Journal of Pure and Applied Algebra 225 (2021) 106488

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Algebraic K-theory of generalized free products and functors Nil

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ARTICLE INFO

Article history:
Received 24 September 2019
Received in revised form 23 April 2020
Available online 23 July 2020
Communicated by C. Weibel

MSC:
18E10; 19D35; 19D50

Keywords:
Algebraic K-theory
Functor Nil
Whitehead groups

ABSTRACT

In this paper, we extend Waldhausen's results on algebraic K-theory of generalized free products in a more general setting and we give some properties of the Nil functors. As a consequence, we get new groups with trivial Whitehead groups.

0. Introduction

Quillen's construction associates to any essentially small exact category $\mathscr{A}$ its algebraic K-theory which is an infinite loop space $K(\mathscr{A})$ and this correspondence is a functor from the category of essentially small exact categories to the category Ωsp_0 of infinite loop spaces.

If $\mathscr{A}$ is the category $\mathscr{P}_A$ of finitely generated projective right modules over some ring A, one gets a functor $A \mapsto K(A) = K(\mathscr{P}_A)$ and this functor can be enriched into a new functor $\underline{K}$ containing also the negative part of the algebraic K-theory. That is $\underline{K}$ is a functor from the category of rings to the category Ωsp of Ω-spectra and the natural transformation $K(A) \to \underline{K}(A)$ induces a homotopy equivalence from $K(A)$ to the 0-th term of $\underline{K}(A)$.

By a left-flat bimodule we mean a pair (A, S) where A is a ring and S is an A-bimodule flat on the left. The left-flat bimodules form a category where a morphism $(A, S) \longrightarrow (B, T)$ is a ring homomorphism $f : A \longrightarrow B$ together with a morphism of A-bimodules $\varphi : S \longrightarrow f^*(T)$.

For each left-flat bimodule (A, S) one has an exact category $\mathscr{N}il(A, S)$ where the objects are the pairs (M, f) where M is an object in $\mathscr{P}_A$ and $f : M \to M \otimes_A S$ is a nilpotent morphism of right A-modules.

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https://doi.org/10.1016/j.jpaa.2020.106488
0022-4049/

The correspondence $M \mapsto (M, 0)$ induces a morphism $K(A) \to K(\mathscr{N}il(A, S))$ and this morphism has a retraction coming from the functor $(M, f) \mapsto M$. Thus there is a functor Nil from the category of left-flat bimodules to Ωsp_0 which is unique up to homotopy such that:

$$K(\mathscr{N}il(A, S)) \simeq K(A) \times Nil(A, S)$$

Theorem 1. *There is a functor $\underline{N}il$ from the category of left-flat bimodules to the category Ωsp of Ω-spectra and a natural transformation $Nil \to \underline{N}il$ such that the following holds for every left-flat bimodule (A, S):*

- *the map $Nil(A, S) \to \underline{N}il(A, S)$ induces a homotopy equivalence from $Nil(A, S)$ to the 0-th term of $\underline{N}il(A, S)$*
- *if R is the tensor algebra of S, then there is a natural homotopy equivalence in Ωsp:*

$$\underline{K}(R) \xrightarrow{\sim} \underline{K}(A) \times \Omega^{-1}(\underline{N}il(A, S))$$

Moreover if A is regular coherent on the right, every spectrum $\underline{N}il(A, S)$ is contractible.

Following a terminology of Waldhausen, a ring homomorphism $\alpha : A \to B$ will be called pure if it is split injective as an A-bimodule homomorphism.

Theorem 2. *Let $\alpha : C \longrightarrow A$ and $\beta : C \longrightarrow B$ be pure ring homomorphisms. Let R be the ring defined by the push-out diagram:*

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d] \\ B \arrow[r] & R \end{tikzcd}$$

and $\underline{K}'(R)$ be the Ω-spectrum defined by the homotopy fibration in Ωsp:

$$\underline{K}(C) \xrightarrow{f} \underline{K}(A) \times \underline{K}(B) \longrightarrow \underline{K}'(R)$$

where f is the map $\underline{K}(\alpha) \times -\underline{K}(\beta)$.

Suppose A and B are C-flat on the left. Then there exist a left-flat bimodule $(C \times C, S)$ and a homotopy equivalence in Ωsp:

$$\underline{K}(R) \xrightarrow{\sim} \underline{K}'(R) \times \Omega^{-1}(\underline{N}il(C \times C, S))$$

Theorem 3. *Let C and A be two rings and α and β be two pure ring homomorphisms from C to A. Let R be the ring generated by A and an invertible element t with the only relations:*

$$\forall c \in C, \quad \alpha(c)t = t\beta(c)$$

and $\underline{K}'(R)$ be the Ω-spectrum defined by the homotopy fibration:

$$\underline{K}(C) \xrightarrow{f} \underline{K}(A) \longrightarrow \underline{K}'(R)$$

where f is the map $\underline{K}(\alpha) - \underline{K}(\beta)$.

Suppose A is C-flat on the left via both α and β. Then there exist a left-flat bimodule $(C \times C, S)$ and a homotopy equivalence in Ωsp:

$$\underline{K}(R) \xrightarrow{\sim} \underline{K}'(R) \times \Omega^{-1}(\underline{N}il(C \times C, S))$$

The connective part of these theorems are generalization of the Waldhausen's results in [8], in the sense that the condition free on the left in [8] may be replaced by the condition flat on the left and (in the polynomial extension case) the condition on the right may be removed. Because of these results, the functor $\underline{N}il$ detects in some sense the default of excision in algebraic K-theory. So it would be useful to know when the spectrum $\underline{N}il(A, S)$ is contractible, especially if A is not regular coherent. Actually we have the following result:

Theorem 4. *Let A and B be two rings, S be an (A, B)-bimodule and T be a (B, A)-bimodule. Suppose S and T are flat on both sides. Using projections $A \times B \longrightarrow A$ and $A \times B \longrightarrow B$, S and T may be considered as $A \times B$-bimodules. Then we have natural homotopy equivalences of spectra:*

$$\underline{N}il(A \times B, S \oplus T) \xrightarrow{\sim} \underline{N}il(A, S \underset{B}{\otimes} T)$$
$$\underline{N}il(A \times B, S \oplus T) \xrightarrow{\sim} \underline{N}il(B, T \underset{A}{\otimes} S)$$

We have other results concerning the spectrum $\underline{N}il(A, S)$ when the bimodule S is a direct sum of bimodules: $S = \underset{i \in I}{\oplus} S_i$.

Let $W(I)$ be the set of words in the set I. This set is the unitary monoid freely generated by I. Let $CW(I)$ the set of cyclic words in I. The set $CW(I)$ is the quotient of $W(I)$ by the equivalence relation $uv \sim vu$ in $W(I)$. A word $u \in W(I)$ is said to be reduced if we have the following:

$$\forall v \in W(I), \quad \forall p > 1, \quad u \neq v^p$$

The set of reduced words is denoted by $W_0(I)$ and its image in $CW(I)$ is denoted by $CW_0(I)$. A subset $X \subset W(I)$ is said to be admissible if the projection $W(I) \longrightarrow CW(I)$ induces a bijection $X \xrightarrow{\sim} CW_0(I)$.

For every $u \in W(I)$, we have a bimodule S_u defined by:

$$u = 1 \implies S_u = A$$
$$u = vi, \text{ with } i \in I, \implies S_u = S_v \underset{A}{\otimes} S_i$$

Using these notations, we have this result:

Theorem 5. *Let A be a ring and S_i, $i \in I$, be a family of A-bimodules. Suppose each bimodule S_i is flat on both sides. Let X be an admissible set in $W(I)$. Then we have a homotopy equivalence of spectra:*

$$\underline{N}il(A, \underset{i}{\oplus} S_i) \xrightarrow{\sim} \underset{u \in X}{\oplus} \underline{N}il(A, S_u)$$

where $\oplus$ is the coproduct in the category of spectra.

Moreover, using theorems 4 and 5, we can deduce this last result:

Theorem 6. *Let A be a ring, S be an A-bimodule and I be a set. For each $i \in I$, let A_i be a right regular coherent ring, E_i be an (A, A_i)-bimodule and F_i be an (A_i, A)-bimodule. Suppose all these bimodules are flat on both sides. Then the inclusion:*

$$S \subset S \oplus \underset{i}{\oplus} E_i \underset{A_i}{\otimes} F_i$$

induces a homotopy equivalence:

$$\underline{N}il(A,S) \xrightarrow{\sim} \underline{N}il(A, S \oplus \bigoplus_i E_i \underset{A_i}{\otimes} F_i)$$

The paper is organized as follows:

In section 1, we construct many categories and functors in such a way they are defined in each of the three cases: the general polynomial extension case, the generalized free product case and the generalized Laurent extension case. We prove also many algebraic properties of these categories and functors.

In section 2, we apply these properties to algebraic K-theory and prove theorems 1, 2 and 3.

The section 3 is devoted to the proof of theorems 4, 5 and 6 about Nil functors.

In the last section we apply all these theorems and get new results about Whitehead spectra. In particular we construct a class Cl_1 bigger that Waldhausen's class Cl such that every group in Cl_1 has trivial Whitehead groups.

1. The categories $\mathscr{V}$ and $\mathscr{M}\mathscr{V}$ and their algebraic properties

In order to simplify the notations, the following writing conventions will often be used: • Convention 1: if $\Phi : \mathscr{A} \to \mathscr{B}$ is a functor, then for every morphism α in $\mathscr{A}$, its image under Φ will be still denoted by α. So, if $\alpha : X \to Y$ is a morphism, we have a morphism $\alpha : \Phi(X) \to \Phi(Y)$. • Convention 2: if E is a right module over some ring A and F is a left module over the same ring, then the module $E \otimes_A F$ will be simply denoted be EF. In the same spirit, if E is an A-bimodule, the tensor product $E \otimes_A \cdots \otimes_A E$ of n copies of E will be denoted by E^n.

Definition 1.1. Let A be a ring and S be an A-bimodule. Let M be a right A-module and $f : M \to MS$ be an A-linear map. So by iteration, we get for each integer $n > 0$ a morphism $f^n : M \to MS^n$. We say that f is nilpotent if every element in M is killed by some power f^n of f.

Lemma 1.2. *Let (A,S) be a left-flat bimodule. Let M be a right A-module and $f : M \to MS$ be an A-linear map. Then f is nilpotent if and only if there is a filtration of M:*

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M$$

by right A-submodules, such that:

- *M is the union of the M_i's*
- *for every $i > 0$, one has: $f(M_i) \subset M_{i-1}S$.*

Proof. If such a filtration exists, then f is clearly nilpotent.

Suppose f is nilpotent. For every integer $n \geq 0$, denote by M_n the kernel of $f^n : M \longrightarrow MS^n$. By construction we have an increasing sequence

$$0 = M_0 \subset M_1 \subset M_2 \subset M_3 \dots$$

and M is the union of the M_i's.

Since S is flat on the left, we have, for every $n \geq 0$, an isomorphism:

$$\mathrm{Ker}(f^n : MS \longrightarrow MS^{n+1}) \simeq \mathrm{Ker}(f^n : M \longrightarrow MS^n)S$$

and then an equality: $M_{n+1} = f^{-1}(M_nS)$. The result follows. $\square$

1.3. The exact category $\mathscr{N}il(A,S)$ and the space $Nil(A,S)$

Let (A,S) be a left-flat bimodule. The pairs (M,f) where M is a right A-module and $f: M \to MS$ is a nilpotent morphism are the objects of a category denoted by $\mathscr{N}il(A,S)^\vee$. Let $\mathscr{A}$ be the category of finitely generated projective right A-modules. The full subcategory of $\mathscr{N}il(A,S)^\vee$ generated by pairs (M,f) with $M \in \mathscr{A}$ will be denoted by $\mathscr{N}il(A,S)$. If $0 \longrightarrow (M,f) \longrightarrow (M',f') \longrightarrow (M'',f'') \longrightarrow 0$ is a sequence in $\mathscr{N}il(A,S)$, we say that this sequence is exact if the following diagram is commutative with exact lines:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & M' & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & {\scriptstyle f''}\downarrow & & \\
0 & \longrightarrow & MS & \longrightarrow & M'S & \longrightarrow & M''S & \longrightarrow & 0
\end{array}$$

With these exact sequences the category $\mathscr{N}il(A,S)$ becomes an exact category in the sense of Quillen and its algebraic K-theory $K(\mathscr{N}il(A,S))$ is a well defined infinite loop space (see [6]).

Actually $\mathscr{N}il$ is a functor from the category of left-flat bimodules to the category of essentially small exact categories and exact functors.

We have two exact functors $M \mapsto (M,0)$ from $\mathscr{A}$ to $\mathscr{N}il(A,S)$ and $(M,f) \mapsto M$ from $\mathscr{N}il(A,S)$ to $\mathscr{A}$ inducing two maps:

$$K(A) \xrightarrow{F} K(\mathscr{N}il(A,S)) \xrightarrow{G} K(A)$$

where G is a retraction of F. Denote by $Nil(A,S)$ the homotopy fiber of G. Then $Nil(A,S)$ is an infinite loop space and we have a decomposition:

$$K(\mathscr{N}il(A,S)) \simeq K(A) \times Nil(A,S)$$

Throughout this paper, we'll consider many categories and functors and, in particular, many exact categories and their abelianizations, where an abelianization of an exact category is defined as follows:

Definition. Let $\mathscr{E}$ be an exact category. We say that a category $\mathscr{E}^\vee$ is an abelianization of $\mathscr{E}$ if the following holds:

- $\mathscr{E}^\vee$ is an abelian category
- $\mathscr{E}$ is a fully exact subcategory of $\mathscr{E}^\vee$ i.e. $\mathscr{E}$ is a full subcategory of $\mathscr{E}^\vee$ and, for every sequence $S = (0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0)$ in $\mathscr{E}$, S is exact in $\mathscr{E}$ if and only if S is exact un $\mathscr{E}^\vee$
- $\mathscr{E}$ is stable in $\mathscr{E}^\vee$ under extension.

Notice that the Gabriel-Quillen embedding theorem produces an abelianization for every essentially small exact category (see [7], thm A.7.1 or [4], prop A.2). Following Waldhausen we consider three situations: the generalized free product case, the generalized Laurent extension case and the generalized polynomial extension case.

In case 1 (i.e. the generalized polynomial extension case), we have a ring C and a C-bimodule S which is flat on the left. In this case the ring R is the tensor algebra of S:

$$R = C \oplus S \oplus S^2 \oplus S^3 \oplus \dots$$

In case 2 (i.e. the generalized free product case) we have two pure morphisms of rings $\alpha : C \to A$ and $\beta : C \to B$ and we suppose that A and B are C-flat on the left. We denote by R the ring defined by the cocartesian diagram:

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & A \\ {\scriptstyle\beta}\downarrow & & \downarrow \\ B & \longrightarrow & R \end{array}$$

In case 3 (i.e. the generalized Laurent extension case) we have two rings C and A and two pure morphisms α and β from C to A. We suppose that A is C-flat on the left via both α and β and we denote by R the ring generated by A and an invertible element t with the only relations:

$$\forall c \in C, \quad \alpha(c)t = t\beta(c)$$

So we have a morphism $\gamma : A \to R$ and $\gamma \circ \alpha$ and $\gamma \circ \beta$ are conjugate.

From now on we will consider α and γ as inclusions. So in each case A, B and R are C-bimodules. We denote by $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{R}$ the categories of finitely generated projective right modules over the rings A, B, C and R respectively. We set also: $\mathscr{D} = \mathscr{C}$ in case 1, $\mathscr{D} = \mathscr{A} \times \mathscr{B}$ in case 2 and $\mathscr{D} = \mathscr{A}$ in case 3. This category is the category of finitely generated projective right modules over the ring C or $A \times B$ or A.

The categories $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{R}$ are contained in the abelian categories $\mathscr{A}^\vee$, $\mathscr{B}^\vee$, $\mathscr{C}^\vee$, $\mathscr{D}^\vee$ and $\mathscr{R}^\vee$ of right-modules over the corresponding rings and these categories are abelianizations of $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{R}$ respectively. Notice that $\mathscr{N}il(A, S)^\vee$ is also an abelianization of $\mathscr{N}il(A, S)$.

We denote by C_2 the ring C in case 1 and $C \times C$ in the other cases and also by $\mathscr{C}_2{}^\vee$ the category of right C_2-modules and by $\mathscr{C}_2$ the subcategory of $\mathscr{C}_2{}^\vee$ generated by finitely generated projective modules. The category $\mathscr{C}_2{}^\vee$ is also an abelianization of $\mathscr{C}_2$.

We will define the C_2-bimodule S and many categories and functors in order to give a common proof of theorems 1, 2 and 3 (at least for the connective part of it).

1.4. The functors s and σ and the bimodule S

Consider the case $C_2 = C \times C$. Let π_1 and π_2 be the two projections $C \times C \to C$. Consider a right C-module M and an integer $i \in \{1, 2\}$. The ring $C \times C$ acts on M via π_i and becomes a right $C \times C$-module M^i. This functor $M \mapsto M^i$ from $\mathscr{C}^\vee$ to $\mathscr{C}_2{}^\vee$ has an adjoint functor (on both sides) from $\mathscr{C}_2{}^\vee$ to $\mathscr{C}^\vee$ denoted by $E \mapsto E_i$. The two functors $M \mapsto M^1$ and $M \mapsto M^2$ induce an equivalence of categories from $\mathscr{C}^\vee \times \mathscr{C}^\vee$ to $\mathscr{C}_2{}^\vee$ and the functors $E \mapsto E_1$ and $E \mapsto E_2$ induce an inverse of it.

The two functors $s_i : M \mapsto M^i$ from $\mathscr{C}$ to $\mathscr{C}_2$ (and also from $\mathscr{C}^\vee$ to $\mathscr{C}_2{}^\vee$) are exact and it is the same for they adjoint functors $\sigma_i : E \mapsto E_i$ from $\mathscr{C}_2$ to $\mathscr{C}$ (and from $\mathscr{C}_2{}^\vee$ to $\mathscr{C}^\vee$). So s_1, s_2 and $s = s_1 \oplus s_2$ are exact functors and their adjoint functors σ_1, σ_2 and $\sigma = \sigma_1 \oplus \sigma_2$ are also exact.

In case 1, s and σ are defined to be identities. Therefore s and σ are well defined in all cases: s is an exact functor from $\mathscr{C}$ to $\mathscr{C}_2$ (and also from $\mathscr{C}^\vee$ to $\mathscr{C}_2{}^\vee$) and σ is an exact functor from $\mathscr{C}_2$ to $\mathscr{C}$ (and also from $\mathscr{C}_2{}^\vee$ to $\mathscr{C}^\vee$).

Moreover, for every module E in $\mathscr{C}_2$ (or $\mathscr{C}_2{}^\vee$) the module $\sigma(E)$ is nothing else but the module E equipped with the C-action induced by the identity or the diagonal map from C to C_2. We can do the same for left modules and we have functors $M \mapsto {}^iM$ and $E \mapsto {}_iE$. In case of bimodules, we gets functors $M \mapsto {}^iM^j$ and $E \mapsto {}_iE_j$. Using these notations we have the following, for every right $C \times C$-module E and left $C \times C$-module F:

$$E \underset{C \times C}{\otimes} F = EF = E_1\, {}_1F \oplus E_2\, {}_2F = \bigoplus_i E_i\, {}_iF$$

In case 1, the C_2-bimodule S is already defined.

Consider the case 2. Since $\alpha : C \to A$ and $\beta : C \to B$ are pure morphisms, A and B have two complements A' and B'. These objects are C-bimodules and we have two decompositions of C-bimodules: $A = \alpha(C) \oplus A'$ and $B = \beta(C) \oplus B'$. Then we define the bimodule S by:

$$ {}_2S_1 = A' \qquad {}_1S_2 = B' \qquad {}_1S_1 = {}_2S_2 = 0 $$

Consider the case 3. Ring homomorphisms α and β induce two left C-actions on A and we get two (C, A)-bimodules ${}_\alpha A$ and ${}_\beta A$. By doing the same on the right, we get four C-bimodules ${}_\alpha A_\alpha$, ${}_\alpha A_\beta$, ${}_\beta A_\alpha$ and ${}_\beta A_\beta$. Moreover, since α and β are pure morphisms, we have two decompositions of C-bimodules:

$$ {}_\alpha A_\alpha = \alpha(C) \oplus A' \qquad {}_\beta A_\beta = \beta(C) \oplus A'' $$

Then we define the bimodule S by:

$$ {}_2S_1 = A' \qquad {}_1S_2 = A'' \qquad {}_1S_1 = {}_\beta A_\alpha \qquad {}_2S_2 = {}_\alpha A_\beta $$

Then in the three cases S is a well defined C_2-bimodule. It is easy to see that S is flat on the left.

1.5. *The categories $\mathscr{D}$, $\mathscr{MV}$ and $\mathscr{V}$ and the functors T, F and $\widehat{F}$*

We have a functor $T : \mathscr{D}^\vee \longrightarrow \mathscr{R}^\vee$ defined as follows:

- in case 1: $T(E) = E \otimes_C R = ER$
- in case 2: $T(E_A, E_B) = E_A \otimes_A R \oplus E_B \otimes_B R = E_A\ R \oplus E_B\ R$
- in case 3: $T(E) = E \otimes_A R = ER$

It is easy to check that T is an exact functor sending $\mathscr{D}$ to $\mathscr{R}$. Let E be an object in $\mathscr{D}^\vee$, M be an object in $\mathscr{C}^\vee$ and $\varphi : T(E) \longrightarrow MR$ be a morphism in $\mathscr{R}^\vee$. We say that φ is admissible if the following holds:

- $\varphi(E) \subset M \oplus MS$ in case 1
- $\varphi(E_A) \subset MA$ and $\varphi(E_B) \subset MB$ in case 2 (with: $E = (E_A, E_B)$)
- $\varphi(E) \subset MA \oplus MtA \subset MR$ in case 3.

The set of admissible morphisms $\varphi : T(E) \longrightarrow MR$ will be denoted by $\mathscr{F}(E, M)$.

Following Waldhausen, we define a splitting diagram as a triple $X = (E, M, \varphi)$ with: $E \in \mathscr{D}^\vee$, $M \in \mathscr{C}^\vee$ and $\varphi \in \mathscr{F}(E, M)$.

The splitting diagram (E, M, φ) is called a Mayer Vietoris presentation (resp. a splitting module) if φ is surjective (resp. bijective). The splitting modules, the Mayer Vietoris presentations and the splitting diagrams define three categories $\mathscr{V}^\vee \subset \overline{\mathscr{MV}} \subset \mathscr{S}^\vee$. Moreover categories $\mathscr{V}^\vee$ and $\mathscr{S}^\vee$ are abelian.

If we replace $\mathscr{D}^\vee$ and $\mathscr{C}^\vee$ by $\mathscr{D}$ and $\mathscr{C}$, we get three subcategories $\mathscr{V} \subset \mathscr{MV} \subset \mathscr{S}$.

The correspondences $(E, M, \varphi) \mapsto E$ and $(E, M, \varphi) \mapsto M$ define two functors $\Phi_2 : \mathscr{S} \longrightarrow \mathscr{D}$ and $\Phi_3 : \mathscr{S} \longrightarrow \mathscr{C}$ (and also from $\mathscr{S}^\vee$ to $\mathscr{D}^\vee$ and from $\mathscr{S}^\vee$ to $\mathscr{C}^\vee$). We have an extra functor Φ_1 sending (E, M, φ) to the kernel of φ.

Consider a sequence $S = (0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0)$ in $\mathscr{V}$ or in $\mathscr{S}$. We say that this sequence is exact if it is sent to an exact sequence under Φ_2 and Φ_3. If S is a sequence in $\mathscr{MV}$, we say that S is exact if it is sent to an exact sequence under Φ_1, Φ_2 and Φ_3.

With these exact sequences, $\mathscr{V}$, $\mathscr{MV}$ and $\mathscr{S}$ become exact categories and the inclusions $\mathscr{V} \subset \mathscr{MV} \subset \mathscr{S}$ are exact functors. Moreover $\Phi_1 : \mathscr{MV} \longrightarrow \mathscr{R}$ is an exact functor. In some sense $\mathscr{V}$ is the kernel of the functor $\Phi_1 : \mathscr{MV} \longrightarrow \mathscr{R}$.

Since $\overline{\mathscr{MV}}$ is not an abelian category, it will be useful to construct an abelian category $\mathscr{MV}^\vee$ containing $\mathscr{MV}$. The category $\mathscr{MV}$ is equivalent to the category of tuples (U, E, M, μ, φ) where (U, E, M) is an object of $\mathscr{R} \times \mathscr{D} \times \mathscr{C}$ and $\mu : U \longrightarrow T(E)$ and $\varphi : T(E) \longrightarrow MR$ are morphisms in $\mathscr{R}$ such that φ is admissible and the sequence:

$$0 \longrightarrow U \xrightarrow{\mu} T(E) \xrightarrow{\varphi} MR \longrightarrow 0$$

is exact. So the category $\mathscr{MV}^\vee$ is defined as the category of tuples (U, E, M, μ, φ) where (U, E, M) is an object of $\mathscr{R}^\vee \times \mathscr{D}^\vee \times \mathscr{C}^\vee$ and $\mu : U \longrightarrow T(E)$ and $\varphi : T(E) \longrightarrow MR$ are morphisms in $\mathscr{R}^\vee$ such that φ is admissible and $\varphi\mu = 0$. It is easy to see that $\mathscr{MV}^\vee$ is an abelian category and the inclusions of exact categories $\mathscr{V} \subset \mathscr{MV}$ and $\mathscr{MV} \subset \mathscr{S}$ extend to functors $\mathscr{V}^\vee \longrightarrow \mathscr{MV}^\vee$ and $\mathscr{MV}^\vee \longrightarrow \mathscr{S}^\vee$. Moreover the categories $\mathscr{V}^\vee$, $\mathscr{MV}^\vee$ and $\mathscr{S}^\vee$ are abelianizations of $\mathscr{V}$, $\mathscr{MV}$ and $\mathscr{S}$ respectively.

We have a functor F from $\mathscr{C}_2{}^\vee$ to $\mathscr{D}^\vee$ defined as follows:

- in case 1, F is the identity
- in case 2, $F(M) = (\sigma_1(M)A, \sigma_2(M)B) \in \mathscr{D}^\vee$
- in case 3, $F(M) = \sigma_1(M)_\alpha A \oplus \sigma_2(M)_\beta A \in \mathscr{D}^\vee$

where ${}_\alpha A$ and ${}_\beta A$ are the module A equipped with the (C, A)-bimodule structure induced by α and β respectively.

