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On finite-dimensional copointed Hopf algebras over dihedral groups ☆

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ABSTRACT

We classify all finite-dimensional Hopf algebras over an algebraically closed field of characteristic zero such that its coradical is isomorphic to the algebra of functions $\Bbbk^{\mathbb{D}_m}$ over a dihedral group $\mathbb{D}_m$, with $m = 4a \geq 12$. We obtain this classification by means of the lifting method, where we use cohomology theory to determine all possible deformations. Our result provides an infinite family of new examples of finite-dimensional copointed Hopf algebras over dihedral groups.

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0. Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic zero. This paper contributes to the classification of finite-dimensional Hopf algebras over $\Bbbk$. We address particularly the case where the coradical of the Hopf algebra is the dual of the group algebra of a non-abelian group G. This kind of algebras are called *copointed* Hopf algebras over G.

Let A be a copointed Hopf algebra over a finite group G. We say that A is *trivial* if $A \simeq \Bbbk^G$ as Hopf algebras. It is known that all copointed Hopf algebras over G are trivial if G is an alternating group $\mathbb{A}_n$, $n \geq 5$, [4], a sporadic group different from Fi_{22}, B, M, [5], or a finite projective special linear group $PSL_n(q)$

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for a certain infinite family of pairs (n, q) [2]. On the other hand, there are few non-trivial examples and the complete classification with non-trivial examples is known only for the symmetric groups $\mathbb{S}_3$ and $\mathbb{S}_4$, see [8] and [18] respectively, and non-abelian groups attached to affine racks [17]. In this paper we give the complete classification of copointed Hopf algebras over the dihedral groups $\mathbb{D}_m$, with $m = 4a \geq 12$. Throughout the paper $\mathbb{D}_m$ will denote the dihedral group of order $2m$.

The case of the dihedral groups was previously treated in [12], where a particular case is studied. In this case, the associated graded algebra $\operatorname{gr} A$ is isomorphic to a bosonization of a quantum line over $\Bbbk^{\mathbb{D}_n}$ with $n \geq 2$. The authors also provide the description in the case where the coradical is the function algebra over a dicyclic group and when the coradical belongs to a certain family of cosemisimple non-pointed self-dual Hopf algebras.

The question of classifying Hopf algebras over $\Bbbk$ is a difficult problem to attack. One of the main reasons lies on the lack of general methods. The best approach to study Hopf algebras with the Chevalley property, *i.e.*, the coradical is a Hopf subalgebra, is the lifting method developed by Andruskiewitsch and Schneider [7]. Any Hopf algebra A has a coalgebra filtration $\{A_n\}_{n\geq 0}$, called *the coradical filtration*, whose first term is the *coradical* A_0. It corresponds to the filtration of A^* given by the powers of the Jacobson radical. If A_0 is a Hopf subalgebra, then $\{A_n\}_{n\geq 0}$ is also an algebra filtration; in particular, its associated graded object $\operatorname{gr} A = \bigoplus_{n\geq 0} A_n/A_{n-1}$ is a graded Hopf algebra, where $A_{-1} = 0$. Let $\pi : \operatorname{gr} A \to A_0$ be the homogeneous projection. It turns out that $\operatorname{gr} A \simeq B\#A_0$ as Hopf algebras, where $B = (\operatorname{gr} A)^{\operatorname{co}\pi} = \{a \in A : \ (\mathrm{id}\otimes\pi)\Delta(a) = a \otimes 1\}$ and $\#$ stands for the Radford–Majid biproduct or *bosonization* of B with A_0. Here B is not a usual Hopf algebra, but a graded connected Hopf algebra in the category ${}^{A_0}_{A_0}\mathcal{YD}$ of Yetter–Drinfeld modules over A_0. It contains the algebra generated by the elements of degree one, called the *Nichols algebra* $\mathfrak{B}(V)$ of V; here $V = B^1$ is a braided vector space whose braiding $c : V \otimes V \to V \otimes V$ is called the *infinitesimal braiding* of A.

Let H be a finite-dimensional cosemisimple Hopf algebra. The main steps to determine all finite-dimensional Hopf algebras A with coradical $A_0 \simeq H$, in terms of the lifting method, are:

(*a*) determine all $V \in {}^H_H\mathcal{YD}$ such that the Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional.
(*b*) for such V, compute all Hopf algebras A such that $\operatorname{gr} A \simeq \mathfrak{B}(V)\#H$. We call A a *lifting* of $\mathfrak{B}(V)$ over H.
(*c*) prove that if A is any finite-dimensional Hopf algebra such that $A_0 \simeq H$ and $\operatorname{gr} A \simeq B\#H$ with B a braided Hopf algebra in ${}^H_H\mathcal{YD}$, then $B \simeq \mathfrak{B}(V)$ for some $V \in {}^H_H\mathcal{YD}$.

By [6, Proposition 2.2.1], we have that ${}^H_H\mathcal{YD} \simeq {}^{H^*}_{H^*}\mathcal{YD}$ as braided tensor categories. Hence, the questions posed in previous steps (*a*) and (*c*) can be answered either in the category ${}^H_H\mathcal{YD}$ or in the category ${}^{H^*}_{H^*}\mathcal{YD}$. Since all pointed Hopf algebras over $\mathbb{D}_m$ were classified in [15] by means of the lifting method, to study copointed Hopf algebras over $\mathbb{D}_m$ we have to deal only with step (b) and (c). To describe all possible liftings, we use the invariant Hochschild cohomology of the Nichols algebra. As a consequence, it turns out that all liftings are cocycle deformations.

Recently, it was proved in [9] that all liftings of Nichols algebras of diagonal type over a cosemisimple Hopf algebra H can be obtained as a 2-cocycle deformation of the bosonization $\mathfrak{B}(V)\#H$, provided V is a principal realization in ${}^H_H\mathcal{YD}$ and the Nichols algebra $\mathfrak{B}(V)$ is finitely presented. In particular, all liftings may be described by generators and relations using the strategy developed in [1]. Although in our case the braiding is diagonal and $\Bbbk^{\mathbb{D}_m}$ is cosemisimple, the infinitesimal braidings can not be described as principal realizations in ${}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$, since $\Bbbk^{\mathbb{D}_m}$ does not coact homogeneously. Hence, the copointed Hopf algebras we introduce might not be obtained directly from their method. The strategy developed in [18] seems to be more appropriate.

A *formal graded deformation* of a graded bialgebra A over $\Bbbk[t]$, consists of a $\Bbbk[t]$-bilinear multiplication $m_t = m + tm_1 + t^2m_2 + \cdots$ and a comultiplication $\Delta_t = \Delta + t\Delta_1 + t^2\Delta_2 + \cdots$ with respect to which the

$\Bbbk[t]$-module $A[t] := A \otimes_{\Bbbk} \Bbbk[t]$ is again a graded bialgebra (where the degree of t is 1). At first glance it might seem that the coefficients are in the formal power series $\Bbbk[[t]]$ (as is the case in the "ungraded" version of the theory), but the fact that structure maps m_t, Δ_t, are homogeneous of degree 0, yields that for every i, the maps m_i, Δ_i are homogeneous of degree $-i$, and hence the images of A and $A \otimes A$ under m_t and Δ_t always lie in $A[t]$ and $(A \otimes A)[t]$. For the sake of simplicity let us now assume that the deformation is a preferred deformation in the sense that $\Delta_t = \Delta$ (formal graded deformations of graded bialgebras satisfying the assumptions of Theorem A are equivalent to such deformations). Given such a deformation of A, let r be the smallest positive integer for which $m_r \neq 0$ (if such an r exists). Then $(m_r, 0)$ is a 2-cocycle in the bialgebra cohomology $\widehat{\mathrm{Z}}^2_b(A)$ called an *infinitesimal deformation*. Every nontrivial deformation is equivalent to one for which the corresponding $(m_r, 0)$ represents a nontrivial cohomology class, see [14] (see also [19] for the ungraded version). Conversely, given a positive integer r and a 2-cocycle $(m', 0)$ in $\widehat{\mathrm{Z}}^2_b(A)$, $m + t^r m'$ is an associative multiplication on the *r-deformation* $A[t]/(t^{r+1})$ of A, making it a bialgebra over $\Bbbk[t]/(t^{r+1})$. There may or may not exist $m_{r+1}, m_{r+2}, \dots$ for which $m + t^r m' + t^{r+1} m_{r+1} + t^{r+2} m_{r+2} + \cdots$ endows $A[t]$ with a bialgebra structure over $\Bbbk[t]$. In principle it is hard to know if an r-deformation corresponds to a formal deformation.

Clearly, any formal deformation U of a graded bialgebra A is a lifting in the sense that $\operatorname{gr} U \simeq A$. Conversely, it is well-known that any lifting corresponds to a formal deformation, see for example [14, Theorem 2.2]. If $\sigma : A \otimes A \to \Bbbk$ is a multiplicative 2-cocycle on A, see Subsection 1.4, then the bialgebra A_σ is a lifting of A and whence corresponds to a formal deformation whose infinitesimal part σ_r is a Hochschild 2-cocycle, see Section 2.

Assume $A = B \# H$ is a bosonization of a quadratic Nichols algebra B over H. If the braiding in ${}^H_H\mathcal{YD}$ is symmetric, then any r-deformation given by a 2-cocycle $(m', 0)$ in $\widehat{\mathrm{Z}}^2_b(A)$ corresponds to a formal deformation, since m' is the infinitesimal part of the multiplicative 2-cocycle given by the exponentiation $\sigma = e^{m'}$ see [16, Corollary 2.6]. Hence, to find all possible deformations it suffices to describe the set $\widehat{\mathrm{H}}^2_b(A)$. For more details on bialgebra deformations see [19], [14], [21].

We will restrict our attention to the H-bitrivial part of cohomology, *i.e.*, pairs (f, g) where $f|_{A_0 \otimes A} = 0 = f|_{A \otimes A_0}$ and $(\pi \otimes \mathrm{id})g = 0 = (\mathrm{id} \otimes \pi)g$, where $\pi \colon A \to A_0$ denotes the canonical projection. We denote the corresponding sets by $\tilde{\mathrm{Z}}^2_b(A)$ and $\tilde{\mathrm{B}}^2_b(A)$, and the relevant cohomology by $\tilde{\mathrm{H}}^2_b(A) = \tilde{\mathrm{Z}}^2_b(A)/\tilde{\mathrm{B}}^2_b(A)$. If $H = A_0$ is a group algebra or a dual of a group algebra, then by [21, Lemma 2.3.1], it follows that $\tilde{\mathrm{H}}^2_b(A) = \widehat{\mathrm{H}}^2_b(A)$. For $\ell \in \mathbb{Z}$, we denote the relevant graded cohomology by $\tilde{\mathrm{H}}^2_b(A)_\ell = \tilde{\mathrm{Z}}^2_b(A)_\ell / \tilde{\mathrm{B}}^2_b(A)_\ell$. Let $\mathrm{H}^2_\varepsilon(B, \Bbbk)$ denote the Hochschild 2-cohomology group of B and $\mathrm{H}^2_\varepsilon(B, \Bbbk)^H$ the H-stable part. Under our hypothesis, it holds that there exists a connecting homomorphism $\tilde{\partial}_\ell \colon \mathrm{H}^2_\varepsilon(B, \Bbbk)^H_\ell \to \tilde{\mathrm{H}}^2_b(A)_\ell$. Suppose $B = \mathrm{T}(V)/I$ for some space $V \in {}^H_H\mathcal{YD}$ and an ideal $I \subseteq \mathrm{T}(V)_{(2)} = \bigoplus_{n \geq 2} V^{\otimes n}$. Set $\mathrm{M}(B) = I/\big(T(V)^+ I + I T(V)^+\big)$ and denote by $\mathcal{YD}(\mathrm{M}(B), V)_\ell$ the ℓ-homogeneous morphisms between $M(B)$ and V in ${}^H_H\mathcal{YD}$. As a particular case of our first result we obtain the following theorem that generalizes [21, Theorem 6.2.1].

Theorem A. *Let $\ell < 0$ and $r = -\ell$. If $\mathcal{YD}(\mathrm{M}(B), \mathcal{P}(B))_\ell = 0$ and B is generated as an algebra by elements of degree at most r, then $\tilde{\partial}_\ell \colon \mathrm{H}^2_\varepsilon(B, \Bbbk)^H_\ell \to \tilde{\mathrm{H}}^2_b(A)_\ell$ is surjective.*

This implies that all homogeneous 2-cocycles $(m', 0) \in \tilde{\mathrm{H}}^2_b(A)_2$, and consequently also all deformations, can be described using H-invariant Hochschild 2-cocycles on B, see Remarks 2.9 and 2.10.

Using the machinery described above, we obtain our main result. It turns out that all finite-dimensional Hopf algebras with coradical A_0 isomorphic to $\Bbbk^{\mathbb{D}_m}$ are liftings of bosonization of Nichols algebras over semisimple Yetter–Drinfeld modules. These semisimple Yetter–Drinfeld modules are given by $M_I = \bigoplus_{1 \leq j \leq r} M_{(i_j, k_j)}$, $M_L = \bigoplus_{1 \leq i \leq r} M_{\ell_i}$ and $M_{I,L} = M_I \oplus M_L$, and are parametrized by sets I, L of tuples satisfying certain properties, see Subsection 3.2. All Nichols algebras over these modules are isomorphic to exterior algebras. The liftings are then encoded by families of parameters ζ_I, μ_L, ν_L, τ_L related to this description.

Theorem B. *Let A be a finite-dimensional Hopf algebra with coradical A_0 isomorphic to $\Bbbk^{\mathbb{D}_m}$ with $m = 4a \geq 12$. Then A is isomorphic to one of the following Hopf algebras:*

(i) $\mathcal{A}(\zeta_I)$ *for some* $I \in \mathcal{I}$ *and some lifting data* ζ_I,
(ii) $\mathcal{B}(\mu_L, \nu_L, \tau_L)$ *for some* $L \in \mathcal{L}$ *and some lifting data* (μ_L, ν_L, τ_L),
(iii) $\mathcal{C}(\zeta_I, \mu_L, \nu_L, \tau_L)$ *for some* $(I, L) \in \mathcal{K}$ *and some lifting data* $(\zeta_I, \mu_L, \nu_L, \tau_L)$.

Conversely, any Hopf algebra above is a lifting of a finite-dimensional Nichols algebra in ${}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$.

The paper is organized as follows. In Section 1 we set the conventions and give the necessary definitions and results used along the paper. Section 2 is devoted to develop the cohomological tools to describe all possible deformations. In Section 3 we begin by recalling the description of the simple objects in ${}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$ and the Nichols algebras associated to them. Then we introduce the deformed algebras listed in Theorem B and we prove they are all the possible finite-dimensional copointed Hopf algebras over $\mathbb{D}_m$.

1. Preliminaries

1.1. Conventions

We work over an algebraically closed field $\Bbbk$ of characteristic zero. Our references for Hopf algebra theory are [22], [24] and [25].

Let A be a Hopf algebra over $\Bbbk$. We denote by Δ, ε and $\mathcal{S}$ the comultiplication, the counit and the antipode, respectively. Comultiplication and coactions are written using the Sweedler–Heynemann notation with summation sign suppressed, *e.g.*, $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for $a \in A$.

