann. Sci. Éc. Norm. Supér. 4 (7) (1974) 181–201.
[2] F. Bogomolov, C. Böhning, Essential dimension, stable cohomological dimension, and stable cohomology of finite Heisenberg groups, arXiv:1405.1394 [abs], 2014.
[3] Jean-Louis Colliot-Thélène, Claire Voisin, et al., Cohomologie non ramifiée et conjecture de Hodge entiere, Duke Math. J. 161 (5) (2012) 735–801.
[4] S. Garibaldi, A. Merkurjev, J.-P. Serre, Cohomological Invariants in Galois Cohomology, University Lecture Series, vol. 28, American Mathematical Society, Providence, 2003.
[5] P. Guillot, Geometric methods for cohomological invariants, arXiv:abs/math/0612323v2, 2007.
[6] Carlo Mazza, Vladimir Voevodsky, Charles Weibel, Lecture Notes on Motivic Cohomology, vol. 2, American Mathematical Soc., 2011.
[7] James S. Milne, Lectures on etale cohomology (v2.21), Available at www.jmilne.org/math/, 2013.
[8] B. Totaro, The Chow ring of a classifying space, in: Proc. Symposia in Pure Math., vol. 67, 1999, pp. 249–281.
[9] B. Totaro, Group Cohomology and Algebraic Cycles, Cambridge Tracts in Mathematics, vol. 204, Cambridge University Press, New York, 2014.
[10] V. Voevodsky, Motivic cohomology with $\mathbb{Z}/l$-coefficients, Ann. Math. 174 (2011) 401–438.
[11] N. Yagita, Chow rings of nonabelian p-groups of order p^3, J. Math. Soc. Jpn. 64 (2) (2012) 507–531.