This functor F is exact and sends $\mathscr{C}_2$ to $\mathscr{D}$. It has a right adjoint functor $\widehat{F}$ from $\mathscr{D}^\vee$ to $\mathscr{C}_2{}^\vee$ and we have:

- in case 1, $\widehat{F}$ is the identity
- in case 2, $\widehat{F}(E_A, E_B) = s_1(E_A) \oplus s_2(E_B)$
- in case 3, $\widehat{F}(E) = s_1(E_\alpha) \oplus s_2(E_\beta)$

where E_α and E_β are the module E equipped with the right C-module structure induced by α and β respectively. In case: $C_2 = C \times C$, we have another functor $M \mapsto \widetilde{M}$ from $\mathscr{C}_2{}^\vee$ (or $\mathscr{C}_2$) to itself defined by:

$$M = (M', M'') \quad \Longrightarrow \quad \widetilde{M} = (M'', M')$$

Lemma 1.6. *Suppose $C_2 = C \times C$. Then for every right $C \times C$-module M, we have natural isomorphisms:*

$$s\sigma(M) \simeq M \oplus \widetilde{M}$$
$$\widehat{F}F(M) \simeq M \oplus \widetilde{M}S$$

Moreover the induced projection $s\sigma(M) \longrightarrow M$ and the induced injections $M \longrightarrow s\sigma(M)$ and $M \longrightarrow \widehat{F}F(M)$ are adjoint to identities.

Proof. Let $M = (M', M'')$ be a module in $\mathscr{C}_2 = \mathscr{C} \times \mathscr{C}$. We have:

$$s(\sigma(M)) = s(M' \oplus M'') = s_1(M' \oplus M'') \oplus s_2(M' \oplus M'') \simeq M \oplus (s_1(M'') \oplus s_2(M')) \simeq M \oplus \widetilde{M}$$

In case 2, we have:

$$\widehat{F}F(M) = \widehat{F}(M'A, M''B) = (M'(C \oplus A'), M''(C \oplus B'))$$
$$\simeq (M', M'') \oplus (M'{}_2S_1, M''{}_1S_2) \simeq M \oplus \widetilde{M}S$$

In the case 3 we have:

$$\widehat{F}F(M) = \widehat{F}(M'{}_\alpha A \oplus M''{}_\beta A) = s_1(M'{}_\alpha A_\alpha \oplus M''{}_\beta A_\alpha) \oplus s_2(M'{}_\alpha A_\beta \oplus M''{}_\beta A_\beta)$$
$$= s_1(M' \oplus M'{}_2S_1 \oplus M''{}_1S_1) \oplus s_2(M'{}_2S_2 \oplus M'' \oplus M''{}_1S_2)$$
$$\simeq M \oplus s_1(\widetilde{M}S_1) \oplus s_2(\widetilde{M}S_2) \simeq M \oplus \widetilde{M}S$$

and the result follows. $\square$

1.7. The module $M[S]$ and the transformations e, ε and τ

For every $M \in \mathscr{C}_2{}^\vee$, we set:

$$M[S] = M \oplus MS \oplus MS^2 \oplus \dots$$

and $M[S]$ is a right C_2-module. In case 1, $M[S]$ is isomorphic to MR.

We have a stabilization map $MS[S] \longrightarrow M[S]$ induced by the identities $MSS^i \longrightarrow MS^{i+1}$.

Lemma 1.8. *There exist natural transformations:*

$$e : TF(P) \xrightarrow{\sim} \sigma(P)R$$

$$\varepsilon : \sigma(\widehat{F}(E)[S]) \xrightarrow{\sim} T(E)$$

$$\tau : \sigma(s(M)[S]) \longrightarrow MR$$

for all $P \in \mathscr{C}_2{}^\vee$, $E \in \mathscr{D}^\vee$ and $M \in \mathscr{C}^\vee$ such that:

- *e is an isomorphism of R-modules*
- *ε is an isomorphism of C-modules*
- *τ is an epimorphism of C-modules and the following diagram is exact (i.e. cartesian and cocartesian):*

$$\begin{tikzcd}
\sigma s(M) \arrow[r] \arrow[d, "\tau_0"'] & \sigma(s(M)[S]) \arrow[d, "\tau"] \\
M \arrow[r] & MR
\end{tikzcd}$$

where the horizontal maps are the canonical inclusions and τ_0 is adjoint to the identity of $s(M)$.

Proof. In case 1, it is easy to see that e, ε, τ and τ_0 can be chosen to be identities.

In the other cases consider a module $P \in \mathscr{C}_2{}^\vee$. So we have two C-modules $M = P_1$ and $N = P_2$.

In case 2, we have:

$$T(F(P)) = T(MA, NB) = MA \otimes_A R \oplus NB \otimes_B R \simeq MR \oplus NR \simeq (M \oplus N)R = \sigma(P)R$$

and we get the isomorphism e.

In case 3, we have:

$$T(F(P)) = T(M_\alpha A \oplus N_\beta A) \simeq M_\alpha R \oplus N_\beta R$$

but the multiplication on the left by t induces a isomorphism of (C, R)-bimodules from ${}_\beta R$ to ${}_\alpha R$. Then we have:

$$T(F(P)) \simeq M_\alpha R \oplus N_\alpha R = \sigma(P)_\alpha R = \sigma(P)R$$

and we get the isomorphism e.

In order to construct the morphism τ, we need to give an explicit description of R as a C-bimodule.

We set:

$${}_\alpha U_\alpha = A' \qquad {}_\beta U_\beta = B' \qquad {}_\alpha U_\beta = {}_\beta U_\alpha = 0$$

in case 2 and:

$$_{\alpha}U_{\alpha} = A' \qquad _{\beta}U_{\beta} = tA''t^{-1} \qquad _{\alpha}U_{\beta} = At^{-1} \qquad _{\beta}U_{\alpha} = tA$$

in case 3.

For each sequence $\Sigma = (i_1, j_1, i_2, j_2, \ldots, i_n, j_n)$ in the set $\{\alpha, \beta\}$, the ring structure on R induces a well defined morphism of C-bimodules:

$$\Phi(\Sigma) : {}_{i_1}U_{j_1\, i_2}U_{j_2} \ldots {}_{i_n}U_{j_n} \longrightarrow R$$

and the sum of these morphisms is an epimorphism. On the other hand it is easy to see that, for each (i, j, k) in $\{\alpha, \beta\}$ the image of $\Phi(i, j, j, k)$ is contained in $C \oplus {}_iU_k$. Therefore, if there is some $k < n$ such that $j_k = i_{k+1}$ in the sequence $\Sigma = (i_1, j_1, i_2, j_2, \ldots, i_n, j_n)$, every element of the image of $\Phi(\Sigma)$ is reducible. As a consequence the sum of the morphisms $\Phi(\Sigma)$, for each sequence $\Sigma = (i_1, j_1, i_2, j_2, \ldots, i_n, j_n)$ such that $j_k \neq i_{k+1}$ for every $k < n$ is still an epimorphism.

Actually this sum is an isomorphism and we have a description of R as C-bimodule:

$$R = C \oplus \bigoplus \Big({}_{i_1}U_{j_1\, i_2}U_{j_2} \ldots {}_{i_n}U_{j_n} \Big)$$

the sum being taken over all non empty sequences $(i_1, j_1, i_2, j_2, \ldots, i_n, j_n)$ in $\{\alpha, \beta\}$ such that $j_k \neq i_{k+1}$ for all $k < n$.

This fact was proven in [8], p. 140 (or in [2]) for the case 2 and in [8], p. 150 (with a suggestion of S. Cappell) for the case 3.

Denote by f (resp. g) the unique bijection from $\{1, 2\}$ to $\{\alpha, \beta\}$ such that $f(1) = \beta$ (resp. $g(1) = \alpha$). Then, because of the definition of S, we have in case 2:

$$\forall i, j \in \{1, 2\}, \quad {}_iS_j = {}_{f(i)}U_{g(j)}$$

In case 3, we check that the multiplication by t or 1 on the left and by t^{-1} or 1 on the right induce for each i, j in $\{1, 2\}$ an isomorphism of C-bimodules ${}_iS_j \xrightarrow{\sim} {}_{f(i)}U_{g(j)}$.

Then in cases 2 and 3 we have an isomorphism of C-bimodules:

$$R \simeq C \oplus \bigoplus \Big({}_{i_1}S_{j_1\, i_2}S_{j_2} \ldots {}_{i_n}S_{j_n} \Big)$$

the sum being taken over all non empty sequences $(i_1, j_1, i_2, j_2, \ldots, i_n, j_n)$ in $\{1, 2\}$ such that $j_k = i_{k+1}$ for all $k < n$. Hence we get an isomorphism of C-bimodules:

$$R \simeq C \oplus \bigoplus_{i,j} ({}_iS_j \oplus {}_i(S^2)_j \oplus {}_i(S^3)_j \oplus \ldots) \simeq C \oplus \bigoplus_{i,j,n>0} {}_i(S^n)_j$$

So we are able to define the morphism τ. If M is a right C-module we have:

$$MR \simeq M \oplus \bigoplus_{i,j,n>0} M_i(S^n)_j \simeq M \oplus \bigoplus_{j,n>0} (s(M)S^n)_j \simeq M \oplus \bigoplus_{n>0} \sigma(s(M)S^n)$$

But we have:

$$\sigma(s(M)[S]) = \sigma s(M) \oplus \bigoplus_{n>0} \sigma(s(M)S^n)$$

Then we define the morphism τ to be the identity on the direct sum of the $\sigma(s(M)S^n)$ and the morphism $\tau_0 : \sigma s(M) \to M$ induced by the adjunction on the first term $\sigma s(M)$.

The last thing to do is to construct the isomorphism ε in cases 2 and 3.

In case 2 with $E = (E_A, E_B)$, we have:

$$T(E) = E_A \otimes_A R \oplus E_B \otimes_B R \simeq E_A(C \oplus B' \oplus B'A' \oplus B'A'B' \oplus \dots) \oplus E_B(C \oplus A' \oplus A'B' \oplus A'B'A' \oplus \dots)$$
$$\simeq E_A \oplus E_B \oplus E_A\big(\underset{j,n>0}{\oplus} {}_1(S^n)_j\big) \oplus E_B\big(\underset{j,n>0}{\oplus} {}_2(S^n)_j\big)$$
$$\simeq \sigma(\widehat{F}(E)) \oplus \big(\underset{n>0}{\oplus} \sigma(\widehat{F}(E)S^n)\big) \simeq \sigma(\widehat{F}(E)[S])$$

which give the isomorphism ε.

In the last case we have:

$$R \simeq C \oplus \bigoplus_{i,j,n>0} {}_i(S^n)_j \simeq (C \oplus {}_2S_1)(C \oplus \bigoplus_{j,n>0} {}_1(S^n)_j) \oplus {}_2S_2(C \oplus \bigoplus_{j,n>0} {}_2(S^n)_j)$$
$$\simeq A(C \oplus \bigoplus_{j,n>0} {}_1(S^n)_j) \oplus At^{-1}(C \oplus \bigoplus_{j,n>0} {}_2(S^n)_j)$$
$$\simeq A_\alpha(C \oplus \bigoplus_{j,n>0} {}_1(S^n)_j) \oplus A_\beta(C \oplus \bigoplus_{j,n>0} {}_2(S^n)_j$$

and we check that the isomorphism from R to this last module is an isomorphism of (A, C)-bimodules. Since E belongs to $\mathscr{A}^\vee$, we have;

$$T(E) \simeq E_\alpha(C \oplus \bigoplus_{j,n>0} {}_1(S^n)_j) \oplus E_\beta(C \oplus \bigoplus_{j,n>0} {}_2(S^n)_j)$$
$$\simeq \sigma(\widehat{F}(E)) \oplus \bigoplus_{n>0} \sigma(\widehat{F}(E)S^n) \simeq \sigma(\widehat{F}(E)[S])$$

which give the isomorphism ε in this last case. $\square$

Remark 1.9. We have an explicit description of the morphism τ and the isomorphism ε.

Consider two modules $M \in \mathscr{C}^\vee$ and $E \in \mathscr{D}^\vee$.

In case 1, set: $\overline{u} = u$ for each $u \in E$.

In case 2, the maps $E_A \longrightarrow E_A R$ and $E_B \longrightarrow E_B R$ induce a map $\sigma(E) \longrightarrow T(E)$ and in case 3, the maps $E_\alpha \longrightarrow ER$ and $E_\beta \longrightarrow EtR \longrightarrow ER$ induce also a map $\sigma(E) \longrightarrow T(E)$. Denote by $v \mapsto \overline{v}$ this map.

So we have a morphism $v \mapsto \overline{v}$ from $\sigma(E)$ to $T(E)$ in the three cases.

Denote also by $s \mapsto \overline{s}$ the isomorphism ${}_iS_j \xrightarrow{\sim} {}_{f(i)}U_{g(j)} \subset R$ in cases 2 or 3 and the identity $S \longrightarrow S$ in case 1.

With these notations, we have the following description of τ and ε:

In case 1, for each integer $n \geq 0$, each $u \in M$, each $v \in E$ and each sequence $(s_1, s_2, \dots, s_n)$ in S we have:

$$\tau(us_1s_2 \dots s_n) = u\ \overline{s_1}\ \overline{s_2} \dots \overline{s_n} \in MR$$
$$\varepsilon(vs_1s_2 \dots s_n) = \overline{v}\ \overline{s_1}\ \overline{s_2} \dots \overline{s_n} \in T(E)$$

In case 2 or 3, for each integer $n \geq 0$, each sequence $(i_0, i_1, \dots, i_n)$ in $\{1, 2\}$, each $u \in M$, each $v \in E_{i_0}$ and each sequence $(s_1, s_2, \dots s_n)$, with $s_k \in {}_{i_{k-1}}S_{i_k}$, we have:

$$\tau(us_1s_2 \dots s_n) = u\ \overline{s_1}\ \overline{s_2} \dots \overline{s_n} \in MR$$
$$\varepsilon(vs_1s_2 \dots s_n) = \overline{v}\ \overline{s_1}\ \overline{s_2} \dots \overline{s_n} \in T(E)$$

Lemma 1.10. *There is a natural transformation:*

$$\Lambda : \mathscr{F}(E, M) \longrightarrow \mathrm{Hom}(\widehat{F}(E), s(M) \oplus s(M)S)$$

for each $(E, M) \in \times \mathscr{D}^\vee \times \mathscr{C}^\vee$ with the following properties:

- *Λ is injective in case 2 or 3 and bijective in case 1*
- *for each $(P, M) \in {\mathscr{C}_2}^\vee \times \mathscr{C}^\vee$, if f is the map $P \longrightarrow \widehat{F}(F(P))$ induced by adjunction, the composite morphism:*

$$\mathscr{F}(F(P), M) \xrightarrow{\Lambda} \mathrm{Hom}(\widehat{F}(F(P)), s(M) \oplus s(M)S) \xrightarrow{f^*} \mathrm{Hom}(P, s(M) \oplus s(M)S)$$

is bijective

- *for each $\varphi \in \mathscr{F}(E, M)$, we have a commutative diagram:*

$$\begin{tikzcd} \sigma(\widehat{F}(E)[S]) \arrow[r, "g"] \arrow[d, "\varepsilon"'] & \sigma(s(M)[S]) \arrow[d, "\tau"] \\ T(E) \arrow[r, "\varphi"] & MR \end{tikzcd}$$

where g is the composite morphism:

$$\sigma(\widehat{F}(E)[S]) \xrightarrow{\Lambda(\varphi)} \sigma((s(M) \oplus s(M)S)[S]) \xrightarrow{h} \sigma(s(M)[S])$$

and h the morphism induced by the identity $s(M)[S] \longrightarrow s(M)[S]$ and the stabilization map $s(M)S[S] \longrightarrow s(M)[S]$.

Proof. Consider the case 1. We have an isomorphism from $\mathrm{Hom}_{\mathscr{R}}(ER, MR)$ to $\mathrm{Hom}_{\mathscr{C}}(E, MR)$ inducing an isomorphism $\mathscr{F}(E, M) \simeq \mathrm{Hom}(E, M \oplus MS)$ and we get the isomorphism $\Lambda : \mathscr{F}(E, M) \xrightarrow{\sim} \mathrm{Hom}(E, M \oplus MS) = \mathrm{Hom}(\widehat{F}(E), s(M) \oplus s(M)S)$.

Consider the case 2. We have $E = (E_A, E_B) \in \mathscr{A} \times \mathscr{B}$ and we get isomorphisms:

$$\mathrm{Hom}_{\mathscr{R}}(T(E), MR) \simeq \mathrm{Hom}_{\mathscr{R}}(E_A R \oplus E_B R, MR) \simeq \mathrm{Hom}_{\mathscr{A}}(E_A, MR) \oplus \mathrm{Hom}_{\mathscr{B}}(E_B, MR)$$

and then isomorphisms:

$$\mathscr{F}(E, M) \simeq \mathrm{Hom}(E_A, MA) \oplus \mathrm{Hom}(E_B, MB) \simeq \mathrm{Hom}(E, (MA, MB)) = \mathrm{Hom}(E, F(s(M)))$$

Consider now the last case. We have:

$$\mathrm{Hom}_{\mathscr{R}}(T(E), MR) \simeq \mathrm{Hom}_{\mathscr{R}}(ER, MR) \simeq \mathrm{Hom}_{\mathscr{A}}(E, MR)$$

and then:

$$\mathscr{F}(E, M) \simeq \mathrm{Hom}(E, MA \oplus MtA) \simeq \mathrm{Hom}(E, M_\alpha A \oplus M_\beta A) \simeq \mathrm{Hom}(E, F(s(M)))$$

Therefore, in case 2 and 3, we have an isomorphism:

$$\mathscr{F}(E, M) \xrightarrow{\sim} \mathrm{Hom}(E, F(s(M))) \qquad (*)$$

On the other hand, the morphism $\widehat{F}$ induces an injection:

$$\mathrm{Hom}(E, F(s(M))) \longrightarrow \mathrm{Hom}(\widehat{F}(E), \widehat{F}(F(s(M)))) \simeq \mathrm{Hom}(\widehat{F}(E), s(M) \oplus s(M)S)$$

(see lemma 1.6) and we get the desired injection

$$\Lambda : \mathscr{F}(E, M) \longrightarrow \mathrm{Hom}(\widehat{F}(E), s(M) \oplus s(M)S)$$

Let (P, M) be an object in $C_2^\vee \times \mathscr{C}^\vee$. In case 1, the morphism $P \longrightarrow \widehat{F}(F(P))$ is the identity and the composite map:

$$\mathscr{F}(F(P), M) \xrightarrow{\Lambda} \mathrm{Hom}(\widehat{F}(F(P)), s(M) \oplus s(M)S) \xrightarrow{f^*} \mathrm{Hom}(P, s(M) \oplus s(M)S)$$

is bijective.

Consider the other cases. Let $\varphi \in \mathscr{F}(F(P), M)$ be an admissible morphism and $\alpha : F(P) \longrightarrow F(s(M))$ be the corresponding morphism (via the isomorphism $(*)$). Let $f : P \longrightarrow \widehat{F}F(P)$ be the morphism adjoint to the identity of $F(P)$. By adjunction, the composite morphism $P \xrightarrow{f} \widehat{F}F(P)) \xrightarrow{\alpha} \widehat{F}F(s(M))$ is the morphism obtained from α by adjunction and we have a bijection $\mathrm{Hom}(F(P), F(s(M))) \simeq \mathrm{Hom}(P, \widehat{F}F(s(M))$. Hence we have a bijection:

$$\mathscr{F}(F(P), M) \simeq \mathrm{Hom}(P, \widehat{F}F(s(M)) \simeq \mathrm{Hom}(P, s(M) \oplus s(M)S)$$

which is nothing else but the map $f^* \circ \Lambda$. Denote by (D) the diagram of the lemma.

In case 1, ε and τ are identities and we have: $g = \varphi$. Hence (D) is commutative.

Consider the other cases. Via the bijection $\mathscr{F}(E, M) \simeq \mathrm{Hom}(E, F(s(M)))$, the morphism $\varphi \in \mathscr{F}(E, M)$ corresponds to a morphism $\widetilde{\varphi} : E \longrightarrow F(s(M))$ and we have a diagram:

$$\begin{tikzcd}
\sigma(\widehat{F}(E)[S]) \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\varepsilon"'] & \sigma(\widehat{F}F(s(M))[S]) \arrow[r, "g_0"] \arrow[d, "\varepsilon"'] & \sigma(s(M)[S]) \arrow[d, "\tau"'] \\
T(E) \arrow[r, "\widetilde{\varphi}"] & T(F(s(M))) \arrow[r, "\varphi_0"] & MR
\end{tikzcd}$$

In this diagram, the square on the left is commutative by naturality and the square on the right (D_0) is the diagram (D) in the case: $E = F(s(M))$. Moreover the total square is the diagram (D). Hence, to prove the commutativity of (D) it is enough to prove that (D_0) is commutative.

In (D_0), the morphism g_0 is induced by the isomorphism $\widehat{F}F(s(M)) \xrightarrow{\sim} s(M) \oplus s(M)S$, the identity $s(M)[S] \longrightarrow s(M)[S]$ and the stabilization map $s(M)S[S] \longrightarrow s(M)[S]$.

The morphism φ_0 is the composite map:

$$T(F(s(M))) \xrightarrow{\sim} (MA \oplus MB)R \simeq MR \oplus MR \xrightarrow{+} MR$$

in case 2 and the composite map:

$$T(F(s(M))) \xrightarrow{\sim} (MA \oplus MtA)R \simeq MR \oplus MR \xrightarrow{+} MR$$

in case 3. Hence φ_0 is the composite map:

$$T(F(s(M))) \underset{\sim}{\xrightarrow{e}} \sigma s(M)R \xrightarrow{a} MR$$

where $a : \sigma s(M) \longrightarrow M$ is adjoint to the identity of $s(M)$.

Consider an element $u \in \widehat{F}F(s(M))$, an integer $n \geq 0$, a sequence $(i_0, i_1, \dots, i_n)$ in $\{1, 2\}$ and a sequence $(s_1, s_2, \dots, s_n)$ with $s_k \in {}_{i_{k-1}}S_{i_k}$. Denote by v the image of u under the isomorphism $\widehat{F}F(s(M)) \simeq s(M) \oplus s(M)S$. If v belongs to $s(M)$ or to $s(M)S_{i_0}$, we have, because of remark 1.9, the following:

$$\varphi_0\varepsilon(us_1s_2 \dots s_n) = v\ \overline{s_1}\ \overline{s_2} \dots \overline{s_n} = \tau g_0(us_1s_2 \dots s_n)$$

and (D_0) is therefore commutative. $\square$

1.11. The functors $\Phi : \mathscr{N}il(C_2, S)^\vee \longrightarrow \mathscr{V}^\vee$ and $\Psi : \mathscr{S}^\vee \longrightarrow \mathscr{N}il(C_2, S)^\vee$

Let H be a module in ${\mathscr{C}_2}^\vee$. Because of lemma 1.10, we have an isomorphism:

$$\zeta : \mathscr{F}(F(H), \sigma(H)) \xrightarrow{\sim} \mathrm{Hom}(H, s\sigma(H) \oplus s\sigma(H)S)$$

Therefore, for every morphism $\theta : H \longrightarrow HS$, we have a unique morphism φ_θ in $\mathscr{F}(F(H), \sigma(H)$ such that $\zeta(\varphi_\theta)$ is the composite morphism:

$$H \xrightarrow{1-\theta} H \oplus HS \xrightarrow{i} s\sigma H \oplus s\sigma HS$$

where $i : H \longrightarrow s\sigma H$ is adjoint to the identity. Thus $\Phi(H, \theta) = (F(H), \sigma(H), \varphi_\theta)$ is a well defined split diagram in $\mathscr{S}^\vee$.

Lemma 1.12. *The correspondence above induces two equivalences of categories $\Phi : \mathscr{N}il(C_2, S) \xrightarrow{\sim} \mathscr{V}$ and $\Phi : \mathscr{N}il(C_2, S)^\vee \xrightarrow{\sim} \mathscr{V}^\vee$.*

Moreover there is a functor $\Psi : \mathscr{S}^\vee \longrightarrow \mathscr{N}il(C_2, S)^\vee$ and a morphism of functors π from $I\Phi\Psi$ to the identity, where $I : \mathscr{V}^\vee \longrightarrow \mathscr{S}^\vee$ is the inclusion, such that the following holds for every $X \in \mathscr{S}^\vee$ (with: $\Psi(X) = (H, \theta)$):

- *we have a natural exact sequence in $\mathscr{C}^\vee$:*

$$0 \longrightarrow \Phi_1(X) \longrightarrow \sigma(H) \xrightarrow{\pi} \Phi_3(X)$$

- *a splitting diagram X belongs to $\overline{\mathscr{M}\mathscr{V}}$ (resp. to $\mathscr{V}^\vee$) if and only if the morphism $\pi : \Phi\Psi(X) \longrightarrow X$ is an epimorphism (resp. an isomorphism).*

Remark. Actually, everything works without any flatness condition from subsection 1.4 to 1.11. But this condition is strongly needed for lemma 1.12, essentially for constructing the functor Ψ.

Proof of lemma 1.12. Let (H, θ) be an object of $\mathscr{N}il(C_2, S)^\vee$. If $\theta = 0$, the morphism $\varphi_0 : T(F(H) \longrightarrow \sigma(H)R$ is nothing else but the isomorphism e defined in lemma 1.8. Therefore $\Phi(H, \theta_\theta)$ belongs to $\mathscr{V}^\vee$ in this case.

Suppose θ is nilpotent. Then, because of lemma 1.2, there is a filtration $0 = H_0 \subset H_1 \subset H_2 \subset \dots$ of H such that H is the union of the H_i's and, for all $i > 0$, $\theta(H_i)$ is contained in $H_{i-1}S$.

On the other hand it is easy to see that the functor Φ is exact. So we get a filtration:

$$0 = \Phi(H_0, \theta) \subset \Phi(H_1, \theta) \subset \Phi(H_2, \theta) \subset \dots \subset \Phi(H, \theta)$$

Because each $\Phi(H_i/H_{i-1}, \theta) = \Phi(H_i/H_{i-1}, 0)$ belongs to $\mathscr{V}^\vee$, each $\Phi(H_i, \theta)$ is also in $\mathscr{V}^\vee$ and then $\Phi(H, \theta)$ is a splitting module.

Therefore Φ is a functor from $\mathscr{N}il(C_2, S)^\vee$ to $\mathscr{V}^\vee$ and also from $\mathscr{N}il(C_2, S)$ to $\mathscr{V}$.