The *coradical* A_0 of A is the sum of all simple sub-coalgebras of A. In particular, if $G(A)$ denotes the group of *group-like elements* of A, we have $\Bbbk G(A) \subseteq A_0$. A Hopf algebra is *pointed* if $A_0 = \Bbbk G(A)$, that is, all simple sub-coalgebras are one-dimensional. We say that A has the *Chevalley property* if A_0 is a Hopf subalgebra of A. In particular, we say that A is *copointed* if A_0 is isomorphic as Hopf algebra to the algebra of functions $\Bbbk^\Gamma$ over some finite group Γ.

Denote by $\{A_i\}_{i\geq 0}$ the *coradical filtration* of A; if A has the Chevalley property, then by $\operatorname{gr} A = \oplus_{n\geq 0} \operatorname{gr} A(n)$ we denote the associated graded Hopf algebra, with $\operatorname{gr} A(n) = A_n/A_{n-1}$, setting $A_{-1} = 0$. For $h, g \in G(A)$, the linear space of (h, g)-*primitives* is:

$$\mathcal{P}_{h,g}(A) := \{x \in A \mid \Delta(x) = x \otimes h + g \otimes x\}.$$

If $g = 1 = h$, the linear space $\mathcal{P}(A) = \mathcal{P}_{1,1}(A)$ is called the set of primitive elements.

Let D be a coalgebra over $\Bbbk$ and denote by $(D_n)_{n\in\mathbb{N}}$ its coradical filtration. Then there exists a coalgebra projection $\pi : D \to D_0$ from D to the coradical D_0 with kernel I, see [22, Theorem 5.4.2]. Define the maps

$$\rho_L := (\pi \otimes \mathrm{id})\Delta : D \to D_0 \otimes D \qquad \text{and} \qquad \rho_R := (\mathrm{id} \otimes \pi)\Delta : D \to D \otimes D_0,$$

and let P_n be the sequence of subspaces defined recursively by

$$\begin{aligned} P_0 &= 0, \\ P_1 &= \{x \in D : \ \Delta(x) = \rho_L(x) + \rho_R(x)\} = \Delta^{-1}(D_0 \otimes I + I \otimes D_0), \\ P_n &= \{x \in D : \ \Delta(x) - \rho_L(x) - \rho_R(x) \in \sum_{1\leq i\leq n-1} P_i \otimes P_{n-i}\}, \quad n \geq 2. \end{aligned}$$

By a result of Nichols, $P_n = D_n \cap I$, $D_n = D_0 \oplus P_n$ for $n \geq 0$, and D_n, P_n are D_0-sub-bicomodules of D via ρ_R and ρ_L.

If M is a left A-comodule via $\delta(m) = m_{(-1)} \otimes m_{(0)} \in A \otimes M$ for all $m \in M$, then the space of *left coinvariants* is ${}^{\mathrm{co}\,\delta}M = \{x \in M \mid \delta(x) = 1 \otimes x\}$. In particular, if $\pi : A \to H$ is a morphism of Hopf algebras, then A is a left H-comodule via $(\pi \otimes \mathrm{id})\Delta$ and in this case

$$ {}^{\mathrm{co}\,\pi}A := {}^{\mathrm{co}\ (\mathrm{id}\otimes\pi)\Delta}A = \{a \in A \mid (\pi \otimes \mathrm{id})\Delta(a) = 1 \otimes a\}.$$

Right coinvariants, written $A^{\mathrm{co}\,\pi}$ are defined analogously.

1.2. Yetter–Drinfeld modules and Nichols algebras

In this subsection we recall the definition of Yetter–Drinfeld modules over Hopf algebras H and we describe the irreducible ones in case H is a group algebra. We also add the definition of Nichols algebras associated with them.

1.2.1. Yetter–Drinfeld modules over Hopf algebras

Let H be a Hopf algebra. A *left Yetter–Drinfeld module* M over H is a left H-module $(M, \cdot)$ and a left H-comodule (M, λ) with $\lambda(m) = m_{(-1)} \otimes m_{(0)} \in H \otimes M$ for all $m \in M$, satisfying the compatibility condition

$$\lambda(h \cdot m) = h_{(1)} m_{(-1)} \mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot m_{(0)} \qquad \forall\ m \in M, h \in H.$$

We denote by ${}^H_H\mathcal{YD}$ the corresponding category. It is a braided monoidal category: for any $M, N \in {}^H_H\mathcal{YD}$, the braiding $c_{M,N} : M \otimes N \to N \otimes M$ is given by

$$c_{M,N}(m \otimes n) = m_{(-1)} \cdot n \otimes m_{(0)} \qquad \forall\ m \in M, n \in N.$$

A *braided* Hopf algebra is a Hopf algebra in ${}^H_H\mathcal{YD}$ such that all structural maps are morphisms in the category.

Suppose H is a finite-dimensional Hopf algebra. Then the category ${}^{H^*}_{H^*}\mathcal{YD}$ is braided equivalent to ${}^H_H\mathcal{YD}$, see [6, Remark 1.1.5 and Theorem 2.2.1]. The functor $F : {}^H_H\mathcal{YD} \to {}^{H^*}_{H^*}\mathcal{YD}$ that yields the braided tensor equivalence is given as follows: for $V \in {}^H_H\mathcal{YD}$ define $F(V)$ to be V as $\Bbbk$-vector space and with its Yetter–Drinfeld module structure defined by

$$f \cdot v = f(\mathcal{S}(v_{(-1)}))v_{(0)}, \quad \lambda(v) = \sum_i \mathcal{S}^{-1}(f_i) \otimes h_i \cdot v \quad \text{for all } v \in V,\ f \in H^*, \tag{1}$$

where $(h_i)_{i\in I}$ and $(f_i)_{i\in I}$ are dual bases of H and H^*.

Assume $H = \Bbbk\Gamma$ with Γ a finite group. In this particular case, a left Yetter–Drinfeld module over $\Bbbk\Gamma$ is a left $\Bbbk\Gamma$-module and left $\Bbbk\Gamma$-comodule M such that

$$\lambda(g \cdot m) = ghg^{-1} \otimes g \cdot m, \qquad \forall\ m \in M_h, g, h \in \Gamma,$$

where $M_h = \{m \in M : \lambda(m) = h \otimes m\}$; clearly, $M = \oplus_{h\in\Gamma} M_h$ and thus M is a Γ-graded vector space such that $g \cdot M_h \subseteq M_{ghg^{-1}}$, for all $g, h \in \Gamma$. We denote this category simply by ${}^\Gamma_\Gamma\mathcal{YD}$. As $\Bbbk\Gamma$ is semisimple, Yetter–Drinfeld modules over $\Bbbk\Gamma$ are completely reducible. The irreducible modules are parameterized by pairs $(\mathcal{O}, \rho)$, where $\mathcal{O}$ is a conjugacy class of Γ and (ρ, V) is an irreducible representation of the centralizer $C_\Gamma(z)$ of a fixed point $z \in \mathcal{O}$. The corresponding Yetter–Drinfeld module is given by $M(\mathcal{O}, \rho) = \mathrm{Ind}_{C_\Gamma(z)}^G V =$

$\Bbbk\Gamma \otimes_{C_\Gamma(z)} V$. Explicitly, let $z_1 = z, \dots, z_n$ be a numeration of $\mathcal{O}$ and let $g_i \in \Gamma$ such that $g_i z g_i^{-1} = z_i$ for all $1 \leq i \leq n$. Then $M(\mathcal{O}, \rho) = \oplus_{1 \leq i \leq n} g_i \otimes V$. The Yetter–Drinfeld module structure is given as follows. Write $g_i v := g_i \otimes v \in M(\mathcal{O}, \rho)$, $1 \leq i \leq n$, $v \in V$. For $v \in V$ and $1 \leq i \leq n$, the action of $g \in \Gamma$ and the coaction are given by

$$g \cdot (g_i v) = g_j(\gamma \cdot v), \qquad \lambda(g_i v) = z_i \otimes g_i v,$$

where $g g_i = g_j \gamma$, for unique $1 \leq j \leq n$ and $\gamma \in C_\Gamma(z)$.

Consider now the function algebra $\Bbbk^\Gamma$ and denote by $\{\delta_g\}_{g \in \Gamma}$ the dual basis of the canonical basis of $\Bbbk\Gamma$. By the braided equivalence described above, we have also a parametrization of the irreducible objects in ${}^{\Bbbk^\Gamma}_{\Bbbk^\Gamma}\mathcal{YD}$. Explicitly, $M(\mathcal{O}, \rho) \in {}^{\Bbbk^\Gamma}_{\Bbbk^\Gamma}\mathcal{YD}$ with the structural maps given by

$$\delta_g \cdot (g_i v) = \delta_{g, z_i^{-1}} g_i v, \qquad \lambda(g_i v) = \sum_{g \in \Gamma} \delta_{g^{-1}} \otimes g \cdot (g_i v).$$

1.2.2. Nichols algebras

The Nichols algebra of a braided vector space (V, c) can be defined in different ways, see [23,6,7,20]. As we are interested in Nichols algebras of braided vector spaces arising from Yetter–Drinfeld modules, we give the explicit definition in this case.

Definition 1.1. [7, Definition 2.1] Let H be a Hopf algebra and $V \in {}^H_H\mathcal{YD}$. A braided $\mathbb{N}$-graded Hopf algebra $B = \bigoplus_{n \geq 0} B(n) \in {}^H_H\mathcal{YD}$ is called the *Nichols algebra* of V if

(i) $\Bbbk \simeq B(0)$, $V \simeq B(1) \in {}^H_H\mathcal{YD}$,
(ii) $B(1) = \mathcal{P}(B) = \{r \in B \mid \Delta_B(b) = b \otimes 1 + 1 \otimes b\}$, the linear space of primitive elements.
(iii) B is generated as an algebra by $B(1)$.

In this case, B is denoted by $\mathfrak{B}(V) = \bigoplus_{n \geq 0} \mathfrak{B}^n(V)$.

For any $V \in {}^H_H\mathcal{YD}$ there is a Nichols algebra $\mathfrak{B}(V)$ associated with it. It can be constructed as a quotient of the tensor algebra $T(V)$ by the largest homogeneous two-sided ideal I satisfying:

- I is generated by homogeneous elements of degree ≥ 2.
- $\Delta(I) \subseteq I \otimes T(V) + T(V) \otimes I$, *i.e.*, it is also a coideal.

In such a case, $\mathfrak{B}(V) = T(V)/I$. See [7, Section 2.1] for details.

An important observation is that the Nichols algebra $\mathfrak{B}(V)$, as algebra and coalgebra, is completely determined just by the braiding.

Let Γ be a finite group. We denote by $\mathfrak{B}(\mathcal{O}, \rho)$ the Nichols algebra associated with the irreducible Yetter–Drinfeld module $M(\mathcal{O}, \rho) \in {}^\Gamma_\Gamma\mathcal{YD}$.

1.3. Bosonization and Hopf algebras with a projection

Let H be a Hopf algebra and B a braided Hopf algebra in ${}^H_H\mathcal{YD}$. The procedure to obtain a usual Hopf algebra from B and H is called the Majid–Radford biproduct or *bosonization*, and it is usually denoted by $B\#H$. As a vector space $B\#H = B \otimes H$, and the multiplication and comultiplication are given by the smash-product and smash-coproduct, respectively. That is, for all $b, c \in B$ and $g, h \in H$

$$(b\#g)(c\#h) = b(g_{(1)} \cdot c)\#g_{(2)}h,$$

$$\Delta(b\#g) = b^{(1)}\#(b^{(2)})_{(-1)}g_{(1)} \otimes (b^{(2)})_{(0)}\#g_{(2)},$$

where $\Delta_B(b) = b^{(1)} \otimes b^{(2)}$ denotes the comultiplication in $B \in {}^H_H\mathcal{YD}$. We identify $b = b\#1$ and $h = 1\#h$; in particular we have $bh = b\#h$ and $hb = h_{(1)} \cdot b\#h_{(2)}$. Clearly, the map $\iota : H \to B\#H$ given by $\iota(h) = 1\#h$ is an injective Hopf algebra map, and the map $\pi : B\#H \to H$ given by $\pi(b\#h) = \varepsilon_B(b)h$ is a surjective Hopf algebra map such that $\pi \circ \iota = \mathrm{id}_H$. Moreover, it holds that $B = (B\#H)^{\mathrm{co}\,\pi}$.

Conversely, let A be a Hopf algebra with bijective antipode and $\pi : A \to H$ a Hopf algebra epimorphism admitting a Hopf algebra section $\iota : H \to A$. Then $B = A^{\mathrm{co}\,\pi}$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ and $A \simeq B\#H$ as Hopf algebras.

1.4. Deforming cocycles

Let A be a Hopf algebra. Recall that a convolution invertible linear map σ in $\mathrm{Hom}_{\Bbbk}(A \otimes A, \Bbbk)$ is a *normalized multiplicative 2-cocycle* if

$$\sigma(b_{(1)}, c_{(1)})\sigma(a, b_{(2)}c_{(2)}) = \sigma(a_{(1)}, b_{(1)})\sigma(a_{(2)}b_{(2)}, c)$$

and $\sigma(a, 1) = \varepsilon(a) = \sigma(1, a)$ for all $a, b, c \in A$, see [22, Section 7.1]. In particular, the inverse of σ is given by $\sigma^{-1}(a, b) = \sigma(\mathcal{S}(a), b)$ for all $a, b \in A$.

Let B be an algebra and consider $\mathrm{Hom}_{\Bbbk}(A^{\otimes n}, B)$ as a $\Bbbk$-algebra with the convolution product given by $(f * g)(a) = f(a_{(1)})g(a_{(2)})$ for all $a \in A^{\otimes n}$. With this notation, the 2-cocycle condition above reads

$$(\varepsilon \otimes \sigma) * [\sigma \circ (\mathrm{id}_A \otimes m)] = (\sigma \otimes \varepsilon) * [\sigma \circ (m \otimes \mathrm{id}_A)]. \tag{2}$$

Using a 2-cocycle σ it is possible to define a new algebra structure on A, which we denote by A_σ, by deforming the multiplication. Moreover, A_σ is indeed a Hopf algebra with $A = A_\sigma$ as coalgebras, deformed multiplication $m_\sigma : A \otimes A \to A$ given by

$$m_\sigma(a, b) = a \cdot_\sigma b = \sigma(a_{(1)}, b_{(1)})a_{(2)}b_{(2)}\sigma^{-1}(a_{(3)}, b_{(3)}) \qquad \text{for all } a, b \in A,$$

and antipode $\mathcal{S}_\sigma : A \to A$ given by (see [13, Theorem 1.6 (b)] for details)

$$\mathcal{S}_\sigma(a) = \sigma(a_{(1)}, \mathcal{S}(a_{(2)}))\mathcal{S}(a_{(3)})\sigma^{-1}(\mathcal{S}(a_{(4)}), a_{(5)}) \qquad \text{for all } a \in A.$$

1.4.1. Deforming cocycles for graded Hopf algebras

Let $A = \bigoplus_{n\geq 0} A_n$ be a $\mathbb{N}_0$-graded Hopf algebra, where A_n denotes the homogeneous component of A of degree n. Let $\sigma\colon A \otimes A \to \Bbbk$ be a normalized multiplicative 2-cocycle and assume that $\sigma|_{A_0\otimes A_0} = \varepsilon \otimes \varepsilon$.

We decompose $\sigma = \sum_{i=0}^{\infty} \sigma_i$ into the (locally finite) sum of homogeneous maps σ_i, with

$$\sigma_i : (A \otimes A)_i = \bigoplus_{p+q=i} A_p \otimes A_q \xrightarrow{\sigma|_{(A\otimes A)_i}} \Bbbk.$$

We set $-i$ for the *degree* of σ_i. Note that due to our assumption $\sigma|_{A_0\otimes A_0} = \varepsilon \otimes \varepsilon$ we have $\sigma_0 = \varepsilon \otimes \varepsilon$. Decomposing $\sigma^{-1} = \sum_{j=0}^{\infty} \eta_j$, where $\eta_j = (\sigma^{-1})|_{(A\otimes A)_j}$, we have that

$$\sum_{i+j=\ell} \sigma_i * \eta_j = \delta_{\ell,0}\, \varepsilon \otimes \varepsilon = \sum_{i+j=\ell} \eta_i * \sigma_j,$$

for $\ell \geq 1$. Note that $\eta_0 = \varepsilon \otimes \varepsilon$ and that for the least positive integer s for which $\sigma_s \neq 0$ we have $\eta_s = -\sigma_s$. Moreover, the cocycle condition (2) implies that

$$\sum_{i+j=\ell} (\varepsilon \otimes \sigma_i) * [\sigma_j \circ (\mathrm{id} \otimes m)] = \sum_{i+j=\ell} (\sigma_i \otimes \varepsilon) * [\sigma_j \circ (m \otimes \mathrm{id})]$$

for all $\ell \geq 1$. In particular,

$$\varepsilon \otimes \sigma_s + [\sigma_s \circ (\mathrm{id} \otimes m)] = \sigma_s \otimes \varepsilon + [\sigma_s \circ (m \otimes \mathrm{id})]$$

which implies that $\sigma_s \colon A \otimes A \to \Bbbk$ is a *Hochschild 2-cocycle* of A. We call σ_s the *graded infinitesimal part* of σ. We denote the Hochschild 2-cohomology group by $\mathrm{H}^2_\varepsilon(A, \Bbbk)$.

2. Computing cohomology

Let $A = \bigoplus_{n \geq 0} A_n$ be a graded Hopf algebra. We say that a filtered Hopf algebra K is a *lifting* of A if $\operatorname{gr} K \simeq A$ as graded Hopf algebras. The aim of this section is to give a recipe for computing the set of A_0-bitrivial bialgebra 2-cocycles $\tilde{\mathrm{H}}^2_b(A)$, see definitions below. For more details on bialgebra cohomology and the definition of the Hochschild and coalgebra differentials ∂^h and ∂^c, respectively, we refer to [21, Section 2].

Remark 2.1. By [14, Theorem 2.2], there is a natural bijection between isomorphisms classes of liftings of a graded bialgebra A and isomorphisms classes of formal deformations of A. Indeed, for a lifting K of A, we denote by m_K the product and we identify K_ℓ with $\oplus_{i=1}^{\ell} A_i$. Since K is a filtered bialgebra, one has that $m_K : A_i \otimes A_j \to K_{i+j}$. Thus, there exist unique homogeneous maps $m_j = \pi_j \circ m_K$ for all $j \geq 0$, such that $m_K(a \otimes b) = \sum_{j \geq 0} m_j(a \otimes b)$; here $\pi_j : K_j \to A_j$ denotes the canonical projection with kernel $\oplus_{i=1}^{j-1} A_i$. Since $\operatorname{gr} K = A$ as graded algebras, one has that $m_0 = m$. The corresponding formal deformation for K is given by $(A[t], m_t)$ with

$$m_t(a \otimes b) = \sum_{j \geq 0} m_j(a \otimes b) t^j, \qquad a, b \in A.$$

Remark 2.2. Let $\sigma \colon A \otimes A \to \Bbbk$ be a normalized multiplicative 2-cocycle. The cocycle deformation A_σ is a filtered bialgebra with the underlying filtration inherited from the grading on A, *i.e.*, the ℓ-th filtered part is $(A_\sigma)_{(\ell)} = \bigoplus_{i=0}^{\ell} A_i$. Note that the associated graded bialgebra $\operatorname{gr} A_\sigma$ can be identified with A, that is, A_σ is a lifting of A. As in Remark 2.1, decomposing the multiplication $m_\sigma = \sigma * m * \sigma^{-1}$ into the sum of homogeneous components m_i of degree $-i$, allows us to identify the filtered $\Bbbk$-linear structure A_σ with a $\Bbbk[t]$-linear structure $(m_\sigma)_t \colon A[t] \otimes A[t] \to A[t]$ induced by $(m_\sigma)_t|_{A \otimes A} = m + tm_1 + t^2 m_2 + \dots$ (as before, we assume that the degree of t is 1). If $\Delta_t \colon A[t] \to A[t] \otimes A[t]$ is the $\Bbbk[t]$-linear map induced by $\Delta_t|_A = \Delta$, then the graded Hopf algebra $A[t]_\sigma = (A[t], (m_\sigma)_t, \Delta_t)$ is a graded formal deformation of A in the sense of [14], see also [19,21]. Note that in case m does not commute with the graded infinitesimal part σ_s of σ, then $(\sigma_s * m - m * \sigma_s, 0)$, is the infinitesimal part of the formal graded deformation (and is a 2-cocycle in $\widehat{\mathrm{Z}}^2_b(A)_s$).

Notation 2.3. Let U, V be graded vector spaces and $f \colon U \to V$ a linear map. For $r \in \mathbb{N}$, define $f_r \colon U \to V$ as the linear map given by $f_r|_{U_r} = f$ and $f_r|_{U_s} = 0$ for $r \neq s$. Similarly we define a linear map $f_{\leq r}$ by $f_{\leq r}|_{U_s} = f$ for $s \leq r$ and $f_{\leq r}|_{U_s} = 0$ for $s > r$. The map $f_{<r}$ is defined by $f_{<r}|_{U_s} = f$ for $s < r$ and $f_{<r}|_{U_s} = 0$ for $s \geq r$.

Recall that the second truncated Gerstenhaber–Schack bialgebra cohomology

$$\widehat{\mathrm{H}}^2_b(A) = \widehat{\mathrm{Z}}^2_b(A)/\widehat{\mathrm{B}}^2_b(A)$$

is given by

$$\begin{aligned}\widehat{\mathrm{Z}}^2_b(A) = \big\{(f,g) \;\big|\; & f\colon A^+\otimes A^+\to A^+,\ g\colon A^+\to A^+\otimes A^+,\\ & af(b,c)+f(a,bc) = f(ab,c)+f(a,b)c,\\ & c_{(1)}\otimes g(c_{(2)}) + (\mathrm{id}\otimes\Delta)g(c) = (\Delta\otimes\mathrm{id})g(c) + g(c_{(1)})\otimes c_{(2)}\\ & (f\otimes m)\Delta(a\otimes b) - \Delta f(a,b) + (m\otimes f)\Delta(a\otimes b) =\\ & \qquad - \Delta(a)g(b) + g(ab) - g(a)\Delta(b)\big\}\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathrm{B}}^2_b(A) = \big\{(f,g) \;\big|\; & \exists h\colon A^+\to A^+,\ f(a,b) = ah(b) - h(ab) + h(a)b,\\ & g(c) = -c_{(1)}\otimes h(c_{(2)}) + \Delta h(c) - h(c_{(1)})\otimes c_{(2)}\big\}.\end{aligned}$$

We view f as a map from $A\otimes A$ to A with the understanding that it is normalized and co-normalized in the sense that $f(\mathrm{id}\otimes u) = 0 = f(u\otimes\mathrm{id})$ and $\varepsilon f = 0$. Similarly, we view g as a map from A to $A\otimes A$ with the understanding that it is normalized and co-normalized in the sense that $gu = 0$ and $(\varepsilon\otimes\mathrm{id})g = 0 = (\mathrm{id}\otimes\varepsilon)g$. We will restrict our attention to the A_0-bitrivial part of the cohomology, *i.e.*, pairs (f,g) where $f|_{A_0\otimes A} = 0 = f|_{A\otimes A_0}$ and $(\pi\otimes\mathrm{id})g = 0 = (\mathrm{id}\otimes\pi)g$. Here $\pi\colon A\to A_0$ denotes the canonical projection. We denote the corresponding sets by $\tilde{\mathrm{Z}}^2_b(A)$ and $\tilde{\mathrm{B}}^2_b(A)$, and the relevant cohomology by

$$\tilde{\mathrm{H}}^2_b(A) = \tilde{\mathrm{Z}}^2_b(A)/\tilde{\mathrm{B}}^2_b(A).$$

We remark that, if A_0 is a group algebra or the dual of a group algebra, then we have that $\tilde{\mathrm{H}}^2_b(A) = \widehat{\mathrm{H}}^2_b(A)$, see [21, Lemma 2.3.1]. For $\ell\in\mathbb{Z}$, we set

$$\tilde{\mathrm{H}}^2_b(A)_\ell = \tilde{\mathrm{Z}}^2_b(A)_\ell/\tilde{\mathrm{B}}^2_b(A)_\ell,$$

for the relevant graded cohomology. It is obtained by restricting to maps of homogeneous degree ℓ. We also point out the following well-known facts: if $f\colon A\otimes A\to A$ is an A_0-trivial Hochschild 2-cocycle, then from the equalities $\partial^h f(h,x,y) = \partial^h f(x,h,y) = \partial^h f(x,y,h)$ and the A_0-triviality it follows that

$$f(hx,y) = hf(x,y), \qquad f(xh,y) = f(x,hy), \qquad f(x,yh) = f(x,y)h,$$

for all $x,y\in A$, $h\in A_0$. If $A_0 = H$ is moreover a Hopf algebra, then one has that $A\simeq B\#H$ as algebras, where $B = A^{\mathrm{co}H}$. Hence, f is H-stable and uniquely determined by its values on $B\otimes B$. This follows from the fact that $f(xh,yk) = f(x,h_{(1)}yS(h_{(2)}))h_{(3)}k$ for all $x,y\in B$ and $h,k\in H$. In particular,

$$\begin{aligned} f(hx,ky) &= hf(x,k_{(1)}yS(k_{(2)}))k_{(3)},\\ f(h_{(1)}xS(h_{(2)}),h_{(3)}yS(h_{(4)})) &= h_{(1)}f(x,y)S(h_{(2)}).\end{aligned}$$

2.1. Liftings as cocycle deformations

Let H be a fixed Hopf algebra with bijective antipode and B a graded connected bialgebra in ${}^H_H\mathcal{YD}$. Set $A = B\#H$ the bosonization of B with H. Then A is a graded Hopf algebra with $A^{\mathrm{co}H} = B$ and $A_0 = H$, see Subsection 1.3. We will study H-bitrivial formal deformations of A. We remark that H-bitriviality is automatic if H is a semisimple and cosemisimple Hopf algebra such that the integrals in H and H^* are cocommutative, see [21, Remark 2.3.2].

Recall that H acts on $\mathrm{H}^2_\varepsilon(B,\Bbbk)$ by $(\eta^h)(x,y) = \eta(h_{(1)}\cdot x, h_{(2)}\cdot y)$ for all $h\in H$, $\eta\in \mathrm{H}^2_\varepsilon(B,\Bbbk)$ and $x,y\in B$. Let $\mathrm{H}^2_\varepsilon(B,\Bbbk)^H$ denote the H-stable part. Then we have a morphism $\tilde{(\ )}\colon \mathrm{H}^2_\varepsilon(B,\Bbbk)^H \to \mathrm{H}^2_\varepsilon(A,\Bbbk)$ given by $\tilde{f}(xh,yk) = f(x,h\cdot y)\varepsilon(k)$. If H is semisimple, then this map is an isomorphism. Moreover, we have a connecting homomorphism [21, Theorem 2.3.7]:

$$\partial\colon \mathrm{H}^2_\varepsilon(A,\Bbbk)\to \widehat{\mathrm{H}}^2_b(A)$$

given by $\partial(f) = (\partial^c f, 0)$, where

$$\partial^c f(x,y) = x_{(1)}y_{(1)}f(x_{(2)},y_{(2)}) - f(x_{(1)},y_{(1)})x_{(2)}y_{(2)}.$$

Note that for $f\in \mathrm{Z}^2_\varepsilon(B,\Bbbk)^H$, one has that $\partial(\tilde{f})\in \tilde{\mathrm{Z}}^2_b(A)$.

Definition 2.4. For an augmented algebra B with multiplication map $\mathrm{m}\colon B\otimes_B B\to B$ define

$$\mathrm{M}(B) := \ker\left(B^+\otimes_B B^+ \xrightarrow{\mathrm{m}} B^+\right).$$

Remark 2.5. It is known that $\mathrm{H}^2_\varepsilon(B,\Bbbk)\simeq \mathrm{Hom}(\mathrm{M}(B),\Bbbk)$, see for example [21, Subsection 4.1]. If $(B^+)^2$ denotes the image of the multiplication $\mathrm{m}\colon B^+\otimes_B B^+\to B^+$ and $\varphi\colon (B^+)^2\to B^+\otimes_B B^+$ is any linear section of the multiplication map, then the isomorphism $\mathrm{Hom}(\mathrm{M}(B),\Bbbk)\xrightarrow{\sim}\mathrm{H}^2_\varepsilon(B,\Bbbk)$ is given by $\hat{\varphi}(f) = f_\varphi$, where $f_\varphi(x\otimes y) = f(x\otimes_B y - \varphi(xy))$.

Let us now fix $\ell<0$ and let $r=-\ell$. We will show that the results and proofs from [21] can be adjusted to show the following: if B is generated as an algebra in degrees at most r and $\mathcal{YD}(\mathrm{M}(A),\mathcal{P}(A))_\ell = 0$, then the connecting homomorphism at degree ℓ,

$$\tilde{\partial}_\ell\colon \mathrm{H}^2_\varepsilon(B,\Bbbk)^H_\ell\to \tilde{\mathrm{H}}^2_b(A)_\ell,$$

given by $\tilde{\partial}_\ell(f) = \partial(\tilde{f})$, is surjective. The following generalizes Lemma 4.2.2 of [21] (see also Theorem 5.3 of [10]). The proof is almost identical.

Lemma 2.6 *(cf. [21, Lemma 4.2.2]). Let $A = B\#H$ be as above and $(F,G)\in \tilde{\mathrm{Z}}^2_b(A)_\ell$ for some $\ell<0$. Set $r=-\ell$. Assume that B is generated as an algebra by elements of degree at most r and that $\mathcal{YD}(\mathrm{M}(B),\mathcal{P}(B))_\ell = 0$. If $F_r = 0$, then $(F,G)\in \tilde{\mathrm{B}}^2_b(A)$.*

Proof. Recall the setting of Notation 2.3. Assume that $F_r = 0$. By degree considerations, one has that $F_{<r} = 0$, hence $F_{\le r} = 0$. Analogously, one deduces that $G_{<r} = 0$. Moreover, also from degree considerations combined with that fact that G is A_0-cotrivial (*i.e.*, $(\pi\otimes \mathrm{id})G = 0 = (\mathrm{id}\otimes\pi)G$), it follows that $G_r = 0$.

Now assume that for some $s>r$, we have that $(F,G)_{<s} = 0$ (as observed above this holds for $s = r+1$). We prove that $(F,G)\sim (F',G')$ where $(F',G')_{\le s} = 0$. The conclusion of the lemma then follows by induction.

We first show that $F_s|_{M(B)}$ lies in $\mathcal{YD}(\mathrm{M}(B), \mathcal{P}(B))_\ell = 0$. Let $u = \sum_i u^i \otimes v^i \in (B \otimes B)_s \cap M(B)$, with all u^i, v^i homogeneous of strictly positive degree, be such that $\mathrm{m}(u) = \sum_i u^i v^i = 0$. Then, by degree considerations and the fact that $\mathrm{m}(u) = 0$, we have that

- $(F \otimes \mathrm{m})\Delta u = F(u) \otimes 1$,
- $(\mathrm{m} \otimes F)\Delta u = \sum_i u^i_{(-1)} v^i_{(-1)} \otimes F(u^i_{(0)}, v^i_{(0)})$,
- $\sum_i \Delta(u^i) G(v^i) = 0$,
- $G(\sum_i u^i v^i) = 0$,
- $\sum_i G(u^i)\Delta(v^i) = 0$.