Let $X = (E, M, \varphi)$ be a splitting diagram with $E \in \mathscr{D}^\vee$ and $M \in \mathscr{C}^\vee$. Denote by P the module $\widehat{F}(E) \in \mathscr{C}_2{}^\vee$. By composing the morphism $\varphi\varepsilon : \sigma(P[S]) \longrightarrow MR$ with the identity $P[S] = \sigma(P[S])$, we get a $\mathbf{Z}$-linear map $f : P[S] \longrightarrow MR$. Denote by H the $\mathbf{Z}$-submodule $f^{-1}(M)$ and by i the inclusion map $H \longrightarrow P[S]$.

Because of lemma 1.10, φ is determined by a morphism $\psi = \Lambda(\varphi)$ from $P = \widehat{F}(E)$ to $s(M) \oplus s(M)S$. Then we have two morphisms $\lambda : P \longrightarrow s(M)$ and $\gamma : P \longrightarrow s(M)S$ such that: $\psi = \lambda - \gamma$. Moreover the composite map $P[S] = \sigma(P[S]) \xrightarrow{\varphi\varepsilon} MR$ is equal to: $\tau(\lambda - \gamma)$.

We have:

$$u \in H \iff \varphi\varepsilon(u) \in M \iff \tau(\lambda - \gamma)(u) \in M$$
$$\iff \forall k \geq 0, \quad \tau(\lambda(u_{k+1}) - \gamma u_k)) = 0$$

But the morphism $\tau : \sigma(s(M)S^k) \longrightarrow MR$ is injective for all $k > 0$ (see lemma 1.8). Then we have:

$$u \in H \iff \forall k \geq 0, \quad \lambda(u_{k+1}) = \gamma(u_k)$$

Since λ and γ are morphisms in $\mathscr{C}_2{}^\vee$, H is a C_2-module and we have an exact sequence in $\mathscr{C}_2{}^\vee$:

$$0 \longrightarrow H \xrightarrow{i} P[S] \xrightarrow{\delta} N[S]$$

where N is the module $s(M)S$ and δ is the morphism sending $u = u_0 + u_1 + u_2 + \dots$ (with $u_k \in PS^k$ for every $k \geq 0$) to:

$$\delta(u_0 + u_1 + u_2 + \dots) = \sum_{k \geq 0} (\lambda(u_{k+1}) - \gamma(u_k))$$

If U is a module in $\mathscr{C}_2{}^\vee$, we have a morphism $\theta_U : U[S] \longrightarrow U[S]S$ sending $u_0 + u_1 + u_2 + \dots \in U[S]$ (with $u_k \in PS^k$ for every $k \geq 0$) to:

$$\theta_U(u_0 + u_1 + u_2 + \dots) = u_1 + u_2 + u_3 + \dots$$

Notice that θ_U^k sends $u_0 + u_1 + u_2 + \dots \in U[S]$ to $u_k + u_{k+1} + \dots$ and θ_U is nilpotent. Hence $(U[S], \theta_U)$ belongs to $\mathscr{N}il(C_2, S)^\vee$.

Consider the following diagram:

$$\begin{array}{ccc} P[S] & \xrightarrow{\delta} & N[S] \\ {\scriptstyle \theta_P}\downarrow & & \downarrow{\scriptstyle \theta_N} \\ P[S]S & \xrightarrow{\delta} & N[S]S \end{array} \qquad (D)$$

Let $n \geq 0$ be an integer, u be an element in P and $s_1, s_2, \dots, s_n$ be elements in S. We have the following:

$$n = 0 \implies \delta\theta_P(u) = \theta_N\delta(u) = 0$$
$$n = 1 \implies \delta\theta_P(us_1) = \theta_N\delta(us_1) = -\gamma(u)s_1$$
$$n > 1 \implies \delta\theta_P(us_1 \dots s_n) = \theta_N\delta(us_1 \dots s_n) = \lambda(u)s_1 \dots s_n - \gamma(u)s_1 \dots s_n$$

and the diagram (D) is commutative.

Since S is flat on the left, $\mathscr{N}il(C_2, S)^\vee$ is an abelian category and there is a unique nilpotent morphism $\theta : H \longrightarrow HS$ such that the following sequence is exact in $\mathscr{N}il(C_2, S)^\vee$:

$$0 \longrightarrow (H, \theta) \xrightarrow{i} (P[S], \theta_P) \xrightarrow{\delta} (N[S], \theta_N)$$

Therefore $\Psi(X) = (H, \theta)$ is a well defined object in $\mathscr{N}il(C_2, S)^\vee$ and we get the desired functor $\Psi : \mathscr{S}^\vee \longrightarrow \mathscr{N}il(C_2, S)^\vee$. Consider the splitting diagram $\Phi\Psi(X) = (F(H), \sigma(H), \varphi_\theta)$. The morphism φ_θ corresponds to the composite morphism $H \xrightarrow{1-\theta} H \oplus HS \longrightarrow s\sigma(H) \oplus s\sigma(H)S$. We have to construct a morphism $\pi : \Phi\Psi(X) \longrightarrow X$ in $\mathscr{S}^\vee$. This morphism is given by two morphisms $\pi_0 : F(H) \longrightarrow E$ and $\pi_1 : \sigma(H) \longrightarrow M$.

For each $u \in H$ we set: $\pi'_0(u) = u_0$, with $i(u) = u_0 + u_1 + \dots$ and $u_k \in PS^k$ for all k. So we have two morphisms $\pi'_0 : H \longrightarrow \widehat{F}(E)$ and $\lambda\pi'_0 : H \longrightarrow s(M)$ and, by adjunction, two morphisms $\pi_0 : F(H) \longrightarrow E$ and $\pi_1 : \sigma(H) \longrightarrow M$.

For $u \in H$, with: $i(u) = u_0 + u_1 + \dots$, we have:

$$\psi\pi'_0(u) = \psi(u_0) = \lambda(u_0) - \gamma(u_0) = \lambda(u_0) - \lambda(u_1) = \lambda\pi'_0(u)) - \lambda\pi'_0\theta(u) = \lambda\pi'_0(1-\theta)(u)$$

and the following diagram is commutative:

$$\begin{tikzcd} H \arrow[r, "\psi_0"] \arrow[d, "\pi'_0"'] & s\sigma(H) \oplus s\sigma(H)S \arrow[d, "\pi_1"'] \\ P \arrow[r, "\psi"] & s(M) \oplus s(M)S \end{tikzcd}$$

Therefore the two morphisms $\pi_0 : F(H) \longrightarrow E$ and $\pi_1 : \sigma(H) \longrightarrow M$ induce a well defined morphism $\pi : \Phi\Psi(X) \longrightarrow X$ and we get a commutative diagram:

$$\begin{tikzcd} T(F(H)) \arrow[r, "\varphi_\theta", "\sim"'] \arrow[d, "\pi_0"'] & \sigma(H)R \arrow[d, "\pi_1"'] \\ T(E) \arrow[r, "\varphi"] & MR \end{tikzcd} \qquad (D_X)$$

We have an exact sequence in $\mathscr{R}^\vee$:

$$0 \longrightarrow \Phi_1(X) \xrightarrow{\mu} T(E) \xrightarrow{\varphi} MR$$

and then an exact sequence in $\mathscr{C}^\vee$:

$$0 \longrightarrow \Phi_1(X) \xrightarrow{\varepsilon^{-1}\mu} \sigma(P[S]) \xrightarrow{\varphi\varepsilon} MR$$

Therefore we have a commutative diagram in $\mathscr{C}^\vee$ with exact lines:

$$\begin{tikzcd} 0 \arrow[r] & \Phi_1(X) \arrow[r] \arrow[d, "="'] & \sigma(H) \arrow[r, "\pi_1"] \arrow[d, "\varepsilon i"'] & M \arrow[d, "j"'] \\ 0 \arrow[r] & \Phi_1(X) \arrow[r, "\mu"] & T(E) \arrow[r, "\varphi"] & MR \end{tikzcd}$$

where $j : M \longrightarrow MR$ is the inclusion, and the top line of this diagram is the desired exact sequence. We have the following equivalences:

$$\begin{aligned} X \in \overline{\mathscr{M}\mathscr{V}} &\iff \text{the morphism } \varphi : T(E) \longrightarrow MR \text{ is surjective} \\ &\iff \text{the image of } \varphi \text{ contains } M \\ &\iff \text{the image of } \sigma(P[S]) \longrightarrow MR \text{ contains } M \\ &\iff \sigma(H) \xrightarrow{\pi_0'} \sigma(P) \xrightarrow{\lambda'} M \text{ is surjective} \end{aligned}$$

where λ' is adjoint to λ. Therefore X belongs to $\overline{\mathscr{M}\mathscr{V}}$ if and only if the morphism $\pi_1 : \sigma(H) \longrightarrow M$ is surjective.

Suppose X belongs to $\overline{\mathscr{M}\mathscr{V}}$. Let E' be the image of $\pi_0 : F(H) \longrightarrow E$. Because of the diagram above, $T(E')$ contains the image of $\mu : \Phi_1(X) \longrightarrow T(E)$ and the image of $\varepsilon i : \sigma(H) \longrightarrow T(E)$. But $T(E')$ is a R-submodule of $T(E)$ and $\varphi(T(E'))$ is a R-submodule of MR containing M. Therefore $\varphi : T(E') \longrightarrow MR$ is surjective and $T(E')$ contains the kernel $\Phi_1(X)$ of φ. Hence we have: $T(E') = T(E)$ and then: $E' = E$.

Consequently, if X belongs to $\overline{\mathscr{M}\mathscr{V}}$, $\pi_1 : \sigma(H) \longrightarrow M$ and $\pi_0 : F(H) \longrightarrow E$ are surjective and $\pi : \Phi\Psi(X) \longrightarrow X$ is an epimorphism (in $\mathscr{S}^\vee$). Conversely, if $\pi : \Phi\Psi(X) \longrightarrow X$ is an epimorphism, $\pi_1 : \sigma(H) \longrightarrow M$ is surjective and X belongs to $\overline{\mathscr{M}\mathscr{V}}$. Suppose X is in $\mathscr{V}^\vee$. Then π_0 and π_1 are epimorphisms. Because of the exact sequence:

$$0 \longrightarrow \Phi_1(X) \longrightarrow \sigma(H) \xrightarrow{\pi_1} M$$

the morphism $\pi_1 : \sigma(H) \longrightarrow M$ is an isomorphism. Therefore in the diagram (D_X), π_0 and φ are isomorphisms and $\pi : \Phi\Psi(X) \longrightarrow X$ is an isomorphism too. As a consequence, the functor Φ induces two equivalences of categories $\mathscr{N}il(C_2, S) \longrightarrow \mathscr{V}$ and $\mathscr{N}il(C_2, S)^\vee \longrightarrow \mathscr{V}^\vee$.

Conversely, if π is an isomorphism, π_0 and π_1 are isomorphisms and φ is an isomorphism too. Therefore X belongs to $\mathscr{V}^\vee$. $\square$

Lemma 1.13. *Let (A, S) be a left-flat bimodule and $X = (M, \theta)$ be an object in $\mathscr{N}il(A, S)^\vee$. Let V be a finitely generated projective right A-module and $f : V \longrightarrow M$ be a morphism. Then there exist an object $Y = (M', \theta') \in \mathscr{N}il(A, S)$, a split injective morphism $f' : V \longrightarrow M'$ and a morphism $g : Y \longrightarrow X$ making the following diagram commutative:*

$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "="'] & M' \arrow[d, "g"] \\ V \arrow[r, "f"] & M \end{tikzcd}$$

Proof. Denote by $\mathscr{A}^\vee$ the category of right A-modules and $\mathscr{A} \subset \mathscr{A}^\vee$ the category of finitely generated projective modules in $\mathscr{A}^\vee$. Because of lemme 1.2 there is a filtration:

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots$$

of M by A-modules such that M is the union of the M_i's and $\theta(M_i) \subset M_{i-1}S$ for every $i > 0$. Since V is finitely generated, there is an integer $n > 0$ such that M_n contains the image of $f : V \longrightarrow M$.

Then we'll construct modules $F_i \in \mathscr{A}$, morphisms $h_i : F_i \longrightarrow M_i$ for $i = 0, 1, \dots, n$ and morphisms $\theta' : F_i \longrightarrow F_{i-1}S$ for $i = 1, 2, \dots, n$ such that:

- $F_n = V$, $F_0 = 0$ and $h_n = f$

- for each $i = 1, 2, \dots, n$, the following diagram is commutative:

$$\begin{tikzcd} F_i \arrow[r, "h_i"] \arrow[d, "\theta'"'] & M_i \arrow[d, "\theta"'] \\ F_{i-1}S \arrow[r, "h_{i-1}"] & M_{i-1}S \end{tikzcd} \tag{D_i}$$

Let p be an integer with $0 \le p \le n$. Denote by $H(p)$ the following property:

- There exist modules $F_i \in \mathscr{A}$ and morphisms $h_i : F_i \longrightarrow E_i$ for $i = p, \dots, n$ and also morphisms $\theta' : F_i \longrightarrow F_{i-1}S$ for $i = p+1, \dots, n$ such that $F_n = V$, $h_n = f$, the diagram (D_i) is commutative for $i = p+1, \dots, n$ and $F_0 = 0$ (if $p = 0$).

This property is clearly true if $p = n$. Suppose $H(p)$ is true with $p > 0$ and consider the composite morphism $\lambda : F_p \longrightarrow M_p \xrightarrow{\theta} M_{p-1}S$. If $p = 1$, this morphism is trivial and we set: $F_0 = 0$. Therefore the property $H(p-1) = H(0)$ is true.

Consider the case $p > 1$. Since F_p is finitely generated, M_{p-1} contains a finitely generated submodule M' such that: $\lambda(F_p) \subset M'S$. Let F_{p-1} be a module in $\mathscr{A}$ and $\mu : F_{p-1} \longrightarrow M'$ be an epimorphism. Since F_p is projective the morphism $F_p \longrightarrow M'S$ factorizes through $F_{p-1}S$ and we have a commutative diagram:

$$\begin{tikzcd} F_p \arrow[r, "\theta'"] \arrow[d, "="'] & F_{p-1}S \arrow[d, "\mu"'] \\ F_p \arrow[r, "\lambda"] & M'S \end{tikzcd}$$

So we define the morphism h_{p-1} as the composite map: $F_{p-1} \xrightarrow{\mu} M' \subset M_{p-1}$ and we have the property $H(p-1)$.

By induction we obtain the property $H(0)$ and all the data are constructed.

Then we set: $M' = F_0 \oplus F_1 \oplus \cdots \oplus F_n$. The morphisms h_i induce a morphism $g : M' \longrightarrow M$ and the morphisms $\theta' : F_i \longrightarrow F_{i-1}S$ induce a morphism $\theta' : M' \longrightarrow M'S$. The lemma is now easy to check. □

Denote by $\mathscr{R}'$ the full subcategory of $\mathscr{R}$ generated by modules on the form $U = VR$ with $V \in \mathscr{C}$. This category is exact and cofinal in $\mathscr{R}$, that is, for each module $M \in \mathscr{R}$, there is a module $M' \in \mathscr{R}$ such that $M \oplus M'$ belongs to $\mathscr{R}'$.

Lemma 1.14. *Let X be a split diagram in $\mathscr{S}^\vee$, V be a module in $\mathscr{R}'$ and $f : V \longrightarrow \Phi_1(X)$ be a morphism in $\mathscr{R}^\vee$. Then there exist an object $Y \in \mathscr{M}\mathscr{V}$, a morphism $g : Y \longrightarrow X$ in $\mathscr{S}^\vee$ and an isomorphism $\varepsilon : V \xrightarrow{\sim} \Phi_1(Y)$ such that the following diagram is commutative:*

$$\begin{tikzcd} V \arrow[r, "\varepsilon", "\sim"'] \arrow[d, "="'] & \Phi_1(Y) \arrow[d, "g"'] \\ V \arrow[r, "f"] & \Phi_1(X) \end{tikzcd}$$

Moreover, if X belongs to $\mathscr{M}\mathscr{V}$, the morphism g can be chosen to be an epimorphism in $\mathscr{S}^\vee$.

Proof. The split diagram X is a triple (E, M, φ) with $E \in \mathscr{D}^\vee$ and $M \in \mathscr{C}^\vee$. Since V belongs to $\mathscr{R}'$, there is a module $W \in \mathscr{C}$ such that: $V = WR$ and we get a morphism $f' : W \longrightarrow \Phi_1(X)$ in $\mathscr{C}^\vee$.

Denote by $K = (H, \theta)$ the object $\Psi(X) \in \mathscr{N}il(C_2, S)^\vee$. Because of lemma 1.12, we have an exact sequence:

$$0 \longrightarrow \Phi_1(X) \xrightarrow{\mu} \sigma(H) \xrightarrow{\pi} M$$

and the morphism $\mu f' : W \longrightarrow \sigma(H)$ has an adjoint $\lambda : s(W) \longrightarrow H$. Because of lemma 1.13, there are an object $K' = (H', \theta') \in \mathscr{N}il(C_2, S)$, a morphism $h : K' \longrightarrow K$ and a split injective morphism $\lambda' : s(W) \longrightarrow H'$ making the following diagram commutative:

$$\begin{tikzcd} s(W) \arrow[r, "\lambda'"] \arrow[d, "="'] & H' \arrow[d, "h"'] \\ s(W) \arrow[r, "\lambda"] & H \end{tikzcd}$$

The morphism $\widetilde{\lambda}' : W \longrightarrow \sigma(H')$ adjoint to λ' is still split injective and we get a commutative diagram with exact lines:

$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r, "\widetilde{\lambda}'"] \arrow[d, "f'"'] & \sigma(H') \arrow[r, "\pi'"] \arrow[d, "h"'] & M' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Phi_1(X) \arrow[r, "\mu"] & \sigma(H) \arrow[r, "\pi"] & M & \end{tikzcd}$$

where M' is the cokernel of $\widetilde{\lambda}'$.

We have now an object $Y' = \Phi(K') = (F(H'), \sigma(H'), \varphi_{\theta''}) \in \mathscr{V}$, an object $Y = (F(H'), M', \pi'\varphi_{\theta'}) \in \mathscr{M}\mathscr{V}$ and an epimorphism $u : Y' \longrightarrow Y$. But the morphism $\Phi(K') \longrightarrow \Phi(K) \xrightarrow{\pi} X$ vanishes on the kernel of u and factorizes by a morphism $g : Y \longrightarrow X$. So we have morphisms $Y' \longrightarrow Y \longrightarrow X$ inducing a commutative diagram with exact lines:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & T(F(H')) \arrow[r] \arrow[d] & \sigma(H')R \arrow[r] \arrow[d, "\pi'"'] & 0 \\ 0 \arrow[r] & WR \arrow[r] \arrow[d, "f"'] & T(F(H')) \arrow[r] \arrow[d] & M'R \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Phi_1(X) \arrow[r] & T(E) \arrow[r] & MR & \end{tikzcd}$$

Hence, we get an isomorphism $\varepsilon : V = WR \longrightarrow \Phi_1(Y)$ and a commutative diagram:

$$\begin{tikzcd} V \arrow[r, "\varepsilon", "\sim"'] \arrow[d, "="'] & \Phi_1(Y) \arrow[d, "g"'] \\ V \arrow[r, "f"] & \Phi_1(X) \end{tikzcd}$$

Suppose X belongs to $\mathscr{M}\mathscr{V}$. Because of lemma 1.12, the morphism $\pi : \Phi\Psi(X) \longrightarrow X$ is an epimorphism in $\mathscr{S}^\vee$ inducing two epimorphisms $F(H) \longrightarrow E$ and $\sigma(H) \longrightarrow M$.

Since E is finitely generated, H contains a finitely generated C_2-submodule $H_0 \subset H$ such that the composite morphism $F(H_0) \longrightarrow F(H) \longrightarrow E$ is an epimorphism. Therefore there exist a module $P \in \mathscr{C}$ and a morphism $u : s(P) \longrightarrow H$ such that the image of u contains the submodule H_0. Because of lemma 1.13, there are an object $K' = (H', \theta') \in \mathscr{N}il(C_2, S)$, a morphism $h : K' \longrightarrow K$ and a split injective morphism $\lambda' \oplus u' : s(W) \oplus s(P) \longrightarrow H'$ making the following diagram commutative:

$$\begin{array}{ccc} s(W) \oplus s(P) & \xrightarrow{\lambda' \oplus u'} & H' \\ = \Big\downarrow & & h \Big\downarrow \\ s(W) \oplus s(P) & \xrightarrow{\lambda \oplus u} & H \end{array}$$

Therefore the composite morphism $F(H') \xrightarrow{h} F(H) \longrightarrow E$ is an epimorphism.

If we continue the construction above by using this morphism $h : K' \longrightarrow K$, we get a morphism $g : Y \longrightarrow X$ such that the morphism $\Phi_2(Y) \longrightarrow \Phi_2(X)$ is isomorphic to the morphism $F(H') \longrightarrow E$ which is an epimorphism. Hence $\Phi_2(Y) \longrightarrow \Phi_2(X)$ is also an epimorphism. On the other hand, we have a commmutative diagram:

$$\begin{array}{ccc} T(\Phi_2(Y)) & \longrightarrow & \Phi_3(Y)R \\ \Big\downarrow & & \Big\downarrow \\ T(E) & \xrightarrow{\varphi} & MR \end{array}$$

where $\varphi : T(E) \longrightarrow MR$ is surjective. Then $\Phi_3(Y)R \longrightarrow MR$ is surjective and $\Phi_3(Y) \longrightarrow M$ is surjective too. The result follows. $\square$

2. Algebraic K-theory of categories $\mathscr{V}$ and $\mathscr{M}\mathscr{V}$

2.1. About Waldhausen K-theory

A Waldhausen category is a category with a zero object and two subcategories: the category of cofibrations and the category of (weak-)equivalences. These categories have to satisfy certain conditions (see [9]). Waldhausen associates to any essentially small Waldhausen category $\mathscr{C}$ an infinite loop space $K(\mathscr{C})$ and K (called the Waldhausen K-theory functor) is a functor from the category of essentially small Waldhausen categories to the category of infinite loop spaces.

An exact category $\mathscr{E}$ may be considered as a Waldhausen category, where a cofibration is an admissible monomorphism of $\mathscr{E}$ (i.e. a morphism f appearing is an exact sequence $0 \longrightarrow X \xrightarrow{f} Y \longrightarrow Z \longrightarrow 0$ in $\mathscr{E}$) and an equivalence is an isomorphism in $\mathscr{E}$. Moreover, if $\mathscr{E}$ is essentially small, we have a natural homotopy equivalence from the Quillen K-theory of $\mathscr{E}$ to the Waldhausen K-theory of $\mathscr{E}$. To every exact category $\mathscr{E}$ we can associate the following category $\mathscr{E}_*$:

The objects of $\mathscr{E}_*$ (called the $\mathscr{E}$-complexes) are the complexes:

$$C = \Big(\cdots \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \xrightarrow{d} \dots \Big)$$

where each C_n is an object of $\mathscr{E}$ and each d is a morphism of $\mathscr{E}$ such that the sum $\bigoplus\limits_n C_n$ exists in $\mathscr{E}$ and each morphism d^2 is zero.

The morphisms in this category are morphisms respecting degrees and differentials. A sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\mathscr{E}_*$ is said to be exact if it induces an exact sequence in $\mathscr{E}$ on each degree. With these exact sequences, $\mathscr{E}_*$ becomes an exact category.

If $\mathscr{E}$ is the category of right modules (resp. the category of finitely generated projective right modules) over a ring A, the $\mathscr{E}$-complexes are called A-complexes (resp. finite A-complexes).

Suppose $\mathscr{E}$ is an exact subcategory of an abelian category $\mathscr{E}^\vee$. Then $\mathscr{E}_*$ is a Waldhausen category, where cofibrations are admissible monomorphisms and equivalences are morphisms inducing an isomorphism in homology (where homologies are computed in $\mathscr{E}^\vee$). Moreover $\mathscr{E}_*$ is saturated and has a cylinder functor satisfying the cylinder axiom (in the sense of Waldhausen [9]). We have the following result ([9], [10]):

Gillet-Waldhausen theorem 2.2. *Let $\mathscr{E}$ be an essentially small exact category contained in an abelian category $\mathscr{E}^\vee$. Suppose $\mathscr{E}$ is stable in $\mathscr{E}^\vee$ by kernel of epimorphisms. Then the inclusion $\mathscr{E} \subset \mathscr{E}_*$ of Waldhausen categories induces a homotopy equivalence in K-theory.*

Let $\mathscr{E}$ be an essentially small exact category and C and C' be two $\mathscr{E}$-complexes. For each integer n, we set:

$$\mathrm{Hom}(C, C')_n = \prod_p \mathrm{Hom}_{\mathscr{E}}(C_p, C'_{n+p})$$

and $\mathrm{Hom}(C, C')$ is a graded $\mathbf{Z}$-module. We have on $\mathrm{Hom}(C, C')$ a natural differential d of degree -1 defined by:

$$\forall f \in \mathrm{Hom}(C, C')_n, \quad d(f) = d \circ f - (-1)^n f \circ d$$

An element of $\mathrm{Hom}(C, C')_n$ is called a linear map of degree n, a cycle in $\mathrm{Hom}(C, C')_n$ is called a morphism of degree n and a boundary of $\mathrm{Hom}(C, C')_n$ is called a homotopy of degree n. The morphisms of degree 0 are the morphisms in the category $\mathscr{E}_*$.

In this category, we have also a notion of n-cone:

Consider a morphism $f : X \longrightarrow Y$ in $\mathscr{E}_*$ and an integer $n \in \mathbf{Z}$. We set: $C = X \oplus Y$. So we have four linear maps: $i : Y \longrightarrow C$, $p : C \longrightarrow X$, $r : C \longrightarrow Y$ and $s : X \longrightarrow C$. The map i is an injection, p is a projection, r is a retraction of i and s is a section of p. There is a unique way to modify degrees and differentials on C such that the following holds:

$$\begin{array}{llll} \partial^\circ i = n & \partial^\circ r = -n & \partial^\circ p = -1 - n & \partial^\circ s = 1 + n \\ d(i) = 0 & d(p) = 0 & d(r) = -(-1)^n fp & d(s) = if \end{array}$$

With these new degrees and differentials, C is an $\mathscr{E}$-complex called the n-cone of f. If $n = 0$, C is the classical mapping cone, the map $i : Y \longrightarrow C$ is a cofibration in $\mathscr{E}_*$ and we have an exact sequence in $\mathscr{E}_*$:

$$0 \longrightarrow X \longrightarrow T(f) \longrightarrow C \longrightarrow 0$$

where $T(f)$ is the cylinder of f.

If $n = -1$, the map $p : C \longrightarrow X$ is also a morphism in $\mathscr{E}_*$. Let $F : \mathscr{A} \longrightarrow \mathscr{B}$ be an exact functor between two Waldhausen categories. We say that F has the approximation property if the following holds:

- (App1) a morphism in $\mathscr{A}$ is an equivalence if and only if its image under F is an equivalence
- (App2) for every $(X, Y) \in \mathscr{A} \times \mathscr{B}$ and every morphism $f : F(X) \longrightarrow Y$, there exist a morphism $\alpha : X \longrightarrow X'$ in $\mathscr{A}$ and a commutative diagram in $\mathscr{B}$:

$$\begin{tikzcd} F(X) \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "="] \\ F(X') \arrow[r, "f'", "\sim"'] & Y \end{tikzcd}$$

where f' is an equivalence. We have the following theorem of Waldhausen ([9]):

Waldhausen approximation theorem 2.3. *Let $F : \mathscr{A} \longrightarrow \mathscr{B}$ be an exact functor between two essentially small saturated Waldhausen categories. Suppose $\mathscr{A}$ has a cylinder functor satisfying the cylinder axiom and F has the approximation property. Then F induces a homotopy equivalence in K-theory.*

Lemma 2.4. *The functor $\Phi_2 \times \Phi_3 : \mathscr{MV} \longrightarrow \mathscr{D} \times \mathscr{C}$ induces a homotopy equivalence in K-theory:*

$$K(\mathscr{MV}) \xrightarrow{\sim} K(\mathscr{D} \times \mathscr{C}) = K(\mathscr{D}) \times K(\mathscr{C})$$

Proof. The proof will be done by using Waldhausen K-theory.

The three exact categories $\mathscr{MV}$, $\mathscr{D}$ and $\mathscr{C}$ are contained in the abelian categories $\mathscr{MV}^\vee$, $\mathscr{D}^\vee$ and $\mathscr{C}^\vee$ respectively. Moreover each of these exact categories are closed by kernel of epimorphisms in the corresponding abelian categories. Therefore, because of Gillet-Waldhausen theorem, it's enough to prove that the functor $\Phi_2 \times \Phi_3 : \mathscr{MV}_* \longrightarrow \mathscr{D}_* \times \mathscr{C}_*$ induces a homotopy equivalence in K-theory.

By replacing the category of equivalences of $\mathscr{MV}_*$ by the category of morphisms $f : X \longrightarrow Y$ inducing a homology equivalence $\Phi_2(X) \xrightarrow{\sim} \Phi_2(Y)$ we get a new Waldhausen category denoted by $\mathscr{MV}'_*$.

Denote also by $\mathscr{MV}^0_*$ the Waldhausen subcategory of $\mathscr{MV}_*$ of objects X with acyclic $\Phi_2(X)$. Because of the fibration theorem (see [9]), the sequence:

$$\mathscr{MV}^0_* \longrightarrow \mathscr{MV}_* \longrightarrow \mathscr{MV}'_*$$

induces a fibration in K-theory.

Denote by $\mathscr{D}^0_*$ the Waldhausen subcategory of $\mathscr{D}_*$ of acyclic complexes in $\mathscr{D}_*$. Since each morphism in $\mathscr{D}^0_*$ is an equivalence, $\mathscr{D}^0_*$ has trivial K-theory. We have a commutative diagram:

$$\begin{array}{ccccc}
\mathscr{MV}^0_* & \longrightarrow & \mathscr{MV}_* & \longrightarrow & \mathscr{MV}'_* \\
\Big\downarrow{\scriptstyle \Phi_2\times\Phi_3} & & \Big\downarrow{\scriptstyle \Phi_2\times\Phi_3} & & \Big\downarrow{\scriptstyle \Phi_2} \\
\mathscr{D}^0_* \times \mathscr{C}_* & \longrightarrow & \mathscr{D}_* \times \mathscr{C}_* & \longrightarrow & \mathscr{D}_*
\end{array}$$

where each line induces a fibration in K-theory. Therefore it will be enough to prove that functors $\Phi_2 : \mathscr{MV}'_* \longrightarrow \mathscr{D}_*$ and $\Phi_2 \times \Phi_3 : \mathscr{MV}^0_* \longrightarrow \mathscr{D}^0_* \times \mathscr{C}_*$ induce homotopy equivalences in K-theory or, equivalently, that $\Phi_2 : \mathscr{MV}'_* \longrightarrow \mathscr{D}_*$ and $\Phi_3 : \mathscr{MV}^0_* \longrightarrow \mathscr{C}_*$ induce homotopy equivalences in K-theory.

Because of the approximation theorem of Waldhausen, in order to prove that $\Phi_2 : \mathscr{MV}'_* \longrightarrow \mathscr{D}_*$ and $\Phi_3 : \mathscr{MV}^0_* \longrightarrow \mathscr{C}_*$ induces a homotopy equivalence in K-theory, it's enough to show that these two functors have the approximation property.

The property (App1) is easy to check. Then the last thing to do is to show that $\Phi_2 : \mathscr{MV}'_* \longrightarrow \mathscr{D}_*$ and $\Phi_3 : \mathscr{MV}^0_* \longrightarrow \mathscr{C}_*$ have the property (App2). Consider an object $X \in \mathscr{MV}'_*$, an object $F \in \mathscr{D}_*$ and a morphism $f : \Phi_2(X) \longrightarrow F$ in $\mathscr{D}_*$. The object X is a triple $X = (E, M, \varphi)$ where $\mathscr{E}$ is a $\mathscr{D}$-complex, M is a $\mathscr{C}$-complex and φ is an element in $\mathscr{F}(E, M)$ inducing a surjective morphism $T(E) \longrightarrow MR$. So f is a morphism from E to F in $\mathscr{D}_*$.

Denote by E_1 the -1-cone of f. So we have linear maps $i : F \longrightarrow E_1$, $p : E_1 \longrightarrow E$, $r : E_1 \longrightarrow F$ and $s : E \longrightarrow E_1$. The map p is an epimorphism in $\mathscr{D}_*$, i is a morphism of degree -1 and we have:

$$d(s) = if \qquad d(r) = fp$$

Consider the triple $(E_1, M, \varphi p)$. Since p is an epimorphism, this triple is an object $X_1 \in \mathscr{MV}_*$ and we have a morphism $g : X_1 \longrightarrow X$. Denote by Y the 0-cone of g. We have a cofibration $j : X \longrightarrow Y$ and four linear maps: $j : E \longrightarrow \Phi_2(Y)$, $q : \Phi_2(Y) \longrightarrow E_1$, $\rho : \Phi_2(Y) \longrightarrow E$ and $\sigma : E_1 \longrightarrow \Phi_2(Y)$. Moreover we have;

$$d(\sigma) = jp \qquad d(\rho) = -pq$$

Consider the linear map $g = f\rho + rq$. The differential $d(g)$ vanishes and g is a morphism from $\Phi_2(Y)$ to F. It is easy to see that g is surjective and its kernel is isomorphic to the 0-cone of the identity of E.

Moreover we have: $gj = f$. Therefore $g : \Phi_2(Y) \longrightarrow F$ is a homology equivalence and the following diagram is commutative:

$$\begin{array}{ccc} \Phi_2(X) & \xrightarrow{f} & F \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle =} \\ \Phi_2(Y) & \underset{\sim}{\xrightarrow{g}} & F \end{array}$$

Then the functor $\Phi_2 : \mathscr{M}\mathscr{V}'_* \longrightarrow \mathscr{D}_*$ has the property (App2) and $\Phi_2 : \mathscr{M}\mathscr{V}'_* \longrightarrow \mathscr{D}_*$ induces a homotopy equivalence in K-theory. Consider an object $X \in \mathscr{M}\mathscr{V}^0_*$, an object $N \in \mathscr{C}_*$ and a morphism $f : \Phi_3(X) \longrightarrow N$ in $\mathscr{C}_*$. The object X is a triple (E, M, φ) where $\mathscr{E}$ is an acyclic $\mathscr{D}$-complex, M is a $\mathscr{C}$-complex and φ is an element in $\mathscr{F}(E, M)$ inducing a surjective morphism $T(E) \longrightarrow MR$. So f is a morphism from M to N.

Let U be the -1-cone of the identity of N. Then U is acyclic and we have an epimorphism $p : U \longrightarrow N$. Consider the composite morphism:

$$\varphi' : T(Fs(U)) \xrightarrow{p} T(Fs(N)) \underset{\sim}{\xrightarrow{e}} \sigma s(N)R \longrightarrow NR$$

This morphism is surjective and the triple $(E \oplus Fs(U), N, f\varphi \oplus \varphi')$ is an object Y in $\mathscr{M}\mathscr{V}^0_*$. Moreover we have a morphism $g : X \longrightarrow Y$ inducing the inclusion $E \subset E \oplus Fs(U)$ and the morphism $f : M \longrightarrow N$, making the following diagram commutative:

$$\begin{array}{ccc} \Phi_3(X) & \xrightarrow{f} & N \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle =} \\ \Phi_3(Y) & \xrightarrow{=} & N \end{array}$$

Hence $\Phi_3 : \mathscr{M}\mathscr{V}^0_* \longrightarrow \mathscr{C}_*$ has the property (App2) and this functor induces a homotopy equivalence in K-theory. $\square$

Denote by $\mathscr{M}\mathscr{V}'$ the full subcategory of $\mathscr{M}\mathscr{V}$ consisting of objects $X \in \mathscr{M}\mathscr{V}$ such that $\Phi_1(X)$ belongs to $\mathscr{R}'$. Then an object $X \in \mathscr{M}\mathscr{V}$ belongs to $\mathscr{M}\mathscr{V}'$ if and only if there is an isomorphism $\Phi_1(X) \simeq VR$ for some $V \in \mathscr{C}$. This category is exact and cofinal in $\mathscr{M}\mathscr{V}$. Moreover the functor Φ_1 sends $\mathscr{M}\mathscr{V}'$ to $\mathscr{R}'$.

We define the following categories:

- The category $\mathscr{E}_0$ of modules in $\mathscr{R}'$ and isomorphisms
- The category $\mathscr{E}_1$ of objects in $\mathscr{M}\mathscr{V}'$ and morphisms inducing isomorphisms under Φ_1
- The category $\mathscr{E}_2$ of objects in $\mathscr{M}\mathscr{V}'$ and epimorphisms inducing isomorphisms under Φ_1
- The category $\mathscr{E}_3$ of objects in $\mathscr{M}\mathscr{V}' \times \mathscr{V}$, where a morphism in $\mathscr{E}_3$ from (X, V) to (Y, W) is an morphism $X \oplus V \longrightarrow Y \oplus W$ in $\mathscr{E}_2$ sending X to Y.

We have four functors $f_1 : \mathscr{E}_1 \longrightarrow \mathscr{E}_0$, $f_2 : \mathscr{E}_2 \longrightarrow \mathscr{E}_1$, $g_1 : \mathscr{E}_3 \longrightarrow \mathscr{E}_1$ and $g_2 : \mathscr{E}_3 \longrightarrow \mathscr{E}_2$ where f_1 is induced by Φ_1, f_2 is the inclusion and g_1 (resp. g_2) is the correspondence $(X, V) \mapsto X$ (resp. $(X, V) \mapsto X \oplus V$).

In order to prove a connective version of theorems 1, 2 and 3, we'll need to prove that the sequence $\mathscr{V} \subset \mathscr{M}\mathscr{V}' \longrightarrow \mathscr{R}'$ induces a fibration in K-theory and, for that, it will be useful to prove that $f_1 f_2$ is a homotopy equivalence.

Lemma 2.5. *The functor f_1 is a homotopy equivalence.*

Proof. It is enough to prove that the fiber category f_1/U is contractible for each object $U \in \mathscr{E}_0 = \mathscr{R}'$.

Consider an object U in $\mathscr{R}'$ and denote by $\mathscr{F}$ the fiber category f_1/U. By applying lemma 1.14 with $X = 0$ and $V = U$, we get an object $Y \in \mathscr{M}\mathscr{V}$ and an isomorphism $U \simeq \Phi_1(Y)$. Hence the category $\mathscr{F}$ is nonempty

Consider two objects (X_1,ε_1) and (X_2,ε_2) in $\mathscr{F}$. For $i=1,2$, ε_i is an isomorphism from $\Phi_1(X_i)$ to U.

By applying lemma 1.14 with $X = X_1\oplus X_2$ and $f=\varepsilon_1^{-1}\oplus\varepsilon_2^{-1} : U \longrightarrow \Phi_1(X)$, we get an object $Y\in\mathscr{M}\mathscr{V}$, a morphism $g : Y\longrightarrow X_1\oplus X_2$ and an isomorphism $\varepsilon : U \xrightarrow{\sim} \Phi_1(Y)$ such that $\varepsilon_1^{-1}\oplus\varepsilon_2^{-1}$ is the composite morphism $U\xrightarrow{\varepsilon}\Phi_1(Y)\xrightarrow{g}\Phi_1(X)$. Therefore (Y,ε^{-1}) is an object in $\mathscr{F}$ and we have two morphisms $(Y,\varepsilon^{-1})\longrightarrow(X_1,\varepsilon_1)$ and $(Y,\varepsilon^{-1})\longrightarrow(X_2,\varepsilon_2)$.

Consider two objects (X_1,ε_1) and (X_2,ε_2) in $\mathscr{F}$ and two morphisms h_1 and h_2 from (X_1,ε_1) to (X_2,ε_2). Denote by X the kernel of h_1-h_2 (in $\mathscr{S}^\vee$).

For $i=1,2$, we have an exact sequence:

$$0\longrightarrow\Phi_1(X_i)\xrightarrow{\mu_i}T\Phi_2(X_i)\xrightarrow{\varphi_i}\Phi_3(X_i)R\longrightarrow 0$$

and then an exact sequence:

$$0\longrightarrow U\xrightarrow{\mu_i\varepsilon_i^{-1}}T\Phi_2(X_i)\xrightarrow{\varphi_i}\Phi_3(X_i)R\longrightarrow 0$$

So we get for each $i=1,2$ a commutative diagram with exact lines:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \xrightarrow{\mu_1\varepsilon_1^{-1}} & T\Phi_2(X_1) & \xrightarrow{\varphi_1} & \Phi_3(X_1)R & \longrightarrow & 0\\
 & & {\scriptstyle =}\big\downarrow & & {\scriptstyle h_i}\big\downarrow & & {\scriptstyle h_i}\big\downarrow & & \\
0 & \longrightarrow & U & \xrightarrow{\mu_2\varepsilon_2^{-1}} & T\Phi_2(X_2) & \xrightarrow{\varphi_2} & \Phi_3(X_2)R & \longrightarrow & 0
\end{array}$$

and then an exact sequence:

$$0\longrightarrow U\longrightarrow T\Phi_2(X)\longrightarrow\Phi_3(X)R$$

inducing an isomorphism $\varepsilon : U\longrightarrow\Phi_1(X)$. By applying lemma 1.14 with ε, we get an object $(Y,u^{-1})\in\mathscr{F}$ and a morphism $k : (Y,u^{-1})\longrightarrow(X_1,\varepsilon_1)$ such that: $h_1k=h_2k$.

Because of these properties, each fiber category $\mathscr{F}$ is cofiltered and then contractible. Hence the functor f_1 is a homotopy equivalence. □

Lemma 2.6. *The functor g_1 is a homotopy equivalence.*

Proof. Let X be an object of $\mathscr{M}\mathscr{V}'$ and $\mathscr{F}$ be the fiber category g_1/X. Applying lemma 1.14 to the morphism $0\longrightarrow\Phi_1(X)$, we get an object $E\in\mathscr{M}\mathscr{V}$, a morphism $\alpha : E\longrightarrow X$ such that: $\Phi_1(E)=0$ and $\Phi_2(E)\longrightarrow\Phi_2(X)$ and $\Phi_3(E)\longrightarrow\Phi_3(X)$ are epimorphisms. Then E belongs to $\mathscr{V}$ and the morphism $\alpha : E\longrightarrow X$ induces epimorphisms on Φ_2 and Φ_3. Therefore, for each morphism $f : Y\longrightarrow X$ in $\mathscr{E}_1$, the morphism $Y\oplus E\xrightarrow{f\oplus\alpha}X$ belongs to $\mathscr{E}_2$.

An object in $\mathscr{F}$ is a triple (Y,V,f) where (Y,V) belongs to $\mathscr{M}\mathscr{V}'\times\mathscr{V}$ and $f : Y\longrightarrow X$ is a morphism in $\mathscr{E}_1$. A morphism $\varphi : (Y,V,f)\longrightarrow(Y',V',f')$ is a morphism $\varphi : Y\oplus V\longrightarrow Y'\oplus V'$ in $\mathscr{E}_2$ sending Y to Y' such that: $f=f'\varphi$.

The category $\mathscr{F}$ is nonempty because it contains the object $(X,0,\mathrm{Id})$.

We have three functors G_0,G_1,G_2 from $\mathscr{F}$ to $\mathscr{F}$ sending each $(Y,V,f)\in\mathscr{F}$ to $G_0(Y,V,f)=(Y,V,f)$, $G_1(Y,V,f)=(Y,V\oplus E,f)$ and $G_2(Y,V,f)=(X,0,\mathrm{Id})$ respectively.

The inclusion $0\subset E$ induces a morphism $G_0\longrightarrow G_1$. The morphism $f\oplus 0\oplus\alpha : Y\oplus V\oplus E\longrightarrow X$ induces a morphism $(Y,V\oplus E,f)\longrightarrow(X,0,\mathrm{Id})$ and we get a morphism $G_1\longrightarrow G_2$. Therefore the identity of $\mathscr{F}$ is homotopic to G_1 and then to G_2 which is constant. Hence $\mathscr{F}$ is contractible and, since each fiber category of g_1 is contractible, g_1 is a homotopy equivalence. □

Lemma 2.7. *The functor g_2 is a homotopy equivalence.*

Proof. Let X be an object in $\mathscr{E}_1$ and $\mathscr{F}$ be the fiber category g_2/X. An object in $\mathscr{F}$ is a triple (Y, V, f) where (Y, V) belongs to $\mathscr{E}_3$ and $f : Y \oplus V \longrightarrow X$ is a morphism in $\mathscr{E}_2$. A morphism $\varphi : (Y, V, f) \longrightarrow (Y', V', f')$ in $\mathscr{F}$ is a morphism $\varphi : Y \oplus V \longrightarrow Y' \oplus V'$ in $\mathscr{E}_2$ such that $\varphi(Y) \subset Y'$ and $f = f'\varphi$. Therefore it is easy to see that, for every $(Y, V, f) \in \mathscr{F}$, we have a unique morphism in $\mathscr{F}$ from (Y, V, f) to $(X, 0, \mathrm{Id})$. Hence $\mathscr{F}$ has a final object and is contractible. Since each fiber category of g_2 is contractible, g_2 is a homotopy equivalence. □

Lemma 2.8. *The functor f_2 is a homotopy equivalence.*

Proof. The inclusion $X \subset X \oplus V$ for all $(X, V) \in \mathscr{E}_3$ induces a morphism from g_1 to f_2g_2 and g_1 is homotopic to f_2g_2. But g_1 and g_2 are homotopy equivalences. Therefore f_2 is a homotopy equivalence too. □

Lemma 2.9. *The following diagram of exact categories:*

$$\mathscr{V} \subset \mathscr{M}\mathscr{V}' \xrightarrow{\Phi_1} \mathscr{R}'$$

induces a fibration in K-theory.

Proof. In [8] (lemma 10.2 p. 206), Waldhausen proved that $Q\mathscr{V} \longrightarrow Q\mathscr{M}\mathscr{V}' \longrightarrow Q\mathscr{F}_R$ is a homotopic fibration in a situation similar to ours. In our situation we'll prove, essentially in the same way, that the sequence $Q\mathscr{V} \longrightarrow Q\mathscr{M}\mathscr{V}' \longrightarrow Q\mathscr{R}'$ is a homotopic fibration.

Following Waldhausen's notations, if $\mathscr{F}$ is an exact subcategory of an exact category $\mathscr{E}$, we have a bicategory $Q^{ep}(\mathscr{E}, \mathscr{F})$ where the horizontal maps form the Quillen's category $Q(\mathscr{E})$ and the vertical morphisms form the category of epimorphisms in $\mathscr{E}$ with kernel in $\mathscr{F}$. In particular we have an equivalence between $Q(\mathscr{E})$ and $Q^{ep}(\mathscr{E}, 0)$ and $Q(\mathscr{E})$ may be considered as a bicategory.

We have a commutative diagram of bicategories:

$$\begin{array}{ccc} Q(\mathscr{V}) & \longrightarrow & Q^{ep}(\mathscr{V}, \mathscr{V}) \\ \big\downarrow & & \big\downarrow \\ Q(\mathscr{M}\mathscr{V}') & \longrightarrow & Q^{ep}(\mathscr{M}\mathscr{V}', \mathscr{V}) \end{array}$$

which is homotopically cartesian. Moreover $Q^{ep}(\mathscr{V}, \mathscr{V})$ is contractible. Therefore the diagram of bicategories:

$$Q(\mathscr{V}) \longrightarrow Q(\mathscr{M}\mathscr{V}') \longrightarrow Q^{ep}(\mathscr{M}\mathscr{V}', \mathscr{V})$$

is a homotopic fibration.

On the other hand the morphism $f = f_1f_2 : \mathscr{E}_2 \longrightarrow \mathscr{R}'$ induces a morphism $f_* : Q^{ep}(\mathscr{M}\mathscr{V}', \mathscr{V}) \longrightarrow Q^{ep}(\mathscr{R}', 0)$ and we want to prove that f_* is a homotopy equivalence. Actually, the proof of lemma 10.2 in [8] works exactly the same in our situation except maybe for the sublemma (p. 209). In this sublemma, we have a filtered object

$$M_1 \subset M_2 \subset \dots M_{n-1} \subset M_n$$

in $\mathscr{M}\mathscr{V}'$ where each quotient M_i/M_{i-1} is in $\mathscr{M}\mathscr{V}'$. Since Φ_1 is exact and each module in $\mathscr{R}'$ is projective, the morphism $\Phi_1(M_n) \longrightarrow \Phi_1(M_n/M_{n-1})$ is surjective and has a section s from $U = \Phi_1(M_n/M_{n-1})$ to $\Phi_1(M_n)$.

Because of lemma 1.14, there are an object $N \in \mathscr{MV}$, a morphism $g : N \longrightarrow M_n$ inducing epimorphisms on Φ_2 and Φ_3, and an isomorphism $U \xrightarrow{\sim} \Phi_1(N)$ making the following diagram commutative:

$$\begin{tikzcd} U \arrow[r, "\sim"] \arrow[d, "="'] & \Phi_1(N) \arrow[d, "g"'] \\ U \arrow[r, "s"] & \Phi_1(M_n) \end{tikzcd}$$

Therefore the morphism $N \longrightarrow M_n$ induces epimorphisms on Φ_2 and Φ_3 and the composite morphism $N \longrightarrow M_n \longrightarrow M_n/M_{n-1}$ is an epimorphism with kernel in $\mathscr{V}$. Hence the sublemma can be proven in our situation and, since f is a homotopy equivalence, the proof of lemma 10.2 applies completely here. The lemma follows. □

Lemma 2.10. *Let $\mathscr{R}''$ be the full subcategory of $\mathscr{R}$ generated by the image of $\Phi_1 : \mathscr{MV} \longrightarrow \mathscr{R}$. Then the diagram:*

$$\mathscr{V} \subset \mathscr{MV} \xrightarrow{\Phi_1} \mathscr{R}''$$

induces a fibration in K-theory.

Proof. Let $X = (E, M, \varphi)$ be an object in $\mathscr{MV}$. Since E is projective in $\mathscr{D}$, there is a module $E_1 \in \mathscr{D}$ such that $E \oplus E_1$ is free in case 1 or 3 and on the form $E \oplus E_1 = (F_A, F_B) \in \mathscr{A} \times \mathscr{B}$ with F_A and F_B free. Therefore $T(E \oplus E_1)$ is free in all cases. Let X_1 be the object $(E_1, 0, 0) \in \mathscr{MV}$. We have an exact sequence in $\mathscr{R}$:

$$0 \longrightarrow \Phi_1(X \oplus X_1) \longrightarrow T(E \oplus E_1) \longrightarrow MR \longrightarrow 0$$

and $\Phi_1(X \oplus X_1)$ is stably in $\mathscr{R}'$. Hence there is another object $X_2 \in \mathscr{MV}$ such that $\Phi_1(X \oplus X_1 \oplus X_2)$ belongs to $\mathscr{R}'$ and $\mathscr{MV}'$ is cofinal in $\mathscr{MV}$.

Consider the following commutative diagram:

$$\begin{tikzcd} \mathscr{MV}' \arrow[r, "\Phi_1"] \arrow[d] & \mathscr{R}' \arrow[d] \\ \mathscr{MV} \arrow[r, "\Phi_1"] & \mathscr{R}'' \end{tikzcd} \qquad (D)$$

where the vertical maps are the canonical cofinal inclusions.

Let K be the fiber product of $K_0(\mathscr{MV})$ and $K_0(\mathscr{R}')$ over $K_0(\mathscr{R}'')$. The commutativity of the diagram $K_0(D)$ induces a map $\lambda : K_0(\mathscr{MV}') \longrightarrow K$. Since the map $K_0(\mathscr{MV}') \longrightarrow K_0(\mathscr{MV})$ is injective (by cofinality) and factorizes through K, the map λ is injective.

Let $w = (u, v)$ be an element in K. Then we have: $u \in K_0(\mathscr{MV})$, $v \in K_0(\mathscr{R}')$ and $\Phi_1(u)$ and v are the same in $K_0(\mathscr{R}'')$.

For every object X in some exact category $\mathscr{A}$, the class of X in the Grothendieck group $K_0(\mathscr{A})$ will be denoted by $[X]$.

So there are two objects X, Y in $\mathscr{MV}$ such that: $u = [X] - [Y]$. Since $\mathscr{MV}'$ is cofinal in $\mathscr{MV}$, there is an object Y_1 in $\mathscr{MV}$ such that $Y \oplus Y_1$ is in $\mathscr{MV}'$. Let us set: $w' = w + \lambda[Y \oplus Y_1]$ and $X' = X \oplus Y_1$. We have: $w' = (u', v')$ with $u' = [X']$ and $[\Phi_1(X')]$ belongs to $K_0(\mathscr{R}')$. Then $\Phi_1(X')$ is stably isomorphic to a module in $\mathscr{R}'$. Up to adding to X (and then to X') an object in $\mathscr{MV}'$ on the form $(E, 0, 0)$, we may as well suppose that $\Phi_1(X')$ belongs to $\mathscr{R}'$ and then that X' belongs to $\mathscr{MV}'$.