Hence, by the compatibility of F and G given by $\partial^c F(u) = -\partial^h G(u)$, or, in expanded form $(F \otimes \mathrm{m})\Delta(u) - \Delta F(u) + (\mathrm{m} \otimes F)\Delta u = -\sum_i \Delta(u^i) G(v^i) + G(m(u)) - \sum_i G(u^i)\Delta(v^i))$, we have

$$\Delta F(u) = F(u) \otimes 1 + \sum_i u^i_{(-1)} v^i_{(-1)} \otimes F(u^i_{(0)}, v^i_{(0)}).$$

Therefore, we conclude that $F(u)$ is in $\mathcal{P}(B)$. Since F is H-trivial and consequently also H-stable, we have that F defines an element of $\mathcal{YD}(\mathrm{M}(B), \mathcal{P}(B))_\ell = 0$. Hence, $F_s|_{(B\otimes B)}$ is an ε-coboundary (see Remark 2.5). Let $T\colon B \to \Bbbk$ be the map given by $F_s(x, y) = T(xy)$ for all $x, y \in B^+$. With no loss of generality, we assume that $T = T_s$. Since F_s is H-stable and the xy's span B_s (generation in degrees at most r), we have that T is H-stable and therefore $F_s = -\partial^h T$. Now set $(F', G') = (F, G) - \partial^b T = (F - \partial^h T, G + \partial^c T)$. By construction, we have that $(F', G') \sim (F, G)$, $F'_{\leq s} = 0$, and $G'_{<s} = 0$. We now prove that $G'_s = 0$, which completes the proof. Let $x \in A_i$, $y \in A_j$ with $i, j > 0$ and $i + j = s$. As $F_{\leq s} = 0$ and $G_{<s} = 0$, by the compatibility of F and G, we get that $G(xy) = 0$. From this follows that $G'_s = 0$, since we are assuming that B, and hence also A, is generated as an algebra by elements of degree at most r, which implies that the xy's span A_s. □

Remark 2.7. Let $A = B\#H$ be as above and let $B = \mathrm{T}(V)/I$ for some $V \in {}^H_H\mathcal{YD}$ and let $I \subseteq \mathrm{T}(V)_{(2)} = \bigoplus_{n\geq 2} V^{\otimes n}$ be a graded ideal; here we assume that B is graded by $\deg(v) = 1$ for $v \in V$ with $v \neq 0$ (and hence generated as an algebra in degree 1). Then

$$\mathrm{M}(B) \simeq I/\big(T(V)^+ I + I T(V)^+\big).$$

Furthermore, if I is generated by $R \in {}^H_H\mathcal{YD}$, then the canonical map $R \to \mathrm{M}(B)$ is an epimorphism of graded ${}^H_H\mathcal{YD}$ modules (see [10, Corollary 4.5]). If R is minimal (in the sense that $R \cap (T(V)^+ I + I T(V)^+) = 0$), then the map in question is an isomorphism. In particular:

(1) If $\mathcal{YD}(R, \mathcal{P}(B))_\ell = 0$ for some $\ell < 0$, then every cocycle pair $(F, G) \in \tilde{\mathrm{Z}}^2_b(A)_\ell$ satisfying $F_{-\ell} = 0$ is cohomologically trivial.
(2) If $B = \mathfrak{B}(V) = \mathrm{T}(V)/\langle R\rangle$ is a Nichols algebra and $\mathcal{YD}(R, V) = 0$, then every cocycle pair $(F, G) \in \tilde{\mathrm{Z}}^2_b(A)_\ell$, $\ell < 0$, satisfying $F_{-\ell} = 0$ is cohomologically trivial.

From now on we assume that there exists a right integral $\lambda\colon H \to \Bbbk$ for H^* (*i.e.*, $\mathrm{id} * \lambda = \lambda$) such that

(1) $\lambda(1) = 1$,
(2) λ is stable under the adjoint action of H.

Note that such a map λ always exists if H is a cosemisimple unimodular Hopf algebra, *i.e.*, the space of right and left integrals in H coincide. In case H is finite-dimensional (we are not assuming this anywhere in this section), the later holds if and only if H is semisimple.

Lemma 2.8. *Let $(F,G) \in \tilde{\mathrm{Z}}^2_b(A)_\ell$ for some $\ell < 0$ and write $r = -\ell$. Then the map $f\colon B\otimes B \to \Bbbk$ given by $f|_{(B\otimes B)_r} = \lambda \circ F$ and $f|_{(B\otimes B)_s} = 0$ for $s \neq r$, is an H-stable ε-cocycle. Moreover, if $\tilde{f}\colon A\otimes A \to \Bbbk$ is the induced ε-cocycle, then $(F + \partial^c \tilde{f})_r = 0$.*

Proof. Since F is H-stable and λ is stable under the adjoint action of H, we have that f is H-stable. Let x, y be homogeneous elements of B of strictly positive degrees such that $x\otimes y \in (B\otimes B)_s$. As in the proof of Lemma 2.6, we have that $F_{<r} = 0$ and $G_{\leq r} = 0$ (by using degree considerations and the fact that G is H-cotrivial). Hence $\partial^h G(x,y) = 0$, and the compatibility between F and G yields that

$$\Delta F(x,y) = x_{(1)}y_{(1)} \otimes F(x_{(2)},y_{(2)}) + F(x_{(1)},y_{(1)}) \otimes x_{(2)}y_{(2)} = x_{(-1)}y_{(-1)} \otimes F(x_{(0)},y_{(0)}) + F(x,y)\otimes 1.$$

Applying $\mathrm{m}(\mathrm{id}\otimes\lambda)$ to this equality gives $(\mathrm{id} * \lambda)(F(x,y)) = x_{(-1)}y_{(-1)}f(x_{(0)},y_{(0)}) + F(x,y)$. Since $\mathrm{id} * \lambda = \lambda$, we get that $F(x,y) = f(x,y) - x_{(-1)}y_{(-1)}f(x_{(0)},y_{(0)})$. Moreover, as $\tilde{f}|_{B\otimes B} = f|_{B\otimes B}$ and $f|_{<r} = 0$, we have that $(F + \partial^c\tilde{f})(x,y) = 0$. Since F and $\partial^c\tilde{f}$ are uniquely determined by their values on $B\otimes B$, we conclude that $(F + \partial^c\tilde{f})_r = 0$. □

We end this subsection with the proof of Theorem A and two technical remarks which are useful for calculations.

Proof of Theorem A. Let $(F,G) \in \tilde{\mathrm{Z}}^2_b(A)_\ell$ for some $\ell < 0$ and let $r = -\ell$ as above. Let $f\colon B\otimes B \to \Bbbk$ be the cocycle from Lemma 2.8 and $\tilde{f}\colon A\otimes A \to \Bbbk$ the induced cocycle on A. Set $(F',G) = (F,G) + \tilde{\partial}_\ell f = (F' + \partial^c\tilde{f}, G)$. By Lemma 2.8, it follows that $F'_r = 0$. Hence, by Lemma 2.6 we have that $(F',G) \in \tilde{\mathrm{B}}^2_b(A)$. Thus, the cohomology classes of (F,G) and $-\tilde{\partial}_\ell f$ coincide and consequently, the cohomology class of (F,G) is in the image of $\tilde{\partial}_\ell$. □

Remark 2.9. Fix $\ell < 0$ and set $r = -\ell$. For $f \in \mathrm{Z}^2_\varepsilon(B,\Bbbk)^H_\ell$ and $(u,v) \in (B^+\otimes B^+)_r$, it holds

$$\partial^c(\tilde{f})(u,v) = u_{(-1)}v_{(-1)}f(u_{(0)},v_{(0)}) - f(u,v).$$

Indeed, assume $u \in B_k$ and $v \in B_\ell$ with $k + \ell = r$. Since $B = A^{\mathrm{co}\,\pi}$, we may assume that $u \in P_k(B)$ and $v \in P_\ell(B)$. Then, $\Delta(u) \in u\otimes 1 + u_{(-1)}\otimes u_{(0)} + \sum_{i=1}^{k-1} A_i\otimes A_{k-i}$ and $\Delta(v) \in v\otimes 1 + v_{(-1)}\otimes v_{(0)} + \sum_{i=1}^{\ell-1} A_i\otimes A_{\ell-i}$. This implies that

$$\partial^c(\tilde{f})(u,v) = u_{(1)}v_{(1)}f(u_{(2)},v_{(2)}) - f(u_{(1)},v_{(1)})u_2v_2 = u_{(-1)}v_{(-1)}f(u_{(0)},v_{(0)}) - f(u,v).$$

Remark 2.10. Fix $\ell < 0$, set $r = -\ell$ and let $f \in \mathrm{Z}^2_\varepsilon(B,\Bbbk)^H_\ell$. If $B = T(V)/\langle R\rangle$ with $R \subseteq V\otimes V$, then the only possible candidate for a lifting corresponding to f is

$$A_f = (\mathrm{T}(V)\#H)/\langle R_f\rangle,$$

where R_f is obtained from R by replacing every relation $x = \sum_i u^i\otimes v^i \in R \subseteq V\otimes V$ by

$$x_f = x + \partial^c(\tilde{f})(x) = x - \sum_i \big(f(u^i_{(1)},v^i_{(1)})u^i_{(2)}v^i_{(2)} - u^i_{(1)}v^i_{(1)}f(u^i_{(2)},v^i_{(2)})\big).$$

This follows from the fact that $m_{A_f}|_{(A\otimes A)_2} = m_A + \partial^c(\tilde{f})$. Note that if $\deg(x) < r$, then $x_f = x$, and in case $\deg(x) = r$, it holds that

$$x_f = x - \sum_i \big(f(u^i, v^i) - u^i_{(-1)} v^i_{(-1)} f(u^i_{(0)}, v^i_{(0)})\big).$$

In principle, it is hard to know if the above is indeed a lifting. This is guaranteed if there is a multiplicative cocycle with infinitesimal part equal to $\tilde{f}$. In the very special case when the braiding is symmetric (and hence R is in degree 2, *i.e.*, $R \subseteq V \otimes V$), this is always the case as such cocycles are produced by $e^{\tilde{f}}$.

2.2. Cohomologically homogeneous liftings

The following results in this subsection are well-known for graded deformations of algebras [11] and it is to some extent implicit in [14]. We include some sketches of proofs for completeness.

Lemma 2.11. *Let $B = (B, m, \Delta)$ be a graded bialgebra and let $U = (B[t], m_t^U, \Delta_t^U)$ be a graded deformation. If $\ell \in \mathbb{N}$ and $f \in \widehat{\mathrm{B}}^2_b(B)_{-\ell}$, then U is equivalent to a deformation $V = (B[t], m_t^V, \Delta_t^V)$ such that for $k < \ell$ we have $(m_k^U, \Delta_k^U) = (m_k^V, \Delta_k^V)$ and $(m_\ell^U, \Delta_\ell^U) = (m_\ell^V, \Delta_\ell^V) + f$. Moreover, we can additionally assume that there is an isomorphism $\Phi\colon U \to V$ such that for $k < \ell$ we have that $\Phi|_{B_k} = \mathrm{id}\,|_{B_k}$.*

Proof. Let $s\colon B \to B$ be the homogeneous map of degree $-\ell$ such that $f = \partial^b s$ and consider the $\Bbbk[t]$-linear map $\Phi\colon B[t] \to B[t]$ given by $\Phi|_B = \mathrm{id} + st^k$. The lemma then follows by setting V to be the unique bialgebra structure on $B[t]$ such that Φ is a bialgebra isomorphism, *i.e.*, $m_t^V = \Phi^{-1} \circ m_t^U \circ (\Phi \otimes \Phi)$ and $\Delta_t^V = (\Phi^{-1} \otimes \Phi^{-1}) \circ \Delta_t^U \circ \Phi$. □

Lemma 2.12 *(Infinitesimal cohomological difference). Let B be a graded bialgebra and let $U = (B[t], m_t^U, \Delta_t^U)$ and $V = (B[t], m_t^V, \Delta_t^V)$ be non-equivalent graded deformations. Then V is equivalent to a deformation $W = (B[t], m_t^W, \Delta_t^W)$ such that for some $\ell \in \mathbb{N}$ we have that $(m_k^U, \Delta_k^U) = (m_k^W, \Delta_k^W)$ for $k < \ell$ and $(m_\ell^U, \Delta_\ell^U) - (m_\ell^W, \Delta_\ell^W) \in \widehat{\mathrm{Z}}^2_b(B)_{-\ell}$ represents a non-trivial cohomology class. The number ℓ and the cohomology class of $(m_\ell^U, \Delta_\ell^U) - (m_\ell^W, \Delta_\ell^W)$ is uniquely determined by the equivalence classes of U and V.*

Proof. If $(m_1^U, \Delta_1^U) - (m_1^V - \Delta_1^V)$ represents a nontrivial cohomology class, then we are done. Otherwise, let $s_1\colon B \to B$ be a homogeneous map of degree -1 such that $(m_1^U, \Delta_1^U) - (m_1^V - \Delta_1^V) = \partial^b s_1$ and let $\Phi_1\colon B[t] \to B[t]$ be given by $\Phi_1 = \mathrm{id} + s_1 t$. Take W_1 as the unique bialgebra structure on $B[t]$ such that Φ_1 is a bialgebra isomorphism; then $(m_1^U, \Delta_1^U) = (m_1^{W_1} - \Delta_1^{W_1})$. If $(m_2^U, \Delta_2^U) - (m_2^{W_1} - \Delta_2^{W_1})$ represents a nontrivial cohomology class, then we are done. On the contrary, take a homogeneous map $s_2\colon B \to B$ of degree -2 such that $(m_2^U, \Delta_2^U) - (m_2^{W_1} - \Delta_2^{W_1}) = \partial^b s_2$, and define $\Phi_2\colon B \to B$ by $\Phi_2|_B = \mathrm{id} + s_1 t + s_2 t^2 = \Phi_1 + t^2 s_2$. Let W_2 be the unique bialgebra structure on $B[t]$ such that Φ_2 is a bialgebra isomorphism. In a similar fashion as above, we may define s_k, Φ_k for $k \geq 3$. We claim that this process eventually stops. Indeed, suppose that it does not. We then define $W = \lim_{k\to\infty} W_k$ to be the unique bialgebra structure on $B[t]$ such that $\Phi\colon B[t] \to B[t]$ given by $\Phi|_B = \mathrm{id} + \sum_{k=1}^\infty s_k = \lim_{k\to\infty} \Phi_k$ is a bialgebra isomorphism. Since the sum $\sum_{k=1}^\infty s_k$ is locally finite it is therefore well-defined. But in such a case, we would have that $U = W$; a contradiction. □

Definition 2.13 *(Infinitesimal difference).* Let B be a graded bialgebra and $U = (B[t], m_t^U, \Delta_t^U)$, $V = (B[t], m_t^V, \Delta_t^V)$ be non-equivalent graded deformations. Let W and ℓ be as in the lemma above. We define the *infinitesimal difference* $U -_\varepsilon V$ to be the cohomological class of $(m_\ell^U, \Delta_\ell^U) - (m_\ell^W, \Delta_\ell^W)$. In such a case, we say that the infinitesimal difference is of degree ℓ. If U and V are equivalent, then we define $U -_\varepsilon V = 0$ and say that $\deg(U -_\varepsilon V) = \infty$. If L_1, L_2 are liftings of B, we define their infinitesimal difference $L_1 -_\varepsilon L_2$ to be the infinitesimal difference between the associated graded deformations.