Therefore we have: $w' - \lambda[X'] = (0, v'')$ where v'' is an element in $K_0(\mathscr{R}')$ killed in $K_0(\mathscr{R}'')$. But the morphism $K_0(\mathscr{R}') \longrightarrow K_0(\mathscr{R}'')$ is injective. So we have:

$$w' - \lambda[X'] = 0 \implies w' = \lambda(X')$$

and w' and then w are in the image of λ. Therefore λ is surjective and then bijective.

Consequently the diagram $K_0(D)$ is exact (cartesian and cocartesian) and, by cofinality, the diagram $K(D)$ is homotopically cartesian. The lemma follows. $\square$

As a consequence we get the following result, which is, in some sense, a connective version of theorems 1, 2 and 3:

Proposition 2.11. *Let X be the homotopy fiber of the map $K(\mathscr{C}_2) \longrightarrow K(\mathscr{D}) \times F(\mathscr{C})$ induced by the functor $F \times \sigma : \mathscr{C}_2 \longrightarrow \mathscr{D} \times \mathscr{C}$. Then there is a natural homotopy equivalence $\Omega K(R) \xrightarrow{\sim} Nil(C_2, S) \times X$.*

Proof. Because of lemma 1.12 the functor $\Phi : \mathscr{N}il(C_2, S) \longrightarrow \mathscr{V}$ is an equivalence of categories and the lemma 2.10 implies that the following diagram:

$$\mathscr{N}il(C_2, S) \xrightarrow{\Phi} \mathscr{M}\mathscr{V} \xrightarrow{\Phi_1} \mathscr{R}''$$

induces a fibration in K-theory. Hence $\Omega K(\mathscr{R}'') \simeq \Omega K(R)$ is homotopically equivalent to the homotopy fiber of the map induced by $\Phi : \mathscr{N}il(C_2, S) \longrightarrow \mathscr{M}\mathscr{V}$ in K-theory.

Because of lemma 2.4, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{N}il(C_2, S) \arrow[r, "\Phi"] \arrow[d, "="'] & \mathscr{M}\mathscr{V} \arrow[d, "\Phi_2 \times \Phi_3"] \\
\mathscr{N}il(C_2, S) \arrow[r, "\Phi'"] & \mathscr{D} \times \mathscr{C}
\end{tikzcd}$$

where the functor $\Phi_2 \times \Phi_3$ induces a homotopy equivalence in K-theory. Therefore $\Omega K(R)$ is homotopically equivalent to the homotopy fiber of the map $\Phi' : K(\mathscr{N}il(C_2, S)) \longrightarrow K(\mathscr{D} \times \mathscr{C})$.

The functor Φ' sends an object $(H, \theta) \in \mathscr{N}il(C_2, S)$ to the pair $(F(H), \sigma(H)) \in \mathscr{D} \times \mathscr{C}$ and Φ' factorizes by the forgetful map $\mathscr{N}il(C_2, S) \longrightarrow \mathscr{C}_2$. Therefore $\Omega K(R)$ is homotopically equivalent to the homotopy fiber of $\Phi'' : K(C_2) \times Nil(C_2, S) \longrightarrow K(\mathscr{D}) \times K(\mathscr{C})$ where Φ'' is trivial on $Nil(C_2, S)$ and induced by the functor $F \times \sigma$ on $K(\mathscr{C}_2)$. The lemma follows. $\square$

2.12. The spectra $\underline{K}$ and $\underline{N}il$

The K-theory of Quillen is a functor K from the category of rings to the category of infinite loop spaces. There are different methods to construct a so called negative K-theory: that is a functor K' from the category of rings to the category Ωsp of Ω-spectra (see [1] and [5]) and a natural homotopy equivalence from $K(A)$ to the 0-th term of $K'(A)$ such that the following sequence is exact for every ring A and every integer $i \in \mathbf{Z}$:

$$0 \longrightarrow K_i(A) \longrightarrow K_i(A[t]) \oplus K_i(A[t^{-1}]) \longrightarrow K_i(A[t, t^{-1}]) \longrightarrow K_{i-1}(A) \longrightarrow 0$$

where $K_i(A)$ is the i-th homotopy group of $K'(A)$. This exact sequence was proven by Bass [1] for $i = 1$ and generalized by Quillen for $i > 0$ [6]. The morphisms of this exact sequence are induced by the inclusions $A \subset A[t] \subset A[t, t^{-1}]$, $A \subset A[t^{-1}] \subset A[t, t^{-1}]$ except for the map $\partial : K_i(A[t, t^{-1}]) \longrightarrow K_{i-1}(A)$. But this map ∂ has a section induced by the multiplication by $t \in K_1(\mathbf{Z}[t, t^{-1}])$. Therefore the exact sequence above is natural in A.

Inspired by the Karoubi-Villamayor method [5], we'll define our version $\underline{K}(A)$ of negative K-theory as follows:

Denote by E the set of infinite square matrices with entries in $\mathbf{Z}$ having only finitely many nonzero entries in each row and each column. This ring has a two-sided ideal $M(\mathbf{Z})$ of matrices having only finitely many nonzero entries. So we set: $\Sigma = E/M(\mathbf{Z})$.

For every ring A, $EA = E \otimes_{\mathbf{Z}} A$ and $\Sigma A = \Sigma \otimes_{\mathbf{Z}} A$ are rings and the morphism $f : EA \longrightarrow \Sigma A$ is a surjective ring homomorphism. It is easy to see that the kernel of f is isomorphism, as a pseudo ring, to $M(A)$ and that EA is a flasque ring. Therefore (see [5]) we have a natural homotopy equivalence:

$$K(A) \xrightarrow{\sim} \Omega K(\Sigma A)$$

and the sequence $K(\Sigma^n A)$ is an Ω-spectrum. This spectrum will be denoted by $\underline{K}(A)$ and $\underline{K}$ is a negative K-theory. For each integer $i \in \mathbf{Z}$ we set also: $K_i(A) = \pi_i(\underline{K}(A))$.

Lemma 2.13. *Let (A, S) be a left-flat bimodule. Then for each integer $n \geq 0$, $(\Sigma^n A, \Sigma^n S)$ is a left-flat bimodule and the sequence $Nil(\Sigma^n A, \Sigma^n S)$ is an Ω-spectrum denoted by $\underline{N}il(A, S)$. Moreover we have a natural homotopy equivalence from $Nil(A, S)$ to the 0-th term of $\underline{N}il(A, S)$ and, for each integer $i \in \mathbf{Z}$, we have an exact sequence which is natural on the left-flat bimodule (A, S):*

$$0 \longrightarrow Nil_i(A, S) \longrightarrow Nil_i(A[t], S[t]) \oplus Nil(A[t^{-1}], S[t^{-1}]) \longrightarrow$$
$$Nil_i(A[t, t^{-1}], S[t, t^{-1}]) \longrightarrow Nil_{i-1}(A, S) \longrightarrow 0$$

where $Nil_i(?)$ is the i-th homotopy group of $\underline{N}il(?)$.

Proof. Let (A, S) be a left-flat bimodule. For every ring B, $BS = B \otimes_Z S$ is a BA-bimodule. Since S is flat on the left S, as a left A-module, is isomorphic to a filtered colimit of free left A-modules E_i and BS is also isomorphic to a filtered colimit of free BA-modules BE_i. Therefore BS is flat on the left and (BA, BS) is a left-flat bimodule. In particular each $(\Sigma^n A, \Sigma^n S)$ is a left-flat bimodule. Moreover we have a natural isomorphism of rings: $\Sigma(A[S]) \simeq (\Sigma A)[\Sigma S]$.

Consider the case 1 (with $C = A$). Because of proposition 2.11, we have a natural homotopy equivalence:

$$\Omega K(A[S]) \xrightarrow{\sim} Nil(A, S) \times X$$

where X is the homotopy fiber of the map: $K(\mathscr{C}_2) \xrightarrow{F \times \sigma} K(\mathscr{D} \times \mathscr{C})$. But in case 1, $\mathscr{C}_2$, $\mathscr{D}$ and $\mathscr{C}$ are equal to the category $\mathscr{A}$ of finitely generated projective right A-modules and functors F and σ are the identity. Therefore X is nothing else but the loop space of $K(A)$. So we get a natural homotopy equivalence:

$$\Omega K(A[S]) \xrightarrow{\sim} Nil(A, S) \times \Omega K(A)$$

By naturality, we get a homotopy equivalence from $Nil(A, S)$ to the homotopy fiber of the map $\Omega K(A[S]) \longrightarrow \Omega K(A)$ induced by the canonical ring homomorphism $A[S] \longrightarrow A$.

We have a commutative diagram:

$$\begin{array}{ccc} K(A[S]) & \longrightarrow & K(A) \\ \sim\downarrow & & \sim\downarrow \\ \Omega K(\Sigma(A[S])) & \longrightarrow & \Omega K(\Sigma A) \end{array}$$

where the horizontal maps are induced by the canonical morphism $A[S] \longrightarrow A$ and the vertical maps are homotopy equivalences.

But $\Sigma(A[S])$ is isomorphic to $(\Sigma A)[\Sigma S]$. So we get a commutative diagram:

$$\begin{tikzcd} \Omega K(A[S]) \arrow[r] \arrow[d, "\sim"'] & \Omega K(A) \arrow[d, "\sim"'] \\ \Omega^2 K((\Sigma A)[\Sigma S])) \arrow[r] & \Omega^2 K(\Sigma A) \end{tikzcd}$$

inducing a homotopy equivalence $Nil(A,S) \xrightarrow{\sim} \Omega Nil(\Sigma A, \Sigma S)$. Then the sequence of spaces $Nil(\Sigma^n A, \Sigma^n, S)$ is a well defined Ω-spectrum $\underline{N}il(A,S)$ and we have a natural decomposition:

$$\Omega\underline{K}(A[S]) \xrightarrow{\sim} \underline{N}il(A,S) \times \Omega\underline{K}(A)$$

Let us set: $Nil_i(A,S) = \pi_i(\underline{N}il(A,S))$. Then we have, for every left-flat bimodule (A,S) and every integer $i \in \mathbf{Z}$ an isomorphism $K_i(A[S]) \simeq Nil_{i-1}(A,S) \oplus K_i(A)$.

For the ring A we have, for every integer $i \in \mathbf{Z}$, the following exact sequence $S_i(A)$:

$$0 \longrightarrow K_i(A) \longrightarrow K_i(A[t]) \oplus K_i(A[t^{-1}]) \longrightarrow K_i(A[t,t^{-1}]) \longrightarrow K_{i-1}(A) \longrightarrow 0$$

For $B = \mathbf{Z}[t]$ or $B = \mathbf{Z}[t^{-1}]$ or $B = \mathbf{Z}[t,t^{-1}]$, the ring $B(A[S])$ is isomorphic to $(BA)[BS]$. Then the sequence $S_{i+1}(A[S])$ decomposes into $S_{i+1}(A)$ and the following sequence:

$$\begin{gathered} 0 \longrightarrow Nil_i(A,S) \longrightarrow Nil_i(A[t],S[t]) \oplus Nil_i(A[t^{-1}],S[t^{-1}]) \longrightarrow \\ Nil_i(A[t,t^{-1}],S[t,t^{-1}]) \longrightarrow Nil_{i-1}(A,S) \longrightarrow 0 \end{gathered}$$

which, as a consequence, is exact. $\square$

2.14. Proofs of theorems 1, 2 and 3

In the proof of lemma 2.13, we have constructed the Ω-spectrum $\underline{N}il(?)$ and the first two properties of theorem 1 have already been proven.

Suppose A is regular coherent on the right, i.e. every finitely presented right A-module has a finite resolution by finitely generated projective modules. The category $\mathscr{A}$ of finitely generated projective right A-modules is contained in the category $\mathscr{A}'$ of finitely presented right A-modules. The category $\mathscr{N}il(A,S)$ is also contained in the category $\mathscr{N}il'(A,S)$ of pairs (H,θ) with $H \in \mathscr{A}'$ and $\theta : H \longrightarrow HS$ nilpotent. Moreover $\mathscr{A}$ is stable in $\mathscr{A}'$ under extension and kernel of admissible epimorphism and the inclusion $\mathscr{N}il(A,S) \subset \mathscr{N}il'(A,S)$ have the same properties. Therefore, by the resolution theorem [6], the inclusions $\mathscr{A} \subset \mathscr{A}'$ and $\mathscr{N}il(A,S) \subset \mathscr{N}il'(A,S)$ induce homotopy equivalences in K-theory.

On the other hand, we have an inclusion $\mathscr{A}' \longrightarrow \mathscr{N}il'(A,S)$ sending H to the pair $(H,0)$ and every object (H,θ) in $\mathscr{N}il'(A,S)$ has a finite filtration with subquotients in $\mathscr{A}'$. Moreover $\mathscr{A}'$ and $\mathscr{N}il'(A,S)$ are abelian categories and $\mathscr{A}'$ is closed in $\mathscr{N}il'(A,S)$ under subobjects and quotients. Therefore, by the devissage theorem [6], the inclusion $\mathscr{A}' \subset \mathscr{N}il'(A,S)$ induces a homotopy equivalence in K-theory. As a consequence the inclusion $\mathscr{A} \subset \mathscr{N}il(A,S)$ induces a homotopy equivalence in K-theory and the space $Nil(A,S)$ is contractible.

Hence for every left-flat bimodule (A,S), with A regular coherent, the space $Nil(A,S)$ is contractible and $Nil_i(A,S)$ is trivial for every $i \geq 0$.

Consider the following property $E(n)$ where n is any integer:

• For every left-flat bimodule (A, S) with A regular coherent on the right, the module $Nil_i(A, S)$ is trivial for every $i \geq n$.

The property $E(0)$ is then satisfied. Suppose $E(n)$ is true and take a left-flat bimodule (A, S) with A regular coherent on the right. Then $A[t]$, $A[t^{-1}]$ and $A[t, t^{-1}]$ are also regular coherent on the right and, in the exact sequence:

$$0 \longrightarrow Nil_n(A, S) \longrightarrow Nil_n(A[t], S[t]) \oplus Nil_n(A[t^{-1}], S[t^{-1}]) \longrightarrow$$
$$Nil_n(A[t, t^{-1}], S[t, t^{-1}]) \longrightarrow Nil_{n-1}(A, S) \longrightarrow 0$$

all the modules are zero except $Nil_{n-1}(A, S)$. Hence the module $Nil_{n-1}(A, S)$ is also trivial and the property $E(n-1)$ is true.

By induction $E(n)$ is satisfied for all n. Hence $\underline{N}il(A, S)$ is contractible for every left-flat bimodule (A, S) with A regular coherent on the right and theorem 1 is proven. Consider the case 2. The ring R is defined by the cocartesian diagram:

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d] \\ B \arrow[r] & R \end{tikzcd} \qquad (D)$$

By tensoring by Σ^n, we get a cocartesian diagram:

$$\begin{tikzcd} \Sigma^n C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \Sigma^n A \arrow[d] \\ \Sigma^n B \arrow[r] & \Sigma^n R \end{tikzcd} \qquad (\Sigma^n D)$$

We remark that the morphism $\alpha : \Sigma^n C \longrightarrow \Sigma^n A$ (resp. $\beta : \Sigma^n C \longrightarrow \Sigma^n B$) is pure with complement $\Sigma^n A'$ (resp. $\Sigma^n B'$). Therefore in this new situation, we get new rings: $\Sigma^n C$, $\Sigma^n A$, $\Sigma^n B$, $\Sigma^n R$, $\Sigma^n C \times \Sigma^n C$ and a new bimodule $\Sigma^n S$.

Because of proposition 2.11, we have a homotopy equivalence between $\Omega K(\Sigma^n R)$ and $Nil(\Sigma^n C \times \Sigma^n C, \Sigma^n S) \times X_n$ where X_n is the homotopy fiber of $f : K(\Sigma^n C \times \Sigma^n C) \longrightarrow K(\Sigma^n A) \times K(\Sigma^n B) \times K(\Sigma^n C)$.

The morphism f is induced by the functor $\mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{A} \times \mathscr{B} \times \mathscr{C}$ sending $(M, M') \in \mathscr{C} \times \mathscr{C}$ to $(MA, M'B, M \oplus M')$. Denote by $f_n(\alpha)$ (resp. $f_n(\beta)$) the map $K(\Sigma^n C) \longrightarrow K(\Sigma^n A)$ (resp. $K(\Sigma^n C) \longrightarrow K(\Sigma^n B)$) induced by α (resp. β). Then X_n is homotopy equivalent to the homotopy fiber of $f_n(\alpha) - f_n(\beta) : K(\Sigma^n C) \longrightarrow K(\Sigma^n A) \times K(\Sigma^n B)$ and we have a homotopy cartesian diagram of spaces:

$$\begin{tikzcd} \Omega H(\Sigma^n C) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \Omega K(\Sigma^n A)) \arrow[d] \\ \Omega K(\Sigma^n B) \arrow[r] & X_n \end{tikzcd}$$

By naturality of the homotopy equivalence: $K(?) \simeq \Omega K(\Sigma ?)$, we get a homotopy equivalence $X_n \longrightarrow \Omega X_{n+1}$ and the sequence of X_n defines an Ω-spectrum X together with a homotopy cartesian diagram of spectra:

$$\begin{tikzcd} \underline{K}(C) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \underline{K}(A) \arrow[d] \\ \underline{K}(B) \arrow[r] & \Omega^{-1} X \end{tikzcd}$$

and that finishes the proof of theorem 2. Consider now the case 3. We proceed as before and we get a homotopy equivalence between $\Omega K(\Sigma^n R)$ and $Nil(\Sigma^n C \times \Sigma^n C, \Sigma^n S) \times X_n$ where X_n is the homotopy fiber of the map $f : K(\Sigma^n C \times \Sigma^n C) \longrightarrow K(\Sigma^n A) \times K(\Sigma^n C)$ induced by the functor $F \times \sigma : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{A} \times \mathscr{C}$ sending $(M, M') \in \mathscr{C} \times \mathscr{C}$ to $(M_\alpha A \oplus M'_\beta A, M \oplus M')$.

Denote by $f_n(\alpha)$ (resp. $f_n(\beta)$) the map $K(\Sigma^n C) \longrightarrow K(\Sigma^n A)$ induced by α (resp. β). Then X_n is homotopy equivalent to the homotopy fiber of $f_n(\alpha) - f_n(\beta)$.

As before, we get a homotopy equivalence $X_n \longrightarrow \Omega X_{n+1}$ and the sequence of X_n defines an Ω-spectrum X which is the homotopy fiber of the map $f(\alpha) - f(\beta) : \underline{K}(C) \longrightarrow \underline{K}(A)$. Therefore we have a homotopy fibration of spectra:

$$\underline{K}(C) \overset{f(\alpha)-f(\beta)}{\longrightarrow} \underline{K}(A) \longrightarrow \Omega^{-1} X$$

and that finishes the proof of theorem 3. $\square$

3. Properties of the functor $\underline{N}il$

Consider two rings A and B. Every right $A \times B$-module M is determined by two right modules M_a and M_b, where M_a is an A-module and M_b is a B-module. By setting: $R_a = A$ and $R_b = B$, we see that M_i is a right R_i-module for each $i \in \{a, b\}$.

If E is an $A \times B$-bimodule, E is determined by four bimodules ${}_aE_a$, ${}_aE_b$, ${}_bE_a$ and ${}_bE_b$, and for each i, j in $\{a, b\}$, ${}_iE_j$ is a (R_i, R_j)-bimodule.

Suppose $f : M \longrightarrow ME$ is a morphism of right $A \times B$-modules. Then f is determined by four morphisms ${}_if_j : M_j \longrightarrow M_i\ {}_iE_j$ and ${}_if_j$ is a morphism of right R_j-modules.

Lemma 3.1. *Let A and B be two rings and E be an $A \times B$-bimodule. Suppose E is flat on the left. Then the correspondences*

$$(M, f) \mapsto (M_b, {}_bf_b)$$
$$(M, f) \mapsto (M_a, {}_af_a + \sum_{k \geq 0} {}_af_b({}_bf_b)^k {}_bf_a)$$

induce two well defined functors:

$$\Phi_1 : \mathscr{N}il(A \times B, E) \longrightarrow \mathscr{N}il(B, {}_bE_b)$$
$$\Phi_2 : \mathscr{N}il(A \times B, E) \longrightarrow \mathscr{N}il(A, {}_aE_a \oplus \bigoplus_{k \geq 0} {}_aE_b({}_bE_b)^k {}_bE_a)$$

Moreover, if E_b is flat on the right, these functors induce a homotopy equivalence of spectra:

$$\underline{N}il(A \times B, E) \overset{\sim}{\longrightarrow} \underline{N}il(B, {}_bE_b) \times \underline{N}il(A, {}_aE_a \bigoplus_{k \geq 0} {}_aE_b({}_bE_b)^k {}_bE_a)$$

This lemma will be proven in 3.7. Using this result, we are able to prove theorem 4. Consider two rings A and B, an (A, B)-bimodule S and a (B, A)-bimodule T. Suppose S and T are flat on both sides. Define the $A \times B$-bimodule E by:

$${}_aE_b = S \qquad {}_bE_a = T \qquad {}_aE_a = {}_bE_b = 0$$

Actually, this bimodule is the bimodule $S \oplus T$, where S and T are considered as $A \times B$-bimodules (via the projections $A \times B \longrightarrow A$ and $A \times B \longrightarrow B$). Moreover $E = S \oplus T$ is flat on both sides.

Applying lemma 3.1, we get two functors:

$$\Phi_1 : \mathscr{N}il(A \times B, E) \longrightarrow \mathscr{N}il(B, 0)$$
$$\Phi_2 : \mathscr{N}il(A \times B, E) \longrightarrow \mathscr{N}il(A, ST)$$

and a homotopy equivalence of spectra:

$$\Phi(S,T) : \underline{N}il(A \times B, E) \xrightarrow{\sim} \underline{N}il(A, ST)$$

By exchanging the roles of A and B, and S and T we get also a homotopy equivalence of spectra:

$$\Phi(T,S) = \underline{N}il(A \times B, E) \xrightarrow{\sim} \underline{N}il(B, TS) \quad \square$$

3.2. Proof of theorem 5

Let A be a ring, I be a set and S_i, $i \in I$ be a family of A-bimodules flat on both sides. The direct sum of the S_i's will be denoted by S.

Let (M, f) be an object of the category $\mathscr{N}il(A, S)$. The morphism $f : M \longrightarrow MS$ decomposes into a finite sum: $f = \sum_{i \in I} f_i$, where f_i is a morphism from M to MS_i.

If $u = i_1 i_2 \dots i_p$ is a word in $W(I)$, we set:

$$f_u = f_{i_1} f_{i_2} \dots f_{i_p} \quad \text{and} \quad S_u = S_{i_1} S_{i_2} \dots S_{i_p}$$

If J is a subset of $W(I)$, we set also:

$$f_J = \sum_{u \in J} f_u \quad \text{and} \quad S_J = \bigoplus_{u \in J} S_u$$

We check that f_u is a morphism from M to MS_u and f_J is a morphism from M to MS_J.

Let u be a non empty word in $W(I)$. Then the correspondence $(M, f) \mapsto (M, f_u)$ induces an exact functor $\varphi_u : \mathscr{N}il(A, S) \longrightarrow \mathscr{N}il(A, S_u)$. If J is a subset of $W(I)$ that does not contains the empty word, the correspondence $(M, f) \mapsto (M, f_J)$ induces also an exact functor $\varphi_J : \mathscr{N}il(A, S) \longrightarrow \mathscr{N}il(A, S_J)$. These functors are compatible with tensor product with any power of Σ and induce morphisms of spectra on K-theory.

Consider two words u and v in $W(I)$. We have four A-bimodules: S_u, S_v, S_{uv} and S_{vu}. Using notations in 1.4, for each A-bimodule T and each $(i,j) \in \{1,2\}^2$ we get an $A \times A$-bimodule ${}^iT^j$ where $A \times A$ acts on the left of T via the i-th projection and on the right via the j-th projection. We have a commutative diagram of exact categories:

$$\begin{array}{ccccc}
\mathscr{N}il(A, S) & \xrightarrow{F} & \mathscr{N}il(A \times A, {}^1S_u^2 \oplus {}^2S_v^1) & \xrightarrow{\psi(u,v)} & \mathscr{N}il(A, S_{uv}) \\
{\scriptstyle =}\Big\downarrow & & {\scriptstyle G}\Big\downarrow{\scriptstyle \sim} & & \\
\mathscr{N}il(A, S) & \xrightarrow{F'} & \mathscr{N}il(A \times A, {}^2S_u^1 \oplus {}^1S_v^2) & \xrightarrow{\psi(v,u)} & \mathscr{N}il(A, S_{vu})
\end{array}$$

where F and F' are defined by:

$$F(M, f) = (M^1 \oplus M^2, f_u + f_v) \qquad F'(M, f) = (M^1 \oplus M^2, f_v + f_u)$$

for each $(M, f) \in \mathscr{N}il(A, S)$ and the morphisms G, $\psi(u, v)$ and $\psi(v, u)$ are defined by sending each $(M, M', f + f')$ (with $M, M' \in \mathscr{A}$, $f : M \longrightarrow M'S_v$ and $f' : M' \longrightarrow MS_u$) to

$$G(M, M', f + f') = (M', M, f + f') \qquad \psi(u, v)(M, M', f + f') = (M, f'f)$$
$$\psi(v, u)(M, M', f + f') = (M', ff')$$

Because of theorem 4, this diagram induces a homotopy commutative diagram of Ω-spectra:

$$\begin{array}{ccccc}
\underline{N}il(A, S) & \xrightarrow{F} & \underline{N}il(A \times A, {}^1S_u^2 \oplus {}^2S_v^1) & \xrightarrow[\sim]{\psi(u,v)} & \underline{N}il(A, S_{uv}) \\
= \big\downarrow & & G \big\downarrow \sim & & H \big\downarrow \sim \\
\underline{N}il(A, S) & \xrightarrow{F'} & \underline{N}il(A \times A, {}^2S_u^1 \oplus {}^1S_v^2) & \xrightarrow[\sim]{\psi(v,u)} & \underline{N}il(A, S_{vu})
\end{array}$$

where $\psi(u, v)$ and $\psi(v, u)$ are homotopy equivalences and H is the map $\psi(u, v)^{-1}G\psi(v, u)$ (up to homotopy).

On the other hand we have: $\psi(u, v)F = \varphi_{uv}$ and $\psi(v, u)F' = \varphi_{vu}$. Then we have a homotopy commutative diagram:

$$\begin{array}{ccc}
\underline{N}il(A, S) & \xrightarrow{\varphi_{uv}} & \underline{N}il(A, S_{uv}) \\
= \big\downarrow & & H \big\downarrow \sim \\
\underline{N}il(A, S) & \xrightarrow{\varphi_{vu}} & \underline{N}il(A, S_{vu})
\end{array}$$

and the homotopy class of the map φ_u depends only on the class of u in $CW(I)$. Therefore, to prove the theorem, its is enough to prove it for some admissible set X.

Lemma 3.3. *Suppose I has exactly two elements i and j. Let $J \subset W(I)$ be the set of elements i^kj, for $k \geq 0$. Then the functors: $\varphi_i : \mathscr{N}il(A, S) \longrightarrow \mathscr{N}il(A, S_i)$ and $\varphi_J : \mathscr{N}il(A, S) \longrightarrow \mathscr{N}il(A, S_J)$ induce a homotopy equivalence of spectra:*

$$\underline{N}il(A, S) \xrightarrow{\sim} \underline{N}il(A, S_i) \times \underline{N}il(A, S_J)$$

Proof. We apply lemma 3.1 in the following case:

$$B = A \qquad {}_aE_a = S_i \qquad {}_bE_b = 0 \qquad {}_bE_a = A \qquad {}_aE_b = S_j$$

and we get a homotopy equivalence of spectra:

$$\underline{N}il(A \times A, E) \xrightarrow{\sim} \underline{N}il(A, S_i \oplus S_j) = \underline{N}il(A, S)$$

induced by the functor $(P, Q, f) \mapsto (P, {}_af_a + {}_af_b\ {}_bf_a)$.

By exchanging the role of a and b, we have also a homotopy equivalence of spectra:

$$\underline{N}il(A \times A, E) \xrightarrow{\sim} \underline{N}il(A, S_i) \times \underline{N}il(A, \bigoplus_{k \geq 0} S_i^k S_j) = \underline{N}il(A, S_i) \times \underline{N}il(A, S_J)$$

and this map is induced by the two functors:

$$(P, Q, f) \mapsto (P, {}_af_a)$$
$$(P, Q, f) \mapsto (Q, \sum_{k \geq 0} {}_bf_a({}_af_a)^k\, {}_af_b)$$

Because of theorem 4, this last functor is, after applying the functor $\underline{N}il$, equivalent to the functor:

$$(P, Q, f) \mapsto (P, \sum_{k \geq 0} ({}_a f_a)^k {}_a f_b \ {}_b f_a)$$

Therefore we get a homotopy equivalence of spectra:

$$\underline{N}il(A, S) \xrightarrow{\sim} \underline{N}il(A, S_i) \times \underline{N}il(A, S_J)$$

induced by the two functors:

$$(P, f) \mapsto (P, f_i)$$
$$(P, f) \mapsto (P, \sum_{k \geq 0} f_i^k f_j)$$

and the lemma follows. $\square$

From now one, the coproduct in the category of spectra will be denoted by $\oplus$ and the trivial spectrum will be denoted by 0. Actually, is E_j is a family of spectra, the spectrum $\bigoplus_j E_j$ is nothing else but the filtered colimit of finite products of the E_j's.

Let X be an admissible set in $W(I)$. Then the map: $X \subset W(I) \xrightarrow{\pi} CW(I)$ induces a bijection $X \xrightarrow{\sim} CW_0(I)$.

For every object $(M, f) \in \mathscr{N}il(A, S)$ there are only finitely many non zero morphisms f_u and the product of the φ_u, for $u \in X$, induces a map:

$$\underline{N}il(A, S) \longrightarrow \prod_{u \in X} \underline{N}il(A, S_u)$$

with values in $\bigoplus_{u \in X} \underline{N}il(A, S_u)$. Hence we have a morphism of spectra:

$$F : \underline{N}il(A, S) \longrightarrow \bigoplus_{u \in X} \underline{N}il(A, S_u)$$

and the last thing to do is to prove that F is a homotopy equivalence.

On the other hand, we have a homotopy commutative diagram:

$$\begin{array}{ccc}
\varinjlim \underline{N}il(A, \bigoplus_{j \in J} S_j) & \longrightarrow & \varinjlim \Big(\bigoplus_{u \in X \cap W(J)} \underline{N}il(A, S_u) \Big) \\
\sim \downarrow & & \sim \downarrow \\
\underline{N}il(A, S) & \longrightarrow & \bigoplus_{u \in X} \underline{N}il(A, S_u)
\end{array}$$

where the limit is taken over all finite subset J of I. Moreover vertical arrows of this diagram are homotopy equivalences and, in order to prove the theorem, it is enough to consider the case where I is finite.

If I has at most 1 elements there is nothing to prove. So we may suppose that I is a finite set with at least 2 elements. If J is a subset of $W(I)$ and j an element of J, we denote by $Z(J, j)$ the set of words in $W(I)$ on the form $j^p j'$, where j' is any element in J distinct from j and p is any non negative integer.

Lemma 3.4. *Suppose I is finite with at least 2 elements. Then there exists a sequence (J_n, j_n), for $n \geq 0$, with the following properties:*

- *for every integer $n \geq 0$, J_n is a subset of $W(I)$, j_n is an element of J_n and J_{n+1} is the set $Z(J_n, j_n)$*
- *the set $Y = \{j_0, j_1, j_2, j_3, \dots\} \subset W(I)$ is admissible*
- *for every integer $p > 0$, there is an integer $m \geq 0$ such that:*

$$\forall n > m,\ \forall u \in J_n, |u| > p$$

where $|u|$ is the length of u.

Proof. In order to define the sequence (J_n, j_n), it's enough to define J_0 and to choose each j_n in J_n. So we set: $J_0 = I$ and, for each n, j_n is chosen to be an element of J_n of minimal length in $W(I)$.

Let $n \geq 0$ be an integer. Set: $J = J_n$, $j = j_n$ and denote by J' the complement of j in J. The inclusion $J_{n+1} = Z(J, j) \subset W(I)$ factorizes through $W(J)$ and J_{n+1} can be considered as a subset of $W(J)$. Every word u in $W(J)$ is written uniquely in the following form:

$$u = j^{n_0} j_1 j^{n_1} j_2 j^{n_2} \dots j_p j^{n_p}$$

with: $p \geq 0$, $n_* \geq 0$, $j_* \in J'$.

Suppose u is reduced.

If $p = 0$, then u is a power of j and, because u is reduced, we have: $u = j$.

If $p > 0$, then u is, up to conjugation, on the form:

$$u = j^{n_0} j_1 j^{n_1} j_2 \dots j^{n_{p-1}} j_p$$

and u is conjugate to an element in $W_0(Z(J, j)) = W_0(J_{n+1})$. Moreover u is reduced in $W(J)$ if and only if u is reduced in $W(I)$.

Therefore every element in $CW_0(J) = CW_0(J_n)$ belongs to the image of the map $CW(J_{n+1}) \longrightarrow CW(J_n)$ except the element $j = j_n$ and the inclusion $J_{n+1} \subset J_n$ induces a bijection $\{j_n\} \coprod CW_0(J_{n+1}) \xrightarrow{\sim} CW_0(J_n)$. As a consequence, we have, for each $n \geq 0$, a bijection:

$$\{j_0, j_1, \dots, j_{n-1}\} \coprod CW_0(J_n) \xrightarrow{\sim} CW_0(I)$$

For each integer $n \geq 0$, denote by p_n the minimal length of the words in J_n. Since J_{n+1} is contained in $W(J_n)$, we have: $p_{n+1} \geq p_n$ and the sequence (p_n) is increasing. We'll prove that this sequence is unbounded.

Let $n \geq 0$ be an integer. For every $u \in J_n$, we have: $|u| \geq p_n$. For every integer m, denote by H_m the set of elements $u \in J_m$ with $|u| = p_n$ and by CH_m its image in $CW(I)$. Since I is finite and every element of H_m is reduced in $W(J_m)$, CH_m is a finite set contained in $CW_0(I)$.

Denote by q the cardinal of CH_n. Since j_n is an element in J_n of minimal length j_n belongs to H_n. Then we have: $q > 0$ and, because of the bijection $\{j_n\} \coprod CW_0(J_{n+1}) \xrightarrow{\sim} CW_0(J_n)$, the cardinal of CH_{n+1} is $q - 1$. For the same reason, we have the following:

$$\forall i \in \{0, 1, \dots, q\},\ \operatorname{card}(CH_{n+i}) = q - i$$

and then:

$$\operatorname{card}(CH_{n+q}) = 0 \ \text{ and } \ p_{n+q} > p_n$$

Therefore the sequence (p_n) is unbounded and the last property of the lemma is proven.

Since the map $\{j_0, j_1, \dots, j_{n-1}\} \coprod CW_0(J_n) \xrightarrow{\sim} CW_0(I)$ is bijective, the map $f : Y \longrightarrow CW_0(I)$ is injective. Let u be an element in $CW_0(I)$ with length p. Since the sequence p_n is unbounded, there is an integer n such that $p_n > p$ and u doesn't belong to the image of $CW_0(J_n) \longrightarrow CW_0(I)$. Hence u belongs to the image of $\{j_0, j_1, \dots, j_{n-1}\} \longrightarrow CW_0(I)$ and f is surjective. Therefore Y is admissible and the lemma is proven. $\square$

For every $n > 0$ we denote by Y_n the set $\{j_0, j_1, \dots, j_{n-1}\}$.

Lemma 3.5. *For all integer $n \geq 0$, the maps $\varphi_u : \underline{N}il(A,S) \longrightarrow \underline{N}il(A,S_u)$, for $u \in Y_n$ and the map $\varphi_{J_n} : \underline{N}il(A,S) \longrightarrow \underline{N}il(A,S_{J_n})$ induce a homotopy equivalence of spectra:*

$$G_n : \underline{N}il(A,S) \xrightarrow{\sim} \bigoplus_{u \in Y_n} \underline{N}il(A,S_u) \oplus \underline{N}il(A,S_{J_n})$$

Proof. We'll prove the lemma by induction on n. The map G_0 is the identity (and then a homotopy equivalence).

Suppose $n \geq 0$ and G_n is a homotopy equivalence. Consider the bimodule $S' = S'_i \oplus S'_j$ with:

$$S'_i = S_{j_n}$$

$$S'_j = S_{J_n \setminus \{j_n\}}$$

By applying lemma 3.3 with this bimodule, we get a homotopy equivalence of spectra:

$$\underline{N}il(A,S') = \underline{N}il(A,S_{J_n}) \xrightarrow{\sim} \underline{N}il(A,S_{j_n}) \oplus \underline{N}il(A,\widehat{S})$$

where $\widehat{S}$ is the following bimodule:

$$\widehat{S} = \bigoplus_{u,k} \left(S_{j_n}\right)^k S_u$$

the sum being taken over all integer $k \geq 0$ and all $u \neq j_n$ in J_n. So we have isomorphisms:

$$\widehat{S} \simeq \bigoplus_{u,k} S_{j_n^k u} \simeq \bigoplus_{v \in J_{n+1}} S_v = S_{J_{n+1}}$$

and then a homotopy equivalence:

$$\underline{N}il(A,S_{J_n}) \xrightarrow{\sim} \underline{N}il(A,S_{j_n}) \oplus \underline{N}il(A,S_{J_{n+1}})$$

Therefore we have homotopy equivalences:

$$\underline{N}il(A,S) \xrightarrow{\sim} \bigoplus_{u \in Y_n} \underline{N}il(A,S_u) \oplus \underline{N}il(A,S_{J_n})$$

$$\xrightarrow{\sim} \bigoplus_{u \in Y_n} \underline{N}il(A,S_u) \oplus \underline{N}il(A,S_{j_n}) \oplus \underline{N}il(A,S_{J_{n+1}})$$

$$\xrightarrow{\sim} \bigoplus_{u \in Y_{n+1}} \underline{N}il(A,S_u) \oplus \underline{N}il(A,S_{J_{n+1}})$$

and G_{n+1} is a homotopy equivalence. The lemma follows by induction. $\square$

We are now able to finish the proof of theorem 5.

As we have said before it is enough to consider the case where I is finite with at least two elements.

Consider the map $\Phi : \underline{N}il(A,S) \longrightarrow \bigoplus_{u\in Y} \underline{N}il(A,S_u)$ induced by the functors $\varphi_u : \mathscr{N}il(A,S) \longrightarrow \mathscr{N}il(A,S_u)$. We have to prove that G is a homotopy equivalence and, for doing that, it will be enough to prove that G induces an isomorphism:

$$\Phi_i : Nil_i(A,S) \longrightarrow \bigoplus_{u\in Y} Nil_i(A,S_u)$$

for every integer $i \in \mathbf{Z}$.

Let $i \in \mathbf{Z}$ be an integer and y be an element in $\bigoplus_{u\in Y} Nil_i(A,S_u)$. There is an integer $n \geq 0$ such that y is in the direct sum of $Nil_i(A,S_u)$, for $u \in Y_n$. Because of lemma 3.5, there is an element $x \in Nil_i(A,S)$ such that:

$$G_n(x) = y \oplus 0 \in \bigoplus_{u\in Y_n} Nil_i(A,S_u) \oplus Nil_i(A,S_{J_n})$$

Hence x is sent to y by Φ_i and Φ_i is surjective.

Suppose $i \geq 0$. Let $x \in Nil_i(A,S)$ be an element killed by Φ_i. This element can be lifted in an element $y \in K_i(\mathscr{N}il(A,S)) = \pi_{i+1}(B(Q\mathscr{N}il(A,S)))$ and there is a finite subcategory $\mathscr{E}$ of the Quillen category $Q\mathscr{N}il(A,S)$ such that y can be lifted in an element $z \in \pi_{i+1}(B\mathscr{E})$. This category $\mathscr{E}$ involves only finitely many objects $(M,f) \in \mathscr{N}il(A,S)$. Denote by $\mathscr{F}$ the set of these morphisms f.

Since each $f \in \mathscr{F}$ is nilpotent, there is an integer p such that f_u is trivial for each $f \in \mathscr{F}$ and each element $u \in W(I)$ of length $\geq p$. Because of the last property of lemma 3.4, the set of integers n such that there is some $u \in J_n$ and some $f \in \mathscr{F}$ with $f_u \neq 0$ is finite. Hence, for n big enough, the composite functor

$$\mathscr{E} \longrightarrow Q\mathscr{N}il(A,S) \xrightarrow{\varphi_{J_n}} Q\mathscr{N}il(A,S_{J_n})$$

factorizes through the category $Q\mathscr{P}_A$ and the composite map

$$\Omega B\mathscr{E} \longrightarrow \Omega BQ\mathscr{N}il(A,S_{J_n}) \longrightarrow Nil(A,S_{J_n})$$

is trivial. Hence z is killed in $Nil_i(A,S_{J_n})$ and $x \in Nil_i(A,S)$ is killed by $\varphi_{J_n} : Nil_i(A,S) \longrightarrow Nil(A,S_{J_n})$.

But x is killed by Φ_i and x is also killed by the map $Nil_i(A,S) \longrightarrow \bigoplus_{u\in Y_n} Nil_i(A,S_u)$. Therefore x is killed by G_n which is a homotopy equivalence and x is zero. Hence the morphism Φ_i is bijective for every integer $i \geq 0$.

If i is a negative integer, we may replace A and S by $\Sigma^p A$ and $\Sigma^p S$ for some $p > -i$ and the bijectivity of Φ_{i+p} for $(\Sigma^p A, \Sigma^p S)$ implies that Φ_i (for (A,S)) is bijective. Hence Φ is a homotopy equivalence of spectra. Moreover this is true for the set Y and then for every admissible set in $W(I)$. Then we get the desired result and the theorem follows. $\square$

3.6. Proof of theorem 6

Denote by S' the following bimodule:

$$S' = S \oplus \bigoplus_i E_i \underset{A_i}{\otimes} F_i$$

Let J be the disjoint union of I and $\{0\}$. We set:

$$\forall i \in I, \quad S_i = E_i \underset{A_i}{\otimes} F_i = E_i\ F_i$$
$$S_0 = S$$

and we have:

$$S' = \underset{j \in J}{\oplus} S_j$$

Because of theorem 5, there exist a family of A-bimodules U_k, $k \in K$ such that the following holds:

- each U_k is flat on both sides
- $\underline{N}il(A, S') \simeq \underline{N}il(A, S) \oplus \underset{k}{\oplus} \underline{N}il(A, U_k)$
- each U_k has the form: $U_k = S_{j_1} S_{j_2} \dots S_{j_n}$, with $j_1, j_2, \dots, j_n$ in J and $j_q \in I$ for some q.

Then, because of theorem 4, there exist a family of A-bimodules V_k, $k \in K$ and elements $i_k \in I$ such that

- each V_k is flat on both sides
- $\underline{N}il(A, S') \simeq \underline{N}il(A, S) \oplus \underset{k}{\oplus} \underline{N}il(A, S_{i_k}\ V_k)$

and we have, for each k:

$$\underline{N}il(A, S_{i_k}\ V_k) = \underline{N}il(A, E_{i_k} F_{i_k} V_k) \simeq \underline{N}il(A_{i_k}, F_{i_k} V_k\ E_{i_k})$$

But A_{i_k} is regular coherent and each spectrum $\underline{N}il(A_{i_k}, F_{i_k} V_k\ E_{i_k})$ is contractible. The result follows. $\square$

3.7. Proof of lemma 3.1

With the notations of lemma 3.1, we denote by $F = B \oplus {}_bE_b \oplus \big({}_bE_b\big)^2 \oplus \dots$ the tensor algebra of ${}_bE_b$ and by $\widehat{E}$ the bimodule $\widehat{E} = {}_aE_a \oplus {}_aE_b\ F\ {}_bE_a$.

For every object $(M, f) \in \mathscr{N}il(A \times B, E)$, we set:

$$g_1 = {}_af_a, \ g_2 = {}_af_b, \ g_3 = {}_bf_a, \ g_4 = {}_bf_b$$

Since f is nilpotent, there is an integer $n > 0$ such that $f^n = 0$. Then, for every integers $i_1, i_2, \dots, i_n$ in $\{1, 2, 3, 4\}$ the morphism $g_{i_1} g_{i_2} \dots g_{i_n}$ is a part of f^n and we have: $g_{i_1} g_{i_2} \dots g_{i_n} = 0$. Hence the two morphisms ${}_bf_b$ and $\widehat{f} = {}_af_a + \sum\limits_{k \geq 0} {}_af_b \big({}_bf_b\big)^k {}_bf_a$ are nilpotent and we have a functor

$$\Phi(A, B, E) : \mathscr{N}il(A \times B, E) \longrightarrow \mathscr{N}il(B, {}_bE_b) \times \mathscr{N}il(A, \widehat{E})$$

sending each object $(M, f) \in \mathscr{N}il(A \times B, E)$ to:

$$\Phi(A, B, E)(M, f) = \Big((M_b, {}_bf_b), (M_a, \widehat{f})\Big)$$

Moreover this functor is exact.

It is easy to see that lemma 3.1 is equivalent to the fact that $\Phi(\Sigma^n A, \Sigma^n, B, \Sigma^n E)$ induces, for all $n \geq 0$, a homotopy equivalence in K-theory. Therefore it is enough to prove that $\Phi(A, B, E)$ induces a homotopy equivalence in K-theory for every left-flat bimodule $(A \times B, E)$ such that E_b is flat on the right and that will be done by using K-theory of Waldhausen categories.

In our situation, we have three exact categories: the categories $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ of finitely generated projective right modules over the rings A, B and $C = A \times B$ respectively. These categories are contained in the corresponding abelian categories $\mathscr{A}^\vee$, $\mathscr{B}^\vee$ and $\mathscr{C}^\vee$ of right modules over the corresponding rings.

If (A, S) is a left-flat bimodule, we have also an exact category $\mathscr{N}il(A, S)$ and a Waldhausen category $\mathscr{N}il(A, S)_*$. Moreover $\mathscr{N}il(A, S)$ is contained in the abelian category $\mathscr{N}il(A, S)^\vee$ (see part 1.3) and, as a subcategory of $\mathscr{N}il(A, S)^\vee$, $\mathscr{N}il(A, S)$ is stable under taking kernel of epimorphisms. Hence the Gillet-Waldhausen theorem applies to the categories $\mathscr{N}il(?, ?)$ and we have a commutative diagram:

$$\begin{tikzcd}
\mathscr{N}il(A \times B, E) \arrow[r, "\Phi(A,B,E)"] \arrow[d] & \mathscr{N}il(B, {}_bE_b) \times \mathscr{N}il(A, \widehat{E}) \arrow[d] \\
\mathscr{N}il(A \times B, E)_* \arrow[r, "\Phi(A,B,E)"] & \mathscr{N}il(B, {}_bE_b)_* \times \mathscr{N}il(A, \widehat{E})_*
\end{tikzcd}$$

of Waldhausen categories where the vertical functors induce homotopy equivalences in K-theory.

Therefore, in order to prove the lemma, it is enough to prove that the functor:

$$\Phi(A, B, E)_* : \mathscr{N}il(A \times B, E)_* \longrightarrow \mathscr{N}il(B, {}_bE_b)_* \times \mathscr{N}il(A, \widehat{E})_*$$

induces a homotopy equivalence in K-theory.

If (A, S) is a left-flat bimodule, it is easy to see that an object of $\mathscr{N}il(A, S)_*$ is nothing else but a pair (M, f), where M is in $\mathscr{A}_*$ and $f : M \longrightarrow MS$ is a nilpotent morphism in $\mathscr{A}_*^\vee$. Then the functor $\Phi(A, B, E)_*$ is given by:

$$\Phi(A, B, E)_*(M, f) = \Big((M_b, {}_bf_b), (M_a, \widehat{f})\Big)$$

for every $M \in \mathscr{C}_* = \mathscr{A}_* \times \mathscr{B}_*$.

Consider the Waldhausen category $\mathscr{E} = \mathscr{N}il(A \times B, E)_*$. We may define a new subcategory of equivalences by saying that $\varphi : (M, f) \longrightarrow (M', f')$ is an equivalence if the induced morphism $M_a \longrightarrow M'_a$ is an isomorphism in homology. With this new equivalences, we get a new Waldhausen category $\mathscr{E}'$. We have also a Waldhausen subcategory $\mathscr{E}_0$ of $\mathscr{E}$ generated by the objects $(M, f) \in \mathscr{E}$ such that M_a is acyclic. Denote also by $\mathscr{N}il_0$ the Waldhausen subcategory of $\mathscr{N}il(A, \widehat{E})_*$ generated by pairs (M, f) with M acyclic. Hence we have a commutative diagram of essentially small Waldhausen categories:

$$\begin{tikzcd}
\mathscr{E}_0 \arrow[r] \arrow[d, "\Phi_0"'] & \mathscr{E} \arrow[r] \arrow[d, "\Phi(A,B,E)"'] & \mathscr{E}' \arrow[d, "\Phi'"'] \\
\mathscr{N}il(B, {}_bE_b)_* \times \mathscr{N}il_0 \arrow[r] & \mathscr{N}il(B, {}_bE_b)_* \times \mathscr{N}il(A, \widehat{E})_* \arrow[r] & \mathscr{N}il(A, \widehat{E})_*
\end{tikzcd}$$

Since every morphism in $\mathscr{N}il_0$ is an equivalence, the category $\mathscr{N}il_0$ has trivial K-theory and, because of the fibration theorem, the two lines of the diagram induce fibrations in K-theory. Hence in order to prove the lemma, it's enough to prove that Φ_0 and Φ' induce homotopy equivalences on K-theory and that's equivalent to show that Φ' and the functor:

$$\Phi'_0 : \mathscr{E}_0 \xrightarrow{\Phi_0} \mathscr{N}il(B, {}_bE_b)_* \times \mathscr{N}il_0 \xrightarrow{pr_1} \mathscr{N}il(B, {}_bE_b)_*$$

induce homotopy equivalences on K-theory. Moreover these two functors have the approximation properties (App1).

Lemma 3.8. *The functor Φ'_0 induces a homotopy equivalence in K-theory.*

Proof. Because of the approximation theorem 2.3, we just have to prove that Φ'_0 has the property (App2).

Let $X = (M, f) \in \mathscr{E}_0$ and $Y = (Q, g) \in \mathscr{N}il(B, {}_bE_b)_*$ be two objects in the corresponding categories. A morphism $\varphi : \Phi'_0(X) \longrightarrow Y$ is represented by a morphism $\varphi : M_b \longrightarrow Q$ in $\mathscr{B}_*$ making the following diagram commutative:

$$\begin{tikzcd} M_b \arrow[r, "\varphi"] \arrow[d, "{}_bf_b"'] & Q \arrow[d, "g"'] \\ M_b\ {}_bE_b \arrow[r, "\varphi"] & Q\ {}_bE_b \end{tikzcd}$$

Denote by C the cylinder of φ and by C' its mapping cone (or its 0-cone as defined in 2.1). The morphisms ${}_bf_b$ and g induce two nilpotent morphisms $\lambda : C \longrightarrow C\ {}_bE_b$ and $\lambda' : C' \longrightarrow C'\ {}_bE_b$. By naturality, we have a commutative diagram in $\mathscr{C}_*^\vee$:

$$\begin{tikzcd} M_b \arrow[r, "i"] \arrow[d, "{}_bf_b"'] & C \arrow[r, "p"] \arrow[d, "\lambda"'] & Q \arrow[d, "g"'] \\ M_b\ {}_bE_b \arrow[r, "i"] & C\ {}_bE_b \arrow[r, "p"] & Q\ {}_bE_b \end{tikzcd}$$

where $i : M_b \longrightarrow C$ is a cofibration and $p : C \longrightarrow Q$ a homotopy equivalence. Moreover we have: $\varphi = pi$.

Define the object $M' \in \mathscr{C}_*$ by: $M'_a = M_a$ and $M'_b = C$. In order to define the desired object $X' \in \mathscr{E}_0$, we have to construct the morphism $f' : M' \longrightarrow M'E$.

We set: ${}_af'_a = {}_af_a$, ${}_bf'_b = \lambda$ and ${}_bf'_a = i\ {}_bf_a$. Since $i : M_b \longrightarrow C$ is a cofibration and $M_a\ {}_aE_b$ is acyclic, there is no obstruction to extend ${}_af_b : M_b \longrightarrow M_a\ {}_aE_b$ to a morphism: ${}_af'_b : C \longrightarrow M_a\ {}_aE_b$. Hence we get a morphism $f' : M' \longrightarrow M'E$.