Note that infinitesimal difference is "associative" in each degree in the following sense: if $\deg(L -_\varepsilon S) = \deg(S -_\varepsilon T) = \ell$, then

(a) $\deg(L -_\varepsilon T) = \ell$ and $L -_\varepsilon T = (L -_\varepsilon S) + (S -_\varepsilon T)$, if $L -_\varepsilon S \neq -(S -_\varepsilon T)$,
(b) $\deg(L -_\varepsilon T) > \ell$, if $L -_\varepsilon S = -(S -_\varepsilon T)$.

Definition 2.14. Let $(L_\lambda)_{\lambda\in\Lambda}$ be a collection of liftings of a graded bialgebra B. We say that the collection is *cohomologically homogeneous*, if for every $\lambda \in \Lambda$, every $\ell \in \mathbb{N}$, and every cohomology class $\alpha \in \widehat{\mathrm{H}}_b(B)_{-\ell}$, there exists a $\mu \in \Lambda$ such that $L_\lambda -_\varepsilon L_\mu = \alpha$.

Definition 2.15. We say that the lifting problem for a graded bialgebra B is *obstruction-free* if every partial graded deformation (*i.e.*, deformation over $\Bbbk[t]/(t^\ell)$ for some $\ell \in \mathbb{N}$) extends to a formal graded deformation.

We end this section with the following theorem.

Theorem 2.16. *Let B be a graded bialgebra such that $\widehat{\mathrm{H}}^2_b(B)_- := \bigoplus_{k\in\mathbb{N}} \widehat{\mathrm{H}}^2_b(B)_{-k}$ is finitely graded,* i.e., *for sufficiently large $k \in \mathbb{N}$ we have $\widehat{\mathrm{H}}^2_b(B)_{-k} = 0$. Then:*

(i) *The lifting problem for B is obstruction-free if and only if there exists a cohomologically homogeneous collection of liftings.*
(ii) *If $(L_\lambda)_{\lambda\in\Lambda}$ is a cohomologically homogeneous collection of liftings, then, up to equivalence, the collection contains all liftings.*

Proof. The only (somewhat) non-obvious part of the theorem is assertion (ii). Suppose on the contrary, that $(L_\lambda)_{\lambda\in\Lambda}$ is a cohomologically homogeneous collection of liftings such that there exists a lifting L that is not equivalent to any L_λ. Let $\ell \in \mathbb{N}$ be the largest number such that $L -_\varepsilon L_\lambda =: \alpha$ is of degree ℓ. By cohomological homogeneity, there exists $\mu \in \Lambda$ such that $L_\lambda -_\varepsilon L_\mu = -\alpha$. But then we either have that L and L_μ are equivalent or that the cohomological difference $L -_\varepsilon L_\mu$ is of degree strictly larger than ℓ; a contradiction. □

3. Copointed Hopf algebras over dihedral groups

3.1. The commutative algebra $\Bbbk^{\mathbb{D}_m}$

As in [15], we use the following presentation by generators and relations for the dihedral group of order $2m$:

$$\mathbb{D}_m := \langle g, h |\ g^2 = 1 = h^m\ ,\ gh = h^{-1}g\rangle. \tag{3}$$

Because of our purposes, we assume that $m = 4a \geq 12$, $n = \frac{m}{2} = 2a$ and we fix ω an m-th primitive root of unity.

A linear basis of the group algebra $\Bbbk\mathbb{D}_m$ is given by $\{e_{ij} = g^i h^j :\ i = 0, 1,\ j = 0, \dots, m-1\}$. Let $\{\varphi_{ij} :\ i = 0, 1,\ j = 0, \dots, m-1\}$ be the corresponding dual basis, *i.e.*, $\varphi_{i,j}(e_{k,\ell}) = \delta_{i,k}\delta_{j,\ell}$. By definition, we have that

- $\varphi_{ij}\varphi_{k\ell} = \delta_{ik}\delta_{j\ell}\varphi_{ij}$.
- $1_{\Bbbk^{\mathbb{D}_m}} = \sum_{i,j} \varphi_{ij}$.
- $\varepsilon(\varphi_{ij}) = \delta_{i,0}\delta_{j,0}$.
- $\Delta(\varphi_{ij}) = \sum_{k,\ell} \varphi_{i+k,(-1)^k(j-\ell)} \otimes \varphi_{k,\ell}$.
- $\mathcal{S}(\varphi_{ij}) = \varphi_{i,(-1)^{i+1}j}$.

Table 1
Linear characters of $\mathbb{D}_m$.

z	h^n	h^b, $1 \leq b \leq n-1$	g	gh
α_0	1	1	1	1
α_1	1	1	-1	-1
α_2	$(-1)^n$	$(-1)^b$	1	-1
α_3	$(-1)^n$	$(-1)^b$	-1	1

Throughout this section we set $H = \Bbbk^{\mathbb{D}_m}$. The group $G(H) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ is given by $\mathrm{Alg}(\Bbbk\mathbb{D}_m, \Bbbk)$, which coincides with the group of multiplicative characters $\widehat{\mathbb{D}_m}$ given by the one-dimensional representations of $\mathbb{D}_m$. These are given by Table 1.

For $0 \leq k \leq 3$, write $\alpha_k = \sum_{i,j} \alpha_k(g^i h^j)\varphi_{ij}$; explicitly,

$$\alpha_0 = \sum_{i,j} \varphi_{ij} = \sum_j \varphi_{0j} + \sum_j \varphi_{1j} = \varepsilon,$$
$$\alpha_1 = \sum_{i,j} (-1)^i \varphi_{ij} = \sum_j \varphi_{0j} - \sum_j \varphi_{1j},$$
$$\alpha_2 = \sum_{i,j} (-1)^j \varphi_{ij} = \sum_j (-1)^j \varphi_{0j} + \sum_j (-1)^j \varphi_{1j},$$
$$\alpha_3 = \sum_{i,j} (-1)^{i+j} \varphi_{ij} = \sum_j (-1)^j \varphi_{0j} - \sum_j (-1)^j \varphi_{1j}.$$

It holds that $\alpha_2\alpha_3 = \alpha_1 = \varepsilon$, $\operatorname{ord}\alpha_i = 2$ with $i \neq 0$ and $G(\Bbbk^{\mathbb{D}_m}) = \langle\alpha_2\rangle \times \langle\alpha_3\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. For $k = 0, 1$ and $0 \leq r \leq n$, write $\theta_{k,r} = \sum_\ell \omega^{r\ell}\varphi_{k,\ell} \in \Bbbk^{\mathbb{D}_m}$. It holds that

$$1 = \varepsilon = \theta_{0,0} + \theta_{1,0}, \quad \alpha_1 = \theta_{0,0} - \theta_{1,0}, \quad \alpha_2 = \theta_{0,n} + \theta_{1,n}, \quad \alpha_3 = \theta_{0,n} - \theta_{1,n}. \tag{4}$$
$$\theta_{0,r}\theta_{1,s} = 0, \quad \theta_{k,r}\theta_{k,s} = \theta_{k,r+s}, \qquad \text{for all } k = 0, 1,\ 0 \leq r, s \leq n. \tag{5}$$

3.2. Yetter–Drinfeld modules and finite-dimensional Nichols algebras over $\Bbbk^{\mathbb{D}_m}$

As we pointed out before, the category ${}^{H^*_{H^*}}_{H^*}\mathcal{YD}$ is braided equivalent to ${}^{H}_{H}\mathcal{YD}$. Since finite-dimensional Nichols algebras in ${}^{\mathbb{D}_m}_{\mathbb{D}_m}\mathcal{YD}$ were classified in [15], all finite-dimensional Nichols algebras in ${}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$ are known. For details concerning simple objects in ${}^{\mathbb{D}_m}_{\mathbb{D}_m}\mathcal{YD}$ see [15, Section 2].

The irreducible Yetter–Drinfeld modules that give rise to finite-dimensional Nichols algebras are associated with the conjugacy classes of h^n and h^i with $1 \leq i < n$, see [15, Table 2]. In the following, we describe these modules explicitly as well as the families of reducible Yetter–Drinfeld modules with finite-dimensional Nichols algebras associated with them.

Recall that the non-trivial conjugacy classes of $\mathbb{D}_m$ and the corresponding centralizers are

- $\mathcal{O}_{h^n} = \{h^n\}$ and $C_{\mathbb{D}_m}(h^n) = \mathbb{D}_m$.
- $\mathcal{O}_{h^i} = \{h^i, h^{m-i}\}$ and $C_{\mathbb{D}_m}(h^i) = \langle h\rangle \simeq \mathbb{Z}/(m)$, for $1 \leq i < n$.
- $\mathcal{O}_g = \{gh^j : j \text{ even}\}$ and $C_{\mathbb{D}_m}(g) = \langle g\rangle \times \langle h^n\rangle \simeq \mathbb{Z}/(2) \times \mathbb{Z}/(2)$.
- $\mathcal{O}_{gh} = \{gh^j : j \text{ odd}\}$ and $C_{\mathbb{D}_m}(gh) = \langle gh\rangle \times \langle h^n\rangle \simeq \mathbb{Z}/(2) \times \mathbb{Z}/(2)$.

3.2.1. Yetter–Drinfeld modules and Nichols algebras associated with $\mathcal{O}_{h^i}$, *with* $1 \leq i < n$

For $0 \leq k < m$, denote by $\Bbbk_{\chi^k}$ the simple representation of $C_{\mathbb{D}_m}(h^i) = \langle h\rangle \simeq \mathbb{Z}/(m)$ given by the character $\chi^k(h) = \omega^k$.

Take e and g as representatives of left cosets in $\mathbb{D}_m/\langle h\rangle$, with $h^i = eh^ie$ and $h^{m-i} = gh^ig$. Then $M_{(i,k)} = M(\mathcal{O}_{h^i}, \chi^k) \in {}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$ is spanned linearly by the elements $y_1^{(i,k)} = e \otimes 1$ and $y_2^{(i,k)} = g \otimes 1$. Its Yetter–Drinfeld module structure is given by

$$\varphi_{rs} \cdot y_1^{(i,k)} = \varphi_{rs}(\mathcal{S}(h^i))y_1^{(i,k)} = \delta_{r,0}\delta_{s,-i}\, y_1^{(i,k)}, \qquad \varphi_{rs} \cdot y_2^{(i,k)} = \varphi_{rs}(\mathcal{S}(h^{-i}))y_2^{(i,k)} = \delta_{r,0}\delta_{s,i}\, y_2^{(i,k)},$$
$$\lambda(y_1^{(i,k)}) = \theta_{0,-k} \otimes y_1^{(i,k)} + \theta_{1,k} \otimes y_2^{(i,k)}, \qquad \lambda(y_2^{(i,k)}) = \theta_{1,-k} \otimes y_1^{(i,k)} + \theta_{0,k} \otimes y_2^{(i,k)}.$$

The irreducible modules with finite-dimensional Nichols algebra are the ones given by the pairs (i,k) satisfying that $\omega^{ik} = -1$. We set $J = \{(i,k) :\ 1 \le i < n, 1 \le k < m \text{ such that } \omega^{ik} = -1\}$. By [3, Theorem 3.1], one has $\mathfrak{B}(\mathcal{O}_{h^i}, \chi^k) \simeq \bigwedge M_{(i,k)}$, for all $(i,k) \in J$, and $\dim \mathfrak{B}(\mathcal{O}_{h^i}, \chi^k) = 4$.

Example 3.1. Consider $M_{i,n} = \Bbbk\{a_1, a_2\} \in {}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$ with i odd and $1 \le i \le n-1$. If we take $x = a_1 + a_2$ and $y = a_1 - a_2$ we have that

$$\lambda(x) = \theta_{0,n} \otimes a_1 + \theta_{1,n} \otimes a_2 + \theta_{1,n} \otimes a_1 + \theta_{0,n} \otimes a_2 = (\theta_{0,n} + \theta_{1,n}) \otimes (a_1 + a_2) = \alpha_2 \otimes x,$$
$$\lambda(y) = \theta_{0,n} \otimes a_1 + \theta_{1,n} \otimes a_2 - \theta_{1,n} \otimes a_1 - \theta_{0,n} \otimes a_2 = (\theta_{0,n} - \theta_{1,n}) \otimes (a_1 - a_2) = \alpha_3 \otimes y,$$

that is, both elements are homogeneous with respect to the coaction of $\Bbbk^{\mathbb{D}_m}$ on $M_{i,n}$. Thus, we may present $M_{i,n} \in {}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$ as the Yetter–Drinfeld modules spanned linearly by x and y with its structure given by

$$\lambda(x) = \alpha_2 \otimes x, \qquad \lambda(y) = \alpha_3 \otimes y,$$
$$\varphi_{k\ell} \cdot x = \varphi_{k\ell} \cdot (a_1 + a_2) = \delta_{k,0}\delta_{i,-\ell}a_1 + \delta_{k,0}\delta_{i,\ell}a_2 = \delta_{k,0}(\delta_{i,-\ell}\frac{x+y}{2} + \delta_{i,\ell}\frac{x-y}{2}),$$
$$\varphi_{k\ell} \cdot y = \varphi_{k\ell} \cdot (a_1 - a_2) = \delta_{k,0}\delta_{i,-\ell}a_1 - \delta_{k,0}\delta_{i,\ell}a_2 = \delta_{k,0}(\delta_{i,-\ell}\frac{x+y}{2} - \delta_{i,\ell}\frac{x-y}{2}),$$
$$\alpha_2 \cdot x = -x, \qquad \alpha_3 \cdot x = -x,$$
$$\alpha_2 \cdot y = -y, \qquad \alpha_3 \cdot y = -y.$$

Consider now the set $\mathcal{I}$ of all sequences of finite length of lexicographically ordered pairs $((i_1,k_1),\dots,(i_r,k_r))$ such that $(i_s,k_s) \in J$ and $\omega^{i_sk_t+i_tk_s} = 1$ for all $1 \le s,t \le r$.

For $I = ((i_1,k_1),\dots,(i_r,k_r)) \in \mathcal{I}$, we define $M_I = \bigoplus_{1\le j\le r} M_{(i_j,k_j)}$. By [15, Proposition 2.5], we have that $\mathfrak{B}(M_I) \simeq \bigwedge M_I$ and $\dim \mathfrak{B}(M_I) = 4^{|I|}$, where $|I| = r$ denotes the length of I.

3.2.2. Yetter–Drinfeld modules and Nichols algebras associated with $\mathcal{O}_{h^n}$

Since h^n is central, $\mathcal{O}_{h^n} = \{h^n\}$ and $C_{\mathbb{D}_m}(h^n) = \mathbb{D}_m$. The irreducible representations of $\mathbb{D}_m$ are of degree 1 or 2. Explicitly, there are:

(i) $n-1 = \frac{m-2}{2}$ irreducible representations of degree 2 given by $\rho_\ell : \mathbb{D}_m \to \mathbf{GL}(V)$ with

$$\rho_\ell(g^ah^b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^a \begin{pmatrix} \omega^\ell & 0 \\ 0 & \omega^{-\ell} \end{pmatrix}^b, \quad \ell \in \mathbb{N} \text{ odd with } 1 \le \ell < n. \tag{6}$$

(ii) 4 irreducible representations of degree 1 given in Table 1.

The irreducible Yetter–Drinfeld modules with finite-dimensional Nichols algebra are the ones given by the two-dimensional representations ρ_ℓ with $\ell \in \mathbb{N}$ odd.

Fix $\ell \in \mathbb{N}$ odd with $1 \leq \ell < n$ and consider the two-dimensional simple representation (ρ_ℓ, V) of $\mathbb{D}_m$ described in (6) above. Then $M_\ell = M(\mathcal{O}_{h^n}, \rho_\ell) \in {}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$ is spanned linearly by the elements $x_1^{(\ell)}, x_2^{(\ell)}$, and its Yetter–Drinfeld module structure is given by

$$\varphi_{ij} \cdot x_k^{(\ell)} = \varphi_{ij}(\mathcal{S}(h^n))x_k^{(\ell)} = \delta_{i,0}\delta_{j,n}x_k^{(\ell)}, \quad k = 1, 2 \qquad \text{and} \tag{7}$$

$$\lambda(x_1^{(\ell)}) = \theta_{0,-\ell} \otimes x_1^{(\ell)} + \theta_{1,\ell} \otimes x_2^{(\ell)}, \qquad \lambda(x_2^{(\ell)}) = \theta_{1,-\ell} \otimes x_1^{(\ell)} + \theta_{0,\ell} \otimes x_2^{(\ell)}.$$

By [3, Theorem 3.1], one has that $\mathfrak{B}(\mathcal{O}_{h^n}, \rho_\ell) \simeq \bigwedge M_\ell$, and consequently $\dim \mathfrak{B}(\mathcal{O}_{h^n}, \rho_\ell) = 4$.

Consider now the set $\mathcal{L}$ of all sequences of finite length $(\ell_1, \dots, \ell_r)$ with $\ell_i \in \mathbb{N}$ odd and $1 \leq \ell_1, \dots, \ell_r < n$. Then, for $L = (\ell_1, \dots, \ell_r) \in \mathcal{L}$ we define $M_L = \bigoplus_{1 \leq i \leq r} M_{\ell_i}$. Clearly, $M_L \in {}^{\mathbb{D}_m}_{\mathbb{D}_m}\mathcal{YD}$ is *reducible* and by [15, Proposition 2.8], we have that $\mathfrak{B}(M_L) \simeq \bigwedge M_L$ and $\dim \mathfrak{B}(M_L) = 4^{|L|}$, where $|L| = r$ denotes the length of L.

3.2.3. Yetter–Drinfeld modules and Nichols algebras associated with mixed classes

Finally, we describe a family of reducible Yetter–Drinfeld modules given by direct sums of the modules described above.

Let $\mathcal{K}$ be the set of all pairs of sequences of finite length (I, L) with $I = ((i_1, k_1), \dots, (i_r, k_r)) \in \mathcal{I}$ and $L = (\ell_1, \dots, \ell_s) \in \mathcal{L}$ such that k_j is odd for all $1 \leq j \leq r$ and $\omega^{i_j \ell_t} = -1$ for all $1 \leq j \leq r$ and $1 \leq t \leq s$.

As before, for $(I, L) \in \mathcal{K}$ we define $M_{I,L} = \left(\bigoplus_{1 \leq j \leq s} M_{(i_j, k_j)}\right) \oplus \left(\bigoplus_{1 \leq t \leq s} M_{\ell_t}\right)$. By [15, Proposition 2.12], we have that $\mathfrak{B}(M_{I,L}) \simeq \bigwedge M_{I,L}$ and $\dim \mathfrak{B}(M_{I,L}) = 4^{|I|+|L|}$.

We end this subsection with the classification of finite-dimensional Nichols algebras over $\Bbbk^{\mathbb{D}_m}$. It follows from the braided equivalence between ${}^{\mathbb{D}_m}_{\mathbb{D}_m}\mathcal{YD}$ and ${}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$.

Theorem 3.2. *[15, Theorem A] Let $\mathfrak{B}(M)$ be a finite-dimensional Nichols algebra in ${}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$. Then $\mathfrak{B}(M) \simeq \bigwedge M$, with M isomorphic either to M_I, or to M_L, or to $M_{I,L}$, with $I \in \mathcal{I}$, $L \in \mathcal{L}$ and $(I, L) \in \mathcal{K}$, respectively.* □

3.3. Finite-dimensional copointed Hopf algebras over $\mathbb{D}_m$

In this section we describe all finite-dimensional Hopf algebras A such that the corradical A_0 is isomorphic to $\Bbbk^{\mathbb{D}_m}$, for $m = 4a \geq 12$. Note that these algebras have the Chevalley property.

Using again the braided equivalence ${}^{\mathbb{D}_m}_{\mathbb{D}_m}\mathcal{YD} \cong {}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$, we have the following result.

Theorem 3.3. *[15, Theorem 3.2] Let A be a finite-dimensional Hopf algebra with $A_0 = \Bbbk^{\mathbb{D}_m}$. Then A is generated in degree one, that is, by its first term A_1 in the coradical filtration.* □

Remark 3.4. To find all possible Hopf algebras A with $A_0 = \Bbbk^{\mathbb{D}_m}$ we will use the theory developed in Section 2. Assume $\operatorname{gr} A \simeq B \# \Bbbk^{\mathbb{D}_m}$ with $B \in {}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$. Let $b \in B$ and denote by $\lambda(b) = b_{(-1)} \otimes b_{(0)}$ the coaction of $b \in {}^{\mathbb{D}_m}_{\mathbb{D}_m}\mathcal{YD}$. Recall that the action of $\Bbbk^{\mathbb{D}_m}$ on B is given by $f \cdot b = f(\mathcal{S}(b_{(-1)}))b_{(0)}$. By the definition of the $\Bbbk^{\mathbb{D}_m}$-action on $\mathrm{Z}^2_\varepsilon(B, \Bbbk)$ given in Subsection 2.1, we have that a 2-cocycle $\eta \in \mathrm{Z}^2_\varepsilon(B, \Bbbk)$ is $\Bbbk^{\mathbb{D}_m}$-invariant if and only if

$$f(\mathcal{S}(y_{(-1)}x_{(-1)}))\eta(x_{(0)}, y_{(0)}) = f(1)\eta(x, y) \qquad \text{for all } f \in \Bbbk^{\mathbb{D}_m},\ x, y \in B.$$

As said before, any 2-cocycle deformation of a coradically graded Hopf algebra A is a lifting. In general, the converse is not known to be true. In case the coradical is $\Bbbk^{\mathbb{D}_m}$, the converse holds by the following result.

Theorem 3.5. *Let A be a finite-dimensional Hopf algebra with $A_0 \simeq \Bbbk^{\mathbb{D}_m}$. Then A is a cocycle deformation of $\mathfrak{B}(M)\#\Bbbk^{\mathbb{D}_m}$ for some finite-dimensional Nichols algebra $\mathfrak{B}(M) \in {}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$.*

Proof. Since the coradical A_0 is a Hopf subalgebra, we know that $\operatorname{gr} A \simeq B\#\Bbbk^{\mathbb{D}_m}$ for some finite-dimensional braided Hopf algebra $B \in {}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$, and A is a lifting of $B\#\Bbbk^{\mathbb{D}_m}$. Moreover, by Theorem 3.2, $B \simeq \mathfrak{B}(M)$ for some $M \in {}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$. By Remark 2.1 and Theorem A, A then corresponds to a formal deformation with infinitesimal part given by $\partial^c(\tilde{\eta})$, for some $\eta \in \mathrm{Z}^2_{\varepsilon}(B,\Bbbk)^{\Bbbk^{\mathbb{D}_m}}$. On the other hand, since the braiding in ${}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$ is symmetric, by [16, Corollary 2.6], the map $\sigma = e^{\tilde{\eta}} = \sum_{i=0}^{\infty} \frac{\tilde{\eta}^{*i}}{i!} \colon \operatorname{gr} A \otimes \operatorname{gr} A \to \Bbbk$ is a normalized multiplicative 2-cocycle with infinitesimal part equal to $\tilde{\eta}$. Hence, by Remark 2.10, we have that $(\operatorname{gr} A)_\sigma \simeq A$ because B is quadratic. □

Remark 3.6. Let $B = \mathfrak{B}(V)$ be a finite-dimensional Nichols algebra in ${}^{\Bbbk^{\mathbb{D}_m}}_{\Bbbk^{\mathbb{D}_m}}\mathcal{YD}$, $\{x_j\}_{j\in J}$ a basis of V and $\{d_j\}_{j\in J}$ its dual basis. Since B is an exterior algebra, by Remark 2.7 we have that $M(B)$ is linearly generated by $\{x_j \otimes x_k + x_k \otimes x_j\}_{j,k\in J}$ and consequently, by Remark 2.5 it follows that $\mathrm{Z}^2_{\varepsilon}(B,\Bbbk)$ is linearly generated by the symmetric functionals $\{d_j \otimes d_k + d_k \otimes d_j\}_{j,k\in J}$.

Let $\operatorname{gr} A = \bigoplus_{i>0} A_i/A_{i-1}$ be the graded Hopf algebra associated with the coradical filtration. Then $\operatorname{gr} A \simeq B\#\Bbbk^{\mathbb{D}_m}$, where $B = (\operatorname{gr} A)^{\mathrm{co}\,\pi}$, and B inherits the gradation of A with $\operatorname{gr} A(n) = B(n)\#A_0$. If $x \in B(1)$, we will denote again $x = x\#1$ if no confusion arises.

Now we introduce three families of quadratic algebras given by deformations of bosonizations of Nichols algebras. Then we prove that these families exhaust the possible Hopf algebras with coradical $\Bbbk^{\mathbb{D}_m}$. We follow the notation of [8].

3.3.1. Deformation of $\mathfrak{B}(M_I)\#\Bbbk^{\mathbb{D}_m}$

For $I \in \mathcal{I}$, define $\zeta_I = (\zeta_{i,k,p,q})_{(i,k),(p,q)\in I}$ as a family of elements in $\Bbbk$ such that $\zeta_{i,k,p,q} = 0$ if $i \neq p$. We call ζ_I a *lifting datum* for I.

Definition 3.7. Let $I \in \mathcal{I}$ and ζ_I a lifting datum for I. We denote by $\mathcal{A}(\zeta_I)$ the $\Bbbk$-algebra given by $(T(M_I)\#\Bbbk^{\mathbb{D}_m})/\mathcal{J}_{\zeta_I}$, where $\mathcal{J}_{\zeta_I}$ is the two-sided ideal generated by the elements

$$y_r^{(i,k)}y_r^{(p,q)} + y_r^{(p,q)}y_r^{(i,k)}, \qquad \text{for all } (i,k),(p,q) \in I,\ r = 1,2,$$
$$y_1^{(i,k)}y_2^{(p,q)} + y_2^{(p,q)}y_1^{(i,k)} - [\zeta_{i,k,p,q}(1-\theta_{0,q-k}) - \zeta_{i,q,p,k}\theta_{1,k-q}], \qquad \text{for all } (i,k),(p,q) \in I.$$

A direct computation shows that $\mathcal{J}_{\zeta_I}$ is a Hopf ideal and thus $\mathcal{A}(\zeta_I)$ is a Hopf algebra. The coalgebra structure is determined for all $(i,k) \in I$ by

$$\Delta(y_1^{(i,k)}) = y_1^{(i,k)} \otimes 1 + \theta_{0,-k} \otimes y_1^{(i,k)} + \theta_{1,k} \otimes y_2^{(i,k)}, \qquad \varepsilon(y_1^{(i,k)}) = 0,$$
$$\Delta(y_2^{(i,k)}) = y_2^{(i,k)} \otimes 1 + \theta_{1,-k} \otimes y_1^{(i,k)} + \theta_{0,k} \otimes y_2^{(i,k)}, \qquad \varepsilon(y_2^{(i,k)}) = 0.$$

Lemma 3.8. *$\mathcal{A}(\zeta_I)$ is a lifting of $\mathfrak{B}(M_I)\#\Bbbk^{\mathbb{D}_m}$ and any lifting of $\mathfrak{B}(M_I)\#\Bbbk^{\mathbb{D}_m}$ is isomorphic to $\mathcal{A}(\zeta_I)$ for some lifting datum ζ_I. In particular, $\mathcal{A}(\zeta_I)_0 = \Bbbk^{\mathbb{D}_m}$ and $\dim \mathcal{A}(\zeta_I) = 4^{|I|}2m$.*

Proof. We begin by proving that $\mathcal{A}(\zeta_I)$ is a lifting of $\mathfrak{B}(M_I)\#\Bbbk^{\mathbb{D}_m}$. To see this, it is enough to show that $\mathcal{A}(\zeta_I)$ is a cocycle deformation of $A_I = \mathfrak{B}(M_I)\#\Bbbk^{\mathbb{D}_m}$. For this, we apply the results of Section 2: To produce the multiplicative cocycle, we look for an element μ in $\tilde{\mathrm{H}}^2_b(A_I)_2$, because the relations in $\mathfrak{B}(M_I)$ are quadratic. By Theorem A, it is enough to look for $\eta \in \mathrm{Z}^2_{\varepsilon}(\mathfrak{B}(M_I),\Bbbk)^{\Bbbk^{\mathbb{D}_m}}$ and take $\mu = \partial^c(\tilde{\eta})$. Since by Remark 3.6, $\mathrm{Z}^2_{\varepsilon}(\mathfrak{B}(M_I),\Bbbk)$ is linearly spanned by linear functionals, by [16, Corollary 2.6] the exponentiation $\sigma = e^{\tilde{\eta}}$ yields a multiplicative cocycle for A_I. To obtain the deformation performed by the multiplicative

cocycle, we need to set only the deformation in degree 2, by Remark 2.10. It turns out that the parameters describing $\mathrm{Z}^2_\varepsilon(\mathfrak{B}(M_I),\Bbbk)^{\Bbbk^{\mathbb{D}_m}}$ are exactly those that produce the lifting data for I.