Let $\overline{C}$ be the finite complex in $\mathscr{C}_*$ defined by: $\overline{C}_a = 0$ and $\overline{C}_b = C'$ and $\overline{\lambda} : \overline{C} \longrightarrow \overline{C}E$ be the morphism in $\mathscr{C}_*^\vee$ associated with λ' under the canonical bijection: $\mathrm{Hom}_{\mathscr{C}_*^\vee}(\overline{C}, \overline{C}E) \simeq \mathrm{Hom}_{\mathscr{B}_*^\vee}(C', C'{}_bE_b)$. Since λ' is nilpotent, $\overline{\lambda}$ is also nilpotent.

We have a commutative diagram in $\mathscr{C}_*^\vee$ with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "f"'] & M' \arrow[r] \arrow[d, "f'"'] & \overline{C} \arrow[r] \arrow[d, "\overline{\lambda}"'] & 0 \\ 0 \arrow[r] & ME \arrow[r] & M'E \arrow[r] & \overline{C}E \arrow[r] & 0 \end{tikzcd}$$

and, since f and $\overline{\lambda}$ are nilpotent, f' is nilpotent also. More precisely, if we have: $f^p = 0$ and $\overline{\lambda}^q = 0$, then we have $f'^{\,p+q} = 0$.

Since M'_a is acyclic, $X' = (M', f')$ is an object in $\mathscr{E}_0$ and the morphism i induces a morphism $\alpha : X \longrightarrow X'$. Moreover we have the following commutative diagram:

$$\begin{tikzcd} \Phi'_0(X) \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & Y \arrow[d, "="'] \\ \Phi'_0(X') \arrow[r, "p"] & Y \end{tikzcd}$$

and the property (App2) is satisfied. The lemma follows. □

So the last thing to do is to prove that the functor $\Phi' : \mathscr{E}' \longrightarrow \mathscr{N}il(A, \widehat{E})_*)$ induces a homotopy equivalence in K-theory. As before, it's enough to prove that Φ' has the property (App2). In order to do that we'll need two technical results:

Lemma 3.9. *Let A and B be two rings and S be an (A, B)-bimodule. Suppose S is flat on the left. Let X be a finite B-complex, Y be an A-complex and $f : X \longrightarrow YS$ be a morphism of B-complexes. Then there exist a finite A-complex Y', a morphism $g : Y' \longrightarrow Y$ and a commutative diagram:*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "="'] & Y'S \arrow[d, "g"'] \\ X \arrow[r, "f"] & YS \end{tikzcd}$$

Lemma 3.10. *Let A be a ring, X and Y two A-complexes and $f : X \longrightarrow Y$ be a morphism. Suppose the module $\bigoplus_n X_n$ is finitely presented and each Y_n is flat. Then f factorizes through a finite A-complex.*

These two lemmas will be proven at the end of the section. As a consequence of these two lemmas we have the following result:

Lemma 3.11. *Let A, B and C be three rings, S be a (C, B)-bimodule and T be a (B, A)-bimodule. Suppose S is flat on the right and T is flat on the left. Let X, Y and Z be finite complexes over the rings A, B and C respectively and $f : X \longrightarrow YT$ and $g : Y \longrightarrow ZS$ be two morphisms. We suppose that the composite morphism:*

$$X \xrightarrow{f} YT \xrightarrow{g} ZST$$

is zero. Then the morphism $g : Y \longrightarrow ZS$ factorizes through a finite complex Y' by a morphism $\lambda : Y \longrightarrow Y'$ in such a way that the composite morphism:

$$X \xrightarrow{f} YT \xrightarrow{\lambda} Y'T$$

is zero.

Proof. Let K be the kernel of g. Since T is flat on the left we have two exact sequences:

$$0 \longrightarrow K \xrightarrow{i} Y \xrightarrow{g} ZS$$

$$0 \longrightarrow KT \xrightarrow{i} YT \xrightarrow{g} ZST$$

and the morphism f factorizes through KT. Because of lemma 3.9 there exist a finite B-complex K', a morphism $j : K' \longrightarrow K$ and a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "="'] & K'T \arrow[d, "ij"'] \\ X \arrow[r, "f"] & YT \end{tikzcd}$$

Denote by Y_1 the cokernel of $ij : K' \longrightarrow Y$. Since gi is the zero morphism, there is a morphism $g_1 : Y_1 \longrightarrow ZS$ making the following diagram commutative:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d] & ZS \arrow[d, "="'] \\ Y_1 \arrow[r, "g_1"] & ZS \end{tikzcd}$$

Since S is flat on the right, ZS is flat and, because of lemma 3.10, g_1 factorizes through a finite complex Y'. The lemma follows. $\square$

Lemma 3.12. *The functor $\Phi' : \mathscr{E}' \longrightarrow \mathscr{N}il(A, \widehat{E})_*)$ has the property (App2).*

Proof. Let's set:

$$H = {}_aE_a \qquad K = {}_bE_b \qquad S = {}_aE_b \qquad T = {}_bE_a \qquad \widehat{K} = F = B \oplus K \oplus K^2 \oplus \dots$$

Let X and Y be two objects in $\mathscr{N}il(A \times B, E)_*$ and $\mathscr{N}il(A, \widehat{E})_*$ respectively and φ be a morphism from $\Phi'(X)$ to Y. By setting: $(U, f) = X$, $(M, g) = Y$, $P = U_a$, $Q = U_b$, we see that P and M are finite A-complexes, Q is a finite B-complex, φ is a morphism from P to M and g is written as a finite sum:

$$g = \lambda + \sum_{k \geq 0} \mu_k$$

where λ is a morphism from M to MH and each μ_k is a morphism from M to MSK^kT. Moreover the following diagrams are commutative:

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "{}_af_a"'] & M \arrow[d, "\lambda"] \\ PH \arrow[r, "\varphi"] & MH \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "{}_af_b({}_bf_b)^k{}_bf_a"'] & M \arrow[d, "\mu_k"] \\ PSK^kT \arrow[r, "\varphi"] & MSK^kT \end{tikzcd}$$

We want to construct an object $X' \in \mathscr{N}il(A \times B, E)_*$ and a morphism $\alpha : X \longrightarrow X'$ such that $\Phi'(\alpha) : \Phi'(X) \longrightarrow \Phi'(X')$ is isomorphic to $\varphi : \Phi'(X) \longrightarrow Y$. So we want to have: $X' = (U', f')$ and $U'_a = M$ and, to determine X', we need to define $Q' = U'_b$, the morphism f' and the morphism $\alpha : Q \longrightarrow Q'$. Actually, Q' will be constructed as a direct sum: $Q' = \bigoplus\limits_{i \geq 0} Q_i$ such that ${}_bf'_b(Q_{i+1})$ is contained in Q_iK for all $i \geq 0$ and is zero for $i = -1$ and ${}_af'_b$ vanishes on each Q_i, for $i > 0$.

So we need to define finite B-complexes Q_i, a morphism $e : Q_0 \longrightarrow MS$ and morphisms $\alpha_i : Q \longrightarrow Q_i$, $\theta_i : Q_{i+1} \longrightarrow Q_iK$ and $\beta_i : M \longrightarrow Q_iT$ (with $Q_i = 0$ for i big enough) and the morphism f' is defined by:

$${}_af'_a = \lambda \qquad {}_bf'_b = \sum_{i \geq 0} \theta_i \qquad {}_bf'_a = \sum_{i \geq 0} \beta_i \qquad {}_af'_b = e\, pr_0$$

where $pr_0 : Q' \longrightarrow Q_0$ is the projection. The morphism α is equal to $\varphi : P \longrightarrow M$ on P and to $\sum\limits_{i \geq 0} \alpha_i :$ $Q \longrightarrow Q'$ on Q.

But we have two conditions: the fact that α is a morphism and the equality: $\Phi'(X') = Y$. These conditions are equivalent to:

$\varphi\ {}_af_b = e\alpha_0$
$\alpha_i\ {}_bf_a = \beta_i\varphi$
$\theta_i\alpha_{i+1} = \alpha_i\ {}_bf_b$
$\mu_i = e\theta_0\theta_1 \dots \theta_{i-1}\beta_i$
for all $i \geq 0$.

For technical reasons, we introduce the morphism $e_i = e\theta_0\theta_1 \dots \theta_{i-1}$ from Q_i to MSK^i. Then we have to construct, for each $i \geq 0$, the complex Q_i and morphisms $e_i : Q_i \longrightarrow MSK^i$, $\alpha_i : Q \longrightarrow Q_i$, $\beta_i : M \longrightarrow Q_iT$ and $\theta_i : Q_{i+1} \longrightarrow Q_iK$, with the following properties:

$A(i)$: $\varphi\ {}_af_b\big({}_bf_b\big)^i = e_i\alpha_i$

$B(i)$: $\alpha_i\ {}_bf_a = \beta_i\varphi$
$C(i)$: $\theta_i\alpha_{i+1} = \alpha_i\ {}_bf_b$
$D(i)$: $\mu_i = e_i\beta_i$
$E(i)$: $e_{i+1} = e_i\theta_i$
for all $i \geq 0$.

Since the sum of the μ_k's is finite, there is an integer $n > 0$ such that: $\mu_i = 0$ for all $i > n$.

So we'll construct $(Q_i, e_i, \alpha_i, \beta_i, \theta_i)$ by induction. Let $i \geq 0$ be an integer and suppose that $(Q_j, e_j, \alpha_j, \beta_j, \theta_j)$ is defined for all $j > i$ such that the properties $A(j)$, $B(j)$, $C(j)$, $D(j)$, $E(j)$, are satisfied for all $j > i$. We begin this induction with $i = n$ by setting: $Q_j = 0$ for all $j > n$.

We have to construct Q_i and morphisms e_i, α_i, β_i and θ_i.

Consider the following morphisms:

$$\varphi\ {}_af_b\big({}_bf_b\big)^i : Q \longrightarrow MSK^i$$
$$\mu_i : M \longrightarrow MSK^iT$$
$$e_{i+1} : Q_{i+1} \longrightarrow MSK^{i+1}$$

These morphisms induce a morphism $h : Q \oplus M \oplus Q_{i+1} \longrightarrow MSK^i(B \oplus T \oplus K)$ and, because of lemma 3.9, there are a finite complex Q_i, a morphism $e_i : Q_i \longrightarrow MSK^i$ and a commutative diagram:

$$\begin{array}{ccc} Q \oplus M \oplus Q_{i+1} & \xrightarrow{h'} & Q_i(B \oplus T \oplus K) \\ {\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle e_i} \\ Q \oplus M \oplus Q_{i+1} & \xrightarrow{h} & MSK^i(B \oplus T \oplus K) \end{array}$$

The morphism h' induces morphisms:

$$\alpha_i : Q \longrightarrow Q_i$$
$$\beta_i : M \longrightarrow Q_iT$$
$$\theta_i : Q_{i+1} \longrightarrow Q_iK$$

and properties $A(i)$, $D(i)$, $(E(i)$ are satisfied. Denote by u and v the defaults of properties $B(i)$ and $C(i)$:

$$u = \beta_i\varphi - \alpha_i\ {}_bf_a$$
$$v = \alpha_i\ {}_bf_b - \theta_i\alpha_{i+1}$$

Because of properties $A(i)$ and $D(i)$, we have:

$$e_iu = e_i\beta_i\varphi - e_i\alpha_i\ {}_bf_a = \mu_i\varphi - \varphi\ {}_af_b\big({}_bf_b\big)^i\ {}_bf_a = 0$$

and, because of properties $A(i)$, $E(i)$ and $A(i+1)$, we have:

$$e_iv = e_i\alpha_i\ {}_bf_b - e_i\theta_i\alpha_{i+1} = \varphi\ {}_af_b\big({}_bf_b\big)^{i+1} - e_{i+1}\alpha_{i+1} = 0$$

Since a tensor product of bimodules which are flat on the right is flat on the right, we can apply the lemma 3.11 to morphisms $u \oplus v : P \longrightarrow Q_i(T \oplus K)$ and $e_i : Q_i \longrightarrow MSK^i$. Thus e_i factorizes through a finite complex Q'_i by a morphism $\varepsilon : Q_i \longrightarrow Q'_i$ such that: $\varepsilon(u \oplus v) = 0$.

Hence, up to replacing Q_i by Q'_i, we may as well suppose that $A(i)$, $B(i)$, $C(i)$, $D(i)$, $E(i)$ are satisfied. Therefore Q_i, e_i, α_i, β_i, θ_i are defined and $A(i)$, $B(i)$, $C(i)$, $D(i)$, $E(i)$ are satisfied for all $i \geq 0$.

Then the finite complex $Q' = \bigoplus_{i\geq 0} Q_i$ is constructed and the morphism f' is defined by:

$$ {}_af'_a = \lambda $$

$$ {}_bf'_b = \bigoplus_{i\geq 0} \theta_i $$

$$ {}_af'_b = e_0 pr_0 $$

$$ {}_bf'_a = \bigoplus_{i\geq 0} \beta_i $$

Hence the desired object X' is constructed and Φ' has the approximation property (App2). The lemma follows and then follow lemma 3.1 and theorems 4, 5 and 6. $\square$

3.13. Proof of lemma 3.9

The situation is the following: (A, S) is a left-flat bimodule, X is a finite A-complex, Y is an A-complex and $f : X \longrightarrow YS$ is a morphism of A-complexes. We want to construct a finite A-complex Y' and a morphism $Y' \longrightarrow Y$ such that $f : X \longrightarrow YS$ factorizes through $Y'S$.

For each integer n, denote by $X(n)$ the n-skeleton of X. Let n be an integer. Suppose the n-skeleton $Y'(n)$ of Y' is constructed in such a way that we have a morphism $g_n : Y'(n) \longrightarrow Y$ and a commutative diagram:

$$\begin{tikzcd} X(n) \arrow[r, "h_n"] \arrow[d, "="'] & Y'(n)S \arrow[d, "g_n"'] \\ X(n) \arrow[r, "f"] & YS \end{tikzcd} \qquad (D_n)$$

If n is small enough, $X(n)$ is null and $Y'(n)$ can be chosen to be zero.

Denote by Z_n the kernel of the morphism $d : Y'_n \longrightarrow Y'_{n-1}$ and by U_{n+1} the module defined by the cartesian square:

$$\begin{tikzcd} U_{n+1} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & Y_{n+1} \arrow[d, "d"'] \\ Z_n \arrow[r, "g_n"] & Y_b \end{tikzcd}$$

The composite morphism $X_{n+1} \xrightarrow{d} X_n \xrightarrow{h_n} Y'_nS$ takes values in Z_nS and induces, together with the morphism $f : X_{n+1} \longrightarrow Y_{n+1}S$, a well defined morphism $\lambda : X_{n+1} \longrightarrow U_{n+1}$. So we get a commutative diagram:

$$\begin{tikzcd} X_{n+1} \arrow[r, "\lambda"] \arrow[d, "d"'] & U_{n+1}S \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & Y_{n+1}S \arrow[d, "d"'] \\ X_n \arrow[r, "h_n"] & Y'_nS \arrow[r, "g_n"] & Y_nS \end{tikzcd}$$

where γ is the composite morphism $U_{n+1} \xrightarrow{\beta} Z_n \subset Y'_n$.

Since X_{n+1} is finitely generated, there is a finitely generated submodule M in U_{n+1} such that $\lambda(X_{n+1})$ is contained in MS. Let Y'_{n+1} be a finitely generated projective A-module and $\mu : Y'_{n+1} \longrightarrow M$ be an epimorphism. Since X_{n+1} is projective, the morphism $\lambda : X_{n+1} \longrightarrow Y'S$ can be lifted in a morphism $X_{n+1} \longrightarrow Y'_{n+1}S$ and we get a commutative diagram:

$$\begin{tikzcd}
X_{n+1} \arrow[r, "h"] \arrow[d, "d"'] & Y'_{n+1}S \arrow[r, "g"] \arrow[d, "d"'] & Y_{n+1} \arrow[d, "d"'] \\
X_n \arrow[r] & Y'_nS \arrow[r] & Y_nS
\end{tikzcd}$$

where $d : Y'_{n+1} \longrightarrow Y'_n$ is the morphism $\gamma\mu$ and $g : Y'_{n+1} \longrightarrow Y_{n+1}$ is the morphism $\alpha\mu$.

Thus we have constructed the complex $Y'(n+1)$ and the commutative diagram (D_{n+1}). By induction we have $Y'(n)$ and the commutative diagram (D_n) for every integer n and, for n big enough, $Y'(n)$ and the diagram (D_n) is a solution of the problem. $\square$

3.14. Proof of lemma 3.10

In the lemma, $f : X \longrightarrow Y$ is a morphism between two A-complexes, the direct sum of the $X'_n s$ if finitely presented and each Y_n is flat.

For every A-complex E, denote by E^n its n-coskeleton i.e. the quotient of E by its $(n-1)$–skeleton.

Let n be an integer. Suppose that the n-coskeleton F^n of a finite complex F is constructed in such a way that the morphism $f : X^n \longrightarrow Y^n$ induced by $f : X \longrightarrow Y$ factorizes through F^n via two morphisms $\alpha : X^n \longrightarrow F^n$ and $\beta : F^n \longrightarrow Y^n$. If n is big enough X^n is trivial and we may set: $F^n = 0$.

We have a commutative diagram:

$$\begin{tikzcd}
X_{n+1} \arrow[r, "\alpha"] \arrow[d, "d"'] & F_{n+1} \arrow[r, "\beta"] \arrow[d, "d"'] & Y_{n+1} \arrow[d, "d"'] \\
X_n \arrow[r, "\alpha"] & F_n \arrow[r, "\beta"] & Y_n
\end{tikzcd}$$

Let E be the A-module defined by the cocartesian square:

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d, "d"'] & F'_n \arrow[d] \\
X_{n-1} \arrow[r] & E
\end{tikzcd}$$

where F'_n is the cokernel of the morphism $d : F_{n+1} \longrightarrow F_n$. Since X_n, X_{n-1} and F'_n are finitely presented, the A-module E is also finitely presented.

We have a commutative diagram:

$$\begin{tikzcd}
X_{n+1} \arrow[r, "\alpha"] \arrow[d, "d"'] & F_{n+1} \arrow[r, "\beta"] \arrow[d, "d"'] & Y_{n+1} \arrow[d, "d"'] \\
X_n \arrow[r, "\alpha"] \arrow[d, "d"'] & F_n \arrow[r, "\beta"] \arrow[d, "\delta"'] & Y_n \arrow[d, "d"'] \\
X_{n-1} \arrow[r] & E \arrow[r] & Y_{n-1}
\end{tikzcd}$$

where the composite morphism $F_{n+1} \xrightarrow{d} F_n \longrightarrow E$ is trivial.

But E is finitely presented and Y_{n-1} is flat. Therefore the morphism $E \longrightarrow Y_{n-1}$ factorizes through a finitely generated projective A-module F_{n-1} and, together with the composite morphism $F_n \xrightarrow{\delta} E \longrightarrow F_{n-1}$, we get the desired finite complex F^{n-1} and a commutative diagram:

$$\begin{array}{ccccc} X^{n-1} & \xrightarrow{\alpha} & F^{n-1} & \xrightarrow{\beta} & Y^{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ X^n & \xrightarrow{\alpha} & F^n & \xrightarrow{\beta} & Y^n \end{array}$$

So we construct the complexes F^n inductively and, for n small enough, the morphism $f : X \longrightarrow Y$ factorizes through the finite complex $F = F^n$. □

Remark. It is not clear that the right flatness condition is necessary in theorems 4, 5 and 6. Actually this condition is only used in order to prove that the functor $\Phi' : \mathscr{E}' \longrightarrow \mathscr{N}il(A, \widehat{E})_*$ (in the proof of lemma 3.1) is a homotopy equivalence. The proof given here needs the right flatness condition (in the lemma 3.10) but another proof without this condition is still possible.

4. Whitehead spectra

If E is an Ω-spectrum and X is a space, we denote by $H(X, E)$ the Ω-spectrum associated to the smash product $X \wedge E$. For every $i \in \mathbf{Z}$, we have:

$$\pi_i(H(X, E)) \simeq H_i(X, E)$$

In [8], section 15, Waldhausen associates to any ring R and any group G an assembly map: $H(BG, K(R)) \longrightarrow K(R[G])$ which is a map of infinite loop spaces. This assembly map induces assembly maps $H(BG, \Sigma^n K(R)) \longrightarrow K(\Sigma^n R[G])$ and then an assembly map $h : H(BG, \underline{K}(R)) \longrightarrow \underline{K}(R[G])$ which is a map of spectra. So we get a fibration of spectra:

$$H(BG, \underline{K}(R)) \xrightarrow{h} \underline{K}(R[G]) \longrightarrow \underline{Wh}^R(G)$$

The spectrum $\underline{Wh}^R(G)$ is called the Whitehead spectrum of G relative to R.

For every integer i, we set:

$$Wh_i(G) = \pi_i(\underline{Wh}^{\mathbf{Z}}(G))$$

For $i < 0$, the group $Wh_i(G)$ is isomorphic to $K_i(Z[G])$ and we have exact sequences:

$$0 \longrightarrow \mathbf{Z} \longrightarrow K_0(Z[G]) \longrightarrow Wh_0(G) \longrightarrow 0$$

$$0 \longrightarrow \mathbf{Z}/2 \oplus H_1(G, \mathbf{Z}) \longrightarrow K_1(\mathbf{Z}[G]) \longrightarrow Wh_1(G) \longrightarrow 0$$

More precisely, $Wh_0(G)$ is the reduced K_0-group of $\mathbf{Z}[G]$, $Wh_1(G)$ is the classical Whitehead group of G and $Wh_2(G)$ is the second Whitehead group of G as defined in [3].

Following Waldhausen, a group G is said to be regular noetherian (resp. regular coherent) if, for every ring R which is regular noetherian on the right, $R[G]$ is regular noetherian (resp. regular coherent) on the right. Since $R^{op}[G]$ is isomorphic to $(R[G])^{op}$, this condition is equivalent to the condition obtained by replacing right by left. We denote also by $\mathscr{G}$ the category of groups and monomorphisms of groups.

Proposition 4.1. *We have the following properties:*

- *If G is the amalgamated free product of a diagram in $\mathscr{G}$:*

$$\begin{array}{ccc} H & \longrightarrow & G_1 \\ \Big\downarrow & & \\ G_2 & & \end{array}$$

where G_1 and G_2 are regular coherent and H regular noetherian, then G is regular coherent.

- *If G is the HNN extension of a diagram in $\mathscr{G}$:*

$$H \underset{\beta}{\overset{\alpha}{\rightrightarrows}} G_1$$

where G_1 is regular coherent and H is regular noetherian, then G is regular coherent.

- *If G is the colimit of a filtered system G_i in $\mathscr{G}$, where each G_i is regular coherent, then G is regular coherent.*
- *A subgroup of a regular coherent group is regular coherent.*

Proof. All these properties are proven in [8] (in theorem 19.1) except the third one.

Let G be the colimit of a filtered system G_i in $\mathscr{G}$ and R be a ring which is regular noetherian on the right. Suppose each G_i is regular coherent. Set: $A = R[G]$ and $A_i = R[G_i]$.

Each ring A_i is regular coherent on the right and for each $i \in I$, the ring A is free on the left over A_i.

Let M be a finitely presented right A-module. We have an exact sequence of right A-modules:

$$F_1 \xrightarrow{f} F_0 \longrightarrow M \longrightarrow 0$$

where F_0 and F_1 are finitely generated free A-modules. The morphism f is represented by a finite matrix with entries in A. Since A is the colimit of the A_i's, there is an element $i \in I$ such that A_i contains all the entries of f. Therefore f comes from a finite matrix with entries in A_i and there exist a finitely presented right A_i-module M' and an isomorphism $M' \otimes_{A_i} A \simeq M$.

Since A_i is regular coherent on the right we have an exact sequence of right A_i-modules:

$$0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow M' \longrightarrow 0$$

where each C_k is finitely generated projective. Since A is free on the left over A_i, we have an exact sequence of right A-modules:

$$0 \longrightarrow C_n \underset{A_i}{\otimes} A \longrightarrow C_{n-1} \underset{A_i}{\otimes} A \longrightarrow \cdots \longrightarrow C_0 \underset{A_i}{\otimes} A \longrightarrow M \longrightarrow 0$$

But each $C_k \underset{A_i}{\otimes} A$ is finitely generated projective right A-module. Therefore every finitely presented A-module has a finite resolution by finitely generated projective A-modules and A is regular coherent. The result follows. □

Let Cl be the class of groups defined by Waldhausen in [8]. This class is the smallest class of groups satisfying the following:

- The trivial group belongs to Cl.

• If G is the amalgamated free product of a diagram in $\mathscr{G}$:

$$\begin{tikzcd} H \arrow[r] \arrow[d] & G_1 \\ G_2 & \end{tikzcd}$$

where G_1 and G_2 are in Cl and H regular coherent, then G belongs to Cl.

• If G is the HNN extension of a diagram in $\mathscr{G}$:

$$H \overset{\alpha}{\underset{\beta}{\rightrightarrows}} G_1$$

where G_1 is in Cl and H is regular coherent, then G belongs to Cl.

• If G is the colimit of a filtered system G_i in $\mathscr{G}$, where each G_i is in Cl, then G belongs to Cl. This class contains free groups, torsion free abelian groups, poly-**Z**-groups, torsion free one-relator groups and fundamental groups of many low-dimensional manifolds. It is also closed under taking subgroups. See theorem 19.5 in [8].

Theorem 4.2. *For every group G in Cl and every ring R which is regular noetherian on the right, the Whitehead spectrum $\underline{W}h^R(G)$ is contractible.*

Proof. This is essentially theorem 19.4 in [8]. We just have to replace spaces $Nil(A, S)$ by spectra $\underline{N}il(A, S)$. Since all these spectra are contractible, the result follows. □

We'll construct a class of groups Cl_1 obtained by replacing the condition "H is regular coherent" (in the definition of Cl) by a weaker condition in such a way that theorem 4.2 is still true for groups in Cl_1. Consider a diagram of groups:

$$\begin{tikzcd} H \arrow[r, "\alpha"] \arrow[d, "\beta"'] & G_1 \\ G_2 & \end{tikzcd}$$

We say that this diagram is regular coherent if the following holds:

• α and β are monomorphisms

• for every $x \in G_1 \setminus \alpha(H)$ and every $y \in G_2 \setminus \beta(H)$, the intersection of the two groups $\alpha^{-1}(x\alpha(H)x^{-1})$ and $\beta^{-1}(y\beta(H)y^{-1})$ is regular coherent.

Consider a diagram of groups:

$$H \overset{\alpha}{\underset{\beta}{\rightrightarrows}} G_1$$

We say that this diagram is regular coherent if the following holds:

• α and β are monomorphisms

• for every $x \in G_1 \setminus \alpha(H)$ and every $y \in G_1 \setminus \beta(H)$, the intersection of the two groups $\alpha^{-1}(x\alpha(H)x^{-1})$ and $\beta^{-1}(y\beta(H)y^{-1})$ is regular coherent.