Recall that M_I is linearly spanned by the elements $y_1^{(i_j,k_j)}, y_2^{(i_j,k_j)}$, with $1 \le j \le s := |I|$, which are homogeneous in ${}^{\mathbb{D}_m}_{\mathbb{D}_m}\mathcal{YD}$. Let $\{d_1^{(i_j,k_j)}, d_2^{(i_j,k_j)}\}_{1\le j\le s}$ denote the dual basis. By Remark 3.6, we know that $\mathrm{Z}^2_\varepsilon(\mathfrak{B}(M_I),\Bbbk)$ is linearly spanned by the elements

$$\begin{aligned}
\eta_{rr}^{ikpq} &= d_r^{(i,k)} \otimes d_r^{(p,q)} + d_r^{(p,q)} \otimes d_r^{(i,k)}, && \text{for } r = 1,2,\ (i,k),(p,q) \in I,\\
\eta_{12}^{ikpq} &= d_1^{(i,k)} \otimes d_2^{(p,q)} + d_2^{(p,q)} \otimes d_1^{(i,k)}, && \text{for all } (i,k),(p,q) \in I.
\end{aligned}$$

Let η be a linear combination of η_{rr}^{ikpq} and η_{12}^{ikpq} with $(i,k),(p,q) \in I$. Note that η is $\Bbbk^{\mathbb{D}_m}$-invariant if and only if the non-zero summands η_{rr}^{ikpq} and η_{12}^{ikpq} are $\Bbbk^{\mathbb{D}_m}$-invariant. Write $\eta_{rr}^{ikpq} = \eta_{rr}$. Then η_{rr} is not $\Bbbk^{\mathbb{D}_m}$-invariant because

$$f \cdot \eta_{rr}(y_r^{(i,k)}, y_r^{(p,q)}) = f(h^{(-1)^r(i+p)})\eta_{rr}(y_r^{(i,k)}, y_r^{(p,q)}),$$

and $f(h^{(-1)^r(i+p)}) \neq f(1)$ for all $f \in \Bbbk^{\mathbb{D}_m}$ since $1 \le i,p < n$, see Remark 3.4. On the other hand, η_{12}^{ikpq} is $\Bbbk^{\mathbb{D}_m}$-invariant if and only if $f(h^{(i-p)}) = f(1)$ for all $f \in \Bbbk^{\mathbb{D}_m}$, that is, $i = p$. Let $(i,k),(i,q) \in I$ and write

$$\eta_{12} = \sum_{(i,k),(p,q)\in I} \frac{\zeta_{i,k,i,q}}{2} \eta_{12}^{ikiq}. \tag{8}$$

Then, any possible deformation in degree 2 is given by $\partial^c(\widetilde{\eta_{12}})$. By Remark 2.10, we have

$$\begin{aligned}
(y_1^{(i,k)})^2 &= \eta_{12}(y_1^{(i,k)}, y_1^{(i,k)}) - \theta_{0,m-k}\theta_{0,m-k}\eta_{12}(y_1^{(i,k)}, y_1^{(i,k)}) - \theta_{0,m-k}\theta_{1,k}\eta_{12}(y_1^{(i,k)}, y_2^{(i,k)})\\
&\quad - \theta_{1,k}\theta_{0,m-k}\eta_{12}(y_2^{(i,k)}, y_1^{(i,k)}) - \theta_{1,k}\theta_{1,k}\eta_{12}(y_2^{(i,k)}, y_2^{(i,k)}) \overset{(5)}{=} 0,\\
(y_2^{(i,k)})^2 &= \eta_{12}(y_2^{(i,k)}, y_2^{(i,k)}) - \theta_{1,m-k}\theta_{1,m-k}\eta_{12}(y_1^{(i,k)}, y_1^{(i,k)}) - \theta_{1,m-k}\theta_{0,k}\eta_{12}(y_1^{(i,k)}, y_2^{(i,k)})\\
&\quad - \theta_{0,k}\theta_{1,m-k}\eta_{12}(y_2^{(i,k)}, y_1^{(i,k)}) - \theta_{0,k}\theta_{0,k}\eta_{12}(y_2^{(i,k)}, y_2^{(i,k)}) \overset{(5)}{=} 0,
\end{aligned}$$

since $\eta_{12}(y_1^{(i,k)}, y_1^{(i,k)}) = 0 = \eta_{12}(y_2^{(i,k)}, y_2^{(i,k)})$ for all $(i,k) \in I$. Now we compute the deformation of $y_1^{(i,k)}y_2^{(p,q)} + y_2^{(p,q)}y_1^{(i,k)}$ for $(i,k),(p,q) \in I$. For the first term we have

$$\begin{aligned}
y_1^{(i,k)}y_2^{(p,q)} &= \eta_{12}(y_1^{(i,k)}, y_2^{(p,q)}) - (y_1^{(i,k)})_{(-1)}(y_2^{(p,q)})_{(-1)}\eta_{12}((y_1^{(i,k)})_{(0)}, (y_2^{(p,q)})_{(0)})\\
&= \eta_{12}(y_1^{(i,k)}, y_2^{(p,q)}) - \theta_{0,m-k}\theta_{1,m-q}\eta_{12}(y_1^{(i,k)}, y_1^{(p,q)}) - \theta_{0,m-k}\theta_{0,q}\eta_{12}(y_1^{(i,k)}, y_2^{(p,q)})\\
&\quad - \theta_{1,k}\theta_{1,m-q}\eta_{12}(y_2^{(i,k)}, y_1^{(p,q)}) - \theta_{1,k}\theta_{0,q}\eta_{12}(y_2^{(i,k)}, y_2^{(p,q)})\\
&= \frac{1}{2}(\zeta_{i,k,p,q} - \zeta_{i,k,p,q}\theta_{0,m-k}\theta_{0,q} - \zeta_{i,q,p,k}\theta_{1,k}\theta_{1,m-q})\\
&= \frac{1}{2}(\zeta_{i,k,p,q} - \zeta_{i,k,p,q}\theta_{0,q-k} - \zeta_{i,q,p,k}\theta_{1,k-q}),
\end{aligned}$$

since $\eta_{12}(y_1^{(i,k)}, y_1^{(p,q)}) = 0 = \eta_{12}(y_2^{(i,k)}, y_2^{(p,q)})$ for all $(i,k),(p,q) \in I$. For the second term,

$$\begin{aligned}
y_2^{(p,q)}y_1^{(i,k)} &= \eta_{12}(y_2^{(p,q)}, y_1^{(i,k)}) - (y_2^{(p,q)})_{(-1)}(y_1^{(i,k)})_{(-1)}\eta_{12}((y_2^{(p,q)})_{(0)}, (y_1^{(i,k)})_{(0)})\\
&= \eta_{12}(y_2^{(p,q)}, y_1^{(i,k)}) - \theta_{1,m-q}\theta_{0,m-k}\eta_{12}(y_1^{(p,q)}, y_1^{(i,k)}) - \theta_{1,m-q}\theta_{1,k}\eta_{12}(y_1^{(p,q)}, y_2^{(i,k)}) -\\
&\quad - \theta_{0,q}\theta_{0,m-k}\eta_{12}(y_2^{(p,q)}, y_1^{(i,k)}) - \theta_{0,q}\theta_{1,k}\eta_{12}(y_2^{(p,q)}, y_2^{(i,k)})
\end{aligned}$$

$$= \frac{1}{2}(\zeta_{i,k,p,q} - \zeta_{i,q,p,k}\theta_{1,m-q}\theta_{1,k} - \zeta_{i,k,p,q}\theta_{0,q}\theta_{0,m-k})$$
$$= \frac{1}{2}(\zeta_{i,k,p,q} - \zeta_{i,q,p,k}\theta_{1,k-q} - \zeta_{i,k,p,q}\theta_{0,q-k}).$$

Consequently, $\partial^c(\widetilde{\eta_{12}})$ produces the deformation

$$y_1^{(i,k)}y_2^{(p,q)} + y_2^{(p,q)}y_1^{(i,k)} = \zeta_{i,k,p,q}(1-\theta_{0,q-k}) - \zeta_{i,q,p,k}\theta_{1,k-q},$$

and $(\zeta_{i,k,p,q})_{(i,k),(p,q)\in I}$ is a lifting datum for $\mathcal{A}(\zeta_I)$. Note that any lifting datum for $\mathcal{A}(\zeta_I)$ defines an element η_{12} in $\mathrm{Z}^2_\varepsilon(\mathfrak{B}(M_I),\Bbbk)^{\Bbbk^{\mathbb{D}_m}}$ as in (8).

Conversely, let A be a Hopf algebra such that $\operatorname{gr} A = \mathfrak{B}(M_I)\#\Bbbk^{\mathbb{D}_m} = A_I$. Then by Remark 2.1, A corresponds to a formal deformation of A_I. Write $\mu \in \widehat{\mathrm{Z}}^2_b(A_I)$ for its infinitesimal part. By Theorem A and the calculations above, we have that $\mu = \partial^c(\tilde{\eta})$ for some $\eta \in \mathrm{Z}^2_\varepsilon(\mathfrak{B}(M_I),\Bbbk)^{\Bbbk^{\mathbb{D}_m}} = \Bbbk\,\eta_{12}$, and the exponentiation $\sigma = e^{\tilde{\eta}}$ is a multiplicative cocycle for A_I. As $\mathfrak{B}(M_I)$ is a quadratic algebra, by Remark 2.10 the deformation corresponding to η coincides with the deformation given by σ. Hence, $A \simeq \mathcal{A}(\zeta_I)$ for the lifting datum associated with η. □

Corollary 3.9. *Let A be a finite-dimensional Hopf algebra with $A_0 \simeq \Bbbk^{\mathbb{D}_m}$ such that its infinitesimal braiding is isomorphic to M_I with $I = (i,k)$ or $|I| \geq 2$ such that $i \neq p$ for all $(i,k),(p,q) \in I$. Then $A \simeq \mathfrak{B}(M_I)\#\Bbbk^{\mathbb{D}_m}$.*

Proof. By assumption, $\operatorname{gr} A \simeq \mathfrak{B}(M_I)\#\Bbbk^{\mathbb{D}_m}$ and A is a lifting of $\mathfrak{B}(M_I)\#\Bbbk^{\mathbb{D}_m}$. Then, by Lemma 3.8, we know that $A \simeq (T(M_I)\#\Bbbk^{\mathbb{D}_m})/\mathcal{J}_{\zeta_I}$, where $\mathcal{J}_{\zeta_I}$ is the ideal generated by

$$y_r^{(i,k)}y_r^{(p,q)} + y_r^{(p,q)}y_r^{(i,k)}, \qquad \text{for all } (i,k),(p,q) \in I,\ r = 1,2,$$
$$y_1^{(i,k)}y_2^{(p,q)} + y_2^{(p,q)}y_1^{(i,k)} - [\zeta_{i,k,p,q}(1-\theta_{0,q-k}) - \zeta_{i,q,p,k}\theta_{1,k-q}], \qquad \text{for all } (i,k),(p,q) \in I.$$

If $I = (i,k)$ or $i \neq p$ for all $(i,k),(p,q) \in I$, by (4) the second relation reduces to $y_1^{(i,k)}y_2^{(p,q)} + y_2^{(p,q)}y_1^{(i,k)}$. Thus, $A \simeq \mathfrak{B}(M_I)\#\Bbbk^{\mathbb{D}_m}$ and the corollary is proved. □

3.3.2. Deformations of $\mathfrak{B}(M_L)\#\Bbbk^{\mathbb{D}_m}$

For $L \in \mathcal{L}$, define $\mu_L = (\mu_{\ell,t})_{\ell\leq t\in L}$, $\nu_L = (\nu_{\ell,t})_{\ell\leq t\in L}$, $\tau_L = (\tau_{\ell,t})_{\ell,t\in L}$ as families of elements in $\Bbbk$. We call (μ_L,ν_L,τ_L) a *lifting data* for L.

Definition 3.10. Let $L \in \mathcal{L}$. Given a lifting data (μ_L,ν_L,τ_L) for L we denote by $\mathcal{B}(\mu_L,\nu_L,\tau_L)$ the algebra $(T(M_L)\#\Bbbk^{\mathbb{D}_m})/\mathcal{J}_{\mu_L,\nu_L,\tau_L}$ where $\mathcal{J}_{\mu_L,\nu_L,\tau_L}$ is the two-sided ideal generated by

$$x_1^{(\ell)}x_1^{(t)} + x_1^{(t)}x_1^{(\ell)} - \mu_{\ell,t}(1-\theta_{0,-\ell-t}) - \nu_{\ell,t}\theta_{1,\ell+t}, \qquad \text{for all } \ell,t \in L,$$
$$x_2^{(\ell)}x_2^{(t)} + x_2^{(t)}x_2^{(\ell)} - \nu_{\ell,t}(1-\theta_{0,\ell+t}) - \mu_{\ell,t}\theta_{1,-\ell-t}, \qquad \text{for all } \ell,t \in L,$$
$$x_1^{(\ell)}x_2^{(t)} + x_2^{(t)}x_1^{(\ell)} - \tau_{\ell,t}(1-\theta_{0,t-\ell}) - \tau_{t,\ell}\theta_{1,\ell-t}, \qquad \text{for all } \ell,t \in L.$$

Here $\pm\ell \pm t$ means $\pm\ell \pm t \mod m$. Recall that $\ell,t < n$ and $2n = m$.

A direct computation shows that $\mathcal{J}_{\mu_L,\nu_L,\tau_L}$ is a Hopf ideal and thus $\mathcal{B}(\mu_L,\nu_L,\tau_L)$ is a Hopf algebra. The coalgebra structure is determined for all $\ell \in L$ by

$$\Delta(x_1^{(\ell)}) = x_1^{(\ell)} \otimes 1 + \theta_{0,-\ell} \otimes x_1^{(\ell)} + \theta_{1,\ell} \otimes x_2^{(\ell)}, \qquad \varepsilon(x_1^{(\ell)}) = 0,$$
$$\Delta(x_2^{(\ell)}) = x_1^{(\ell)} \otimes 1 + \theta_{1,-\ell} \otimes x_1^{(\ell)} + \theta_{0,\ell} \otimes x_2^{(\ell)}, \qquad \varepsilon(x_1^{(\ell)}) = 0.$$

Lemma 3.11. *$\mathcal{B}(\mu_L, \nu_L, \tau_L)$ is a lifting of $\mathfrak{B}(M_L)\#\Bbbk^{\mathbb{D}_m}$ and any lifting of $\mathfrak{B}(M_L)\#\Bbbk^{\mathbb{D}_m}$ is isomorphic to $\mathcal{B}(\mu_L, \nu_L, \tau_L)$ for some lifting datum (μ_L, ν_L, τ_L). In particular, $\mathcal{B}(\mu_L, \nu_L, \tau_L)_0 = \Bbbk^{\mathbb{D}_m}$ and* $\dim \mathcal{B}(\mu_L, \nu_L, \tau_L) = 4^{|L|}2m$.

Proof. We proceed as in the proof of Lemma 3.8. That is, we prove that $\mathcal{B}(\mu_L, \nu_L, \tau_L)$ is a lifting of $\mathfrak{B}(M_L)\#\Bbbk^{\mathbb{D}_m}$, by showing that it is a cocycle deformation. Similarly as before, the multiplicative 2-cocycle is given by $\sigma = e^{\tilde{\eta}}$ where $\eta \in \mathrm{Z}^2_\varepsilon(\mathfrak{B}(M_L), \Bbbk)^{\Bbbk^{\mathbb{D}_m}}$. Since the relations in $\mathfrak{B}(M_L)$ are quadratic, we need to describe only the deformation in degree 2, by Remark 2.10.

We know that M_L is linearly spanned by the elements $x_1^{(\ell)}, x_2^{(\ell)}$ with $\ell \in L$ which are homogeneous in ${}^{\mathbb{D}_m}_{\mathbb{D}_m}\mathcal{YD}$ of degree h^n. Let $\{d_1^{(\ell)}, d_2^{(\ell)}\}_{\ell \in L}$ denote the dual basis. By Remark 3.6, we know that $\mathrm{Z}^2_\varepsilon(\mathfrak{B}(M_L), \Bbbk)$ is linearly spanned by the elements

$$\begin{aligned} \eta_{rr}^{(\ell),(t)} &= d_r^{(\ell)} \otimes d_r^{(t)} + d_r^{(t)} \otimes d_r^{(\ell)}, \qquad && \text{for } r = 1, 2,\ \ell, t \in L \\ \eta_{12}^{(\ell,t)} &= d_1^{(\ell)} \otimes d_2^{(t)} + d_2^{(t)} \otimes d_1^{(\ell)}, && \text{for all } \ell, t \in L. \end{aligned}$$

It follows that $\eta_{rr}^{(\ell),(t)}$ and $\eta_{12}^{(\ell,t)}$ are $\Bbbk^{\mathbb{D}_m}$-invariant for all $r = 1, 2$ and $\ell, t \in L$; in particular, $\mathrm{Z}^2_\varepsilon(\mathfrak{B}(M_L), \Bbbk)^{\Bbbk^{\mathbb{D}_m}} = \mathrm{Z}^2_\varepsilon(\mathfrak{B}(M_L), \Bbbk)$. Define

$$\eta = \frac{1}{2}\left(\sum_{\ell \leq t \in L} \left(\mu_{\ell,t}\eta_{11}^{(\ell),(t)} + \nu_{\ell,t}\eta_{22}^{(\ell),(t)} \right) + \sum_{\ell,t \in L} \tau_{\ell,t}\eta_{12}^{(\ell,t)} \right).$$

Then η represents a generic element in $\mathrm{Z}^2_\varepsilon(\mathfrak{B}(M_L), \Bbbk)^{\Bbbk^{\mathbb{D}_m}}$ and all the possible 2-deformations are given by $\partial^c(\widetilde{\eta})$. Following Remark 2.10, for $\ell \leq t \in L$ we have

$$\begin{aligned} x_1^{(\ell)}x_1^{(t)} &= \eta(x_1^{(\ell)}, x_1^{(t)}) - \theta_{0,-\ell}\theta_{0,-t}\eta(x_1^{(\ell)}, x_1^{(t)}) - \theta_{0,-\ell}\theta_{1,t}\eta(x_1^{(\ell)}, x_2^{(t)}) - \\ &\quad - \theta_{1,\ell}\theta_{0,-t}\eta(x_2^{(\ell)}, x_1^{(t)}) - \theta_{1,\ell}\theta_{1,t}\eta(x_2^{(\ell)}, x_2^{(t)}) \\ &= \frac{1}{2}(\mu_{\ell,t} - \theta_{0,-\ell}\theta_{0,-t}\mu_{\ell,t} - \theta_{0,-\ell}\theta_{1,t}\tau_{\ell,t} - \theta_{1,\ell}\theta_{0,-t}\tau_{t,\ell} - \theta_{1,\ell}\theta_{1,t}\nu_{\ell,t}) \\ &= \frac{1}{2}[\mu_{\ell,t}(1 - \theta_{0,-\ell-t}) - \theta_{1,\ell+t}\nu_{\ell,t}]. \end{aligned}$$

Analogously,

$$\begin{aligned} x_2^{(\ell)}x_2^{(t)} &= \eta(x_2^{(\ell)}, x_2^{(t)}) - \theta_{1,-\ell}\theta_{1,-t}\eta(x_1^{(\ell)}, x_1^{(t)}) - \theta_{1,-\ell}\theta_{0,t}\eta(x_1^{(\ell)}, x_2^{(t)}) - \\ &\quad - \theta_{0,\ell}\theta_{1,-t}\eta(x_2^{(\ell)}, x_1^{(t)}) - \theta_{0,\ell}\theta_{0,t}\eta(x_2^{(\ell)}, x_2^{(t)}) \\ &= \frac{1}{2}(\nu_{\ell,t} - \mu_{\ell,t}\theta_{1,-\ell-t} - \nu_{\ell,t}\theta_{0,\ell+t}) = \frac{1}{2}[\nu_{\ell,t}(1 - \theta_{0,\ell+t}) - \mu_{\ell,t}\theta_{1,-\ell-t}]. \end{aligned}$$

Finally, for the relation involving $x_1^{(\ell)}$ and $x_2^{(t)}$ we have that

$$\begin{aligned} x_1^{(\ell)}x_2^{(t)} + x_2^{(t)}x_1^{(\ell)} &= \eta(x_1^{(\ell)}, x_2^{(t)}) - (x_1^{(\ell)})_{(-1)}(x_2^{(t)})_{(-1)}\eta((x_1^{(\ell)})_{(0)}, (x_2^{(t)})_{(0)}) + \\ &\quad + \eta(x_2^{(t)}, x_1^{(\ell)}) - (x_2^{(t)})_{(-1)}(x_1^{(\ell)})_{(-1)}\eta((x_2^{(t)})_{(0)}, (x_1^{(\ell)})_{(0)}). \end{aligned}$$

For the first term on the right hand side we have

$$\begin{aligned} &\eta(x_1^{(\ell)}, x_2^{(t)}) - (x_1^{(\ell)})_{(-1)}(x_2^{(t)})_{(-1)}\eta((x_1^{(\ell)})_{(0)}, (x_2^{(t)})_{(0)}) = \\ &= \eta(x_1^{(\ell)}, x_2^{(t)}) - \theta_{0,-\ell}\theta_{1,-t}\eta_{12}(x_1^{(\ell)}, x_1^{(t)}) - \theta_{0,-\ell}\theta_{0,t}\eta(x_1^{(\ell)}, x_2^{(t)}) \end{aligned}$$

$$- \theta_{1,\ell}\theta_{1,-t}\eta(x_2^{(\ell)}, x_1^{(t)}) - \theta_{1,\ell}\theta_{0,t}\eta(x_2^{(\ell)}, x_2^{(t)})$$
$$= \frac{1}{2}(\tau_{\ell,t} - \tau_{\ell,t}\theta_{0,-\ell+t} - \tau_{t,\ell}\theta_{1,\ell-t}) = \frac{1}{2}[\tau_{\ell,t}(1-\theta_{0,-\ell+t}) - \tau_{t,\ell}\theta_{1,\ell-t}],$$

and a similar computation for the second term yields

$$\eta(x_2^{(t)}, x_1^{(\ell)}) - (x_2^{(t)})_{(-1)}(x_1^{(\ell)})_{(-1)}\eta((x_2^{(t)})_{(0)}, (x_1^{(\ell)})_{(0)}) = \frac{1}{2}[\tau_{\ell,t}(1-\theta_{0,t-\ell}) - \tau_{t,\ell}\theta_{1,\ell-t}].$$

Hence, the deformation of $\mathfrak{B}(M_L)\#\Bbbk^{\mathbb{D}_m}$ performed by $\partial^c(\tilde{\eta})$ is isomorphic to $\mathcal{B}(\mu_L, \nu_L, \tau_L)$ for some lifting data (μ_L, ν_L, τ_L) associated with the parameters of η.

Conversely, let A be a Hopf algebra such that $\operatorname{gr} A = \mathfrak{B}(M_L)\#\Bbbk^{\mathbb{D}_m} = A_L$. Then by Remark 2.1, A corresponds to a formal deformation of A_L. Write $\mu \in \widehat{\mathrm{Z}}^2_b(A_L)$ for its infinitesimal part. By Theorem A and the calculations above, we have that $\mu = \partial^c(\tilde{\eta})$ for some $\eta \in \mathrm{Z}^2_\varepsilon(\mathfrak{B}(M_L), \Bbbk)^{\Bbbk^{\mathbb{D}_m}}$, and the exponentiation $\sigma = e^{\tilde{\eta}}$ is a multiplicative cocycle for A_L. By Remark 2.10, the deformation corresponding to η coincides with the deformation given by σ. Hence, $A \simeq \mathcal{B}(\mu_L, \nu_L, \tau_L)$ for some lifting datum (μ_L, ν_L, τ_L). □

3.3.3. Deformations of $\mathfrak{B}(M_{I,L})\#\Bbbk^{\mathbb{D}_m}$

We introduce the last families of deformed algebras corresponding to the finite-dimensional Nichols algebras associated with mixed classes.

Definition 3.12. Let $(I, L) \in \mathcal{K}$. Given lifting data ζ_I and (μ_L, ν_L, τ_L) for I and L, respectively, we denote by $\mathcal{C}(\zeta_I, \mu_L, \nu_L, \tau_L)$ the algebra $(T(M_{I,L})\#\Bbbk^{\mathbb{D}_m})/\mathcal{J}_{\zeta_I,\mu_L,\nu_L,\tau_L}$ where $\mathcal{J}_{\zeta_I,\mu_L,\nu_L,\tau_L}$ is the two-sided ideal generated by the elements

$$\begin{aligned}
y_r^{(i,k)}y_r^{(p,q)} + y_r^{(p,q)}y_r^{(i,k)}, &\qquad \forall\ (i,k),(p,q) \in I,\ r = 1,2,\\
y_1^{(i,k)}y_2^{(p,q)} + y_2^{(p,q)}y_1^{(i,k)} - [\zeta_{i,k,p,q}(1-\theta_{0,q-k}) - \zeta_{i,q,p,k}\theta_{1,k-q}], &\qquad \forall\ (i,k),(p,q) \in I,\\
x_1^{(\ell)}x_1^{(t)} + x_1^{(t)}x_1^{(\ell)} - \mu_{\ell,t}(1-\theta_{0,-\ell-t}) - \nu_{\ell,t}\theta_{1,\ell+t}, &\qquad \forall\ \ell,t \in L,\\
x_2^{(\ell)}x_2^{(t)} + x_2^{(t)}x_2^{(\ell)} - \nu_{\ell,t}(1-\theta_{0,\ell+t}) - \mu_{\ell,t}\theta_{1,-\ell-t}, &\qquad \forall\ \ell,t \in L,\\
x_1^{(\ell)}x_2^{(t)} + x_2^{(t)}x_1^{(\ell)} - \tau_{\ell,t}(1-\theta_{0,t-\ell}) - \tau_{t,\ell}\theta_{1,\ell-t}, &\qquad \forall\ \ell,t \in L.\\
y_r^{(i,k)}x_s^{(\ell)} + x_s^{(\ell)}y_r^{(i,k)}, &\qquad \forall\ (i,k) \in I,\ \ell \in L,\ r,s = 1,2.
\end{aligned}$$

A direct computation shows that $\mathcal{J}_{\zeta_I,\mu_L,\nu_L,\tau_L}$ is a Hopf ideal and thus $\mathcal{C}(\zeta_I, \mu_L, \nu_L, \tau_L)$ is a Hopf algebra. As before, for $(I, L) \in \mathcal{K}$, we call $(\zeta_I, \mu_L, \nu_L, \tau_L)$ a *lifting data* for (I, L).

Following the same lines of the proofs of Lemmata 3.8 and 3.11 we have

Lemma 3.13. *$\mathcal{C}(\zeta_I, \mu_L, \nu_L, \tau_L)$ is a lifting of $\mathfrak{B}(M_{I,L})\#\Bbbk^{\mathbb{D}_m}$ and any lifting of $\mathfrak{B}(M_{I,L})\#\Bbbk^{\mathbb{D}_m}$ is isomorphic to $\mathcal{C}(\zeta_I, \mu_L, \nu_L, \tau_L)$ for some lifting datum $(\zeta_I, \mu_L, \nu_L, \tau_L)$. In particular, we have $\mathcal{C}(\zeta_I, \mu_L, \nu_L, \tau_L)_0 = \Bbbk^{\mathbb{D}_m}$ and $\dim \mathcal{C}(\zeta_I, \mu_L, \nu_L, \tau_L) = 4^{|I|+|L|}\, 2m$.*

Proof. We begin by looking for the $\Bbbk^{\mathbb{D}_m}$-invariant cocycles in $\mathrm{Z}^2_\varepsilon(\mathfrak{B}(M_{I,L}), \Bbbk)$. Since $M_{I,L} = \left(\bigoplus_{1\le j\le s} M_{(i_j,k_j)}\right) \oplus \left(\bigoplus_{1\le t\le s} M_{\ell_t}\right)$, it is linearly spanned by the elements $y_1^{(i_j,k_j)}, y_2^{(i_j,k_j)}$, with $1 \le j \le s = |I|$, and $x_1^{(\ell)}, x_2^{(\ell)}$ with $\ell \in L$. Let $\left\{d_1^{(i_j,k_j)}, d_2^{(i_j,k_j)}\right\}_{1\le j\le s} \bigcup \left\{d_1^{(\ell)}, d_2^{(\ell)}\right\}_{\ell\in L}$ denote the dual basis.

By the proof of the Lemmata 3.8 and 3.11, we know that η_{12}^{ikiq}, $\eta_{11}^{(\ell),(t)}$, $\eta_{22}^{(\ell),(t)}$ and $\eta_{12}^{(\ell),(t)}$ belong to $\mathrm{Z}^2_\varepsilon(\mathfrak{B}(M_{I,L}), \Bbbk)^{\Bbbk^{\mathbb{D}_m}}$. On the other hand, the elements in $\mathrm{Z}^2_\varepsilon(\mathfrak{B}(M_{I,L}), \Bbbk)$ given by

$$\eta_{rr}^{ik\ell} = d_r^{(i,k)} \otimes d_r^{(\ell)} + d_r^{(\ell)} \otimes d_r^{(i,k)}, \qquad \text{for } r = 1, 2,\ (i,k), \in I, \ell \in L,$$
$$\eta_{12}^{ik\ell} = d_1^{(i,k)} \otimes d_2^{(\ell)} + d_2^{(\ell)} \otimes d_1^{(i,k)}, \qquad \text{for all } (i,k) \in I, \ell \in L,$$
$$\eta_{21}^{ik\ell} = d_2^{(i,k)} \otimes d_1^{(\ell)} + d_1^{(\ell)} \otimes d_2^{(i,k)}, \qquad \text{for all } (i,k) \in I, \ell \in L,$$

are not $\Bbbk^{\mathbb{D}_m}$-invariant because

$$f \cdot \eta_{rr}^{ik\ell}(y_r^{(i,k)}, x_r^{(\ell)}) = f(h^{(-1)^r(i+\ell)})\, \eta_{rr}^{ik\ell}(y_r^{(i,k)}, x_r^{(\ell)}),$$
$$f \cdot \eta_{12}^{ik\ell}(y_1^{(i,k)}, x_2^{(\ell)}) = f(h^{(-1)^r(i+\ell)})\, \eta_{12}^{ik\ell}(y_r^{(i,k)}, x_2^{(\ell)}),$$
$$f \cdot \eta_{21}^{ik\ell}(y_2^{(i,k)}, x_1^{(\ell)}) = f(h^{(-1)^r(i+\ell)})\, \eta_{21}^{ik\ell}(y_r^{(i,k)}, x_1^{(\ell)}),$$

and $f(h^{(-1)^r(i+\ell)}) \neq f(1)$ for all $f \in \Bbbk^{\mathbb{D}_m}$ since $1 \leq i, \ell < n$, see Remark 3.4. Hence, the deformation of $\mathfrak{B}(M_{I,L})\#\Bbbk^{\mathbb{D}_m}$ performed by $\partial^c(\tilde{\eta}_{12})$ and $\partial^c(\tilde{\eta})$ is isomorphic to $\mathcal{C}(\zeta_I, \mu_L, \nu_L, \tau_L)$ for some lifting data $(\zeta_I, \mu_L, \nu_L, \tau_L)$ associated with the parameters of η_{12} and η.

The converse follows *mutatis mutandis* using the same arguments of the last paragraph of the proof of Lemma 3.8. □

We end this paper with the proof of Theorem B.

Proof of Theorem B. It is clear that two algebras from different families are not isomorphic as Hopf algebras since their infinitesimal braidings are not isomorphic as Yetter–Drinfeld modules.

Let A be a finite-dimensional Hopf algebra with coradical $A_0 \simeq \Bbbk^{\mathbb{D}_m}$. Then by Theorems 3.3 and 3.2, we have that $\operatorname{gr} A \simeq \mathfrak{B}(M)\#\Bbbk^{\mathbb{D}_m}$, where M isomorphic either to M_I, or to M_L, or to $M_{I,L}$, with $I \in \mathcal{I}$, $L \in \mathcal{L}$ and $(I, L) \in \mathcal{K}$, respectively. Hence, by Lemmata 3.8, 3.11 and 3.13 the result follows. □

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