• for every $x \in G_1$, the group $\beta^{-1}(x\alpha(H)x^{-1})$ is regular coherent.

Since the condition "regular coherent" is stable under taking subgroups, it is easy to see that diagrams above are regular coherent if the subgroup H is regular coherent. So we define the class Cl_1 as the smallest class of groups satisfying the following:

- The trivial group belongs to Cl_1.
- If G is the amalgamated free product of a diagram D:

$$\begin{tikzcd} H \arrow[r] \arrow[d] & G_1 \\ G_2 & \end{tikzcd}$$

where G_1 and G_2 are in Cl_1 and D is regular coherent, then G belongs to Cl_1.
- If G is the HNN extension of a diagram D':

$$\begin{tikzcd} H \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & G_1 \end{tikzcd}$$

where G_1 is in Cl_1 and D' is regular coherent, then G belongs to Cl_1.
- If G is the colimit of a filtered system G_i in $\mathscr{G}$, where each G_i is in Cl_1, then G belongs to Cl_1.

Theorem 4.3. *Let R be a ring which is regular noetherian on the right and G be the amalgamated free product of a regular coherent diagram of groups:*

$$\begin{tikzcd} H \arrow[r, "\alpha"] \arrow[d, "\beta"'] & G_1 \\ G_2 & \end{tikzcd}$$

Then this diagram induces a homotopically cartesian diagram of spectra:

$$\begin{tikzcd} \underline{W}h^R(H) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \underline{W}h^R(G_1) \arrow[d] \\ \underline{W}h^R(G_2) \arrow[r] & \underline{W}h^R(G) \end{tikzcd}$$

Theorem 4.4. *Let R be a ring which is regular noetherian on the right and G be the HNN extension of a regular coherent diagram of groups:*

$$\begin{tikzcd} H \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & G_1 \end{tikzcd}$$

Then this diagram induces a homotopy fibration of spectra:

$$\underline{W}h^R(H) \xrightarrow{f} \underline{W}h^R(G_1) \longrightarrow \underline{W}h^R(G)$$

where f is the difference (in Ωsp) of maps induced by α and β.

Proofs of theorems 4.3 and 4.4. In the amalgamated case, we have a commutative diagram of spectra:

$$\begin{tikzcd} H(BH, \underline{K}(R)) \arrow[r, "\alpha\oplus-\beta"] \arrow[d] & H(BG_1, \underline{K}(R)) \oplus H(BG_2, \underline{K}(R)) \arrow[r] \arrow[d] & H(BG, \underline{K}(R)) \arrow[d] \\ \underline{K}(R[H]) \arrow[r, "\alpha\oplus-\beta"] & \underline{K}(R[G_1]) \oplus \underline{K}(R[G_2]) \arrow[r] & \underline{K}(R[G])' \end{tikzcd}$$

where horizontal lines are fibrations and vertical maps are assembly maps. Moreover theorem 2 implies a homotopy equivalence:

$$\underline{K}(R[G]) \simeq \underline{K}(R[G])' \oplus \Omega^{-1}\underline{N}il(R[H] \times R[H], S)$$

for some $R[H] \times R[H]$ bimodule S. Therefore we have a fibration:

$$\underline{W}h^R(H) \longrightarrow \underline{W}h^R(G_1) \oplus \underline{W}h^R(G_2) \longrightarrow \underline{W}h^R(G)'$$

and a homotopy equivalence:

$$\underline{W}h^R(G) \simeq \underline{W}h^R(G)' \oplus \Omega^{-1}\underline{N}il(R[H] \times R[H], S)$$

We can do the same for the HNN extension and we get a fibration:

$$\underline{W}h^R(H) \xrightarrow{f} \underline{W}h^R(G_1) \longrightarrow \underline{W}h^R(G)'$$

and a homotopy equivalence:

$$\underline{W}h^R(G) \simeq \underline{W}h^R(G)' \oplus \Omega^{-1}\underline{N}il(R[H] \times R[H], S)$$

for some $R[H] \times R[H]$ bimodule S.

Hence the only thing to do is to prove that $\underline{N}il(R[H]\times R[H], S)$ is contractible. Let's denote by C the ring $R[H]$. With the notations of 1.4, the bimodule S is determined by four C-bimodules ${}_iS_j$ (with $i, j \in \{1, 2\}$). In order to describe these bimodules we'll introduce the following terminology:

Let G be a group. A G-biset is a set X equipped with two compatible actions of G, one on the left and the other one on the right. We say that a G-biset X is free if both actions on X are free. If R is a ring, $R[X]$ is naturally a $R[G]$-bimodule and, if X is free, $R[X]$ is free on both sides. For any free G-biset X and any ring R, we set:

$$\underline{N}il^R(G, X) = \underline{N}il(R[G], R[X])$$

If G_1 and G_2 are two groups, we can also define a (G_1, G_2)-biset as a set equipped with two compatible actions: a left action of G_1 and a right action of G_2. Then, for every ring R and any (G_1, G_2)-biset X, $R[X]$ is a $(R[G_1], R[G_2])$-bimodule. Consider the amalgamated case. We have two monomorphisms $\alpha : H \longrightarrow G_1$ and $\beta : H \longrightarrow G_2$. Denote by X the complement of $\alpha(H)$ in G_1 and by Y the complement of $\beta(H)$ in G_2. The group H acts on both sides on X and Y and X and Y are free H-bisets. Moreover we have:

$${}_2S_1 = R[X] \qquad {}_1S_2 = R[Y] \qquad {}_1S_1 = {}_2S_2 = 0$$

and then:

$$S = {}^2R[X]^1 \oplus {}^1R[Y]^2$$

In the HNN extension case we have two monomorphisms $\alpha : H \longrightarrow G_1$ and $\beta : H \longrightarrow G_1$. Denote by X the complement of $\alpha(H)$ in G_1 and by Y the complement of $\beta(H)$ in G_1. Denote also by U (resp. V) the set G_1 where H acts on the left by β and on the right by α (resp. H acts on the right by β and on the left by α). These sets X, Y, U and V are free H-bisets. In this case, the bimodule S is characterized by the conditions:

$$ {}_2S_1 = R[X] \qquad {}_1S_2 = R[Y] \qquad {}_1S_1 = R[U] \qquad {}_2S_2 = R[V] $$

and then:

$$ S = {}^2R[X]^1 \oplus {}^1R[Y]^2 \oplus {}^1R[U]^1 \oplus {}^2R[V]^2 $$

Consider the HNN extension case. For every $x \in G_1$ we set:

$$ \Gamma(x) = \beta^{-1}(x\alpha(H)x^{-1}) \qquad \Gamma'(x) = \alpha^{-1}(x\beta(H)x^{-1}) $$

For each $x \in G_1$, $\Gamma(x)$ and $\Gamma'(x)$ are subgroups of H and $\Gamma(x)$ is regular coherent. Moreover, we have a group homomorphism $\lambda_x : \Gamma(x) \longrightarrow H$ such that:

$$ \forall \gamma \in \Gamma(x), \quad \beta(\gamma) = x\alpha(\lambda_x(\gamma))x^{-1} $$

It is easy to see that λ_x is an isomorphism from $\Gamma(x)$ to $\Gamma'(x^{-1})$. Thus groups $\Gamma'(x)$ are also regular coherent.

Let:

$$ U = \coprod_i U_i $$

be the decomposition of U by orbits. Then, for every i, there exists an element $x \in U$ such that:

$$ U_i = \beta(H)x\alpha(H) $$

Let H_1 (resp. ${}_xH$) be the $(H, \Gamma(x))$-biset (resp. the $(\Gamma(x), H)$-biset) H, where H acts in the standard way on the left (resp. on the right) and $\Gamma(x)$ acts by the inclusion on the right (resp. by the morphism λ_x on the left). Then the map: $(u, v) \mapsto \beta(u)x\alpha(v)$ from $H \times H$ to $\beta(H)x\alpha(H)$ induces an isomorphism of H-bisets:

$$ H_1 \underset{\Gamma(x)}{\times} {}_xH \simeq \beta(H)x\alpha(H) $$

and we have:

$$ \begin{gathered} R[U_i] \simeq R[H_1 \underset{\Gamma(x)}{\times} {}_xH] \\ \Longrightarrow {}^1R[U_i]^1 \simeq {}^1R[H_1] \underset{R[\Gamma(x)]}{\otimes} R[{}_xH]^1 \end{gathered} $$

Moreover the ring $R[\Gamma(x)]$ is regular coherent.

Hence, because of theorem 6, we have a homotopy equivalence of spectra:

$$ \begin{aligned} &\underline{N}il(C \times C, {}^2R[X]^1 \oplus {}^1R[Y]^2 \oplus {}^2R[V]^2) \xrightarrow{\sim} \\ &\qquad \underline{N}il(C \times C, {}^2R[X]^1 \oplus {}^1R[Y]^2 \oplus {}^1R[U]^1 \oplus {}^2R[V]^2) \end{aligned} $$

We proceed the same with the biset V and we get a homotopy equivalence of spectra:

$$ \underline{N}il(C \times C, {}^2R[X]^1 \oplus {}^1R[Y]^2) \xrightarrow{\sim} \underline{N}il(C \times C, {}^2R[X]^1 \oplus {}^1R[Y]^2 \oplus {}^2R[V]^2) $$

Hence, in both amalgamated case and HNN extension case, we have a homotopy equivalence:

$$ \underline{N}il(C \times C, {}^2R[X]^1 \oplus {}^1R[Y]^2)) \xrightarrow{\sim} \underline{N}il(C \times C, S) $$

and, because of theorem 4, we have a homotopy equivalence:

$$\underline{N}il(C \times C, S) \simeq \underline{N}il(C, R[X \underset{H}{\times} Y])$$

Denote by Z_j the orbits of the biset $X \underset{X}{\times} Y$. Then, for each j, there is an element $(x, y) \in X \times Y$ such that:

$$Z_j = \alpha(H)xy\beta(H)$$

For each $x \in X$ and each $y \in Y$ we have the groups:

$$\Gamma_1(x) = \alpha^{-1}(x\alpha(H)x^{-1}) \qquad \Gamma_2(y) = \beta^{-1}(y\beta(H)y^{-1}) \qquad H(x, y) = \Gamma_1(x) \cap \Gamma_2(y)$$

We have group homomorphisms $\lambda_x : \Gamma_1(x) \longrightarrow H$ and $\mu_y : \Gamma_2(y) \longrightarrow H$ defined by:

$$\forall \gamma \in \Gamma_1(x), \quad \alpha(\lambda_x(\gamma)) = x\alpha(\gamma)x^{-1}$$
$$\forall \gamma \in \Gamma_2(y), \quad \beta(\gamma) = y\beta(\mu_y(\gamma))y^{-1}$$

Denote by H_x the $(H, H(x, y))$-biset H where H acts in the standard way on the left and $H(x, y)$ acts via λ_x on the right. Denote also by ${}_yH$ the $(H(x, y), H)$-biset where H acts in the standard way on the right and $H(x, y)$ acts via μ_y in the left. Then the map $(u, v) \mapsto \alpha(u)xy\beta(v)$ from $H \times H$ to $\alpha(H)xy\beta(H)$ induces an isomorphism:

$$H_x \underset{H(x,y)}{\times} {}_yH \xrightarrow{\sim} \alpha(H)xy\beta(H)$$

where $H(x, y)$ is regular coherent. Hence, because of theorem 6, the spectrum $\underline{N}il(R[H], R[X \underset{H}{\times} Y])$ is contractible and so is $\underline{N}il(R[H] \times R[H], S)$. $\square$

Theorem 4.5. *For every group G in Cl_1 and every ring R which is regular noetherian on the right, the Whitehead spectrum $\underline{W}h^R(G)$ is contractible.*

Proof. Denote by Cl_2 the class of groups G such that $\underline{W}h^R(G)$ is contractible for every ring R which is regular noetherian on the right. Because of theorem 4.2, Cl_2 contains the class Cl.

Since the functor $\underline{W}h$ commutes with filtered colimits, the class Cl_2 is stable under filtered colimits. Therefore it's enough to prove that Cl_2 is stable under taking amalgamated free products and HNN extensions of regular coherent diagrams. But that follows directly from theorems 4.3 and 4.4. $\square$

Example 4.6. Consider the group H with two generators x and t and the following relations:

$$\forall n \in \mathbf{Z}, \quad xt^nxt^{-n} = t^nxt^{-n}x$$

For every integer $p \neq 0$, the correspondence $x \mapsto x$ and $t \mapsto t^p$ induces a monomorphism $f_p : H \longrightarrow H$. Consider a monomorphism of groups $\alpha : H \longrightarrow G$ and denote by Γ the amalgamated free product of the diagram:

$$\begin{array}{ccc} H & \xrightarrow{f_p} & H \\ {\scriptstyle\alpha}\downarrow & & \\ G & & \end{array}$$

Proposition 4.7. *For every ring R which is regular noetherian on the right, the morphism $G \longrightarrow \Gamma$ induces a homotopy equivalence of spectra: $\underline{W}h^R(G) \xrightarrow{\sim} \underline{W}h^R(\Gamma)$.*

Moreover, if G belongs to Cl_1, then the group Γ is also in Cl_1.

Proof. Denote by H' the normal closure of x in H. This group is commutative and freely generated by the elements: $x_n = t^n x t^{-n}$ for $n \in \mathbf{Z}$. The correspondence. $x_n \mapsto x_{n+1}$ is an automorphism $\tau : H' \xrightarrow{\sim} H'$ and H is the semidirect product of H' and $\mathbf{Z}$, or equivalently, the HNN extension of H' with morphisms Id, $\tau : H' \longrightarrow H'$. On the other hand, $R[H']$ is regular coherent on the right (but not noetherian) and H belongs to the class Cl. Hence the Whitehead spectrum $\underline{W}h^R(H)$ is contractible.

Denote by H_p the image of $f_p : H \longrightarrow H$ and by X its complement in H. For every $z \in H$, denote by $\Gamma(z)$ the subgroup $f_p^{-1}(zf_p(H)z^{-1})$ of H. We have the following formula:

$$\Gamma(f_p(a)zf_p(b)) = a\Gamma(z)a^{-1}$$

for every a, b, z in H and the conjugacy class of $\Gamma(z)$ depends only on the class of z in the set $Y = H_p\backslash H/H_p$. Let z be an element in X. A direct computation shows the following:

- if z is congruent in Y to an element in H' then $\Gamma(z)$ is the group H'
- if z is congruent in Y to a power of t then $\Gamma(z)$ is conjugate to the subgroup of H generated by t
- in the other cases $\Gamma(z)$ is the trivial group.

Therefore $\Gamma(z)$ is always a free abelian group. Hence, for every y in $G \setminus \alpha(H)$, the group $\Gamma(z, y) = \Gamma(z) \cap \alpha^{-1}(y\alpha(H)y^{-1})$ is also a free abelian group and the ring $R[\Gamma(z, y)]$ is regular coherent on the right. Then theorem 4.3 applies and the result follows. □

The class Cl_1 seems to be strictly bigger than the class Cl. For example the amalgamated free product Γ of the diagram:

$$\begin{array}{ccc} H & \xrightarrow{f_p} & H \\ {\scriptstyle f_q}\big\downarrow & & \\ H & & \end{array}$$

with $p, q > 1$, belongs to the class Cl_1. But in this case, Waldhausen's theorems cannot be used to prove that Γ belongs to the class Cl because of the following result:

Proposition 4.8. *The ring $\mathbf{Z}[H]$ is not regular coherent.*

Proof. Let $f : \mathbf{Z}[H] \oplus \mathbf{Z}[H] \longrightarrow \mathbf{Z}[H]$ be the following morphism:

$$(U, V) \mapsto f(U, V) = (1 - t + tx)U - (1 - t + t^2xt^{-1})V$$

and K be its kernel. We'll prove that K is not finitely generated and that will imply that $\mathbf{Z}[H]$ is not coherent and therefore not regular coherent.

Denote by A the ring $\mathbf{Z}[H']$. This ring is the ring of Laurent polynomials in the x_i's. Then A is an integral domain and every element $u \in \mathbf{Z}[H]$ can be written in a unique way on a finite sum:

$$u = \sum_{i \in \mathbf{Z}} t^i u_i$$

with each u_i in A. So u may be considered as a Laurent polynomial in t and has a valuation $\nu(u)$ and a degree $\partial^\circ u$ (at least if u is not zero). If $u = 0$, we set: $\nu(u) = +\infty$ and $\partial^\circ u = -\infty$.

Define the elements y_i and z_i in A by:

$$\forall i \in \mathbf{Z}, \quad y_i = 1 - x_i = 1 - t^i x t^{-i} \quad z_i = y_i - y_{i-1} = x_{i-1} - x_i$$

and, for every integer $n \geq 0$, we have the following elements in $\mathbf{Z}[H]$:

$$U_n = z_{-n} - t^{n+1} z_1 y_0 y_{-1} \dots y_{-n}$$

$$V_n = z_{-n} + \sum_{0<i\leq n} t^i z_{-n} z_1 y_0 y_{-1} \dots y_{2-i} - t^{n+1} z_1 y_0 y_{-1} \dots y_{1-n} y_{-1-n}$$

An explicit computation shows that, for each $n \geq 0$, $W_n = (U_n, V_n)$ is killed by f and each W_n belongs to K. $\square$

Lemma 4.9. *The set $\{W_0, W_1, W_2, \dots\}$ is a generating set of K.*

Proof. Denote by E the $\mathbf{Z}[H]$-submodule of $\mathbf{Z}[H] \oplus \mathbf{Z}[H]$ generated by the set $\{W_0, W_1, W_2, \dots\}$. Since each W_n is in K, we have an inclusion $E \subset K$ and we have to prove that this inclusion is an equality.

For each integer $n \geq 0$, denote by K_n the set of the elements (U, V) in K satisfying the following:

$$\nu(U) \geq 0 \qquad \partial^\circ U \leq n \qquad\qquad \nu(V) \geq 0 \qquad \partial^\circ V \leq n$$

We denote also by I_n the ideal $(z_0, z_{-1}, z_{-2}, \dots, z_{1-n}) \subset A$ and by J_n the right ideal of $\mathbf{Z}[H]$ generated by I_n.

The quotient $B_n = A/I_n$ is the quotient of A by the relations

$$x_0 = x_{-1} = x_{-2} = \dots = x_{-n}$$

and B_n is a Laurent polynomial ring where z_1, z_{-n} and all y_i are not zero.

Suppose we have proven that K_{n-1} is contained in E. Let $U = \sum_{0\leq i\leq n} t^i u_i$ and $V = \sum_{0\leq i\leq n} t^i v_i$ be two elements in $\mathbf{Z}[H]$, with u_i and v_i in A. For $W = (U, V)$ we have the following equivalences:

$$W \in K_n \iff (1 - ty_0)U = (1 - ty_1)V$$

$$\iff \sum_i t^i (u_i - v_i) = ty_0 \sum_i t^i u_i - ty_1 \sum_i t^i v_i$$

$$\iff \sum_i t^i (u_i - v_i) = \sum_i t^{i+1} y_{-i} u_i - \sum_i t^{i+1} y_{1-i} v_i$$

$$\iff \forall i, \; u_i - v_i = y_{1-i} u_{i-1} - y_{2-i} v_{i-1}$$

And these conditions are equivalent to the following:

$$v_0 = u_0$$

$$v_1 = u_1 + z_1 u_0$$

$$v_2 = u_2 + z_0 u_1 + z_1 y_0 u_0$$

$$\dots$$

$$v_n = u_n + \sum_{0\leq i<n} z_{1-i} y_{-i} y_{-1-i} \dots y_{2-n} u_i$$

$$0 = \sum_{0 \le i \le n} z_{1-i} y_{-i} y_{-1-i} \dots y_{1-n} u_i$$

This last relation implies the following relation in B_n:

$$z_1 y_0 y_{-1} \dots y_{1-n} u_0 \equiv 0 \ \in \ B_n$$

and we have: $u_0 \equiv 0$ in B_n because B_n is an integral domain.

Suppose W is in K_n. Then u_0 is in I_n and there are elements $a_0, a_1, \dots a_{n-1}$ in A such that:

$$u_0 = z_0 a_0 + z_{-1} a_1 + \dots + z_{1-n} a_{n-1}$$

Set:

$$W' = W - (W_0 a_0 + W_1 a_1 + W_2 a_2 + \dots W_{n-1} a_{n-1})$$

Since $W_0, W_1, \dots, W_{n-1}$ are in K_n, W' belongs to K_n and there exist elements u'_i and v'_i in A such that:

$$W' = (\sum_{0 \le i \le n} t^i u'_i, \sum_{0 \le i \le n} t^i v'_i)$$

Moreover, because W' is in K_n, we have $u'_0 = v'_0$ and:

$$u'_0 = u_0 - (z_0 a_0 + z_{-1} a_1 + \dots + z_{1-n} a_{n-1}) = 0$$

So we have: $u'_0 = v'_0 = 0$ and $W' t^{-1}$ belongs to $K_{n-1} \subset E$. Therefore W' and then W belong to E and we have: $K_n \subset E$.

Thus K_n is contained in E for every $n \ge 0$. On the other hand, for every $W \in K$, there is some integer p such that Wt^p belongs to some K_n. Hence K is contained in E and the lemma is proven. □

Lemma 4.10. *The module K is not finitely generated.*

Proof. For each integer $n > 0$, denote by E_n the submodule of E generated by $\{W_0, W_1, W_2, \dots, W_{n-1}\}$.

Let $F_n : \mathbf{Z}[H]^n \longrightarrow \mathbf{Z}[H] \oplus \mathbf{Z}[H]$ be the morphism:

$$(c_0, c_1, c_2, \dots, c_{n-1}) \mapsto \sum_{0 \le i < n} W_i c_i$$

The image if F_n is the module E_n. Denote by R_n the kernel of F_n and by π_n the last projection $\mathbf{Z}[H]^n \longrightarrow \mathbf{Z}[H]$. We have an exact sequence of right $\mathbf{Z}[H]$-modules:

$$0 \longrightarrow R_n \longrightarrow \mathbf{Z}[H]^n \xrightarrow{F_n} E_n \longrightarrow 0$$

Hence we have a commutative diagram of right $\mathbf{Z}[H]$-modules with exact lines:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}[H]^{n-1} \arrow[r] \arrow[d, "F_{n-1}"'] & \mathbf{Z}[H]^n \arrow[r, "\pi_n"] \arrow[d, "F_n"'] & \mathbf{Z}[H] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E_{n-1} \arrow[r] & E_n \arrow[r] & \mathbf{Z}[H]/J'_n \arrow[r] & 0
\end{tikzcd}$$

with: $J'_n = \pi_n(R_n)$. We'll prove that $\mathbf{Z}[H]/J'_n$ is not trivial.

An easy computation shows that, for every integers p, q with $0 \le p < q$, the following element:

$$W_q z_{-p} - W_p z_{-q} - W_{q-p-1} t^{p+1} z_1 y_0 y_{-1} \dots y_{-p}$$

is zero in K and induces a well defined element $X(p,q) \in R_n$ (for every $n > q$) and, for every p with $0 \le p < n-1$, we have: $\pi_n(X(p, n-1)) = z_{-p}$.

Let $n > 0$ be an integer. Suppose J'_n is not contained in J_n. Then there is an element $X_0 \in R_n$ such that $\pi_n(X_0)$ doesn't belong to J_n. Set: $d = \partial^\circ \pi_n(X_0)$ and denote by Z the set of $X \in R_n$ such that: $\partial^\circ \pi_n(X) = d$ and $\pi_n(X) - \pi_n(X_0) \in J_n$.

For each $X = (c_0, c_1, \dots, c_{n-1}) \in Z$ we can associate three integers α, β, γ defined this way:

- $\alpha = \nu(c_{n-1})$
- β is the lowest $\nu(c_k)$, for $k = 0, 1, \dots, n-1$
- γ is the highest integer k such that $\nu(c_k) = \beta$.

The triple $\chi(X) = (\alpha, \beta, \gamma)$ will be called the complexity of X. This complexity belongs to the set C of triple $(\alpha, \beta, \gamma) \in \mathbf{Z}^3$ satisfying the following conditions:

$$\beta \le \alpha \le d \quad \text{and} \quad 0 \le \gamma < n$$

The lexicographical order of $(d - \alpha, \alpha - \beta, \gamma)$ induces a well order relation on C and we have:

$$(\alpha, \beta, \gamma) < (\alpha', \beta', \gamma') \Longleftrightarrow \alpha > \alpha' \text{ or } \alpha = \alpha' \text{ and } \beta > \beta' \text{ or } (\alpha, \beta) = (\alpha', \beta') \text{ and } \gamma < \gamma'$$

Since C is well ordered, there is an element in Z with a minimal complexity. Let $X = (c_0, c_1, \dots, c_{n-1})$ be such an element.

For each integer $k \in \{0, 1, \dots, n-1\}$, we have a decomposition:

$$c_k = \sum_{\beta \le i} c_{ki} t^i$$

with $c_{ki} \in A$.

The condition $X \in R_n$ implies the following:

$$\sum_{0 \le k \le \gamma} z_{-k} c_{k\beta} = 0$$

and that implies the congruence $z_{-\gamma} c_{\gamma\beta} \equiv 0$ in $B_\gamma = A/I_\gamma$. But $z_{-\gamma}$ is not a zero divisor in B_γ and $c_{\gamma\beta}$ belongs to I_γ. So we have a decomposition in A:

$$c_{\gamma\beta} = \sum_{0 \le j < \gamma} z_{-j} a_j$$

and we get a new element in R_n:

$$X' = X - \sum_{0 \le j < \gamma} X(j, \gamma) a_j t^\beta$$

It is easy to see that $\pi_n(X') \equiv \pi_n(X) \equiv \pi_n(C_0) \bmod J_n$ and that X' belongs to Z. Moreover we have the following: $\chi(X') < \chi(X)$. But that's impossible because X was chosen with a minimal complexity.

Hence we get a contradiction and the module J'_n is contained in J_n. As a consequence, by killing all the z_i's, we get epimorphisms:

$$\mathbf{Z}[H]/J'_n \longrightarrow \mathbf{Z}[H]/J_n \longrightarrow \mathbf{Z}[x^{\pm 1}, t^{\pm 1}]$$

and $\mathbf{Z}[H]/J'_n$ is not trivial. Hence the sequence $E_0 \subset E_1 \subset E_2 \subset \dots$ is strictly increasing and $E = \mathrm{Ker}(f)$ is not finitely generated. Therefore the category of finitely presented right $\mathbf{Z}[H]$-modules is not abelian and $\mathbf{Z}[H]$ is not coherent. The result follows. □

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